AIAA Journal
Vol. 41, No. 5, May 2003

Symmetric State-Space Method for a Class of Nonviscously Damped Systems

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Multiple-degree-of-freedom linear nonviscously damped systems are considered. It is assumed that the nonviscous damping forces depend on the past history of velocities via convolution integrals over exponentially decaying kernel functions. The traditional state-space approach, well known for viscously damped systems, is extended to such nonviscously damped systems using a set of internal variables. Suitable numerical examples are provided to illustrate the proposed approach.

I. Introduction

VISCOUS damping is the most common model for the modeling of vibration damping in linear systems. This model, first introduced by Rayleigh,[1] assumes that the instantaneous generalized velocities are the only relevant variables that determine damping. Viscous damping models are used widely for their simplicity and mathematical convenience, even though the behavior of real structural materials is, at best, poorly mimicked by simple viscous models. For this reason, it is well recognized that, in general, a physically realistic model of damping will not be viscous. Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities are nonviscous damping models. Mathematically, any causal model that makes the energy dissipation functional nonnegative is a possible candidate for a nonviscous damping model. Clearly, a wide range of choice is possible, either based on the physics of the problem or by selecting a model a priori and fitting its parameters from experiments. Here, we will use a particular type of damping model that is not viscous, and throughout the paper the terminology nonviscous damping will refer to this specific model only.

Possibly the most general way to model damping within the linear range is to use nonviscous damping models that depend on the past history of motion via convolution integrals over kernel functions.[2] The equations of motion of an N-degree-of-freedom linear system with such damping can be expressed by

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\boldsymbol{u}}(\tau)\,\mathrm{d}\tau + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{f}(t) \qquad (1)$$

together with the initial conditions

$$\boldsymbol{u}(t=0) = \boldsymbol{u}_0 \in \mathbb{R}^N, \qquad \dot{\boldsymbol{u}}(t=0) = \dot{\boldsymbol{u}}_0 \in \mathbb{R}^N \qquad (2)$$

Here, $\boldsymbol{u}(t) \in \mathbb{R}^N$ is the displacement vector, $\boldsymbol{f}(t) \in \mathbb{R}^N$ is the forcing vector, $\boldsymbol{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ is the stiffness matrix, and $\boldsymbol{\mathcal{G}}(t-\tau)$ is the matrix of damping kernel functions. The kernel functions $\boldsymbol{\mathcal{G}}(t-\tau)$ are known as retardation functions, heredity functions, after-effect functions, or relaxation functions in the context of different subjects. Note that the convolution integral approach for the material property modeling is not new. Early work in this area can be traced back to Biot[3] in the context of viscoelastic materials. A main difference between Eq. (1) and corresponding equations for viscoelastic structures is that the kernel function is associated with damping and not stiffness. Although mathematically both are similar, conceptually they are somewhat different. For example, in the limit when $\boldsymbol{\mathcal{G}}(t-\tau) = \boldsymbol{C}\delta(t-\tau)$, where $\delta(t)$ is the Dirac delta function, Eq. (1) reduces to the case of viscous damping, whereas for the viscoelastic case the equivalent would be the usual elastic constant (stiffness matrix).

Equation (1) is very general, and, for any engineering application, some specific form of $\boldsymbol{\mathcal{G}}(t)$ has to be assumed. A wide variety of mathematical expressions could be used for the kernel functions $\boldsymbol{\mathcal{G}}(t)$ as long as the rate of energy dissipation is nonnegative. Some of the damping functions used in the literature are shown in Table 1 (Refs. 4–10).

A well-known damping model, known as the fractional derivative model,[4,11–13] appears as a specific case when the Laplace transform of $\boldsymbol{\mathcal{G}}(t)$ is selected as

$$s\boldsymbol{G}(s) = \sum_j s^{\nu_j}\boldsymbol{g}_j$$

where $\boldsymbol{g}_j$ are complex constant matrices and ν_j are fractional powers. Here, we will use a damping model for which the kernel function matrix has the special form

$$\boldsymbol{\mathcal{G}}(t) = \sum_{k=1}^{n} \mu_k e^{-\mu_k t}\boldsymbol{C}_k \qquad (3)$$

or, in the Laplace domain,

$$\boldsymbol{G}(s) = \sum_{k=1}^{n} \frac{\mu_k}{s+\mu_k}\boldsymbol{C}_k \qquad (4)$$

The constants $\mu_k \in \mathbb{R}^+$ are known as the relaxation parameters, and n denotes the number of relaxation parameters used to describe the damping behavior. This model will also be referred as the exponential damping model (models 1, 3, and 4 in Table 1) for obvious reasons. A physical justification (using the principles of mechanics and thermodynamics) as to why a general structure should always have this type of damping is hard to provide. From this point of view, this damping model is on a similar footing to that of the viscous model. However, based on engineering judgement and intuition, several reasons behind the selection of this model could be given:

1) In the context of viscoelastic materials, the physical basis for exponential models has been well established. In the words of Cremer and Heckl,[14] "Of the many after-effect functions that are possible in principle, only one—the so-called relaxation function—is physically meaningful."

2) In a large engineering structure it is possible to have different damping in different parts of a structure. For example, various

Received 26 September 2001; revision received 21 October 2002; accepted for publication 22 October 2002.

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Table 1 Some nonviscous damping functions in the Laplace domain

Model number	Damping function	Author(s) (year of publication)
1	$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$	Biot[8] (1955)
2	$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta} \quad (0 < \alpha, \beta < 1)$	Bagley and Torvik[4] (1983)
3	$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \frac{s^2 + 2\xi_k \omega_k s}{s^2 + 2\xi_k \omega_k s + \omega_k^2} \right]$	Golla and Hughes[5] (1985) and McTavish and Hughes[6] (1993)
4	$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori[7] (1990)
5	$G(s) = c \frac{1 - e^{-st_0}}{st_0}$	Adhikari[9] (1998)
6	$G(s) = \frac{c}{st_0} \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari[9] (1998)
7	$G(s) = c e^{s^2/4\mu} \left[1 - \mathrm{erf}\left(\frac{s}{2\sqrt{\mu}} \right) \right]$	Adhikari and Woodhouse[10] (2001)

members of a space frame may have different damping properties, each characterized by a relaxation parameter μ_k. Then the associated coefficient matrix $\boldsymbol{C}_k$ would have nonzero blocks corresponding to the relevant elements only. One could perform experiments for individual members and use the finite element method to obtain the element damping matrix, for example, $\boldsymbol{C}_k^{(e)}$. Using a standard approach it is possible to assemble all of the element matrices associated with relaxation parameter μ_k to obtain a global damping matrix $\boldsymbol{C}_k$. This procedure may be repeated for all damping types present in the structure to obtain μ_k and $\boldsymbol{C}_k$ for all k.

3) In a recent work, Adhikari and Woodhouse[15] proposed a method to identify μ_k and $\boldsymbol{C}_k$ from vibration measurements when $n = 1$ in Eq. (3). It was also noted[10] that, when the damping is nonviscous, forceful fitting of viscous damping may produce a nonphysical result (for example, a nonsymmetric coefficient matrix). Thus, from a parameter estimation point of view, the damping model in Eq. (3) gives additional flexibility to fit measured data obtained from modal testing.

4) A mathematical rationalization of this model can be given in terms of the Laplace transform of $\mathcal{G}(t)$. The matrix $\boldsymbol{G}(s)$ in Eq. (4) is, in general, a matrix of complex functions. From the theory of complex variables, it is well known that a wide range of complex functions can be represented in the "pole-residue" form. From Eq. (4), it is easy to see that λ_k and $\boldsymbol{C}_k$ are directly related to the poles and residues of $\boldsymbol{G}(s)$. Therefore, many damping models (except the fractional derivative model, which has branch points due to the fractional powers) can essentially be represented in the form of Eq. (3).

Because most vibration analysis textbooks, finite element packages, and modal analysis software only allow viscous damping, it is useful to relate the exponential damping model with the viscous damping model. From Eq. (3), observe that in the limit when $\mu_k \to \infty$, $\forall k$, the exponential model reduces to a viscous damping model with an equivalent viscous damping matrix

$$\boldsymbol{C} = \sum_{k=1}^{n} \boldsymbol{C}_k \tag{5}$$

The aim of this paper is to develop a state-space based approach analogous to viscously damped systems, with a view toward treating the exponentially damped system as a simple extension of the familiar viscously damped system.

II. State-Space Formalism

The state-space methods, or first-order methods, have been used extensively in the literature for viscously damped systems with nonproportional damping (see Newland,[16,17] for example). The purpose of this section is to extend the state-space approach to linear systems with an exponential damping model. The proposed method is based on the introduction of a set of internal variables. Bagley and Torvik[4] have used an extended state-space approach for linear systems with fractional derivative damping models. They have expressed the extended state vector in terms of various fractional-order differentials of the displacement vector. Golla and Hughes[5] and McTavish and Hughes[6] have used an internal variables based approach, the Golla–Hughes–McTavish (GHM) method, in the context of viscoelastic structures. Another approach, known as the anelastic displacement field (ADF) method, has been developed by Lesieutre and Mingori[7] and Lesieutre and Bianchini.[18] Like GHM, the ADF method is also an internal variable based viscoelastic model, but it is distinguished from GHM in that it is first order in time, not second order. Muravyov and Hutton[19] and Muravyov[20] have proposed an extended state-space method for systems with exponential kernels associated with a stiffness operator. Here, the formulation is presented for two cases, namely, 1) when all $\boldsymbol{C}_k$ matrices are of full rank and 2) when $\boldsymbol{C}_k$ matrices have rank deficiency.

A. Case A: All $\boldsymbol{C}_k$ Matrices are of Full Rank

We assume that

$$\mathrm{rank}(\boldsymbol{C}_k) = N, \qquad \forall k = 1, \ldots, n \tag{6}$$

The exponential function is an eigenfunction in the sense that the application of the operator $\mathrm{d}/\mathrm{d}t$ to the exponential term $e^{\mu t}$ produces $\mu e^{\mu t}$. Therefore, we introduce the internal variables $\boldsymbol{y}_k(t) \in \mathbb{R}^N$, $\forall k = 1, \ldots, n$, through following relationship:

$$\boldsymbol{y}_k(t) = \int_0^t \mu_k e^{-\mu_k (t-\tau)} \dot{\boldsymbol{u}}(\tau)\, \mathrm{d}\tau \tag{7}$$

Applying Leibniz's rule for differentiation of an integral to Eq. (7), one obtains

$$\dot{\boldsymbol{y}}_k(t) = \int_0^t -\mu_k^2 e^{-\mu_k (t-\tau)} \dot{\boldsymbol{u}}(\tau)\, \mathrm{d}\tau + \mu_k \dot{\boldsymbol{u}}(t) \tag{8}$$

Multiplying Eq. (7) by the relaxation parameter μ_k, then adding it to Eq. (8), yields the so-called evolution equation[21]

$$\dot{\boldsymbol{y}}_k(t) + \mu_k \boldsymbol{y}_k(t) = \mu_k \dot{\boldsymbol{u}}(t) \tag{9}$$

When the kernel function matrix (3) is accounted for, Eq. (1) can be rewritten as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \sum_{k=1}^{n} \mathbf{C}_k \left\{ \int_0^t \mu_k e^{-\mu_k (t-\tau)} \dot{\boldsymbol{u}}(\tau)\, \mathrm{d}\tau \right\} + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{f}(t) \tag{10}$$

With Eq. (7), the preceding equation leads to

$$M\ddot{u}(t) + \sum_{k=1}^{n} C_k y_k(t) + Ku(t) = f(t) \tag{11}$$

Substituting y_k from Eq. (9) into the preceding equation, one obtains

$$M\ddot{u}(t) + \sum_{k=1}^{n}\left[C_k\dot{u}(t) - \frac{1}{\mu_k}C_k\dot{y}_k(t)\right] + Ku(t) = f(t) \tag{12}$$

or

$$M\ddot{u}(t) + \left(\sum_{k=1}^{n} C_k\right)\dot{u}(t) - \sum_{k=1}^{n}\frac{1}{\mu_k}C_k\dot{y}_k(t) = -Ku(t) + f(t) \tag{13}$$

Premultiplying Eq. (9) by C_k, dividing by μ_k^2, and rearranging, we get

$$-(1/\mu_k)C_k\dot{u}(t) + \left(1/\mu_k^2\right)C_k\dot{y}_k(t) = -(1/\mu_k)C_k y_k(t), \qquad \forall k \tag{14}$$

Now, by the use of additional state variables

$$v(t) = \dot{u}(t) \tag{15}$$

Eqs. (13), (15), and (14) can be represented in the first-order form as

$$B\dot{z}(t) = Az(t) + r(t) \tag{16}$$

where

$$B = \begin{bmatrix} \sum_{k=1}^{n} C_k & M & -\frac{C_1}{\mu_1} & \cdots & -\frac{C_n}{\mu_n} \\ M & O & O & O & O \\ -\frac{C_1}{\mu_1} & O & \frac{C_1}{\mu_1^2} & O & O \\ \vdots & O & O & \ddots & O \\ -\frac{C_n}{\mu_n} & O & O & O & \frac{C_n}{\mu_n^2} \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{17}$$

$$A = \begin{bmatrix} -K & O & O & O & O \\ O & M & O & O & O \\ O & O & -\frac{C_1}{\mu_1} & O & O \\ O & O & O & \ddots & O \\ O & O & O & O & -\frac{C_n}{\mu_n} \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{18}$$

$$r(t) = \begin{Bmatrix} f(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \in \mathbb{R}^{m} \tag{19}$$

$$z(t) = \begin{Bmatrix} u(t) \\ v(t) \\ y_1(t) \\ \vdots \\ y_n(t) \end{Bmatrix} \in \mathbb{R}^{m} \tag{20}$$

In the preceding equations, $z(t)$ is the extended state vector, A and B are the system matrices in the extended state-space, $r(t)$ is the force vector in the extended state-space, and O is an $N \times N$ null matrix. Clearly, the order of the system m is given by

$$m = 2N + nN \tag{21}$$

Because it is assumed that M, K, and C_k, $\forall k$, are symmetric matrices, B is a symmetric matrix, and A is a block-diagonal and, therefore, also a symmetric matrix.

It is useful to consider the viscous damping limit at this stage. When $\mu_k \to \infty$, $\forall k$, dividing Eq. (9) by μ_k, it is easy to show that

$$y_k(t) = \dot{u}(t), \qquad \forall k \tag{22}$$

This implies that in the viscous damping limit all internal variables reduce to the velocity vector. For this reason, the $n \times N$ equations after the first $2N$ rows in Eq. (16) become trivial and can be deleted from the formulation. Under these conditions, it is easy to see that the equations of motion (16) reduce to the standard Duncan form (see Ref. 22) for viscously damped systems with

$$B = \begin{bmatrix} C & M \\ M & O \end{bmatrix}, \qquad A = \begin{bmatrix} -K & O \\ O & M \end{bmatrix}$$
$$r(t) = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}, \qquad z(t) = \begin{Bmatrix} u(t) \\ \dot{u}(t) \end{Bmatrix} \tag{23}$$

where C is given by Eq. (5). This shows that the representation of the equations of motion by Eq. (16) is a natural generalization of the standard state-space formulation for viscously damped systems.

B. Case B: C_k Matrices are Rank Deficient

In this section, we assume that, in general,

$$r_k = \mathrm{rank}(C_k) \le N, \qquad \forall\, k = 1, \ldots, n \tag{24}$$

This implies that the number of nonzero eigenvalues of C_k is r_k. It is useful to decompose the C_k matrices to full-rank matrices of smaller dimensions, similar to those used by Golla and Hughes[5] and McTavish and Hughes[6] in the context of viscoelastic structures.

Because C_k is a symmetric matrix, there exists an orthogonal matrix $U_k \in \mathbb{R}^{N\times N}$ whose columns are the eigenvectors of C_k, such that

$$U_k^T C_k U_k = \begin{bmatrix} d_k & O_{1k} \\ O_{1k}^T & O_{2k} \end{bmatrix} \tag{25}$$

In the preceding equation, $d_k \in \mathbb{R}^{r_k\times r_k}$ is a diagonal matrix consisting of only the nonzero eigenvalues of C_k. $O_{1k} \in \mathbb{R}^{r_k\times(N-r_k)}$, and $O_{2k} \in \mathbb{R}^{(N-r_k)\times(N-r_k)}$ are the null matrices. For convenience, partition U_k as

$$U_k = [U_{1k} \mid U_{2k}] \tag{26}$$

where the columns of $U_{1k} \in \mathbb{R}^{N\times r_k}$ are the eigenvectors corresponding to the nonzero block d_k, and the columns of $U_{2k} \in \mathbb{R}^{N\times(N-r_k)}$ are the eigenvectors corresponding to the remaining $(N - r_k)$ number of zero eigenvalues. When a rectangular transformation matrix is defined as

$$R_k = U_{1k} \in \mathbb{R}^{N\times r_k} \tag{27}$$

it is easy to show that

$$R_k^T C_k R_k = d_k \tag{28}$$

Therefore, the matrix R_k in Eq. (27) transforms the originally rank-deficient matrix C_k to a full-rank (diagonal) matrix of rank r_k. Note that the choice of R_k can be arbitrary and, in general, R_k may be expressed as

$$R_k = U_k \times \begin{bmatrix} Q_{r_k} \\ O_{3k} \end{bmatrix} \tag{29}$$

where $Q_{r_k} \in \mathbb{R}^{r_k\times r_k}$ is any orthogonal matrix and $O_{3k} \in \mathbb{R}^{(N-r_k)\times r_k}$ is a null matrix. The matrix R_k appearing in Eq. (27) can be obtained as a special case when Q_{r_k} is an identity (therefore, orthogonal) matrix.

Now, define a set of internal variables of reduced dimension $\tilde{\boldsymbol{y}}_k(t) \in \mathbb{R}^{r_k}$ using the rectangular transformation matrix $\boldsymbol{R}_k$ as

$$\boldsymbol{y}_k(t) = \boldsymbol{R}_k \tilde{\boldsymbol{y}}_k(t) \tag{30}$$

From this equation, it immediately follows that

$$\dot{\boldsymbol{y}}_k(t) = \boldsymbol{R}_k \dot{\tilde{\boldsymbol{y}}}_k(t) \tag{31}$$

where $\boldsymbol{y}_k(t)$ is defined in Eq. (7). With these relationships, Eqs. (13) and (14) can be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \left(\sum_{k=1}^{n} \boldsymbol{C}_k\right)\dot{\boldsymbol{u}}(t) - \sum_{k=1}^{n} \frac{1}{\mu_k}\boldsymbol{C}_k\boldsymbol{R}_k\dot{\tilde{\boldsymbol{y}}}_k(t) = -\boldsymbol{K}\boldsymbol{u}(t) + \boldsymbol{f}(t) \tag{32}$$

$$-\frac{1}{\mu_k}\boldsymbol{C}_k\dot{\boldsymbol{u}}(t) + \frac{1}{\mu_k^2}\boldsymbol{C}_k\boldsymbol{R}_k\dot{\tilde{\boldsymbol{y}}}_k(t) = -\frac{1}{\mu_k}\boldsymbol{C}_k\boldsymbol{R}_k\tilde{\boldsymbol{y}}_k(t) \tag{33}$$

Because Eq. (33) still represents a set of N equations, we premultiply this by $\boldsymbol{R}_k^T$ to obtain a reduced set of r_k equations:

$$\begin{aligned} &-(1/\mu_k)\boldsymbol{R}_k^T\boldsymbol{C}_k\dot{\boldsymbol{u}}(t) + \left(1/\mu_k^2\right)\boldsymbol{R}_k^T\boldsymbol{C}_k\boldsymbol{R}_k\dot{\tilde{\boldsymbol{y}}}_k(t) \\ &\quad = -(1/\mu_k)\boldsymbol{R}_k^T\boldsymbol{C}_k\boldsymbol{R}_k\tilde{\boldsymbol{y}}_k(t), \qquad \forall k \end{aligned} \tag{34}$$

Now, Eqs. (32), (15), and (34) can be combined into a first-order form as

$$\tilde{\boldsymbol{B}}\dot{\tilde{\boldsymbol{z}}}(t) = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{z}}(t) + \tilde{\boldsymbol{r}}(t) \tag{35}$$

where

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \sum_{k=1}^{n}\boldsymbol{C}_k & \boldsymbol{M} & -\frac{\boldsymbol{C}_1\boldsymbol{R}_1}{\mu_1} & \cdots & -\frac{\boldsymbol{C}_n\boldsymbol{R}_n}{\mu_n} \\ \boldsymbol{M} & \boldsymbol{O}_{N,N} & \boldsymbol{O}_{N,r_1} & \cdots & \boldsymbol{O}_{N,r_n} \\ -\frac{\boldsymbol{R}_1^T\boldsymbol{C}_1}{\mu_1} & \boldsymbol{O}_{N,r_1}^T & \frac{\boldsymbol{R}_1^T\boldsymbol{C}_1\boldsymbol{R}_1}{\mu_1^2} & \cdots & \boldsymbol{O}_{r_1,r_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\boldsymbol{R}_1^T\boldsymbol{C}_n}{\mu_n} & \boldsymbol{O}_{N,r_n}^T & \boldsymbol{O}_{r_1,r_n}^T & \cdots & \frac{\boldsymbol{R}_n^T\boldsymbol{C}_n\boldsymbol{R}_n}{\mu_n^2} \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{36}$$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -\boldsymbol{K} & \boldsymbol{O}_{N,N} & \boldsymbol{O}_{N,r_1} & \cdots & \boldsymbol{O}_{N,r_n} \\ \boldsymbol{O}_{N,N} & \boldsymbol{M} & \boldsymbol{O}_{N,r_1} & \cdots & \boldsymbol{O}_{N,r_n} \\ \boldsymbol{O}_{N,r_1}^T & \boldsymbol{O}_{N,r_1}^T & -\frac{\boldsymbol{R}_1^T\boldsymbol{C}_1\boldsymbol{R}_1}{\mu_1} & \cdots & \boldsymbol{O}_{r_1,r_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O}_{N,r_n}^T & \boldsymbol{O}_{N,r_n}^T & \boldsymbol{O}_{r_1,r_n}^T & \cdots & -\frac{\boldsymbol{R}_n^T\boldsymbol{C}_n\boldsymbol{R}_n}{\mu_n} \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{37}$$

$$\tilde{\boldsymbol{r}}(t) = \begin{Bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{0}_N \\ \boldsymbol{0}_{r_1} \\ \vdots \\ \boldsymbol{0}_{r_n} \end{Bmatrix} \in \mathbb{R}^m \tag{38}$$

$$\tilde{\boldsymbol{z}}(t) = \begin{Bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{v}(t) \\ \tilde{\boldsymbol{y}}_1(t) \\ \vdots \\ \tilde{\boldsymbol{y}}_n(t) \end{Bmatrix} \in \mathbb{R}^m \tag{39}$$

In the preceding equations

$$m = 2N + \sum_{k=1}^{n} r_k \tag{40}$$

is the order of the system, $\boldsymbol{O}_{ij}$ are $i \times j$ null matrices, and $\boldsymbol{0}_j$ are vectors of j zeros. The terms with tildes correspond to the terms in parentheses defined in Eq. (16). Again, note that the system matrices $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ are symmetric. When all $\boldsymbol{C}_k$ matrices are of full rank, that is, when $r_k = N$, $\forall k$, then one can choose each $\boldsymbol{R}_k$ matrix as the identity matrix, and Eq. (35) reduces to Eq. (16). With this symmetric state-space representation, the system response can be obtained easily by the mode superposition method, which is very similar to what is normally used for undamped or viscously damped systems.

III. Numerical Examples

A. Example 1: Single-Degree-of-Freedom (DOF) System

Consider a simple single-degree-of-freedom (DOF) system with nonviscous damping. The equation of motion is given by

$$m\ddot{u}(t) + f_d(t) + ku(t) = f(t) \tag{41}$$

where $f_d(t)$, the damping force due to the nonviscous damper, is assumed to be of the form

$$f_d(t) = \int_0^t \left\{c_1\mu_1 e^{-\mu_1(t-\tau)} + c_2\mu_2 e^{-\mu_2(t-\tau)}\right\}\dot{u}(\tau)\,\mathrm{d}\tau \tag{42}$$

This problem belongs to case A in Sec. II.A, that is, there is no rank deficiency. By the use of Eqs. (16–20), the symmetric state-space form is given by

$$\begin{aligned} &\begin{bmatrix} c_1+c_2 & m & -c_1/\mu_1 & -c_2/\mu_2 \\ m & 0 & 0 & 0 \\ -c_1/\mu_1 & 0 & c_1/\mu_1^2 & 0 \\ -c_2/\mu_2 & 0 & 0 & c_1/\mu_1^2 \end{bmatrix} \begin{Bmatrix} \dot{u}(t) \\ \ddot{u}(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{Bmatrix} \\ &= \begin{bmatrix} -k & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -c_1/\mu_1 & 0 \\ 0 & 0 & 0 & -c_1/\mu_1 \end{bmatrix} \begin{Bmatrix} u(t) \\ \dot{u}(t) \\ y_1(t) \\ y_2(t) \end{Bmatrix} + \begin{Bmatrix} f(t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} \end{aligned} \tag{43}$$

B. Example 2: Three-DOF System

A three-DOF system, shown in Fig. 1, with nonviscous damping, is considered. Three masses, each of mass m_u, are connected by springs of stiffness k_u. The nonviscous damping model of the system is composed of two exponential damping models, as shown in Fig. 1. The equations of motion of this model system can be represented by Eq. (1). The mass and the stiffness matrices of the system are given by

$$\boldsymbol{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix} \tag{44}$$

$$\boldsymbol{K} = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{45}$$

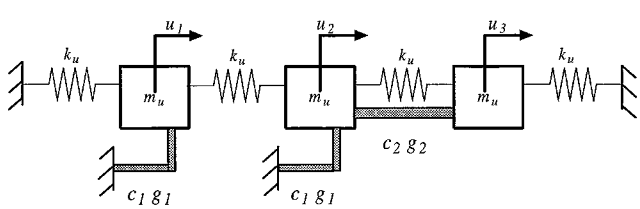

Fig. 1 Three-DOF system with nonviscous damping; shaded bars represent the nonviscous dampers with damping functions given by $g_i(t-\tau) = \mu_i e^{-\mu_i(t-\tau)}$, $i = 1, 2$.

The matrix of the damping function can be expressed by Eq. (3) with $n = 2$, and the coefficient matrices are given by

$$\boldsymbol{C}_1 = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix} \tag{46}$$

Both of the matrices have rank deficiencies because one can easily verify that

$$r_1 = \operatorname{rank}(\boldsymbol{C}_1) = 2 \leq 3 \tag{47}$$

$$r_2 = \operatorname{rank}(\boldsymbol{C}_2) = 1 \leq 3 \tag{48}$$

The order of the system matrices in the state-space m, expressed by Eq. (40), can be obtained using r_1 and r_2 given by Eqs. (47) and (48). From Eq. (40), it can be easily shown that $m = 2 \times 3 + (2 + 1) = 9$. The transformation matrices $\boldsymbol{R}_k$, $k = 1, 2$, appearing in Eqs. (36) and (37), can be obtained from Eq. (27):

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{R}_2 = \begin{bmatrix} 0 \\ -1\big/\sqrt{2} \\ 1\big/\sqrt{2} \end{bmatrix} \tag{49}$$

With these, the system matrices in the state-space, given by Eqs. (36) and (37), are obtained as

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} c_1 & 0 & 0 & m_u & 0 & 0 & 0 & -c_1/\mu_1 & 0 \\ 0 & c_1 + c_2 & -c2 & 0 & m_u & 0 & -c_1/\mu_1 & 0 & c_2\sqrt{2}\big/\mu_2 \\ 0 & -c_2 & c2 & 0 & 0 & m_u & 0 & 0 & -c_2\sqrt{2}\big/\mu_2 \\ m_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_1/\mu_1 & 0 & 0 & 0 & 0 & c_1\big/\mu_1^2 & 0 & 0 \\ -c_1/\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & c_1\big/\mu_1^2 & 0 \\ 0 & c_2\sqrt{2}\big/\mu_2 & -c_2\sqrt{2}\big/\mu_2 & 0 & 0 & 0 & 0 & 0 & 2c_2/\mu_2^2 \end{bmatrix} \tag{50}$$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -2k_u & k_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_u & -2k_u & k_u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_u & -2k_u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_1/\mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_1/\mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c_2/\mu_2 \end{bmatrix} \tag{51}$$

Once these system matrices are obtained, the dynamic response of the system can be readily obtained using a mode superposition method.

IV. Conclusions

Multiple-DOF linear systems with exponentially decaying damping memory kernels are considered. The proposed method is based on an extended state-space representation of the equations of motion. This approach, in contrast to the usual state vector for viscously damped systems, utilizes a set of internal variables in addition to the displacement and the velocity as the state vector. Two cases, namely, 1) when all of the damping coefficient matrices are of full rank and 2) when the damping coefficient matrices have rank deficiency, have been presented. For both of the cases, the equations of motion can be represented in terms of two symmetric matrices. The formulation proposed here is very similar to the standard Duncan form used in the context of viscously damped systems. The only difference is that the dimension of the system matrices is now more than $2N$ and depends on the rank of the damping coefficient matrices. Because of this similarity, it may be possible to extend many system identification, model updating, optimization, and control algorithms available for viscously damped systems to such nonviscously damped systems.

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