Symmetric Sum-Free Partitions and Lower Bounds for Schur Numbers

Harold Fredricksen
Naval Postgraduate School,
Department of Mathematics,
Monterey, CA 93943, USA
halF@nps.navy.mil

Melvin M. Sweet
Institute for Defense Analyses,
Center for Communications Research,
4320 Westerra Court,
San Diego, CA 92121, USA
sweet@ccrwest.org

Submitted:May 9, 2000; Accepted:May 18, 2000

Abstract

We give new lower bounds for the Schur numbers $S(6)$ and $S(7)$. This will imply new lower bounds for the Multicolor Ramsey Numbers $R_6(3)$ and $R_7(3)$. We also make several observations concerning symmetric sum-free partitions into 5 sets.

Introduction

A set of integers is said to be *sum-free* if for all i and $j \geq i$ in the set the sum $i + j$ is not in the set. The Schur number $S(k)$ is defined to be the maximum integer n for which the interval $[1, n]$ can be partitioned into k sum-free sets. [1] I. Schur [7] proved that $S(k)$ is finite, and that

$$S(k) \geq 3S(k-1) + 1 \; . \tag{1}$$

In the framework of Ramsey theory, Schur's proof yields the inequality

$$S(k) \leq R_k(3) - 2, \tag{2}$$

where $R_k(n)$ denotes the k-color Ramsey number, defined to be the smallest integer such that any k-color edge coloring of the complete graph on $R_k(n)$ vertices has at least one complete subgraph all of whose edges have the same color.

[1] Some authors define $S(k)$ to be the smallest n for which there are no sum-free partitions into k sets.

It is easily verified that $S(1) = 1$, $S(2) = 4$, and $S(3) = 13$; and L.D. Baumert [1] showed in 1961 with the aid of a computer that $S(4) = 44$. The best known bounds for $S(5)$ are

$$160 \leq S(5) \leq 315 \tag{3}$$

The first inequality in (3) is due to G. Exoo [4], and the second follows from (2) and the bounds for $R_5(3)$ given in S. Radziszowski's survey paper [6]. For earlier work on lower bounds for $S(k)$ see H. Fredricksen [5] and A. Beutelspacher and W. Brestovansky [2].

The best previously known lower bound $S(6) \geq 481$ for $S(6)$ follows from (1). At the end of this note we list constructions that show

$$S(6) \geq 536$$
$$S(7) \geq 1680.$$

Using (2) we obtain the following lower bounds for the Ramsey numbers $R_6(3)$ and $R_7(3)$:

$$R_6(3) \geq 538$$
$$R_7(3) \geq 1682.$$

This improves the bounds given in Radziszowski's survey paper. Note that the bound for $R_7(3)$ is better than the lower bound of 1662 that can be obtained by using F. Chung's [3] inequality

$$R_{k+1}(3) \geq 3R_k(3) + R_{k-2}(3) - 3, \qquad \text{for} \qquad k \geq 3,$$

with our new bound for $R_6(3)$.

The constructions and results in this paper focus on symmetric partitions, where a sum-free partition P of the interval $[1, n]$ is said to be *symmetric* if all symmetric pairs i and $(n+1) - i$ are in the same set; except in the case when $n + 1$ is divisible by 3, we must allow $(n + 1)/3$ and $2(n + 1)/3$ to be in different sets. There are no real theorems in this note, only observations and conjectures based on a computer study of symmetric sum-free partitions. We have restricted our focus to such partitions because the search space is smaller, and the concepts of equivalence and depth which are explained below provide additional useful structure for restricting the search.

It is easy to see that if a symmetric sum-free partition P is multiplied $mod(n + 1)$ by an integer m that is relatively prime to $n + 1$, then $m\,P$ is also a symmetric sum-free partition. We will say two such partitions are *equivalent*.

Given a partition P of $[1, n]$, we define the *depth*, denoted $d(P)$, to be the largest of the set minimums; and for a symmetric sum-free partition we define the *e-depth*, denoted $D(P)$, to be the maximum of $d(P')$ over all partitions P' equivalent to P. We define $D_k(n)$ to be the maximum of $D(P)$ over all symmetric sum-free partitions P of $[1, n]$ into k sets. Note that we must have

$$D_k(n) \leq S(k - 1) + 1.$$

In a later section we will give exact values of $D_5(n)$ for many values of n. These values and other studies described later for partitions into 5 sets have led us to make the following conjectures:

Conjecture 1 *The largest integer for which there is a symmetric sum-free partition into 5 sets is 160; and perhaps* $S(5) = 160$.

Conjecture 2 *All symmetric sum-free partitions of 160 into 5 sets have e-depth 44.*

Conjecture 3 *There are no symmetric sum-free partitions of 155 or 158 into 5 sets.*

Note 155 and 158 are the special cases in the definition of symmetric, and that it is not obvious that a symmetric partition of n into k sets implies that there is a symmetric partition of m into k sets for $m < n$.

The Search Algorithm

The search algorithm we used is the obvious branching scheme modified with various heuristics. We successively place an integer that is "most blocked" (in set membership) by earlier placements, and then backtrack when a branch dies or an answer is found. The heuristic used to decide which of the most blocked integers to select was to choose the smallest. Choosing the correct heuristic can make a huge difference in search time. Using this algorithm with the condition $d(P) \geq m$, it is possible to completely exhaust the search tree in less than a minute for partitions of $[1, n]$ into 5 sets with n near 160 and m near to $44 = S(4)$. For smaller depths the running time required to exhaust becomes excessive; but by limiting the total number of placements made beyond a certain level it is possible to probabilistically prune the search tree and have a good probability of finding an answer when one exists.

Our search algorithm and heuristics were motivated by the following three observations that evolved during the development of the algorithm:

1. It is easy to exhaustively find all (not necessarily symmetric) sum-free partitions into 4 sets.
2. With high probability, it is easy to complete in a few steps a symmetric sum-free partition if only about 20% of the elements are known.
3. Many partitions are equivalent to partitions with large depth which can be build by first finding partitions in fewer sets.

The partitions P of $k = 6$ and 7 sets given at the end of this paper were found using a probabilistic algorithm starting with a partition P' of $k-1$ sets with small $d(P')$ and by restricting $d(P)$ to be near $S(k-1)$.

We believe that it should be possible to find a larger lower bound for $S(6)$ by looking for symmetric sum-free partitions that have large depth and have one other set with large e-depth. We currently have programs running to do this.

Conjectures and Observations for 5 Sets

This section describes observations and conjections resulting from a computer study of symmetric sum-free partitions into 5 sets.

By an exhaustive branching program, we were able to determine $D_5(n)$ for all $n \leq 154$. For $102 \leq n \leq 154$, $D_5(n)$ is always either 44 or 45; for $86 \leq n \leq 101$, $D_5(n)$ is between 41 and 45; and for $45 \leq n \leq 85$, $D_5(n) = [(n+1)/2]$. We were also able to determine that $D_5(160) = D_5(157) = D_5(156) = 44$, but that $D_5(159) = 39$. We were only able to show exhaustively that $D_5(155) < 36$ and that $D_5(158) < 36$. A probabilistic search for symmetric sum-free partitions of 158 into 5 parts, leads us to conjecture that there may be no such partitions; it is not obvious that a symmetric partition of n into k sets implies that there is a symmetric partition of m into k sets for $m < n$. A similar statement may apply to 155, but we have not examined it as carefully. For $160 < n \leq 315$ we exhaustively searched for symmetric partitions into 5 sets with depth ≥ 44 and found that there are none. We did the same for other more relaxed depth restrictions, but for smaller n, with the same result; to summarize

$$
\begin{array}{ll}
D_5(n) < 36 & \text{for } 161 \leq n \leq 190 \\
D_5(n) < 39 & \text{for } 191 \leq n \leq 210 \\
D_5(n) < 44 & \text{for } 211 \leq n \leq 315
\end{array}
$$

This, various probabilistic searches, and the upper bound in (3) leads us to conjecture that 160 is the largest integer for which there is a symmetric sum-free partition into 5 sets, and perhaps that $S(5) = 160$.

We found that there are symmetric sum-free partitions of 157 into 5 sets for each of the e-depths 44,31,and 26. For partitions of 159 into 5 sets, we found examples with $D(P) = 39$ and examples with $D(P) = 26$. The only e-depth found for partitions of 160 into 5 sets was 44. We conjecture that no other e-depths are possible for 160 into 5 sets. This would mean that all partitions of 160's would be equivalent to the 5840 we found exhaustively for 160 with $d = 44$; of these, 768 are equivalent to exactly one other partition with $d(P) = 44$ (interestingly, the equivalence multiplier is always 30 or 59 $(= -1/30 \; mod \; 161)$), the rest were equivalent to no others with $d(P) = 44$. So we conjecture that there are $(4304 + (768/2)) * \phi(161)/2 = 309,408$ symmetric sum-free partitions of 160 into 5 sets.

An interesting note for depth 44 symmetric sum-free partitions of 160 into 5 sets is that the shortest distance between one and a multiple of another is small. Here the distance between partitions is defined as the smallest number of elements that are in different sets between the partitions, after a relabeling of sets for one of the partitions. Considering only multiplication by 1, 30, and 59 the maximum distance is 19. This means that there is essentially only one cluster of depth 44 partitions, and if our conjecture is correct, then every symmetric partition is equivalent to something in this cluster.

The situation in the last paragraph for 160 is similar to what happens for the largest partitions into 4 and 3 sets; further evidence that 160 may be maximal for 5 set partitions. For partitions of 44 into 4 sets all $24 = \phi(45)$ symmetric sum-free partitions of the 273 sum-free partitions are equivalent (note that, since 3 divides 45, multiplying by 1 and -1 are not the same), and 6 of the 24 have depth 13. For sum-free partitions of 13 into 3 sets all three partitions are symmetric and equivalent.

Another interesting aside, for depth 44 symmetric sum-free partitions of 160 into 5 sets is that the sizes of the sets have the following properties:

1. There are 128 different set size configurations.
2. The largest set size is 44, and the smallest is 24.
3. The largest difference between smallest and largest set in a partition is 20, and the smallest is 6. For the largest difference, the largest set is always 44. For the smallest difference, the largest set is always 34.
4. The most likely set size configuration is {26,28,32,34,40},

Constructions

The constructions exhibited below are symmetric, and so we only list the smallest of a symmetric pair.

Partition of 536 into 6 symmetric sumfree sets:

```
Set  1:   1   5   8  11  14  24  27  30  33  36  40  43  46  49
 52  65  71  77  81  84  90  93  99 103 109 112 115 125 128 131
134 137 144 147 150 153 160 163 166 169 172 181 185 188 191 194
201 204 207 213 220 223 226 229 232 235 238 242 245 248 251 254
264 267 358

Set  2:   2  12  19  25  26  34  41  57  58  63  72  79  85  86
 95  96 102 118 123 124 140 141 145 146 155 156 162 173 183 193
200 206 211 215 216 222 233 239 244 253 260 261 266

Set  3:   3  10  16  22  23  29  35  42  48  56  60  62  67  68
 69  74  75  80  87  88  94 100 101 106 107 113 114 119 121 126
133 139 151 152 158 159 164 165 171 178 184 192 197 198 203 205
210 217 237 241 243 249 250 255 256

Set  4:   4  13  20  28  31  38  50  61  64  73  83  91  98 108
110 117 120 132 135 142 143 154 161 168 177 179 187 195 209 212
214 219 221 224 231 236 246 258 265

Set  5:   6   9  17  21  32  39  44  51  54  55  66  70  82  89
 92 104 111 127 129 130 149 167 175 189 190 202 225 227 247 252
262 263

Set  6:   7  15  18  37  45  47  53  59  76  78  97 105 116 122
136 138 148 157 170 174 176 180 182 186 196 199 208 218 228 230
234 240 257 259 268
```

$(D(P) = 161)$

We also have constructions of 6 set partitions for the following n and respective e-depths D:

$$n = 499, 506, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 523, 525, 526, 531$$
$$D = 160, 161, 160, 161, 161, 161, 144, 147, 158, 144, 142, 161, 161, 161, 155, 160, 161.$$

Partition of 1680 into 7 symmetric sumfree sets:

```
Set  1:    1  15  17  38  52  56  80  82  85  87  89  96  98 109
129 133 135 140 142 149 151 153 158 182 184 186 190 200 206 208
230 237 239 241 250 253 255 261 274 283 308 318 320 334 336 352
371 373 389 391 415 424 428 444 446 470 477 479 486 488 493 499
512 521 523 525 532 541 543 546 565 567 572 574 576 583 590 596
627 629 643 664 666 682 684 708 717 737 750 759 761 768 770 779
781 800 803 810 812 814 816 821 834

Set  2:    2  29  36  42  53  67  73  86  91 100 117 124 138 147
155 171 188 202 209 235 240 268 280 290 291 299 306 311 317 324
337 350 355 361 362 383 388 393 394 421 426 432 443 459 473 476
481 490 506 514 528 537 544 555 561 575 588 594 599 621 626 658
659 665 670 681 696 697 703 714 728 740 747 752 778 785 799 832

Set  3:    3   8  10  12  21  23  28  30  32  50  61  63  68  70
 72  79  81  83  99 101 103 105 110 112 114 121 123 130 132 134
141 143 150 152 154 156 161 163 165 174 176 183 185 194 196 201
203 205 214 221 223 232 236 238 243 245 252 254 256 258 265 267
272 276 278 285 287 294 296 303 305 307 314 323 325 327 329 338
345 347 356 358 367 374 376 380 385 387 396 398 405 407 409 420
427 431 438 440 447 449 456 458 460 465 467 469 471 478 480 487
489 491 498 507 509 511 513 518 520 522 529 531 540 542 547 551
558 560 562 569 573 578 580 582 591 593 600 602 609 611 613 620
622 624 631 633 640 644 649 662 671 673 680 691 693 695 700 706
711 713 715 726 731 733 735 744 746 753 757 762 773 775 777 782
784 793 795 797 802 806 808 813 815 817 822 824 826 828 835 837

Set  4:    4  11  18  20  33  34  35  62  64  65  71  88 115 118
120 167 168 170 173 217 218 220 227 242 249 270 271 273 295 297
300 302 309 312 326 339 348 353 370 379 406 408 411 423 425 452
455 461 503 505 508 533 550 564 577 605 608 617 632 646 647 661
663 690 699 729 741 743 780 788 789 796 811

Set  5:    5  14  39  46  47  49 102 104 106 136 139 189 191 192
199 224 233 259 262 277 284 288 315 321 340 341 342 343 344 359
377 403 412 429 430 437 439 441 462 474 494 496 497 515 526 527
538 549 556 559 579 581 597 612 614 615 623 641 650 667 668 675
676 678 679 694 709 712 732 734 749 764 765 767 776 794 819 820
829 831

Set  6:    6   9  24  25  26  27  41  43  44  45  55  57  58  59
 74  76  77  92  94  95  97 108 111 126 127 144 145 146 159 162
164 177 178 179 180 193 195 197 211 212 213 215 226 229 244 246
247 263 264 279 282 293 330 332 333 335 346 349 364 365 366 368
```

```
382 384 397 399 400 402 414 416 417 418 433 434 435 448 450 451
453 464 466 468 482 483 484 485 500 501 502 517 534 535 536 552
553 568 570 571 584 585 587 603 606 618 619 634 635 637 638 652
653 655 656 672 685 687 688 702 704 705 718 720 721 722 723 725
738 755 756 758 771 772 774 787 790 791 805 809 823 825 838 840

Set  7:   7  13  16  19  22  31  37  40  48  51  54  60  66  69
 75  78  84  90  93 107 113 116 119 122 125 128 131 137 148 157
160 166 169 172 175 181 187 198 204 207 210 216 219 222 225 228
231 234 248 251 257 260 266 269 275 281 286 289 292 298 301 304
310 313 316 319 322 328 331 351 354 357 360 363 369 372 375 378
381 386 390 392 395 401 404 410 413 419 422 436 442 445 454 457
463 472 475 492 495 504 510 516 519 524 530 539 545 548 554 557
563 566 586 589 592 595 598 601 604 607 610 616 625 628 630 636
639 642 645 648 651 654 657 660 669 674 677 683 686 689 692 698
701 707 710 716 719 724 727 730 736 739 742 745 748 751 754 760
763 766 769 783 786 792 798 801 804 807 818 827 830 833 836 839
```

$(D(P) = 537)$

We also have constructions of 7 set partitions for

$n =$	1615,	1620,	1630,	1640,	1650,	1660,	1665
$D =$	532,	537,	536,	537,	537,	537,	537.

References

[1] L. D. Baumert and S.W. Golomb. Backtrack Programming. *J. Ass. Comp. Machinery*, 12:516–524, 1965.

[2] Albrecht Beutelspacher and Walter Brestovansky. Generalized Schur Numbers. In *Combinatorial Theory, Proceedings of a Conference Held at Schloss Rauischholzhausen, May 6-9, 1982*, volume 969 of *Lecture Notes in Mathematics*, pages 30–38. Springer-Verlag, Berlin-New York, 1982.

[3] Fan Rong K. Chung. On the Ramsey Numbers $N(3, 3, ...3; 2)$. *Discrete Mathematics*, 5:317–321, 1973.

[4] Geoffrey Exoo. A Lower Bound for Schur Numbers and Multicolor Ramsey Numbers of K_3 . *The Electonic Journal of Combinatorics, http://www.combinatorics.org/*, #R8, 1: 3 pages, 1994.

[5] Harold Fredricksen. Schur Numbers and the Ramsey Numbers $N(3, 3, \ldots, 3; 2)$. *J. Combinatorial Theory Ser A*, 27:371–379, 1979.

[6] Stainislaw P. Radziszowski. Small Ramsey Numbers. Dynamic Survey DS1. *Electronic J. Combinatorics*, 1:35 pages, 1984 Revision #6 July 5, 1999.

[7] I. Schur. Uber die Kongruenz $x^m + y^m \equiv z^m \ (mod\, p)$. *Jahresber. Deutsch. Math.-Verin.*, 25:114–116, 1916.