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Symmetrical Properties of Two-Dimensional Ising Lattices

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Some symmetrical properties on the statistics of two-dimensional Ising lattices are studied. It is shown that such considerations give us very useful suggestions even for cases in which we know no exact solution. We investigate especially the cases of the square, triangular, honeycomb and generalized square lattices, with respect to the expressions of spontaneous magnetizations and susceptibilities.

§ 1. Introduction

As is well known, the exact treatments of the statistics for the two-dimensional Ising lattices have been obtained under zero magnetic field. But under finite magnetic field, it is very difficult to solve the problem. We know the internal energy, short range order, specific heat, entropy, etc., for the several types of lattices, but on their magnetic properties we know only Yang's[1] work on the spontaneous magnetization. It seems to us very difficult to extend the exact theory to know the spontaneous magnetization or the susceptibilities for other types of lattices.

The thermodynamic properties of one Ising lattice can be transformed to those of another type of lattice. These transformations enable us to know the unknown solution from the known one. In fact, when there are no magnetic fields, exact solutions for several types of lattices are derived from the known one. Under the external magnetic field, such transformations have not been successfully used.

One of us[2] and Fisher[3] derived the relations on the spontaneous magnetizations and the susceptibilities between the honeycomb lattice and the triangular lattice. But it is not sufficient for knowing the solution. Fisher[4] have obtained the exact solution for a special model by a transformation method, but his model seems to us too specialized to know a general feature of the Ising lattice. Basing on the symmetry consideration, Potts[5] derived the expression for the spontaneous magnetization for a triangular lattice from that of a square lattice.

Considerations on the symmetric properties will be very useful for the case in which we know no exact solution.

§ 2. Y-Δ transformation, quasi-spherical triangle

Y-Δ transformation for the anisotropic case seems very complicated at first sight.[6] But there are interesting symmetry properties between parameters. The equation for the Y-Δ transformation is

$$\sum_{\mu_0=\pm1} \exp[\mu_0(H_1\mu_1+H_2\mu_2+H_3\mu_3)]=A\exp(T_1\mu_2\mu_3+T_2\mu_3\mu_1+T_3\mu_1\mu_2), \tag{1}$$

where H_1, H_2, H_3 and T_1, T_2, T_3, are the interaction parameters for the honeycomb lattice and the triangular lattice respectively, and $\mu_i(i=0, 1, 2, 3)$ are the Ising spin variables. For the possible values of $\mu_i=\pm1$ we get the relations

$$\left.\begin{aligned} 2\,\mathrm{ch}(H_1+H_2+H_3)&=A\exp(T_1+T_2+T_3)\\ 2\,\mathrm{ch}(H_1-H_2+H_3)&=A\exp(-T_1+T_2-T_3)\\ 2\,\mathrm{ch}(H_1+H_2-H_3)&=A\exp(-T_1-T_2+T_3)\\ 2\,\mathrm{ch}(-H_1+H_2+H_3)&=A\exp(T_1-T_2-T_3)\end{aligned}\right\}, \tag{2}$$

which are easily deformed to

$$\left.\begin{aligned} A^4&=4(s_1^2+s_2^2+s_3^2+2c_1c_2c_3+2)\\ \coth 2T_1&=(c_1+c_2c_3)/s_2s_3\\ \coth 2T_2&=(c_2+c_3c_1)/s_3s_1\\ \coth 2T_3&=(c_3+c_1c_2)/s_1s_2\end{aligned}\right\}, \tag{3}$$

or

$$\left.\begin{aligned} 1/\mathrm{sh}^2\,2T_1&=A^4/4(s_2s_3)^2\\ 1/\mathrm{sh}^2\,2T_2&=A^4/4(s_3s_1)^2\\ 1/\mathrm{sh}^2\,2T_3&=A^4/4(s_1s_2)^2\end{aligned}\right\}, \tag{4}$$

where $s_i=\mathrm{sh}\,2H_i$, $c_i=\mathrm{ch}\,2H_i$. We can write these in symmetric form

$$\frac{1}{\mathrm{sh}^2\,2T_1\cdot\mathrm{sh}^2\,2H_1}=\frac{1}{\mathrm{sh}^2\,2T_2\cdot\mathrm{sh}^2\,2H_2}=\frac{1}{\mathrm{sh}^2\,2T_3\cdot\mathrm{sh}^2\,2H_3}$$
$$=\frac{s_1^2+s_2^2+s_3^2+2c_1c_2c_3+2}{(s_1s_2s_3)^2}. \tag{5}$$

By the dual transformation from the triangular lattice (T_1, T_2, T_3), we get again the honeycomb lattice with interaction parameters (H_1', H_2', H_3').

$$\mathrm{sh}\,2H_i'\cdot\mathrm{sh}\,2T_i=\mathrm{ch}\,2H_i'\cdot\mathrm{th}\,2T_i=\mathrm{th}\,2H_i'\cdot\mathrm{ch}\,2T_i=1. \tag{6}$$

With (3) and (6), we have

$$\left.\begin{aligned} \mathrm{ch}\,2H_1'&=(c_1+c_2c_3)/(s_2s_3)\\ \mathrm{ch}\,2H_2'&=(c_2+c_3c_1)/(s_3s_1)\\ \mathrm{ch}\,2H_3'&=(c_3+c_1c_2)/(s_1s_2)\end{aligned}\right\}. \tag{7}$$

Rearranging the terms, we get

$$\left.\begin{aligned} c_1 &= -c_2c_3 + s_2s_3c_1' \\ c_2 &= -c_3c_1 + s_3s_1c_2' \\ c_3 &= -c_1c_2 + s_1s_2c_3' \end{aligned}\right\}, \tag{8}$$

where $c_i' = \operatorname{ch} 2H_i'$.

Putting $s_i' = \operatorname{sh} 2H_i'$, with (5) and (6) we have

$$\frac{s_1'^2}{s_1^2} = \frac{s_2'^2}{s_2^2} = \frac{s_3'^2}{s_3^2} = \frac{s_1^2 + s_2^2 + s_3^2 + 2c_1c_2c_3 + 2}{(s_1s_2s_3)^2} \equiv k_H^2. \tag{9}$$

Eqs. (8) and (9) have similarity to the "cosine law" and the "sine law" of a spherical triangle. But the relations between H_1, H_2, H_3 and H_1', H_2', H_3' are symmetric:

$$\left.\begin{aligned} c_1' &= -c_2'c_3' + s_2's_3'c_1 \\ c_2' &= -c_3'c_1' + s_3's_1'c_2 \\ c_3' &= -c_1'c_2' + s_1's_2'c_3 \end{aligned}\right\}, \tag{10}$$

$$\begin{aligned} \frac{s_1'^2}{s_1^2} = \frac{s_2'^2}{s_2^2} = \frac{s_3'^2}{s_3^2} &= \frac{(s_1's_2's_3')^2}{s_1'^2 + s_2'^2 + s_3'^2 + 2c_1'c_2'c_3' + 2} \\ &= \frac{s_1^2 + s_2^2 + s_3^2 + 2c_1c_2c_3 + 2}{(s_1s_2s_3)^2} = k_H^2. \end{aligned} \tag{11}$$

From (8), we can get

$$c_1'c_2'c_3' + 1 = k_H^2(c_1c_2c_3 + 1). \tag{12}$$

(11) and (12) give the symmetric relations

$$\frac{s_1'^2}{s_1^2} = \frac{s_2'^2}{s_2^2} = \frac{s_3'^2}{s_3^2} = \frac{c_1'c_2'c_3' + 1}{c_1c_2c_3 + 1} = \frac{c_1'c_2'c_3' + 1 - (s_1's_2' + s_2's_3' + s_3's_1')}{c_1c_2c_3 + 1 - (s_1s_2 + s_2s_3 + s_3s_1)} = k_H^2. \tag{13}$$

According to Potts' conjecture, the spontaneous magnetization I_h for the honeycomb lattice will be

$$I_h^8 = (1 - k_H^2). \tag{14}$$

The similar relations for the triangular lattice will be easily obtained. Let the dual lattice of the honeycomb lattice (H_1, H_2, H_3) be the triangular lattice (T_1', T_2', T_3'). Then we have

$$\operatorname{sh} 2H_i \operatorname{sh} 2T_i' = \operatorname{ch} 2H_i \operatorname{th} 2T_i' = \operatorname{th} 2H_i \operatorname{ch} 2T_i' = 1. \tag{15}$$

With (11), (13) and (15), putting $c_i' = \operatorname{ch} 2T_i'$, $c_i = \operatorname{ch} 2T_i$, we have

$$\frac{s_1'^2}{s_1^2} = \frac{s_2'^2}{s_2^2} = \frac{s_3'^2}{s_3^2} = \frac{1}{(s_1s_2)^2 + (s_2s_3)^2 + (s_3s_1)^2 + 2(s_1s_2s_3)^2 + 2c_1c_2c_3s_1s_2s_3}$$

$$= (s_1's_2')^2 + (s_2's_3')^2 + (s_3's_1')^2 + 2(s_1's_2's_3')^2 + 2c_1'c_2'c_3's_1's_2's_3'$$

$$= \left(\frac{c_1'c_2'c_3' + s_1's_2's_3'}{c_1c_2c_3 + s_1s_2s_3}\right)^2 = \left(\frac{c_1'c_2'c_3' + s_1's_2's_3' - (s_1' + s_2' + s_3')}{c_1c_2c_3 + s_1s_2s_3 - (s_1 + s_2 + s_3)}\right)^2 \equiv k_T^2. \quad (16)$$

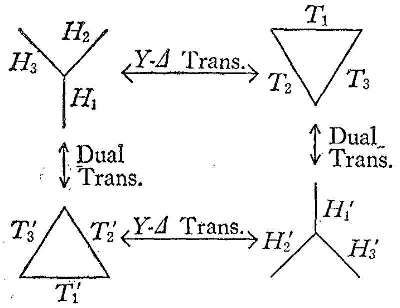


Fig. 1. Transformation of lattices.

Potts proposed that the spontaneous magnetization for the triangular lattice should be

$$I_t^8 = (1 - k_T^2). \quad (17)$$

(17) and (14) satisfy the condition derived by one of us: $I_h(H_1, H_2, H_3) = I_t(T_1, T_2, T_3)$. When $H_3 \to \infty$ or $T_3 \to 0$, (14) or (17) is reduced to the spontaneous magnetization for a square lattice:

$$I_s^8 = \left(1 - \frac{1}{s_1^2 s_2^2}\right). \quad (18)$$

§ 3. Spontaneous magnetization of the generalized square lattice

As a special case of the so-called chequer-type lattice, Utiyama[7] treated the partition function of a generalized square lattice. This lattice is reduced to a triangular, honeycomb or square lattice according as one of the parameters, for example, L_4, tends to infinity, zero or $L_4 = L_2$, and $L_1 = L_3$. As a generalization of (14) and (17) we arrive at the expression for the spontaneous magnetization for a generalized square lattice:

$$I_{g.s} = \left(1 - \frac{\Sigma s_i^2 + 2c_1c_2c_3c_4 + 2s_1s_2s_3s_4 + 2}{\Sigma(s_is_js_k)^2 + 2(s_1s_2s_3s_4)^2 + 2c_1c_2c_3c_4s_1s_2s_3s_4 + 2s_1s_2s_3s_4}\right)^{1/8}, \quad (19)$$

where the summations extend for i, j, k from 1 to 4, excluding the congruence of them. Hereafter we will omit this notice on the summations which will appear frequently. Or with parameters for a dual lattice (L_1^*, L_2^*, L_3^*, L_4^*), we can rewrite (19) as a symmetric form

$$I_{g.s} = \left(1 - \frac{s_1^*s_2^*s_3^*s_4^*}{s_1s_2s_3s_4} \cdot \frac{(\Sigma s_i^2 + 2c_1c_2c_3c_4 + 2s_1s_2s_3s_4 + 2)}{(\Sigma s_i^{*2} + 2c_1^*c_2^*c_3^*c_4^* + 2s_1^*s_2^*s_3^*s_4^* + 2)}\right)^{1/8}. \quad (20)$$

This symmetry on duality will suggest correctness of (19). There are several reasons to believe the correctness of (19).

i) At the Curie point of a generalized square lattice

$$c_1c_2c_3c_4 + s_1s_2s_3s_4 + 1 = \Sigma s_i s_j,$$

or in terms of Gudermannian functions $\mathrm{gd}(2L) = \arctan(\mathrm{sh}\, 2L)$,

$$\mathrm{gd}(2L_1) + \mathrm{gd}(2L_2) + \mathrm{gd}(2L_3) + \mathrm{gd}(2L_4) = \pi,$$

the expression (19) tends to zero, and at absolute zero of temperature (19) tends

to unity.
ii) When $L_4 \to 0$, (19) tends to (14), the spontaneous magnetization of a honeycomb lattice.
iii) When $L_4 \to \infty$, (19) tends to (17), the spontaneous magnetization of a triangular lattice.
iv) When $L_2 = L_4$ and $L_1 = L_3$, (19) becomes (18), the spontaneous magnetization of anisotropic square lattice.
v) Expanding (19) in power series of $x_i = \exp(-2L_i)$, we have

$$I_{g.s} = 1 - 2(x_1x_2x_3x_4) - 2\Sigma x_i^2 x_j^2 x_k^2 + 2(x_1x_2x_3x_4)^2 - 6(x_1x_2x_3x_4)\Sigma x_i^2 x_j^2$$
$$+ 4(x_1x_2x_3x_4)\Sigma x_i^2 x_j^2 x_k^2 - 4\Sigma x_i^2 x_j^4 x_k^4 - 30(x_1x_2x_3x_4)^2 \Sigma x_i^2 + O(x^{12}), \quad (21)$$

which coincides with the exact low temperature expansion.

§ 4. Susceptibility of one-dimensional lattice

On the susceptibility of a two-dimensional Ising lattice, there is no exact derivation. As stated in the Introduction, some of the transformations will be useful for the derivation of susceptibility, but, unless, at least, one exact solution is found for a standard Ising lattice, such transformation will hardly be used. Domb and Sykes,[8] basing on the exact high temperature expansion of the susceptibility, considered the critical behavior of the ferromagnetic susceptibility by ex trapolation. And also Fisher,[9] basing upon the Onsager-Kaufman's work, concluded that ferromagnetic susceptibility has the singularity of the following type, in a critical region,

$$\chi \simeq \frac{N\mu^2}{kT} \frac{C}{(1-T_c/T)^{7/4}}, \quad (22)$$

where C is a constant which depends on lattice type and k, T, N, μ have usual meanings. Domb and Sykes[8] obtained the numerical values of C, as 0.9684 for square lattice and 0.9295 for triangular lattice.

As an introduction, we want to summarize the results for one-dimensional Ising lattice which can be solved exactly. As is well known, the susceptibility of a linear chain with interaction parameter L_1 is, putting $v_1 \equiv \text{th}\, L_1$,

$$\chi_l = \frac{N\mu^2}{kT} \frac{1+v_1}{1-v_1}. \quad (23)$$

Or substitution of $c_1 = \text{ch}\, 2L_1$, $s_1 = \text{sh}\, 2L_1$, gives

$$\chi_l = \frac{N\mu^2}{kT} \frac{1}{c_1 - s_1}. \quad (24)$$

The denominator of (24) is closely related to the expression for the partition function of a linear chain:

$$\log \lambda_l \sim \int_0^\pi \log (c_1 - s_1 \cos \omega)\, d\omega. \tag{25}$$

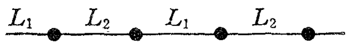


Fig. 2.

To assure the correspondence between (24) and (25), we calculate the susceptibility χ_l' for a linear chain with two interaction parameters L_1, L_2 alternatively.

$$\chi_l' = \frac{N\mu^2}{kT} \frac{c_1 + c_2 + s_1 + s_2}{c_1 c_2 + 1 - s_1 s_2}, \tag{26}$$

and the partition function is

$$\log \lambda_l' \sim \int_0^\pi \log (c_1 c_2 + 1 - s_1 s_2 \cos \omega)\, d\omega. \tag{27}$$

In this case also, there occurs a close relationship between the expressions (26) and (27). In the so-called "semi-ferromagnetic" case, in which the magnetic atom and non-magnetic atom alternate, we have

$$\chi_{l,semi}' = \frac{N\mu^2}{kT} \frac{c_1 + c_2}{c_1 c_2 + 1 - s_1 s_2}. \tag{28}$$

§ 5. A closed form related to the susceptibility of square and triangular lattices

Taking into consideration the relations (24), (25) and Onsager's[10] partition function,

$$\log \lambda_s \sim \int_0^\pi \int_0^\pi \log (c_1 c_2 - s_1 \cos \omega_1 - s_2 \cos \omega_2)\, d\omega_1 d\omega_2,$$

we can get a rough estimate of susceptibility of square lattice,

$$\frac{N\mu^2}{kT} \frac{1}{c_1 c_2 - s_1 - s_2}. \tag{29}$$

(The expression similar to this can be suggested from the combinatory consideration, for example, by Temperley.[11]) The expression (29) has the singularity at the critical temperature, $s_1 s_2 = 1$, but its character differs from (22). We modify (29) as

$$\chi_s = \frac{N\mu^2}{kT} \frac{(1 - s_1^2 s_2^2)^{1/4}}{c_1 c_2 - s_1 - s_2}. \tag{30}$$

Notice that the numerator of (30) is closely related to the long-range order of the dual lattice. We can rewrite (30) as

$$\chi_s = \frac{N\mu^2}{kT} \frac{(1 - v_1^2)(1 - v_2^2)}{(1 - v_1 - v_2 - v_1 v_2)^2} \left(1 - \frac{16 v_1^2 v_2^2}{(1 - v_1^2)^2 (1 - v_2^2)^2}\right)^{1/4}, \tag{31}$$

where

$$v_i = \tanh L_i, \quad i=1, 2.$$

Naturally, if one of the parameters v_2 tends to zero, (31) tends to Eq. (23) for the one-dimensional lattice. Expanding (31) in power series of v_i, we have

$$\begin{aligned}\chi_s = (N\mu^2/kT)\,\{&1+2(v_1+v_2)+2(v_1^2+v_2^2)+8v_1v_2+2(v_1^3+v_2^3)+16(v_1^2v_2+v_1v_2^2)\\&+2(v_1^4+v_2^4)+24(v_1^3v_2+v_1v_2^3)+48v_1^2v_2^2+2(v_1^5+v_2^5)+32(v_1^4v_2+v_1v_2^4)\\&+104(v_1^3v_2^2+v_1^2v_2^3)+2(v_1^6+v_2^6)+40(v_1^5v_2+v_1v_2^5)+180(v_1^4v_2^2+v_1^2v_2^4)\\&+296v_1^3v_2^3+2(v_1^7+v_2^7)+48(v_1^6v_2+v_1v_2^6)+280(v_1^5v_2^2+v_1^2v_2^5)\\&+656(v_1^4v_2^3+v_1^3v_2^4)+O_s(v^8)\}.\end{aligned} \tag{32}$$

This coincides with the exact high temperature expansion up to the order of v^7. In the isotropic case where $v_1=v_2\equiv v$, we have

$$\chi_s = \frac{N\mu^2}{kT}\,\frac{(1-v^2)^2}{(1-2v-v^2)^2}\left(1-\frac{16v^4}{(1-v^2)^4}\right)^{1/4} \tag{33}$$

$$= (N\mu^2/kT)\,(1+4v+12v^2+36v^3+100v^4+276v^5+740v^6+1972v^7+O_s(v^8)).$$

In the critical region, near the Curie temperature, we have

$$\chi_s \simeq \frac{N\mu^2}{kT}\,\frac{C_s}{(1-T_c/T)^{7/4}}, \qquad C_s=\frac{1}{8}\left(\frac{\sqrt{2}}{L_c}\right)^{7/4} \doteqdot 0.9617, \quad (L_c=0.4407), \tag{34}$$

which has the singularity of the type (22), and the numerical value of C_s has good agreement with that given by Domb and Sykes[8] 0.9684.

For the triangular lattice, after the same reasoning, we have

$$\chi_t = \frac{N\mu^2}{kT}\,\frac{[1-(\Sigma s_i^2s_j^2+2c_1c_2c_3s_1s_2s_3+2s_1^2s_2^2s_3^2)]^{1/4}}{c_1c_2c_3+s_1s_2s_3-s_1-s_2-s_3}$$

$$= \frac{N\mu^2}{kT}\,\frac{(1-v_1^2)^2(1-v_2^2)^2(1-v_3^2)^2}{(1-v_1-v_2-v_3-v_1v_2-v_2v_3-v_3v_1+v_1v_2v_3)^2}$$

$$\times\left[1-16\,\frac{(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)}{(1-v_1^2)^2(1-v_2^2)^2(1-v_3^2)^2}\right]^{1/4}. \tag{35}$$

The expression (35) is reduced to the susceptibility of square lattice (30) or (31), when one of the parameters tends to zero. Expanding (35) in power series of v_i, we have

$$\begin{aligned}\chi_t = (N\mu^2/kT)\,\{&1+2\Sigma v_i+2\Sigma v_i^2+8\Sigma v_iv_j+2\Sigma v_i^3+16\Sigma v_i^2v_j+36v_1v_2v_3\\&+2\Sigma v_i^4+24\Sigma v_i^3v_j+48\Sigma v_i^2v_j^2+104\Sigma v_i^2v_jv_k+2\Sigma v_i^5+32\Sigma v_i^4v_j\\&+104\Sigma v_i^3v_j^2+212\Sigma v_i^3v_jv_k+376\Sigma v_i^2v_j^2v_k+O_t(v^6)\},\end{aligned} \tag{36}$$

which coincides with the exact high temperature expansion up to the order of v^5. In the isotropic case, we have

$$\chi_t=\frac{N\mu^2}{kT}\frac{(1-v^2)^3}{(1-3v-3v^2+v^3)^2}\left(1-\frac{16v^3(1+v^3)}{(1-v)^3(1-v^2)^3}\right)^{1/4}$$

$$=(N\mu^2/kT)\,(1+6v+30v^2+138v^3+606v^4+2586v^5+O_t(v^6)). \quad (37)$$

At the neighborhood of the critical temperature, we have

$$\chi_t\simeq\frac{N\mu^2}{kT}\frac{C_t}{(1-T_c/T)^{7/4}},\quad C_t=\frac{1}{6\sqrt{3}\,L_c^{7/4}}\doteqdot 0.9235,\quad (L_c=0.2747). \quad (38)$$

The numerical value of C_t has good agreement with that given by Domb and Sykes[8] (0.9295).

§ 6. Extension to the honeycomb lattice

Fisher[3] derived a useful relation which connects the susceptibility of honeycomb lattice with that of triangular lattice,

$$\chi_t(T_1,\ T_2,\ T_3)=\chi_{h,semi}\,(H_1,\ H_2,\ H_3)=\frac{1}{2}\{\chi_h(H_1,\ H_2,\ H_3)+\chi_h(-H_1,\ -H_2,\ -H_3)\}, \quad (39)$$

where T_1, T_2, T_3 and H_1, H_2, H_3 are connected by Y-$\varDelta$ transformation (2). The middle part of (39) is the susceptibility of semi-ferromagnetic honeycomb-lattice. From (35) and (39), we can get

$$\chi_{h,semi}=\frac{N\mu^2}{kT}\frac{(\Sigma s_i^2+2c_1c_2c_3+2)^{1/2}}{c_1c_2c_3+1-\Sigma s_i\,s_j}\left(1-\frac{s_1^2s_2^2s_3^2}{\Sigma s_i^2+2c_1c_2c_3+2}\right)^{1/4}. \quad (40)$$

Extension to honeycomb lattice will be suggested by the case of linear chain. We have, comparing (40) with (26) and (28), the following expression:

$$\chi_h=\frac{N\mu^2}{kT}\frac{(\Sigma s_i^2+2c_1c_2c_3+2)^{1/2}+\Sigma s_i}{c_1c_2c_3+1-\Sigma s_is_j}\left(1-\frac{s_1^2s_2^2s_3^2}{\Sigma s_i^2+2c_1c_2c_3+2}\right)^{1/4}$$

$$=\frac{N\mu^2}{kT}\Bigg[\Big\{(1+v_1v_2+v_2v_3+v_3v_1)\,(1+v_1v_2-v_2v_3-v_3v_1)\,(1-v_1v_2+v_2v_3-v_3v_1)$$

$$\times(1-v_1v_2-v_2v_3+v_3v_1)\Big\}^{1/2}+\sum v_i(1-v_j^2)(1-v_k^2)\Bigg]\Big/(1-v_1v_2-v_2v_3-v_3v_1)^2$$

$$\times\Big[1-16\cdot v_1^2\cdot v_2^2\cdot v_3^2\Big/\Big\{(1+v_1v_2+v_2v_3+v_3v_1)\,(1+v_1v_2-v_2v_3-v_3v_1)$$

$$\times(1-v_1v_2+v_2v_3-v_3v_1)\,(1-v_1v_2-v_2v_3+v_3v_1)\Big\}\Big]^{1/4}. \quad (41)$$

This satisfies the relation (39) and, when one of the parameters H_3 tends to zero, it is reduced to the susceptibility of a linear chain:

$$\chi_l' = \frac{N\mu^2}{kT} \frac{(s_1^2+s_2^2+2c_1c_2+2)^{1/2}+s_1+s_2}{c_1c_2+1-s_1s_2} = \frac{N\mu^2}{kT} \frac{c_1+c_2+s_1+s_2}{c_1c_2+1-s_1s_2}. \tag{42}$$

Moreover, when one of the interaction parameters H_3 tends to infinity, (41) is reduced to that of a square lattice (30) or (31). In fact, the expression (41) has a singularity of the type of (22) and has high temperature expansion:

$$\begin{aligned}\chi_h = (N\mu^2/kT)\,\{&1+\Sigma v_i+2\Sigma v_iv_j+\Sigma v_i^2v_j+6v_1v_2v_3+2\Sigma v_i^2v_j^2+6\Sigma v_i^2v_jv_k\\ &+\Sigma v_i^3v_j^2+2\Sigma v_i^3v_jv_k+12\Sigma v_i^2v_j^2v_k+2\Sigma v_i^3v_j^3+10\Sigma v_i^3v_j^2v_k+24v_1^2v_2^2v_3^2\\ &+\Sigma v_i^4v_j^3+3\Sigma v_i^4v_j^2v_k+18\Sigma v_i^3v_j^3v_k+30\Sigma v_i^3v_j^2v_k^2+2\Sigma v_i^4v_j^4+14\Sigma v_i^4v_j^3v_k\\ &+22\Sigma v_i^4v_j^2v_k^2+54\Sigma v_i^3v_j^3v_k^2+\Sigma v_i^5v_j^4+4\Sigma v_i^5v_j^3v_k+6\Sigma v_i^5v_j^2v_k^2\\ &+24\Sigma v_i^4v_j^4v_k+63\Sigma v_i^4v_j^3v_k^2+102v_1^3v_2^3v_3^3+2\Sigma v_i^5v_j^5+18\Sigma v_i^5v_j^4v_k\\ &+40\Sigma v_i^5v_j^3v_k^2+96\Sigma v_i^4v_j^4v_k^2+152\Sigma v_i^4v_j^3v_k^3+O_h(v^{11})\}\,,\end{aligned} \tag{43}$$

which coincides with the exact one up to the order of v^{10}. Especially, in the isotropic case, we have

$$\begin{aligned}\chi_h &= \frac{N\mu^2}{kT} \frac{((1-v^2)^3(1+3v^2))^{1/2}+3v(1-v^2)^2}{(1-3v^2)^2}\left(1-\frac{16v^6}{(1-v^2)^3(1+3v^2)}\right)^{1/4}\\ &= (N\mu^2/kT)\,(1+3v+6v^2+12v^3+24v^4+48v^5+90v^6+168v^7\\ &\qquad +318v^8+600v^9+1098v^{10}+O_h(v^{11}))\,.\end{aligned} \tag{44}$$

§ 7. Expression for generalized square lattice and consistency examination

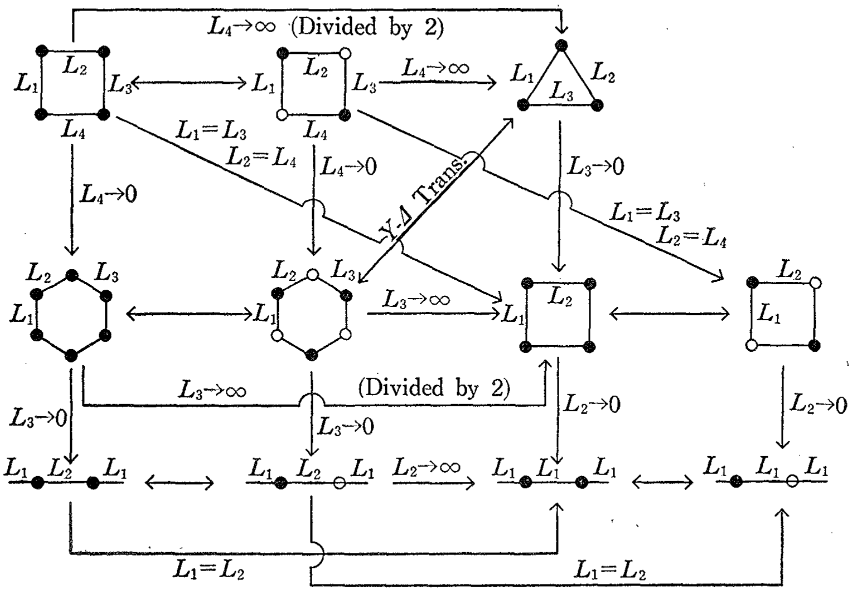
Extension to the generalized square lattice, which describe the properties for the square, honeycomb and triangular lattices, is now easy. We have

$$\begin{aligned}\chi_{g.s} &= \frac{N\mu^2}{kT} \frac{(\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2)^{1/2}+\Sigma s_i}{c_1c_2c_3c_4+s_1s_2s_3s_4+1-\Sigma s_i s_j}\\ &\quad \times\left(1-\frac{\Sigma s_i^2s_j^2s_k^2+2s_1^2s_2^2s_3^2s_4^2+2s_1s_2s_3s_4(c_1c_2c_3c_4+1)}{\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2}\right)^{1/4}.\end{aligned} \tag{45}$$

Of course, we have, for the semi-ferromagnetic generalized square lattice,

$$\begin{aligned}\chi_{g\cdot s,\,semi} &= \frac{N\mu^2}{kT} \frac{(\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2)^{1/2}}{c_1c_2c_3c_4+s_1s_2s_3s_4+1-\Sigma s_is_j}\\ &\quad \times\left(1-\frac{\Sigma s_i^2s_j^2s_k^2+2s_1^2s_2^2s_3^2s_4^2+2s_1s_2s_3s_4(c_1c_2c_3c_4+1)}{\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2}\right)^{1/4}.\end{aligned} \tag{46}$$

To show that these expressions are satisfactory in the scope of our consideration, we give a graphical representation of their consistency.



● indicates magnetic atom
○ indicates non-magnetic atom

Fig. 3. Graphical representation for the consistency examination of our expressions.

§ 8. Discussion on antiferromagnetic case

The formula mentioned above cannot be applied to the antiferromagnetic cases. As a trial to fit the expressions to the antiferromagnetic cases, we replace

$$\text{the factor}\quad (1-k^2)^{1/4} \text{ by } \frac{2}{\pi}\mathrm{E}(k)=\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\omega)^{1/2}d\omega, \tag{47}$$

the complete elliptic integral of the second kind. For example, we have for the case of antiferromagnetic square lattice

$$\chi_s=\frac{N\mu^2}{kT}\frac{1}{c_1c_2+s_1+s_2}\cdot\frac{2}{\pi}\mathrm{E}(k),\quad k=s_1s_2. \tag{48}$$

These replacements of our formulae do not affect the range of coincidence for the high temperature expansions (32), (36) and (43) and the consistency examinations. At the Curie point, the replaced expressions remain finite, unlike (30), (41) and (45) which tend to zero for the antiferromagnetic cases. Furthermore, the expressions behave, in the critical region, as

$$\chi\simeq\frac{N\mu^2}{kT}\{\xi_c-D(T-T_c)\log|T-T_c|\}, \tag{49}$$

which can be shown by the well-known properties of elliptic functions,

$$\frac{\partial\mathrm{E}}{\partial k}=\frac{\mathrm{E}-\mathrm{K}}{k}, \text{ and for } k\to1,$$

$$K(k)\simeq\log 4/|k'|,\quad k'^2=1-k^2, \tag{50}$$

where K is the complete elliptic integral of the first kind. This type of critical behavior for the antiferromagnetic case has been conjectured by Fisher,[9] and Sykes and Fisher[12] and the critical value $\xi_c=1/2\pi=0.159$ obtained from (48) for the square lattice has also good agreement with that given by them $(kT\chi/N\mu^2=0.156\sim0.158)$. For the honeycomb lattice, we have $\xi_c=2/3\sqrt{3}\pi=0.123$. On the other hand, for the triangular lattice which has no transition point, the expression decrease monotonically and has value $\chi_t=0.339N\mu^2/kT$ at the temperature corresponding to its ferromagnetic Curie point.

However, these expressions diverge to infinity as $(1-T_c/T)^{-2}$, not as $-7/4$ power, if applied to ferromagnetic cases. Below the critical temperature, we must change the expressions, about which we have no satisfactory results.

§ 9. Conclusion

There are no reasons to doubt the correctness of the expressions for the spontaneous magnetizations. However, it is obvious that the expressions for susceptibilities cannot be exact. In fact, the high temperature expansions diverge slightly as in Table I. Moreover, the discrepancy between ferromagnetic and antiferromagnetic expressions is troublesome. Although these expressions are not exact, these will be a milestone for the derivation of exact expressions for the susceptibilities of two-dimensional Ising lattices.

Table I

	Square Lattice	Triangular Lattice	Honeycomb Lattice
Exact Value	$5172v^8+13492v^9+\cdots$	$10818v^6+44574v^7+\cdots$	$2007v^{11}+3696v^{12}+\cdots$
Ferro. Mag. Expr.	$5168v^8+13492v^9+\cdots$	$10814v^6+44550v^7+\cdots$	$2004v^{11}+3692v^{12}+\cdots$
Antiferro. Mag. Expr.	$5180v^8+13540v^9+\cdots$	$10826v^6+44694v^7+\cdots$	$2004v^{11}+3704v^{12}+\cdots$

References

1) C. N. Yang, Phys. Rev. **85** (1952), 808.
2) S. Naya, Prog. Theor. Phys. **11** (1954), 53.
3) M. E. Fisher, Phys. Rev. **113** (1959), 969.
4) M. E. Fisher, Proc. Roy. Soc. A **254** (1960), 66.
5) R. B. Potts, Phys. Rev. **88** (1952), 392.
6) R. M. F. Houtappel, Physica **16** (1950), 425.
7) T. Utiyama, Prog. Theor. Phys. **6** (1951), 907.
8) C. Domb and M. F. Sykes, Proc. Roy. Soc. A **240** (1957), 214.
9) M. E. Fisher, Physica **25** (1959), 521.
10) L. Onsager, Phys. Rev. **65** (1944), 117.
11) H. N. V. Temperley, Phys. Rev. **103** 1956, 1.
12) M. F. Sykes and M. E. Fisher, Phys. Rev. Letters **1** (1958), 321.