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SYMMETRIC STATISTICS, POISSON POINT PROCESSES, AND MULTIPLE WIENER INTEGRALS

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The asymptotic behaviour of symmetric statistics of arbitrary order is studied. As an application we describe all limit distributions of square integrable U-statistics. We use as a tool a randomization of the sample size. A sample of Poisson size N_λ with $EN_\lambda = \lambda$ can be interpreted as a Poisson point process with intensity λ, and randomized symmetric statistics are its functionals. As $\lambda \to \infty$, the probability distribution of these functionals tend to the distribution of multiple Wiener integrals. This can be considered as a stronger form of the following well-known fact: properly normalized, a Poisson point process with intensity λ approaches a Gaussian random measure, as $\lambda \to \infty$.

1. Outline of results.

1.1 Let $X_1, X_2, \cdots$ be independent and identically distributed observations with values in an arbitrary measurable space $(\mathscr{X}, \mathscr{B})$. To every symmetric function $h(x_1, \cdots, x_k)$ there corresponds a statistic

$$\sigma_k^n(h) = \sum h(X_{s_1}, \cdots, X_{s_k}) \tag{1.1}$$

where the sum is taken over all $1 \le s_1 < \cdots < s_k \le n$; we put $\sigma_k^n = 0$ for $k > n$. Every integrable symmetric statistic has a unique representation of the form

$$S(X_1, \cdots, X_n) = \sum_{k=0}^\infty \sigma_k^n(h_k) \tag{1.2}$$

where $h_k(x_1, \cdots, x_k)$ are symmetric functions subject to the condition

$$Eh(x_1, \cdots, x_{k-1}, X_k) = \int_{\mathscr{X}} h(x_1, \cdots, x_{k-1}, y)\nu(dy) = 0 \tag{1.3}$$

where ν is the probability distribution of X_k. We call such h_k canonical. (Formula (1.2) is proved in [4] (cf. [1], [8]). For the convenience of the reader we give a short proof in subsection 0.1 of the Appendix.) If only a finite number of the h_k do not vanish, then S is a statistic of finite order. There exists an extensive literature on the limit behaviour of such statistics. In this paper, symmetric statistics of infinite order are investigated. This leads not only to a generalization of the theory but also to its simplification: a class of statistics which are the easiest to investigate have infinite order; they correspond to

$$h_0^\phi = 1, \quad h_k^\phi(x_1, \cdots, x_k) = \phi(x_1) \cdots \phi(x_k), \quad k = 1, 2, \cdots \tag{1.4}$$

with a fixed function ϕ; the general case can be reduced to this particular one.

1.2 The systematic investigation of symmetric statistics of finite order has been initiated by Von Mises [10] and Hoeffding [3], and further references can be found in Serfling [9]. The most complete results are due to Rubin and Vitale [8]. They have proved

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that, for every canonical square integrable $h_1, \cdots, h_m$,

$$\sum_{k=0}^{m} n^{-k/2}\sigma_k^n(h_k)$$

converges in distribution as $n \to \infty$, and they have described the limit distribution in terms of infinite series of products of Hermite polynomials of normal random variables. (This is a special case of the situation studied by Rubin and Vitale who considered, in fact, a triangular array $h_{k,n}, X_{k,n}, n = 1, 2, \cdots, k = 1, \cdots, n$.)

1.3 We shall use the abbreviation $\nu(\phi)$ for the integral of a function $\phi(x)$ with respect to the measure ν. Hence $E\phi(X_k) = \nu(\phi)$.

Let H stand for the set of all sequences $h = (h_0, h_1(x_1), \cdots, h_k(x_1, \cdots, x_k), \cdots)$ where h_k are canonical and

$$\| h \|^2 = \sum_{k=0}^{\infty} \frac{1}{k!} Eh_k^2(X_1, \cdots, X_k) < \infty.$$

Let

$$Y_n(h) = \sum_{k=0}^{\infty} n^{-k/2}\sigma_k^n(h_k). \tag{1.5}$$

To describe the limiting distribution of $Y_n(h)$ we use a Gaussian family $I_1(\phi)$, $\nu(\phi^2) < \infty$ with moments

$$EI_1(\phi) = 0, \quad EI_1(\phi)I_1(\psi) = \nu(\phi\psi).$$

(The Gaussian family $I_1(\phi)$ is defined on a different probability space than the one on which the observations $X_1, \cdots, X_k, \cdots$ are defined. For typographical convenience we use the same letter E for expectations in both spaces.)

The random variable $I_1(\phi)$ is called the Wiener integral of the function ϕ. We also use multiple Wiener integrals. The multiple Wiener integral of order k is a linear mapping $I_k(h_k)$ from the space of symmetric functions h_k, subject to the condition $Eh_k^2(X_1, \cdots, X_k) < \infty$, to the space of random variables which are functionals of the Gaussian family $I_1(\phi)$. It is defined uniquely by the following two conditions:

1.3.A For functions of the form (1.4) with $\nu(\phi^2) = 1$,

$$I_k(h_k^{\phi}) = H_k(I_1(\phi))$$

where H_k is the Hermite polynomial of degree k with leading coefficient 1.

1.3.B $$EI_k(h_k)^2 = k!Eh_k^2.$$

We give a construction of $I_k(h_k)$ in subsection 2.2. It is rather close to the construction in [7]. A different approach is used in [5].

Our main result is

THEOREM 1. *For $h \in H$, the sequence of random variables $Y_n(h)$ converges in distribution, as $n \to \infty$, to*

$$W(h) = \sum_{k=0}^{\infty} \frac{1}{k!} I_k(h_k). \tag{1.6}$$

To get Theorem 1, we consider first a sample of the size N_λ where N_λ has a Poisson distribution with mean λ and is independent of the $X_1, X_2, \cdots$.

THEOREM 2. *For $h \in H$, the random variables*

$$Z_\lambda(h) = \sum_{k=0}^{\infty} \lambda^{-k/2}\phi_k^{N_\lambda}(h_k) \tag{1.7}$$

converge in distribution, as $\lambda \to \infty$, *to* $W(h)$ *described by* (1.6). *For every* $h, g \in H$.

$$EZ_\lambda(h)Z_\lambda(g) = EW(h)W(g). \tag{1.8}$$

We prove Theorem 2 in subsections 2.2 and 2.3. We get Theorem 1 by showing that $Y_n(h) - Z_n(h) \to 0$ in the mean square as $n \to \infty$ (subsection 2.4).

A measurable function $\phi(x)$ is called elementary if it takes a finite number of values and if $\nu(\phi) = 0$. Let Φ stand for the set of all elementary functions. The proof of Theorem 2 is based on the following lemma (proved in subsection 2.1).

LEMMA 1. *For every* $\phi, \psi \in \Phi$,

$$EZ_\lambda(h^\phi)Z_\lambda(h^\psi) = E\varepsilon(\phi)\varepsilon(\psi) \tag{1.9}$$

where

$$\varepsilon(\phi) = e^{I_1(\phi)-(1/2)\nu(\phi^2)}.$$

For every $\phi_1, \cdots, \phi_m \in \Phi$, the joint probability distribution of $(Z_\lambda(h^{\phi_1}), \cdots, Z_\lambda(h^{\phi_m}))$ converges to that of $(\varepsilon(\phi_1), \cdots, \varepsilon(\phi_m))$, as $\lambda \to \infty$.

Since $\varepsilon(\phi) = W(h^\phi)$, (1.9) is a special case of (1.8) which, in fact, implies (1.8). Once this is established, Theorem 2 follows easily.

1.4 A special type of symmetric statistics are the U-statistics introduced by Hoeffding [3]. The U-statistic of order m that corresponds to a symmetric function $h(x_1, \cdots, x_m)$ is defined by

$$U_m^n(h) = \frac{1}{\binom{n}{m}} \sigma_m^n(h), \quad n \geq m \geq 1.$$

It follows from (1.2) that

$$U_m^n(h) = \textstyle\sum_{k=0}^m U_k^n(h_k)$$

where $h_1, \cdots, h_m$ are canonical. (This was first proved in [4]. We give a proof in subsection 0.2 of the Appendix.) Let $c \geq 1$ be the first integer for which $h_c \neq 0$. We have

1.4.A The statistic $U_m^n(h)$ has the asymptotic expansion

$$U_m^n(h) \sim h_0 + n^{-c/2}I_c(h_c) + \cdots + n^{-m/2}I_m(h_m).$$

1.4.B The normalized U-statistic

$$n^{c/2}(U_m^n(h) - h_0)$$

converges in distribution to $I_c(h_c)$, as $n \to \infty$.

The statements 1.4.A, B are closely related to von Mises' Taylor expansion of differentiable statistics. The case $c = 1$, in which the limiting distribution is normal, is the one considered in [3]. The case $c = 2$ is treated by a different method in [9], where examples and further references are given. The form of the limiting distribution in [9] follows from 1.4.B by using a special representation of multiple Wiener integrals of order 2. A canonical square integrable function $h_2(x_1, x_2)$ can be decomposed as

$$h_2(x_1, x_2) = \textstyle\sum_{k=1}^\infty \lambda_k\phi_k(x_1)\phi_k(x_2)$$

where $\nu(\phi_k) = 0$, $\nu(\phi_k^2) = 1$, $\nu(\phi_k\phi_\ell) = 0$ if $k \neq \ell$, $\sum_{k=1}^\infty \lambda_k^2 < \infty$ (see, for example, Proposition 6.18 in [7]). Since $H_2(z) = z^2 - 1$, we have $I_2(h_2) = \sum_{k=1}^\infty \lambda_k(Z_k^2 - 1)$, where $Z_k = I_1(\phi_k)$.

Hence:

1.4.C The random variables $(1/n)\sigma_2^n(h_2)$ converge in distribution, as $n \to \infty$, to

$$\sum_{k=1}^{\infty} \lambda_k(Z_k^2 - 1)$$

where $Z_1, Z_2, \cdots$ are independent standard normal, and $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$.

1.5 The Wiener integral $I_1(\phi)$ can be interpreted as a stochastic integral $\int_{\mathcal{X}} \phi(x) W(dx)$ with respect to the Gaussian random measure $W(B) = I_1(1_B)$, $B \in \mathscr{B}$ ($1_B(x)$ is the indicator function of the set B).

A sample $X_1, \cdots, X_{N_\lambda}$ can be viewed as a Poisson point process. Let $\sigma_\lambda(B)$, $B \in \mathscr{B}$ be the number of elements in the sample which belong to the set B. For large λ, the random measure $\sigma_\lambda(B)$ (normalized in a proper way) is close to the Gaussian random measure $W(B)$. Theorem 2 can be considered as a stronger form of this well-known fact: a wide class of functionals of W (given by (1.6)) is approximated, preserving the covariance structure, by functionals (1.7) of the Poisson point process.

All square integrable functionals of W can be represented in the form (1.6) if we drop the condition (1.3) from the definition of H. Theorem 1 and 2 remain valid if we replace the statistics (1.1) by

$$\sum \beta_{s_1} \cdots \beta_{s_k} h(X_{s_1}, \cdots, X_{s_k}) \tag{1.10}$$

where $\beta_1, \cdots, \beta_n, \cdots$ are random variables taking the values ± 1 with equal probabilities, mutually independent and independent of $X_1, \cdots, X_n, \cdots$ and N_λ (see the remark at the end of subsection 2.2).

1.6 Poisson randomization of the sample size has been used as early as 1949 by Kac [6] to investigate the limit behavior of the empirical distribution function. Multiple stochastic integrals with respect to the Brownian bridge appear in Filippova's paper [2] to describe the limiting distribution of Von Mises statistics. The functionals h^ϕ and $\varepsilon(\phi)$ arise in a natural way as exponential elements in the symmetric tensor algebras related to Poisson point processes and to Gaussian random measures W. The idea to use these elements as generators is due to Neveu [7].

1.7 The authors are indebted to R. Farrell for very helpful discussions.

2. Proofs.

2.1 The proof of Lemma 1 is based on the relation

$$E\varepsilon(\phi)\varepsilon(\psi) = e^{\nu(\phi\psi)}, \tag{2.1}$$

and the following representation

$$Z_\lambda(h^\phi) = \prod_{i=1}^{N_\lambda} \left(1 + \frac{\phi(X_i)}{\sqrt{\lambda}}\right). \tag{2.2}$$

If $\nu(\phi) = \nu(\psi) = 0$ and ϕ, ψ are square integrable, then

$$\begin{aligned} EZ_\lambda(h^\phi)Z_\lambda(h^\psi) &= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \prod_{i=1}^{n} E\left(1 + \frac{\phi(X_i)}{\sqrt{\lambda}}\right)\left(1 + \frac{\psi(X_i)}{\sqrt{\lambda}}\right) \\ &= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \left(1 + \frac{\nu(\phi\psi)}{\lambda}\right)^n = e^{\nu(\phi\psi)}. \end{aligned} \tag{2.3}$$

It follows from (2.1) and (2.3) that (1.9) holds for all elementary functions ϕ, i.e. the mapping $Z_\lambda(h^\phi) \to \varepsilon(\phi)$ is an isometry.

Now we show that for $\phi_1, \cdots, \phi_m \in \Phi$, the distribution of $(\log Z_\lambda(h^{\phi_1}), \cdots, \log Z_\lambda(h^{\phi_m}))$ converges to that of $(I_1(\phi_1) - \frac{1}{2}\nu(\phi_1^2), \cdots, I_1(\phi_m) - \frac{1}{2}\nu(\phi_m^2))$, as $\lambda \to \infty$. First we note that $N_\lambda/\lambda \to 1$ in probability, as $\lambda \to \infty$. From Taylor's expansion of $\log(1 + x)$ and the representation (2.2) we get

$$\log Z_\lambda(h^{\phi_k}) = A_k^\lambda - B_k^\lambda + \varepsilon_k^\lambda$$

where $A_k^\lambda = \lambda^{-1/2}\,\sigma_1^{N_\lambda}(\phi_k)$, $B_k = (2\lambda)^{-1}\sigma_1^{N_\lambda}(\phi_k^2)$ and $\varepsilon_k^\lambda \to 0$ in probability, as $\lambda \to \infty$. By the central limit theorem, the distribution of $(A_1^\lambda, \cdots, A_m^\lambda)$ converges to that of $(I_1(\phi_1), \cdots, I_1(\phi_m))$, as $\lambda \to \infty$. By the law of large numbers, B_k^λ converges in probability to $\nu(\phi_k^2)$, $k = 1, \cdots, m$. This completes the proof of Lemma 1.

2.2 Now we prove Theorem 2 (the multiple Wiener integral will be constructed in the process). The isometry $Z_\lambda(h^\phi) \to \varepsilon(\phi)$ can be extended to an isometry I from the minimal Hilbert space V_λ which contains $\{Z_\lambda(h^\phi), \phi \in \Phi\}$ onto the minimal Hilbert space G which contains $\{\varepsilon(\phi), \phi \in \Phi\}$. For every real number t we have

$$I(Z_\lambda^{t\phi}) = \varepsilon(t\phi).$$

The function $\varepsilon(t\phi)$ is infinitely differentiable in t in the mean square, hence so is $Z_\lambda^{t\phi}$ and

$$k!\lambda^{-k/2}\sigma_k^{N_\lambda}(h_k^\phi) = \frac{d^k}{dt^k} Z_\lambda^{t\phi}\bigg|_{t=0} \overset{I}{\to} \frac{d^k}{dt^k}\varepsilon(t\phi)\bigg|_{t=0}.$$

In particular, $\sigma_k^{N_\lambda}(h_k^\phi) \in V_\lambda$ for every $\phi \in \Phi$. We deduce from here that $\sigma_k^{N_\lambda}(h_k) \in V_\lambda$ for every square integrable canonical h_k. It suffices to show that if

$$E\sigma_k^{N_\lambda}(h_k)\sigma_k^{N_\lambda}(h_k^\phi) = 0 \tag{2.4}$$

for all $\phi \in \Phi$, then $h_k = 0$ a.e. To this end we note that for any canonical h_k and g_k, $Eh_k(X_{s_1}, \cdots, X_{s_k})g_k(X_{r_1}, \cdots, X_{r_k}) = 0$ if $\{s_1, \cdots, s_k\} \neq \{r_1, \cdots, r_k\}$, hence

$$E\sigma_k^{N_\lambda}(h_k)\sigma_k^{N_\lambda}(g_k) = \frac{\lambda^k}{k!} Eh_kg_k, \tag{2.5}$$

so (2.4) implies that $Eh_kh_k^\phi = 0$ for all $\phi \in \Phi$. Using polarization and taking into account the symmetry of h_k, we conclude that

$$Eh_k\phi_1(X_1) \cdots \phi_k(X_k) = 0 \quad \text{for all} \quad \phi_1, \cdots, \phi_k \in \Phi. \tag{2.6}$$

By (1.3), formula (2.6) holds for all simple $\phi_1, \cdots, \phi_k$. This implies that $h_k = 0$ a.e.

We observe that the random variables

$$I_k(h_k) = I(k!\lambda^{-k/2}\sigma_k^{N_\lambda}(h_k))$$

depend linearly on h_k and have the following properties:

2.2.A $\qquad EI_k(h_k) = 0, \quad k \geq 1.$

2.2.B $\qquad EI_k(h_k)I_k(g_k) = k!Eh_kg_k.$

2.2.C $\qquad EI_k(h_k)I_\ell(g_\ell) = 0$ if $k \neq \ell$.

This follows from analogous relations for $k!\lambda^{-k/2}\sigma_k^{N_\lambda}(h_k)$ and the fact that I is an isometry. Now we see that, for every $h \in H$, $Z_\lambda(h) \in V_\lambda$, $I(Z_\lambda(h)) = W(h)$ and $EZ_\lambda(h)^2 = EW(h)^2 = \|h\|^2$. It is easy to check that $V_\lambda = \{Z_\lambda(h), h \in H\}$, $G = \{W(h), h \in H\}$.

Finally we show that $I_k(h_k)$ satisfies the requirements of the axiomatic definition of the multiple Wiener integral given in subsection 1.3. Property 1.3.B is a particular case of 2.2.B. To check 1.3.A we use the relation

$$\varepsilon(t\phi) = e^{tI_1(\phi)-(1/2)t^2\nu(\phi^2)} = \sum_{k=0}^\infty \frac{t^k}{k!} I_k(h_k^\phi),$$

and the fact that $\varepsilon(t\phi)$ is the generating function of the Hermite polynomials, namely

$$\varepsilon(t\phi) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(I_1(\phi))$$

when $\nu(\phi^2) = 1$.

REMARK. To prove the statement at the end of 1.5, it is sufficient to note that: (i) if

$$Z_\lambda(h^\phi) = \prod_{i=1}^{N_\lambda} \left(1 + \beta_i \frac{\phi(X_i)}{\sqrt{\lambda}}\right),$$

then (2.3) holds even if the condition $\nu(\phi) = \nu(\psi) = 0$ is violated; (ii) $\varepsilon(\phi)$ with simple ϕ are everywhere dense in the space of square integrable functionals of W. In this way $I_k(h)$ can be defined without the assumption (1.3).

2.3 To conclude the proof of Theorem 2 we show that:

2.3.A For every $h \in H$, $Ee^{itZ_\lambda(h)}$ converges to $Ee^{itW(h)}$ as $\lambda \to \infty$.
By Lemma 1, this is so for h of the form

$$h = \sum_{k=1}^{m} t_k h^{\phi_k}. \tag{2.7}$$

Using (1.8) and the inequality $|e^{it} - 1| \leq |t|$, we get

$$|Ee^{itW(h)} - Ee^{itW(g)}|^2 \leq t^2 E(W(h) - W(g))^2 \tag{2.8}$$

$$|Ee^{itZ_\lambda(h)} - Ee^{itZ_\lambda(g)}|^2 \leq t^2 E(W(h) - W(g))^2 \tag{2.9}$$

for every $h, g \in H$. Since $W(H)$ with h of the form (2.7) are dense in G and since 2.3.A. holds for such h, we get 2.3.A. for an arbitrary $h \in H$ by (2.8), (2.9) and the triangle inequality.

2.4 We deduce Theorem 1 from Theorem 2 by showing that, for $h \in H$, $E(Z_n(h) - Y_n(h))^2 \to 0$ as $n \to \infty$. Since $EZ_n(h)^2 = \|h\|^2$, it is sufficient to prove

$$EY_n(h)^2 \to \|h\|^2, \quad \text{as} \quad n \to \infty, \tag{2.10}$$

$$EY_n(h)Z_n(h) \to \|h\|^2, \quad \text{as} \quad n \to \infty. \tag{2.11}$$

If h_k is square integrable and canonical, then

$$E\sigma_k^m(h_k)\sigma_\ell^n(h_\ell) = \binom{n \wedge m}{k} Eh_k^2 \quad k = \ell$$
$$= 0 \qquad k \neq \ell \tag{2.12}$$

where $n \wedge m$ is the minimum of n and m. By (2.12)

$$EY_n(h)^2 = \sum_{k=0}^{\infty} n^{-k} E\sigma_k^n(h_k)^2 = \sum_{k=0}^{\infty} \frac{Eh_k^2}{k!} a_{n,k}$$

where $a_{n,k} = \binom{n}{k} n^{-k} k!$. Since $a_{n,k} \leq 1$ and $a_{n,k} \to 1$ as $n \to \infty$, (2.10) follows from the dominated convergence theorem.

Using (2.12) again we get

$$EY_n(h)Z_n(h) = \sum_{m=0}^{\infty} e^{-n} \frac{n^m}{m!} \sum_{k=0}^{\infty} n^{-k} E\sigma_k^n(h_k)\sigma_k^m(h_k) = \sum_{k=0}^{\infty} \frac{Eh_k^2}{k!} b_{n,k}$$

where

$$b_{n,k} = \sum_{m=0}^{\infty} e^{-n} \frac{n^m}{m!} \binom{n \wedge m}{k} \frac{k!}{n^k} = \sum_{m=k}^{\infty} e^{-n} \frac{n^{m-k}}{(m-k)!} \binom{n \wedge m}{k} \Big/ \binom{m}{k}.$$

Clearly $b_{n,k} \leq 1$. We note also that for $n \geq k$,

$$b_{n,k} = P\{N_n \leq n - k\} + a_{n,k}P\{N_n \geq n + 1\}$$

and by the central limit theorem, $P\{N_n \leq n - k\} \to \frac{1}{2}$ as $n \to \infty$, $k = 0, 1, \cdots$. Hence $b_{n,k} \to 1$ as $n \to \infty$ and (2.11) holds by the dominated convergence theorem.

APPENDIX

0.1 Let $Q_i(S)$ be the conditional expectation of S given $(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$. Define

$$h_k(X_1, \cdots, X_k) = (I - Q_1) \cdots (I - Q_k)Q_{k+1} \cdots Q_n(S)$$

where I is the identity operator. Formula (1.2) follows from the relation

$$S = [(I - Q_1) + Q_1] \cdots [(I - Q_n) + Q_n](S)$$

and the fact that $Q_1, \cdots, Q_n$ commute.

0.2 Let $\mathcal{S}^n$ be the σ-algebra of symmetric sets in $(\mathcal{X}^n, \mathcal{B}^n)$. Then

$$U_m^n(h) = E[h(X_{s_1}, \cdots, X_{s_m})|\mathcal{S}^n].$$

Using (1.2), we represent h as

$$\textstyle\sum_{k=0}^m \sigma_k^{m'}(\tilde{h}_k).$$

Taking conditional expectation with respect to $\mathcal{S}^n$, we get Hoeffding's decomposition with $h_k = \binom{m}{k}\tilde{h}_k$.

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