SYMMETRIC VISCOUS FLOW BETWEEN TWO ROTATING POROUS DISCS—MODERATE ROTATION*

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1. Introduction. The flow between porous discs has recently been studied by several authors. As in the case of porous pipes and porous channels, the Navier–Stokes equations reduce to a set of nonlinear ordinary differential equations. Dorfman [1], Elkouh [2, 3, 4] and Naranyana [6] obtained solutions only for small cross-flow Reynolds numbers. Rasmussen [7] found analytic solutions for large Reynolds numbers in the case of two nonrotating porous discs. In this paper we shall study the case of two rotating porous discs. The rate of rotation is assumed to be comparable to the cross flow, i.e. $\beta = \Omega L/W = O(1)$. Analytic solutions will be found for both small and large Reynolds numbers. The results will be compared to those from numerical integration.

2. Formulation. Fig. 1 shows two coaxial porous discs situated at $z = \pm L$ and rotating with the same angular velocity Ω. Fluid is withdrawn from both discs with velocity W (injection if W is negative). We shall assume the gap with $2L$ is small compared to the diameter of the discs so that end effects can be neglected. The flow field is symmetric about the $z = 0$ plane and about the z axis.

The incompressible axisymmetric Navier–Stokes equations in cylindrical polar coordinates are

$$uu_r + wu_z - v^2/r = -p_r/\rho + \nu(\nabla^2 u - u/r^2), \tag{2.1}$$

$$uv_r + wv_z + uv/r = \nu(\nabla^2 v - v/r^2), \tag{2.2}$$

$$uw_r + ww_z = -p_z/\rho + \nu\nabla^2 w, \tag{2.3}$$

$$(ru)_r + rw_z = 0, \tag{2.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2.5}$$

and u, v, w are velocity components in the directions r, θ, z respectively. The boundary conditions are

$$\text{at} \quad z = \pm L \qquad u = 0, \qquad v = r\Omega, \qquad w = \pm W. \tag{2.6}$$

Utilizing the symmetry of the problem, we set

$$u = rf'(\eta)W/L, \qquad v = rg(\eta)W/L, \qquad w = -2f(\eta)W, \tag{2.7}$$

$$p = -\rho r^2 AW^2/(2L^2) + \rho P(\eta), \tag{2.8}$$

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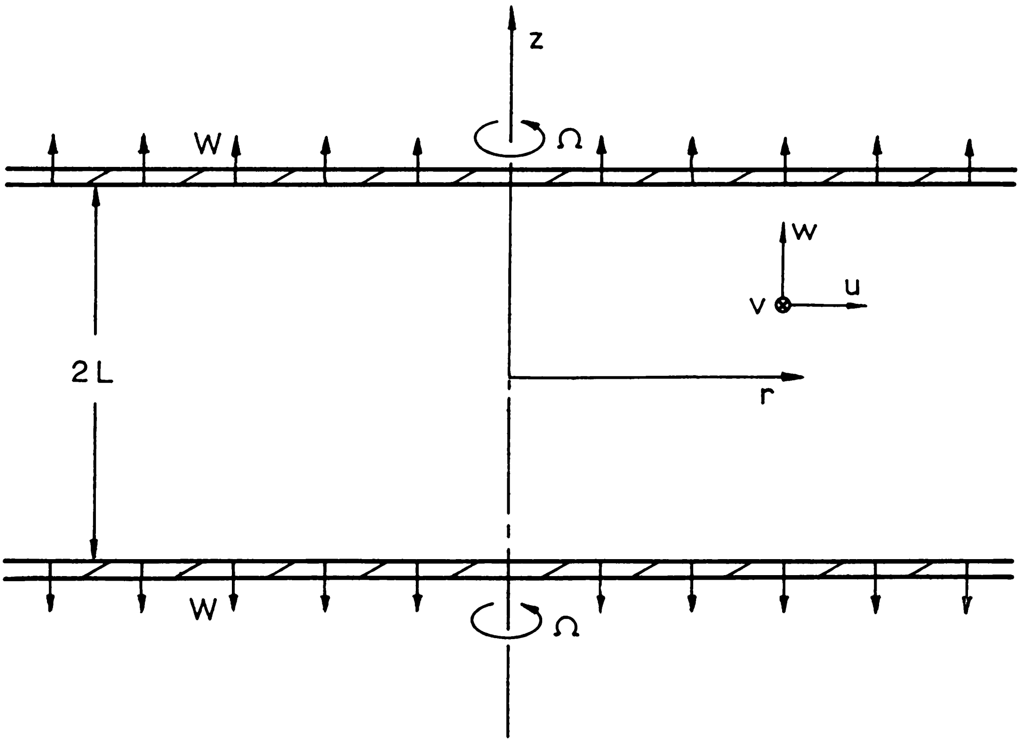

FIG. 1. The coordinate system.

where $\eta = z/L$ and A is a constant to be determined. Eqs. (2.1)–(2.4) become

$$(f')^2 - 2ff'' - g^2 = A + \frac{1}{R} f''' \tag{2.9}$$

or, after differentiating once,

$$-2(ff''' + gg') = \frac{1}{R} f'''', \tag{2.10}$$

$$2(f'g - fg') = \frac{1}{R} g'', \tag{2.11}$$

$$P(\eta) = -2f^2W^2 - 2\nu Wf'/L + B. \tag{2.12}$$

Here $R = WL/\nu$ is the cross-flow Reynolds number. The constant B is determined, say, from the pressure at the edge of the discs. The boundary conditions are

$$f'(1) = f(0) = f''(0) = 0, \qquad f(1) = -\tfrac{1}{2} \tag{2.13}$$

$$g'(0) = 0, \qquad g(1) = \Omega L/W \equiv \beta. \tag{2.14}$$

Eqs. (2.10)–(2.11), a set of nonlinear ordinary differential equations, are to be solved with the boundary conditions Eqs. (2.13)–(2.14). After $f(\eta)$ is obtained, the constant

A is determined by setting

$$A = -2f(1)f''(1) - \beta^2 - \frac{1}{R} f'''(1). \tag{2.15}$$

We shall assume $\beta = O(1)$.

3. Small Reynolds number. We expand the dependent variables in terms of the small parameter R:

$$f(\eta) = \varphi_0 + R\varphi_1 + R^2\varphi_2 + \cdots \tag{3.1}$$

$$g(\eta) = \chi_0 + R\chi_1 + R^2\chi_2 + \cdots \tag{3.2}$$

Eqs. (2.10)–(2.11) yield

$$\varphi_0'''' = 0, \qquad \chi_0'' = 0, \tag{3.3}$$

$$\varphi_1'''' = -2(\varphi_0\varphi_0''' + \chi_0\chi_0'), \qquad \chi_1'' = 2(\varphi_0'\chi_0 - \varphi_0\chi_0'), \tag{3.4}$$

$$\varphi_2'''' = -2(\varphi_0\varphi_1''' + \varphi_1\varphi_0''' + \chi_0\chi_1' + \chi_1\chi_0'),$$
$$\chi_2'' = 2(\varphi_1'\chi_0 + \varphi_0'\chi_1 - \varphi_1\chi_0' - \varphi_0\chi_1'). \tag{3.5}$$

The boundary conditions are

$$\varphi_0'(1) = \varphi_0(0) = \varphi_0''(0) = 0, \qquad \varphi_0(1) = -\tfrac{1}{2},$$
$$\chi_0'(0) = 0, \qquad \chi_0(1) = \beta, \tag{3.6}$$

$$\varphi_n(1) = \varphi_n'(1) = \varphi_n(0) = \varphi_n''(0) = \chi_n'(0) = \chi_n(1) = 0, \qquad n \neq 0. \tag{3.7}$$

After some algebra, the solutions are

$$\varphi_0 = \frac{1}{4}(\eta^3 - 3\eta), \qquad \chi_0 = \beta, \tag{3.8}$$

$$\varphi_1 = \frac{1}{1120}(-\eta^7 + 21\eta^5 - 39\eta^3 + 19\eta), \tag{3.9}$$

$$\chi_1 = \frac{\beta}{8}(\eta^4 - 6\eta^2 + 5), \tag{3.10}$$

$$\varphi_2 = \frac{-1}{560}\left(\frac{-3}{440}\eta^{11} + \frac{1}{6}\eta^9 - \frac{177}{140}\eta^7 + \frac{17}{10}\eta^5 - \frac{443}{1848}\eta^3 - \frac{137}{385}\eta\right),$$
$$- \beta^2\left(\frac{1}{840}\eta^7 - \frac{1}{40}\eta^5 + \frac{13}{280}\eta^3 - \frac{19}{840}\eta\right), \tag{3.11}$$

$$\chi_2 = \beta\left(\frac{-3}{2240}\eta^8 + \frac{1}{80}\eta^6 - \frac{111}{3360}\eta^4 - \frac{253}{560}\eta^2 + \frac{3183}{6720}\right). \tag{3.12}$$

When $\beta = 0$, the solution for $f(\eta)$ reduces to those obtained by Elkouh [2] who studied the small Reynolds number flow between two non-rotating porous discs.

4. Large-suction Reynolds number. In this case R is large and positive. Eqs. (2.10)–(2.11) show a boundary layer exists on $\eta = \pm 1$. We shall use the method of matched asymptotic expansions. Let $R = 1/\epsilon \gg 1$ and we expand, for the interior flow,

$$f = F_0(\eta) + \epsilon F_1(\eta) + \cdots , \tag{4.1}$$

$$g = G_0(\eta) + \epsilon G_1(\eta) + \cdots , \tag{4.2}$$

The zeroth-order equations are

$$F_0 F_0''' + G_0 G_0' = 0, \tag{4.3}$$

$$F_0' G_0 - F_0 G_0' = 0, \tag{4.4}$$

The boundary conditions are

$$F_0(0) = F_0''(0) = G_0'(0) = 0, \qquad F_0(1) = -\tfrac{1}{2}. \tag{4.5}$$

The only solution is

$$F_0 = -\tfrac{1}{2}\eta, \qquad G_0 = 0. \tag{4.6}$$

This shows that although both discs are rotating in the same sense, the interior is primarily not rotating. For the boundary layer at $\eta = 1$ we set

$$\eta = 1 - \epsilon\xi, \tag{4.7}$$

$$f = -\tfrac{1}{2} + \epsilon f_1(\xi) + \cdots , \tag{4.8}$$

$$g = g_0(\xi) + \epsilon g_1(\xi) + \cdots . \tag{4.9}$$

Eqs. (2.10)–(2.11) yield

$$-f_1''' = f_1'''', \tag{4.10}$$

$$-g_0' = g_0''. \tag{4.11}$$

Together with the boundary conditions $f_1(0) = f_1'(0) = 0$, $g_0(0) = \beta$ and matching the outer solution, we find

$$f_1 = \tfrac{1}{2}[\xi - 1 + \exp(-\xi)], \tag{4.12}$$

$$g_0 = \beta \exp(-\xi). \tag{4.13}$$

Similarly the higher-order solutions are

$$F_1 = -\eta/2, \qquad G_1 = 0, \tag{4.14}$$

$$g_1 = \beta\xi\left(1 + \frac{\xi}{2}\right)\exp(-\xi), \tag{4.15}$$

A uniformly valid composite solution is thus obtained:

$$f = -\frac{\eta}{2} + \frac{1}{R}\left(\frac{-\eta}{2} + \exp(-\xi)\right) + O\left(\frac{1}{R^2}\right), \tag{4.16}$$

$$g = \beta \exp(-\xi) + \frac{1}{R}\beta\xi\left(1 + \frac{\xi}{2}\right)\exp(-\xi) + O\left(\frac{1}{R^2}\right). \tag{4.17}$$

5. Large-injection Reynolds number. If R is large and negative, we expect the boundary layer to be at the center plane $\eta = 0$. We set $R = -1/\epsilon \leqq -1$. An expansion similar to Eqs. (4.1)–(4.2) yields Eqs. (4.3)–(4.4), albeit with different boundary conditions:

$$F_0(0) = F_0'(1) = 0 \qquad G_0(1) = \beta, \qquad F_0(1) = -1/2. \tag{5.1}$$

The solutions are unique:

$$F_0 = \frac{-(1 - \cos 2\beta\eta - \tan 2\beta \sin 2\beta\eta)}{2(1 - \sec 2\beta)}, \tag{5.2}$$

$$G_0 = \frac{\beta(1 - \cos 2\beta\eta - \tan 2\beta \sin 2\beta\eta)}{(1 - \sec 2\beta)}. \tag{5.3}$$

As $\beta \to 0$, F_0 reduces to the quadratic found by Rasmussen [7] for non-rotating discs:

$$\lim_{\beta \to 0} F_0 = \tfrac{1}{2}\eta^2 - \eta. \tag{5.4}$$

The singularity at the boundary layer is much weaker than the suction case, since the discontinuity is in the derivative of G_0 and the second derivative of F_0. We set

$$\eta = \sqrt{\epsilon}\, \zeta, \tag{5.5}$$

$$f = \sqrt{\epsilon}\, f_0(\zeta) + \cdots, \tag{5.6}$$

$$g = \sqrt{\epsilon}\, g_0(\zeta) + \cdots. \tag{5.7}$$

The governing equations become

$$2 f_0 f_0''' = f_0'''', \tag{5.8}$$

$$2(f_0' g_0 - g_0' f_0) = -g_0''. \tag{5.9}$$

The boundary conditions are

$$f_0(0) = f_0''(0) = 0, \qquad f_0(\infty) \to -C\zeta, \tag{5.10}$$

$$g'(0) = 0, \qquad g_0(\infty) \to 2\beta C \zeta, \tag{5.11}$$

where

$$C = \frac{\beta \sin 2\beta}{1 - \cos 2\beta} > 0. \tag{5.12}$$

These are obtained by expressing the interior solution in terms of the boundary layer variable ζ. The solutions are

$$f_0 = -C\zeta, \tag{5.13}$$

$$g_0 = 2\beta \sqrt{\frac{C}{\pi}} \left[2C\zeta \int_0^{\zeta} \exp(-C\zeta^2)\, d\zeta + \exp(-C\zeta^2) \right]. \tag{5.14}$$

The composite solution is then

$$f = \frac{-(1 - \cos 2\beta\eta - \tan 2\beta \sin 2\beta\eta)}{(1 - \sec 2\beta)} + O\frac{1}{(\sqrt{-R})}, \tag{5.15}$$

$$g = \frac{\beta(1 - \cos 2\beta\eta - \tan 2\beta \sin 2\beta\eta)}{(1 - \sec 2\beta)} + \frac{2\beta}{\sqrt{-R}} \sqrt{\frac{C}{\pi}} \exp(-C\zeta^2) + O\left(\frac{1}{\sqrt{-R}}\right). \tag{5.16}$$

6. Numerical integration. An exact solution to the Navier–Stokes equations is obtained if we can integrate Eqs. (2.10, 2.11, 2.13, 2.14) numerically. This is, however, a difficult two-point boundary-value problem. Instead of using a three-parameter shooting method, we shall use a scheme first utilized by Terrill [9] who studied the porous flow in a channel. Let

$$\eta = \gamma\xi \tag{6.1}$$

$$f = \frac{F(\xi)}{2R\gamma}, \qquad g = \frac{G(\xi)}{2R\gamma^2}. \tag{6.2}$$

The governing equations and boundary conditions become

$$F'''' + FF''' + GG' = 0, \tag{6.3}$$

$$G'' + FG' - GF' = 0, \tag{6.4}$$

$$F'(1/\gamma) = F(0) = F''(0) = G'(0) = 0, \tag{6.5}$$

$$F(1/\gamma) = -R\gamma, \qquad G(1/\gamma) = 2\beta R\gamma^2. \tag{6.6}$$

Starting with a guessed $F'(0)$, $F'''(0)$ and $G(0)$, whose values are guided by the approximate formulae from the previous sections, we integrate Eqs. (6.3), (6.4) by the variable-step Runge–Kutta algorithm until $F'(\xi)$ reaches zero, say at $\xi = \xi^*$. Then we set $\gamma = 1/\xi^*$, $R = -F(\xi^*)/\gamma$, $\beta = G(\xi^*)/(2R\gamma^2)$. Whereas the values of R and β are determined a posteriori, one integration is sufficient to solve the nonlinear problem.

7. Discussion. Table 1 shows the results from the numerical integration compared to those from the small Reynolds number expansion (Eqs. (3.1, 3.2)) and those from the large Reynolds number expansions (Eqs. (4.16, 4.17, 5.15, 5.16)). It is seen that the small Reynolds number expansion is good when $|R| \leq 1$ for $f(\eta)$ while $g(\eta)$ is less accurate

TABLE 1. Comparison of starting values from numerical integration with those from approximate methods.

		$f'(0)$		$f'''(0)$		$g(0)$	
R	β	numerical	approximate	numerical	approximate	numerical	approximate
			(Eq. (4.16))		(Eq. (4.16))		(Eq. (4.17))
20.345	3.6648	−0.54398	−0.5245	0.00698	0.000	−0.00108	0.000
8.5679	0.00000	−0.60236	−0.5583	0.24870	0.000	−0.00000	0.000
4.0866	0.65478	−0.67462		0.69747		−0.16865	
			(Eq. (3.1))		(Eq. (3.1))		(Eq. (3.2))
1.0021	0.04101	−0.73198	−0.73232	1.28873	1.29272	0.14639	0.08622
0.31438	1.17726	−0.73988	−0.74150	1.37682	1.39640	1.47976	1.46370
0.13151	0.68501	−0.74754	−0.74757	1.46991	1.47030	0.74754	0.74693
0.044253	0.72837	−0.74922	−0.74922	1.49045	1.49046	0.74922	0.74920
−0.033438	0.015323	−0.75056	−0.75056	1.50698	1.50698	0.01501	0.01501
−0.58530	0.20341	−0.75951	−0.75939	1.62068	1.61921	0.15190	0.16201
−1.23220	0.25527	−0.76899	−0.76770	1.74881	1.73377	0.15379	0.24227
−4.39563	0.43440	−0.79913		2.24572		0.15982	
			(Eq. (5.15))		(Eq. (5.15))		(Eq. (5.16))
−11.8240	1.14699	−0.69147	−0.517	0.45228	0.000	0.27659	0.270
−39.0691	0.98224	−0.70736	−0.655	0.00156	0.000	0.14147	0.143

for the same range. The asymptotic large Reynolds number expansion is good if $|R|$ is above twenty or thirty. Numerical integration should be used for the intermediate values.

Figs. 2, 3, 4 show the normalized velocity distributions $f(\eta)$, $f'(\eta)$ and $g(\eta)$ for various cross-flow Reynolds numbers and rotation using numerical integration. The boundary layer character for large Reynolds numbers is evident. There is very little change in the normal velocity profiles $f(\eta)$, while the radial velocity profile $f'(\eta)$ shows more variation for positive (suction) Reynolds numbers. Of some interest is the azimuthal velocity profile $g(\eta)$. Depending on R and β, the interior may rotate slower or faster than the discs. For intermediate and large positive R, $g(\eta)$ changes sign. For instance, we see from Fig. 4 that when $R = 4.0866$, $\beta = 0.65478$ the rotation is *retrograde* in the interior $-.57 \leq \eta \leq .57$ and *prograde* near the discs.

The present study considers two discs rotating with the same angular velocity in the same sense. If suction or injection is not present ($R = 0$), the interior rotates rigidly

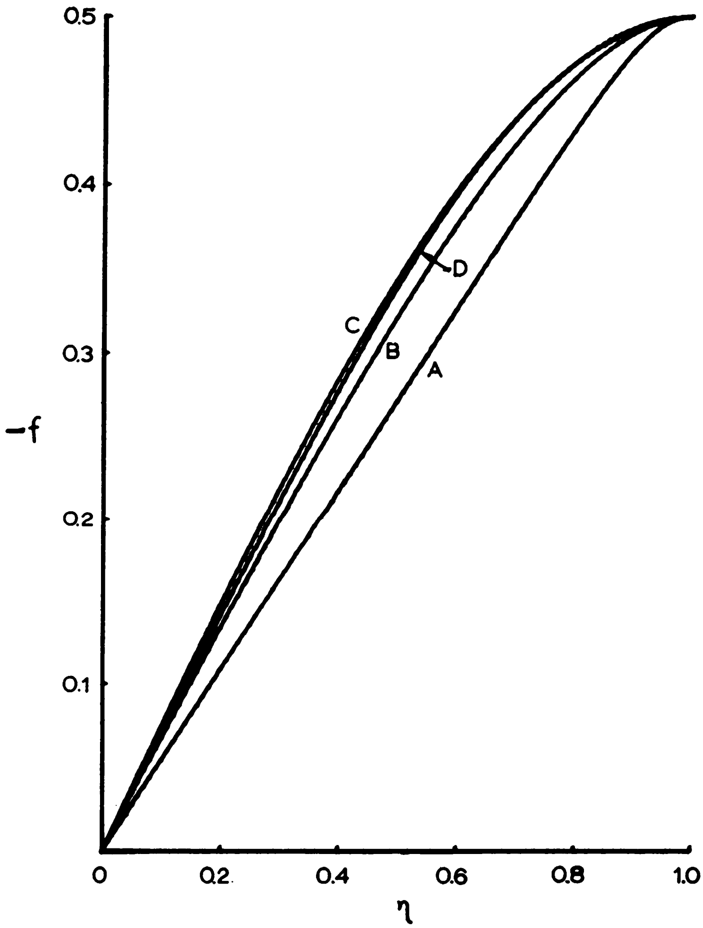

FIG. 2. Normal velocity distribution: A: R = 20.345, β = 3.6648; B: R = 4.0866, β = 0.65478; C: R = 0.033438, β = 0.015323; D: R = −39.0691, β = 0.98224.

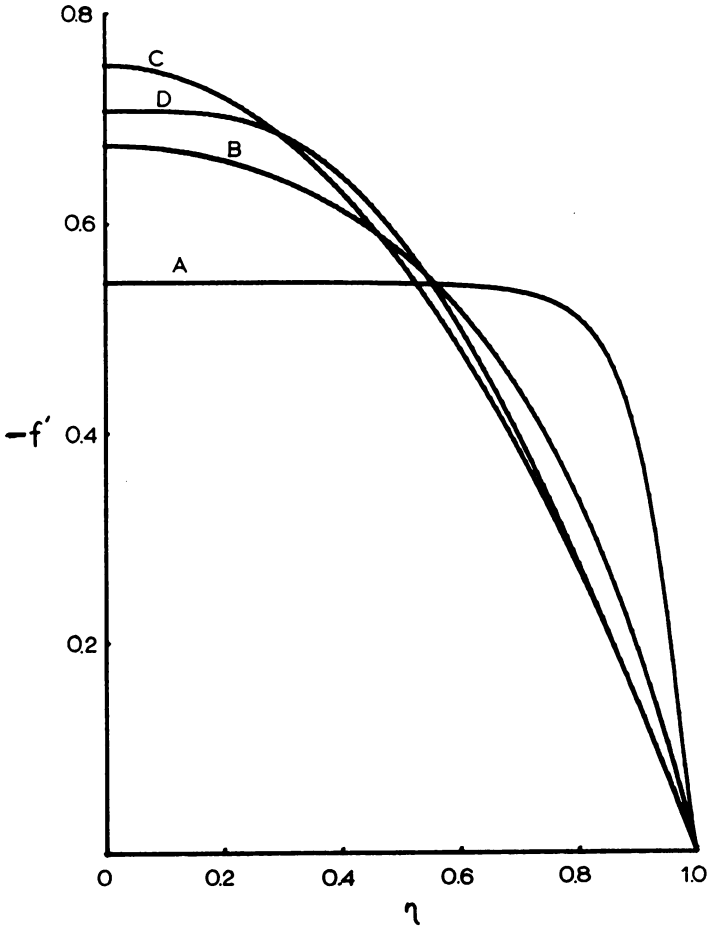

FIG. 3. Radial velocity distribution: A: $R = 20.345$, $\beta = 3.6648$; B: $R = 4.0866$, $\beta = 0.65478$; C: $R = 0.033438$, $\beta = 0.015323$; D: $R = -39.0691$, $\beta = 0.98224$.

with the same angular velocity. Since this problem can be regarded as porous flow in a rotating system, why not formulate our equations in a rotating frame. The reason for not doing so lies in the difficulty of numerical integration. As we recall, the numerical integration was started midway between the discs, where all *known* boundary conditions are zero. We integrated out to the disc and accepted whatever values of suction R and rotation β result. In a rotating frame the above integration scheme does not work and the tedious shooting method must be used.

In this analysis $\beta = \Omega L/W$ is assumed to be of order unity, i.e. suction is as important as rotation. Actually suction must be present in order for the flow to differ from rigid rotation. Also, two discs are required in order for suction to have a non-trivial effect on interior rotation. The motion therefore is not driven primarily by rotation, as in the case of single porous discs studied by Stuart [8] and Kuiken [5].

The assumption $\beta = O(1)$ of course includes the case when $\beta \ll O(1)$. In particular, when $\beta = 0$ we recover the non-rotating solutions obtained by Elkouh [2] and Rasmussen [7]. For low Reynolds numbers, the radial velocities φ_0 and φ_1 are identical with Elkouh's.

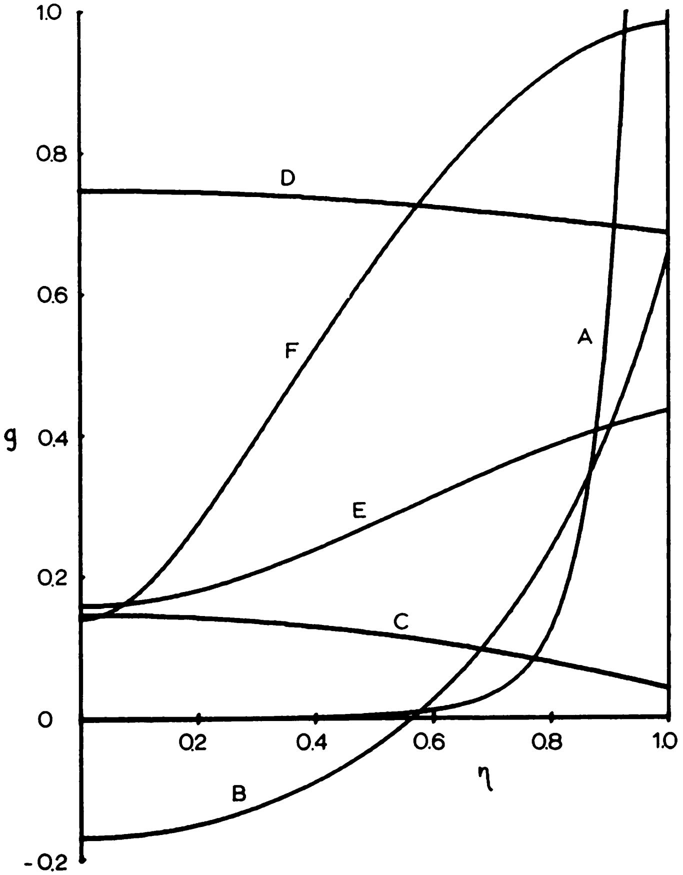

FIG. 4. Azimuthal velocity distribution: A: $R = 20.345$, $\beta = 3.6648$; **B**: $R = 4.0866$, $\beta = 0.65478$; C: $R = 1.0021$, $\beta = 0.04101$; D: $R = 0.13151$, $\beta = 0.68501$; E: $R = -4.39563$, $\beta = 0.43440$; F: $R = -39.0691$, $\beta = 0.98224$.

The correction due to rotation in φ_2 is of order $R^2\beta^2$. Therefore our solution for small Reynolds numbers breaks down when $\beta = O(1/\mathrm{R})$ or larger.

For large suction Reynolds numbers rotation does not affect the radial velocity to the order considered, and the solution is identical with Rasmussen's. The azimuthal velocity, of order β, was not found by Rasmussen. On the other hand, for large injection Reynolds numbers rotation has a primary effect on both radial and azimuthal velocities. The solution differs entirely from Rasmussen's, although his result could be recovered by taking the proper limits.

When β is much larger than $O(1)$ the problem becomes a perturbation on rigid rotation and a rotating frame of reference should be used. Although both analysis and numerical integration become more complicated, interesting phenomena studied here such as the reversal of interior rotation of the fluid do not occur.

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