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Symmetries in the reflection and transmission of elastic waves

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Summary. The symmetry relations between the reflection and transmission coefficients for plane elastic waves incident upon an arbitrary horizontally stratified medium are derived by a novel approach. Previous results, particularly for a single interface, are obtained as special cases of this treatment.

In addition, for perfectly elastic media, projection operators for travelling and evanescent waves are introduced and used to derive a number of new relationships between the reflection and transmission coefficients.

1 Introduction

The reflection and transmission of elastic waves at a plane interface is a problem which has attracted considerable attention for a long time. Green (1838) formulated the correct boundary conditions for the reflection and refraction of plane waves at an interface of two elastic media in a study of light propagation. It was however left to Knott (1899) to derive the reflection and transmission coefficients using energy arguments and an alternative formulation due to Zoeppritz (1919) obtained the coefficients in terms of amplitudes. Since that time these coefficients have been presented by a number of authors, but the results have in many cases been marred by minor errors and misprints.

Some symmetry relations between the reflection and transmission coefficients for a single interface have been presented by Frazier (1970) and a more restricted set is given by Červený & Ravindra (1971) who also present accurate expressions for the coefficients. The generalization of the symmetry properties to multilayered media was made by Lapwood & Hudson (1975) and derived by an alternative approach by Woodhouse (1974a).

The purpose of this paper is to present a unified treatment of the symmetry properties of the reflection and transmission coefficients for elastic waves in multilayered media. This will allow us to obtain all the previous results as special cases.

An alternative development exploiting the unitarity properties of the matrix of reflection and transmission coefficients leads to the introduction of projection operators for travelling and evanescent waves and a number of new relations between the reflection and transmission coefficients.

2 Elastic wave propagation in a layered medium

We will consider plane harmonic elastic waves with displacements of the form

$$\mathbf{u}(x, z, t) = \bar{\mathbf{u}}(k, \omega, z) \exp [i(kx - \omega t)], \tag{1}$$

propagating in a horizontally stratified medium composed of isotropic elastic layers within which the elastic wave speed for P waves (α), for S waves (β) and the density ρ depend only on the depth coordinate z.

We will restrict our attention to coupled P–SV wave propagation. For each plane wave component the evolution of the horizontal and vertical components of displacement ($\bar{u}, \bar{w}$) and the associated stresses ($\bar{\tau}_{xz}, \bar{\tau}_{zz}$) are described by the differential equations (see, e.g. Gilbert & Backus 1966; Kennett 1972),

$$\partial \mathbf{B}(k, \omega, z)/\partial z = \mathbf{A}(k, \omega, z)\, \mathbf{B}(k, \omega, z), \tag{2}$$

where the stress–displacement vector $\mathbf{B}$ is defined as

$$\mathbf{B}(k, \omega, z) = [\bar{u}, \bar{w}, \bar{\tau}_{xz}, \bar{\tau}_{zz}]^T, \tag{3}$$

where T denotes a transpose, and the matrix $\mathbf{A}$ takes the form

$$\mathbf{A}(k, \omega, z) = \begin{pmatrix} 0 & -ik & (\rho\beta^2)^{-1} & 0 \\ -ik(1 - 2\beta^2/\alpha^2) & 0 & 0 & (\rho\alpha^2)^{-1} \\ \nu k^2 - \rho\omega^2 & 0 & 0 & -ik(1 - 2\beta^2/\alpha^2) \\ 0 & -\rho\omega^2 & -ik & 0 \end{pmatrix}, \tag{4}$$

with $\nu = 4\rho\beta^2(1 - \beta^2/\alpha^2)$. The stress–displacement vector $\mathbf{B}$ has the convenient property of remaining continuous across planes or interfaces z = constant.

In order to relate the stress–displacement vector $\mathbf{B}$ more directly to the elastic wave field we follow Dunkin (1965) and make a transformation

$$\mathbf{B} = \mathbf{TV}, \tag{5}$$

where $\mathbf{T}$ is the eigenvector matrix for $\mathbf{A}$, i.e. such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{\Lambda}, \tag{6}$$

where $\mathbf{\Lambda}$ is diagonal. The new column vector $\mathbf{V}$ satisfies the equation

$$\partial \mathbf{V}/\partial z = [\mathbf{T}^{-1}\mathbf{A}\mathbf{T} - \mathbf{T}^{-1}\partial \mathbf{T}/\partial z]\, \mathbf{V}. \tag{7}$$

If the elastic properties are locally uniform then $\mathbf{T}$ is there independent of z so that we have

$$\partial \mathbf{V}/\partial z = \mathbf{\Lambda V} \tag{8}$$

with the solution $\mathbf{V}(z) = \exp[\mathbf{\Lambda}(z - z_0)]\, \mathbf{V}(z_0)$. The diagonal entries of $\mathbf{\Lambda}$ are just the eigenvalues of the matrix $\mathbf{A}$ and thus

$$\mathbf{\Lambda} = \mathrm{diag}\,[-i\nu_\alpha, -i\nu_\beta, i\nu_\alpha, i\nu_\beta],$$

where

$$\left.\begin{aligned} \nu_\alpha &= (\omega^2/\alpha^2 - k^2)^{1/2}, \\ \nu_\beta &= (\omega^2/\beta^2 - k^2)^{1/2}. \end{aligned}\right\} \quad \mathrm{Im}\, \nu_\alpha, \nu_\beta \geqslant 0. \tag{9}$$

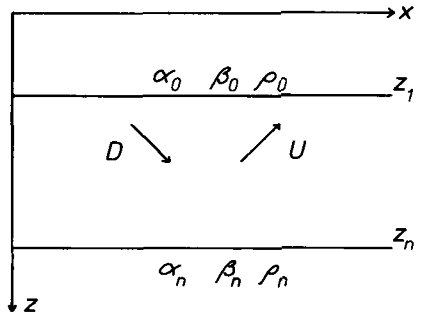

Figure 1. Structure considered for reflection and transmission problems, a layer sequence sandwiched between two uniform half spaces in $z < z_1$ and $z > z_n$. Also illustrated is the convention for up and downgoing waves.

Thus the phase factor appearing in the solution of (8) has the form

$$\exp[\mathbf{\Lambda}(z-z_0)] = \mathrm{diag}[\![\exp[-i\nu_\alpha(z-z_0)], \exp[-i\nu_\beta(z-z_0)], \exp[i\nu_\alpha(z-z_0)], \exp[i\nu_\beta(z-z_0)]]\!] \tag{10}$$

and since z increases with increasing depth we see that the exponentials correspond to the phase differences to be expected for up and downgoing P and S waves. We may therefore identify the elements of $\mathbf{V}$

$$\mathbf{V} = [\phi_U, \psi_U, \phi_D, \psi_D]^T = [\mathbf{v}_U, \mathbf{v}_D]^T \tag{11}$$

where ϕ, ψ are the amplitudes of P and S waves respectively and the suffices U, D represent upgoing and downgoing waves (as in Kennett (1974a) – see Fig. 1). The columns of the matrix $\mathbf{T}$ are the eigenvectors of the matrix $\mathbf{A}$, and from our identification of the elements of $\mathbf{V}$ these correspond to 'elementary' stress–displacement vectors for the different wave types. We write

$$\mathbf{T} = [\mathbf{b}_P^U, \mathbf{b}_S^U, \mathbf{b}_P^D, \mathbf{b}_S^D] \tag{12}$$

and the vectors $\mathbf{b}$ take the form

$$\begin{aligned}\mathbf{b}_P^{U,D} &= \epsilon_\alpha^{U,D}[k, \mp\nu_\alpha, \mp 2i\mu k\nu_\alpha, i\mu\Gamma]^T,\\ \mathbf{b}_S^{U,D} &= \epsilon_\beta^{U,D}[\pm i\nu_\beta, ik, \mu\Gamma, \pm 2\mu k\nu_\beta]^T,\end{aligned} \tag{13}$$

with $\mu\Gamma = \rho\omega^2 - 2\mu k^2$, $\mu = \rho\beta^2$. We have a free choice of the scaling parameters ϵ_α and ϵ_β and therefore arrange to normalize the $\mathbf{b}$ vectors with respect to the energy flux in the z direction, i.e. across planes z = constant,

$$\mathscr{F}(\mathbf{B}) = \left(\frac{i\omega}{4}\right)[\bar{u}\bar{\tau}_{xz}^* + \bar{w}\bar{\tau}_{zz}^* - \bar{u}^*\bar{\tau}_{xz} - \bar{w}^*\bar{\tau}_{zz}]. \tag{14}$$

We take

$$\epsilon_\alpha^U = \epsilon_\alpha^D = \left(\frac{2}{\rho\omega^3\nu_\alpha}\right)^{1/2}, \quad \epsilon_\beta^U = \epsilon_\beta^D = \left(\frac{2}{\rho\omega^3\nu_\beta}\right)^{1/2}, \tag{15}$$

so that for propagating waves

$$\mathscr{F}(\mathbf{b}_{P,S}^U) = -1, \quad \mathscr{F}(\mathbf{b}_{P,S}^D) = 1, \tag{16}$$

whilst for evanescent waves

$$\mathscr{F}(\mathbf{b}_{P,S}^U) = 0, \quad \mathscr{F}(\mathbf{b}_{P,S}^D) = 0, \tag{17}$$

confirming that an evanescent wave carries no energy in the z direction.

3 Basic propagation invariants

For convenience in subsequent notation we will introduce the 2×2 matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{18}$$

and the 4×4 composite matrices

$$\mathbf{M} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ -\boldsymbol{\sigma}_2 & 0 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ -\boldsymbol{\sigma}_1 & 0 \end{pmatrix}. \tag{19}$$

The propagation characteristics of the stress–displacement vector $\mathbf{B}$ are governed by the coefficient matrix $\mathbf{A}$ (4) through the differential equation (2). The propagation invariants are thus determined by the properties of the matrix $\mathbf{A}$.

3.1 DISSIPATIVE MEDIA

Dissipation may be introduced into the seismic wave equations by allowing the seismic wave velocities to take on complex values. However, even in the presence of dissipation the coefficient matrix $\mathbf{A}$ satisfies

$$\mathbf{M}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{M} = 0 \tag{20}$$

a relation which depends only on the nature of the equations of motion and the stress–strain equations. Thus if we introduce a composition $\mathscr{G}$ of two wavefields $\mathbf{B}$ and $\mathbf{B}'$, which both satisfy the differential equation (2),

$$\mathscr{G}(\mathbf{B}, \mathbf{B}') = \frac{i\omega}{4} \mathbf{B}^{\mathrm{T}}\mathbf{M}\mathbf{B}' \tag{21}$$

we have

$$\begin{aligned} \frac{4}{i\omega}\, \partial\mathscr{G}(\mathbf{B}, \mathbf{B}')/\partial z &= (\partial\mathbf{B}^{\mathrm{T}}/\partial z)\,\mathbf{M}\mathbf{B}' + \mathbf{B}^{\mathrm{T}}\mathbf{M}(\partial\mathbf{B}'/\partial z) \\ &= \mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{M} + \mathbf{M}\mathbf{A})\,\mathbf{B}' = 0, \end{aligned} \tag{22}$$

using (21). Since both $\mathbf{B}$ and $\mathbf{B}'$ will be continuous across planes z = constant, including planes of discontinuity in material properties, so also will be $\mathscr{G}(\mathbf{B}, \mathbf{B}')$. Thus if we consider any two levels z_1 and z_n in the horizontally stratified medium, the constancy of $\mathscr{G}$ implies that

$$\mathscr{G}(\mathbf{B}, \mathbf{B}')|_{z_1} = \mathscr{G}(\mathbf{B}, \mathbf{B}')|_{z_n}. \tag{23}$$

Further the eigenvector matrix $\mathbf{T}$ has the property for a uniform medium

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = -\frac{4}{i\omega}\mathbf{N} \tag{24}$$

as may be verified directly from the definition of $\mathbf{T}$ in terms of the elementary wave vectors

b. Thus if the levels z_1 and z_n lie within regions of uniform material properties so that we may write

$$\begin{aligned} \mathbf{B}(z_1) &= \mathbf{T}_0\mathbf{V}_0, \quad \mathbf{B}'(z_1) = \mathbf{T}_0\mathbf{V}_0' \\ \mathbf{B}(z_n) &= \mathbf{T}_n\mathbf{V}_n, \quad \mathbf{B}'(z_n) = \mathbf{T}_n\mathbf{V}_n' \end{aligned} \tag{25}$$

in terms of the decomposition of the wavefields into up and downgoing parts, then

$$\mathbf{V}_0^{\mathrm{T}}\mathbf{N}\mathbf{V}_0'|_{z_1} = \mathbf{V}_n^{\mathrm{T}}\mathbf{N}\mathbf{V}_n'|_{z_n}. \tag{26}$$

3.2 PERFECTLY ELASTIC MEDIA

If all velocities are required to be real, we have an additional property for the coefficient matrix **A**, that

$$\mathbf{N}\mathbf{A} + \mathbf{A}^{\dagger}\mathbf{N} = 0 \tag{27}$$

where the dagger (†) indicates the Hermitian adjoint, i.e. the complex conjugate of the transpose. This relation is a consequence of the derivation of **A** from a real Hamiltonian for perfectly elastic media (Kennett 1974b; Woodhouse 1974b). We introduce a second composition $\mathscr{H}$ of the wavefields **B** and **B**$'$

$$\mathscr{H}(\mathbf{B}, \mathbf{B}') = \frac{\mathrm{i}\omega}{4}\,\mathbf{B}^{\dagger}\mathbf{N}\mathbf{B}' \tag{28}$$

which for $\mathbf{B} = \mathbf{B}'$ reduces to the energy flux $\mathscr{F}$ in the z direction, and we may show by analogous reasoning to (22) that $\mathscr{H}$ is a constant for all levels in the medium and thus

$$\mathscr{H}(\mathbf{B}, \mathbf{B}')|_{z_1} = \mathscr{H}(\mathbf{B}, \mathbf{B}')|_{z_n}. \tag{29}$$

The corresponding property of the eigenvector matrix **T** is more involved, and reflects as we might expect from (16), (17) the distinction between travelling and evanescent waves. We find that for a uniform medium

$$\mathbf{T}^{\dagger}\mathbf{N}\mathbf{T} = -\frac{4}{\mathrm{i}\omega}\,\mathbf{E} \tag{30}$$

where if both P and S waves are travelling, i.e. ν_α and ν_β are real

$$\mathbf{E} = \begin{pmatrix} -\boldsymbol{\sigma}_1 & 0 \\ 0 & \boldsymbol{\sigma}_1 \end{pmatrix} \tag{31a}$$

but if the P wave becomes evanescent whilst the S wave still propagates, i.e. ν_α imaginary and ν_β real

$$\mathbf{E} = \tfrac{1}{2}\begin{pmatrix} -(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) & \mathrm{i}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \\ -\mathrm{i}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) & (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \end{pmatrix}. \tag{31b}$$

If however both P and S waves are evanescent (ν_α and ν_β are imaginary) only off diagonal terms are present

$$\mathbf{E} = \begin{pmatrix} 0 & \mathrm{i}\boldsymbol{\sigma}_1 \\ -\mathrm{i}\boldsymbol{\sigma}_1 & 0 \end{pmatrix}. \tag{31c}$$

If as in (25) we assume that the levels z_1 and z_n lie in uniform regions and again make a decomposition in terms of the amplitude vectors $\mathbf{V}$ for the up and downgoing wave components we find

$$\mathbf{V}_0^\dagger \mathbf{E}_0 \mathbf{V}_0'|_{z_1} = \mathbf{V}_n^\dagger \mathbf{E}_n \mathbf{V}_n'|_{z_n}. \tag{32}$$

4 Symmetry relations for reflection and transmission

We consider a sequence of isotropic elastic layers bounded by isotropic half spaces above the plane $z = z_1$ and below the plane $z = z_n$. For our single plane wave component the stress-displacement fields at the top and bottom of the sequence are related by (Gilbert & Backus 1966; Kennett 1972)

$$\mathbf{B}(k, z_1) = \mathbf{P}(k, z_1, z_n)\, \mathbf{B}(k, z_n) \tag{33}$$

where $\mathbf{P}$ is the resultant propagator which may be decomposed into layer contributions

$$\mathbf{P}(k, z_1, z_n) = \mathbf{P}_1(k, z_1, z_2)\, \mathbf{P}_2(k, z_2, z_3) \ldots \mathbf{P}_{n-1}(k, z_{n-1}, z_n). \tag{34}$$

For uniform layers these layer propagators may be found from equation (8) and are identical to the Haskell layer matrices.

The stress–displacement vectors at the top and bottom of the sequence may each be expressed in terms of upgoing and downgoing waves so that

$$\mathbf{T}_0(k)\, \mathbf{V}_0(k, z_1 -) = \mathbf{P}(k, z_1, z_n)\, \mathbf{T}_n(k)\, \mathbf{V}_n(k, z_n +) \tag{35}$$

and thus from (11)

$$\begin{bmatrix} \mathbf{v}_\mathrm{U}(k, z_1 -) \\ \mathbf{v}_\mathrm{D}(k, z_1 -) \end{bmatrix} = \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \begin{bmatrix} \mathbf{v}_\mathrm{U}(k, z_n +) \\ \mathbf{v}_\mathrm{D}(k, z_n +) \end{bmatrix} \tag{36}$$

in terms of the 2×2 partitions of the matrix

$$\mathbf{Q} = \mathbf{T}_0^{-1}(k)\, \mathbf{P}(k, z_1, z_n)\, \mathbf{T}_n(k). \tag{36a}$$

As in Kennett (1974a) we introduce matrices of reflection and transmission coefficients, e.g.

$$R_\mathrm{D} = \begin{pmatrix} r_{PP}^\mathrm{D} & r_{PS}^\mathrm{D} \\ r_{SP}^\mathrm{D} & r_{SS}^\mathrm{D} \end{pmatrix}, \quad T_\mathrm{D} = \begin{pmatrix} t_{PP}^\mathrm{D} & t_{PS}^\mathrm{D} \\ t_{SP}^\mathrm{D} & t_{SS}^\mathrm{D} \end{pmatrix} \tag{37}$$

and in terms of the subpartitions of $\mathbf{Q}$

$$\begin{aligned} T_\mathrm{D} &= Q_{22}^{-1}, \\ R_\mathrm{D} &= Q_{12} Q_{22}^{-1}, \\ T_\mathrm{U} &= Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}, \\ R_\mathrm{U} &= -Q_{22}^{-1} Q_{21}. \end{aligned} \tag{38}$$

Alternatively we may look at the reflection and transmission properties more directly in terms of the stress–displacement field by making use of the conservation relation (23), applied at the top and bottom of the sequence. In terms of the amplitude vectors $\mathbf{V}$ from (26) we have

$$\mathbf{V}_0^\mathrm{T} \mathbf{N} \mathbf{V}_0'|_{z_1} = \mathbf{V}_n^\mathrm{T} \mathbf{N} \mathbf{V}_n'|_{z_n} \tag{39}$$

for even dissipative media. This relation is sufficient to enable us to determine the symmetries in reflection and transmission coefficients by taking specific choices for the amplitude vectors **V**. If we consider a plane wave incident on the interface $z = z_1$ from above, then corresponding to incident *P* and *S* waves we may construct

$$\mathbf{V}_P^{\mathrm{D}}(z_1) = [r_{PP}^{\mathrm{D}}, r_{SP}^{\mathrm{D}}, 1, 0]^{\mathrm{T}} \qquad (40a)$$
$$\mathbf{V}_S^{\mathrm{D}}(z_1) = [r_{PS}^{\mathrm{D}}, r_{SS}^{\mathrm{D}}, 0, 1]^{\mathrm{T}}$$

and at the bottom of the sequence

$$\mathbf{V}_P^{\mathrm{D}}(z_n) = [0, 0, t_{PP}^{\mathrm{D}}, t_{SP}^{\mathrm{D}}]^{\mathrm{T}} \qquad (40b)$$
$$\mathbf{V}_S^{\mathrm{D}}(z_n) = [0, 0, t_{PS}^{\mathrm{D}}, t_{SS}^{\mathrm{D}}]^{\mathrm{T}}.$$

A similar set of vectors can be constructed for incidence from below z_n, e.g.

$$\mathbf{V}_P^{\mathrm{U}}(z_1) = [t_{PP}^{\mathrm{U}}, t_{SP}^{\mathrm{U}}, 0\ 0]^{\mathrm{T}} \qquad (41)$$
$$\mathbf{V}_P^{\mathrm{U}}(z_n) = [1, 0, r_{PP}^{\mathrm{U}}, r_{SP}^{\mathrm{U}}]^{\mathrm{T}}.$$

We now choose $\mathbf{V}_0, \mathbf{V}_0'$ to be any pair of vectors from the set $[\mathbf{V}_P^{\mathrm{D}}(z_1), \mathbf{V}_S^{\mathrm{D}}(z_1), \mathbf{V}_P^{\mathrm{U}}(z_1), \mathbf{V}_S^{\mathrm{U}}(z_1)]$ and $\mathbf{V}_n, \mathbf{V}_n'$ to be the corresponding pair from $[\mathbf{V}_P^{\mathrm{D}}(z_n), \mathbf{V}_S^{\mathrm{D}}(z_n), \mathbf{V}_P^{\mathrm{U}}(z_n), \mathbf{V}_S^{\mathrm{U}}(z_n)]$ and employ the relation (39). Thus for example if we take $\mathbf{V}_S^{\mathrm{D}}$ and $\mathbf{V}_P^{\mathrm{D}}$ at each of the interfaces we obtain

$$[r_{PS}^{\mathrm{D}} - r_{SP}^{\mathrm{D}}]|_{z_1} = 0. \qquad (42)$$

Similarly using the other pairs of vectors we obtain the symmetry relations for reflection and transmission through an arbitrary velocity structure between the planes $z = z_1$ and $z = z_n$

$$r_{PS}^{\mathrm{D}} = r_{SP}^{\mathrm{D}}$$
$$r_{PS}^{\mathrm{U}} = r_{SP}^{\mathrm{U}}$$

and

$$t_{PP}^{\mathrm{D}} = t_{PP}^{\mathrm{U}} \qquad (43)$$
$$t_{SP}^{\mathrm{D}} = t_{PS}^{\mathrm{U}}$$
$$t_{PS}^{\mathrm{D}} = t_{SP}^{\mathrm{U}}$$
$$t_{SS}^{\mathrm{D}} = t_{SS}^{\mathrm{U}}.$$

The complete symmetry of these relations arises from the choice of elementary **b** vectors to correspond to the propagation directions of upgoing and downgoing *P* and *S* waves and the normalization employed to refer all these solutions to a common energy flux in the *z* direction.

These relations may be shown to be equivalent to those presented by Lapwood & Hudson (1975), but their choice of up and downgoing *P*- and *S*-wave solutions was less convenient.

The symmetry relations (43) can also be represented in terms of the reflection and transmission coefficient matrices introduced in (37), so that

$$\boldsymbol{R}_{\mathrm{D}} = \boldsymbol{R}_{\mathrm{D}}^{\mathrm{T}}$$
$$\boldsymbol{R}_{\mathrm{U}} = \boldsymbol{R}_{\mathrm{U}}^{\mathrm{T}} \qquad (44)$$
$$\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{T}_{\mathrm{U}}^{\mathrm{T}}$$

and these relations will hold for both travelling and evanescent waves in a dissipative medium.

The effects of change of normalization of the elementary stress–displacement vectors **b** on these reflection and transmission coefficients is discussed in the Appendix.

5 Reflection and transmission at a single interface

We now specialize the preceding results to the case of a single interface separating two elastic media with properties $\alpha_0, \beta_0, \rho_0$ and $\alpha_1, \beta_1, \rho_1$. In this case equation (31) reduces to

$$\mathbf{V}_0(k, z_1-) = \mathbf{T}_0^{-1}(k)\,\mathbf{T}_1(k)\,\mathbf{V}_1(k, z_1+) = \mathbf{Q}\mathbf{V}_1(k, z_1+) \tag{45}$$

and as we have seen the reflection and transmission coefficient matrices may be related to the 2×2 subpartitions of $\mathbf{Q}$ by equation (36). For simplicity we will write

$$\mathbf{T}_0^{-1} = \mathbf{U}, \quad \mathbf{T}_1 = \mathbf{T} \tag{46}$$

and then in terms of the partitions of the matrices, $\mathbf{Q}, \mathbf{T}, \mathbf{U}$ we have

$$\begin{aligned} Q_{11} &= U_{11}T_{11} + U_{12}T_{21}, \quad Q_{22} = U_{21}T_{12} + U_{22}T_{22},\\ Q_{12} &= U_{11}T_{12} + U_{12}T_{22}, \quad Q_{21} = U_{21}T_{11} + U_{22}T_{21}. \end{aligned} \tag{47}$$

However from the properties of the 'elementary' **b** vectors (13)

$$\begin{aligned} &T_{11} = \boldsymbol{\sigma}_2 T_{12}, \quad T_{12} = \boldsymbol{\sigma}_2 T_{11}\\ &T_{22} = -\boldsymbol{\sigma}_2 T_{21}, \; T_{21} = -\boldsymbol{\sigma}_2 T_{22}\\ &U_{21} = U_{11}\boldsymbol{\sigma}_2, \quad U_{22} = -U_{12}\boldsymbol{\sigma}_2 \end{aligned} \tag{48}$$

where $\boldsymbol{\sigma}_2$ is the matrix introduced in (18). On substituting these relations into (47) we find

$$\begin{aligned} Q_{22} &= U_{11}\boldsymbol{\sigma}_2 T_{12} + U_{12}(-\boldsymbol{\sigma}_2)\,T_{22} = U_{11}T_{11} + U_{12}T_{21}\\ Q_{21} &= U_{11}\boldsymbol{\sigma}_2 T_{11} + U_{12}(-\boldsymbol{\sigma}_2)\,T_{21} = U_{11}T_{12} + U_{12}T_{22}, \end{aligned} \tag{49}$$

i.e. we have

$$Q_{11} = Q_{22}, \quad Q_{12} = Q_{21}. \tag{50}$$

Thus using the relations (34) we have for a single interface

$$\begin{aligned} &\boldsymbol{R}_{\mathrm{U}} = -Q_{22}^{-1}Q_{21} = -T_{\mathrm{D}}Q_{12} = -T_{\mathrm{D}}\boldsymbol{R}_{\mathrm{D}}T_{\mathrm{D}}^{-1}\\ &\text{and}\\ &T_{\mathrm{U}}T_{\mathrm{D}} = Q_{11}Q_{22}^{-1} - Q_{12}Q_{22}^{-1}Q_{21}Q_{22}^{-1} = \boldsymbol{I} - \boldsymbol{R}_{\mathrm{D}}\boldsymbol{R}_{\mathrm{D}} \end{aligned} \tag{51}$$

where $\boldsymbol{I} = \boldsymbol{\sigma}_1$ is the 2×2 unit matrix. For a single interface we therefore have the usual symmetries

$$\begin{aligned} &\boldsymbol{R}_{\mathrm{D}} = \boldsymbol{R}_{\mathrm{D}}^{\mathrm{T}}, \; \boldsymbol{R}_{\mathrm{U}} = \boldsymbol{R}_{\mathrm{U}}^{\mathrm{T}}, \; T_{\mathrm{D}} = T_{\mathrm{U}}^{\mathrm{T}}\\ &\text{and in addition}\\ &\boldsymbol{R}_{\mathrm{U}} = -T_{\mathrm{D}}\boldsymbol{R}_{\mathrm{D}}T_{\mathrm{D}}^{-1}; \quad T_{\mathrm{U}}T_{\mathrm{D}} = \boldsymbol{I} - \boldsymbol{R}_{\mathrm{D}}\boldsymbol{R}_{\mathrm{D}}. \end{aligned} \tag{52}$$

These relations were first derived by Frazier (1970) by a rather different and less general treatment.

The interface matrix **Q** which may in general be written as (*cf.* Kennett 1974a)

$$\mathbf{Q} = \left(\begin{array}{c|c} T_{\mathrm{U}} - R_{\mathrm{D}} T_{\mathrm{D}}^{-1} R_{\mathrm{U}} & R_{\mathrm{D}} T_{\mathrm{D}}^{-1} \\ \hline -T_{\mathrm{D}}^{-1} R_{\mathrm{U}} & T_{\mathrm{D}}^{-1} \end{array}\right) \tag{53}$$

reduces by virtue of the symmetry relations (52) to

$$\mathbf{Q} = \left(\begin{array}{c|c} T_{\mathrm{D}}^{-1} & -T_{\mathrm{D}}^{-1} R_{\mathrm{U}} \\ \hline -T_{\mathrm{D}}^{-1} R_{\mathrm{U}} & T_{\mathrm{D}}^{-1} \end{array}\right) \tag{54}$$

with our choice of energy normalized reflection and transmission coefficients.

6 Unitary relations for reflection and transmission

We will consider again the model of a sequence of layers bounded by uniform half spaces above the plane $z = z_1$ and below the plane $z = z_n$, used in our discussion of symmetry relations.

For a perfectly elastic medium we use the conservation relation (29) for the form $\mathscr{H}(\mathbf{B}, \mathbf{B}')$ for any two stress–displacement vectors $\mathbf{B}$ and $\mathbf{B}'$. We will apply the conservation relation at the top and bottom of the sequence of layers. In terms of the amplitude vectors $\mathbf{V}$ from (32) we have

$$\mathbf{V}_0^{\dagger} \mathbf{E}_0 \mathbf{V}_0' |_{z_1} = \mathbf{V}_n^{\dagger} \mathbf{E}_n \mathbf{V}_n' |_{z_n} \tag{55}$$

where the matrix $\mathbf{E}$ is defined as in (31). By analogy with the treatment for the symmetry relation we choose for $\mathbf{V}, \mathbf{V}'$ pairs of vectors from the amplitude vectors appropriate to incident down or upgoing P and S waves $\mathbf{V}_P^{\mathrm{D}}, \mathbf{V}_S^{\mathrm{D}}, \ldots$ (40).

We will define matrices whose columns are the vectors (40)

$$\begin{aligned} \boldsymbol{\Phi}_0 &= [\mathbf{V}_P^{\mathrm{D}}(z_1),\ \ \mathbf{V}_S^{\mathrm{D}}(z_1),\ \ \mathbf{V}_P^{\mathrm{U}}(z_1),\ \ \mathbf{V}_S^{\mathrm{U}}(z_1)] \\ \boldsymbol{\Phi}_n &= [\mathbf{V}_P^{\mathrm{D}}(z_n),\ \ \mathbf{V}_S^{\mathrm{D}}(z_n),\ \ \mathbf{V}_P^{\mathrm{U}}(z_n),\ \ \mathbf{V}_S^{\mathrm{U}}(z_n)] \end{aligned} \tag{56}$$

and also a matrix $\mathscr{R}$ composed of the complete set of reflection and transmission coefficients for the layer sequence

$$\mathscr{R} = \begin{bmatrix} R_{\mathrm{D}} & T_{\mathrm{U}} \\ T_{\mathrm{D}} & R_{\mathrm{U}} \end{bmatrix} \tag{57}$$

which will be symmetric, i.e. $\mathscr{R}^{\mathrm{T}} = \mathscr{R}$ by virtue of the symmetry relations (44). We will introduce the factorization matrices

$$\begin{aligned} J_{11} &= \begin{pmatrix} \boldsymbol{\sigma}_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad J_{12} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ 0 & 0 \end{pmatrix}, \\ J_{21} &= \begin{pmatrix} 0 & 0 \\ \boldsymbol{\sigma}_1 & 0 \end{pmatrix}, \qquad J_{22} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\sigma}_1 \end{pmatrix}, \end{aligned} \tag{58}$$

and then from the definition of the $\mathbf{V}_P^{\mathrm{D}}, \mathbf{V}_S^{\mathrm{D}} \ldots$ vectors (40) we may write the matrix $\boldsymbol{\Phi}_0$ as

$$\boldsymbol{\Phi}_0 = J_{11} \mathscr{R} + J_{21}$$

and (59)

$$\boldsymbol{\Phi}_n = J_{22} \mathscr{R} + J_{12}.$$

The process of selecting pairs of vectors from the set of vectors (40) can be written in the form

$$\boldsymbol{\Phi}_0^\dagger \mathrm{E}_0 \boldsymbol{\Phi}_0 = \boldsymbol{\Phi}_n^\dagger \mathrm{E}_n \boldsymbol{\Phi}_n \tag{60}$$

so that using the representations (59) we have

$$(\mathscr{R}^\dagger J_{11} + J_{12})\, \mathrm{E}_0 (J_{11}\mathscr{R} + J_{21}) = (\mathscr{R}^\dagger J_{22} + J_{21})\, \mathrm{E}_n (J_{22}\mathscr{R} + J_{12}). \tag{61}$$

We may rearrange this equation into the following form

$$\begin{aligned}\mathscr{R}^\dagger (J_{22}\mathrm{E}_n J_{22} - J_{11}\mathrm{E}_0 J_{11})\mathscr{R} + (J_{21}\mathrm{E}_n J_{22} - J_{12}\mathrm{E}_0 J_{11})\mathscr{R} + \mathscr{R}^\dagger (J_{22}\mathrm{E}_n J_{12} - J_{11}\mathrm{E}_0 J_{21})\\ + (J_{21}\mathrm{E}_n J_{12} - J_{12}\mathrm{E}_0 J_{21}) = 0.\end{aligned} \tag{62}$$

Each of the expressions in brackets may be represented in terms of a single matrix $\mathbf{J}$

$$\begin{aligned} J_{22}\mathrm{E}_n J_{22} - J_{11}\mathrm{E}_0 J_{11} &= \mathbf{J}\\ J_{21}\mathrm{E}_n J_{22} - J_{12}\mathrm{E}_0 J_{11} &= \mathrm{i}\bar{\mathbf{J}}\\ J_{22}\mathrm{E}_n J_{12} - J_{11}\mathrm{E}_0 J_{21} &= -\mathrm{i}\bar{\mathbf{J}}\\ J_{21}\mathrm{E}_n J_{12} - J_{12}\mathrm{E}_0 J_{21} &= -\mathbf{J}\end{aligned} \tag{63a}$$

where $\mathbf{J}$ is a diagonal matrix with entries which are either 1 or 0, determined by the conditions

$$\begin{aligned} j_{11} &= 1 \quad \text{iff} \quad \nu_{\alpha 0} \text{ is real,}\\ j_{22} &= 1 \quad \text{iff} \quad \nu_{\beta 0} \text{ is real,}\\ j_{33} &= 1 \quad \text{iff} \quad \nu_{\alpha n} \text{ is real,}\\ j_{44} &= 1 \quad \text{iff} \quad \nu_{\beta n} \text{ is real,}\end{aligned} \tag{63b}$$

where iff indicates a necessary and sufficient condition and

$$\bar{\mathbf{J}} = \mathbf{I} - \mathbf{J} \tag{63c}$$

with $\mathbf{I}$ the 4×4 unit matrix.

In terms of the matrix $\mathbf{J}$ the conservation relation (62) takes the form

$$\mathscr{R}^\dagger \mathbf{J} \mathscr{R} + \mathrm{i}(\bar{\mathbf{J}}\mathscr{R} - \mathscr{R}^\dagger \bar{\mathbf{J}}) = \mathbf{J} \tag{64}$$

which is our basic unitarity relation.

Since the radicals $\nu_{\alpha 0}$, $\nu_{\beta 0}$, $\nu_{\alpha n}$, $\nu_{\beta n}$ will only be real when the corresponding *P* or *S* wave is a travelling wave rather than being evanescent we see that $\mathbf{J}$ has the role of a projection operator onto travelling waves and $\bar{\mathbf{J}}$ that of a projection operator onto evanescent waves. Thus using these projectors we may isolate particular types of behaviour. We may note that, as we would expect, the joint operator

$$\begin{aligned}&\mathbf{J}\bar{\mathbf{J}} = \bar{\mathbf{J}}\mathbf{J} = 0\\ &\text{whilst}\\ &\mathbf{J}\mathbf{J} = \mathbf{J}, \quad \bar{\mathbf{J}}\bar{\mathbf{J}} = \bar{\mathbf{J}}.\end{aligned} \tag{65}$$

On applying the projector $\mathbf{J}$ to the basic unitarity relation (64) we obtain

$$(\mathbf{J}\mathscr{R}\mathbf{J})^\dagger (\mathbf{J}\mathscr{R}\mathbf{J}) = \mathbf{J} \tag{66}$$

which shows that the subpartition of the overall reflection and transmission matrix $\mathscr{R}$ corresponding to travelling waves is unitary and this reflects the conservation of energy amongst the travelling waves. Similarly applying the evanescent projector $\bar{\mathbf{J}}$ to the basic relation (64) we find that

$$(\mathbf{J}\mathscr{R}\bar{\mathbf{J}})^\dagger(\mathbf{J}\mathscr{R}\bar{\mathbf{J}}) = \mathrm{i}\,\{(\bar{\mathbf{J}}\mathscr{R}\bar{\mathbf{J}})^\dagger - (\bar{\mathbf{J}}\mathscr{R}\bar{\mathbf{J}})\} = 2\,\mathrm{Im}\,(\bar{\mathbf{J}}\mathscr{R}\bar{\mathbf{J}}) \tag{67}$$

where we have used the symmetry of the matrix $\mathscr{R}$. Additional relations may also be derived by using both projectors

$$(\mathbf{J}\mathscr{R}\mathbf{J})^\dagger(\mathbf{J}\mathscr{R}\bar{\mathbf{J}}) = \mathrm{i}\,(\bar{\mathbf{J}}\mathscr{R}\mathbf{J})^\dagger = \mathrm{i}\,(\mathbf{J}\mathscr{R}\bar{\mathbf{J}})^*$$

and (68)

$$(\mathbf{J}\mathscr{R}\bar{\mathbf{J}})^\dagger(\mathbf{J}\mathscr{R}\mathbf{J}) = -\mathrm{i}\,(\bar{\mathbf{J}}\mathscr{R}\mathbf{J}).$$

The set of equations (66–68) then enable us to establish a range of interconnections between the reflection and transmission coefficients for a sequence of perfectly elastic layers.

6.1 TRAVELLING WAVES

If both the P and S waves for the horizontal wavenumber k are travelling waves at the top and base of the stack of layers then $\nu_{\alpha 0}, \nu_{\beta 0}, \nu_{\alpha n}, \nu_{\beta n}$ will all be real and so

$$\mathbf{J} = \mathbf{I}, \quad \bar{\mathbf{J}} = 0,$$

with the result that (64) becomes (69)

$$\mathscr{R}^\dagger\mathscr{R} = \mathbf{I}.$$

Thus the reflection and transmission coefficient matrix $\mathscr{R}$ is unitary, a result first obtained by Woodhouse (1974a).

6.2 EVANESCENT WAVES

If on the other hand for the wavenumber k of the plane wave component under consideration both P waves and S waves are evanescent throughout the layer seqence all the radicals $\nu_{\alpha 0}, \nu_{\beta 0}, \nu_{\alpha n}, \nu_{\beta n}$ are imaginary so that

$$\mathbf{J} = 0, \quad \bar{\mathbf{J}} = \mathbf{I}$$

and from (64) (70)

$$\mathscr{R} - \mathscr{R}^\dagger = 0.$$

Since in addition the matrix $\mathscr{R}$ is symmetric the whole matrix $\mathscr{R}$ is real, i.e. all the reflection and transmission coefficients are real.

6.3 TURNING POINTS FOR BOTH P AND S WAVES

When for the wavenumber k both P and S waves incident at the top of the layer sequence are turned back by the velocity structure (Fig. 2a) $\nu_{\alpha n}, \nu_{\beta n}$ will be imaginary and so

$$\mathbf{J} = J_{11}, \quad \bar{\mathbf{J}} = J_{22}.$$

Thus from equation (64) we have (71)

$$\boldsymbol{R}_\mathrm{D}^\dagger\boldsymbol{R}_\mathrm{D} = \boldsymbol{I} = \boldsymbol{\sigma}_1$$

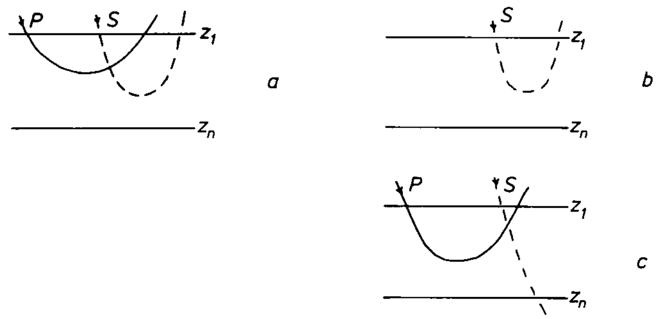

Figure 2. Schematic representation of propagation configurations. (a) Turning point for both P and S waves. (b) Evanescent P wave, turning point for S wave. (c) Turning point for P wave, travelling S wave.

using the definition of the matrix $\mathscr{R}$ (57), thus the reflection coefficients for downward propagation form a unitary matrix, and from the symmetry relations (44) the matrix $\mathbf{R}_\mathrm{D}$ is symmetric.

The unitary condition (71) implies that

$$|r^\mathrm{D}_{PP}|^2 + |r^\mathrm{D}_{SP}|^2 = |r^\mathrm{D}_{PS}|^2 + |r^\mathrm{D}_{SS}|^2 = 1$$

and (72)

$$|r^\mathrm{D}_{PP}| = |r^\mathrm{D}_{SS}|.$$

Alternatively from equations (67) and (68), using the symmetries (44) we find

$$\begin{aligned} \mathbf{T}^*_\mathrm{D}\mathbf{T}^\mathrm{T}_\mathrm{D} &= 2\,\mathrm{Im}\,\mathbf{R}_\mathrm{U} \\ \mathbf{R}^*_\mathrm{D}\mathbf{T}_\mathrm{U} &= \mathrm{i}\mathbf{T}^*_\mathrm{U}. \end{aligned} \tag{73}$$

6.4 EVANESCENT P, TURNING POINT FOR S

If only an S wave can travel at the top of the stack for the particular horizontal wavenumber k and is turned back by the velocity structure (Fig. 2b).

$$\mathbf{J} = \mathrm{diag}\,\{0, 1, 0, 0\}, \quad \bar{\mathbf{J}} = \mathrm{diag}\,\{1, 0, 1, 1\}$$

and from (66) (74)

$$|r^\mathrm{D}_{SS}| = 1.$$

Alternatively we find from equation (68) that

$$r^{\mathrm{D}*}_{SS}[r^\mathrm{D}_{SP}, t^\mathrm{D}_{PS}, t^\mathrm{D}_{SS}] = \mathrm{i}\,[r^{\mathrm{D}*}_{SP}, t^{\mathrm{D}*}_{PS}, t^{\mathrm{D}*}_{SS}]$$

and thus (75)

$$\arg(r^\mathrm{D}_{SP}) = \arg(t^\mathrm{D}_{PS}) = \arg(t^\mathrm{D}_{SS}) = \pi/4 + \tfrac{1}{2}\arg(r^\mathrm{D}_{SS})$$

where the phase is only determined to within a multiple of 2π; and from (67)

$$|r^\mathrm{D}_{SP}|^2 = 2\,\mathrm{Im}\,r^\mathrm{D}_{PP}$$

$$|t^\mathrm{D}_{PS}|^2 = 2\,\mathrm{Im}\,r^\mathrm{U}_{PP}, \quad |t^\mathrm{D}_{SS}|^2 = 2\,\mathrm{Im}\,r^\mathrm{U}_{SS}$$

and (76)

$$|t^\mathrm{D}_{PS}|\,|r^\mathrm{D}_{SP}| = 2\,|\mathrm{Im}\,t^\mathrm{D}_{PP}|, \quad |t^\mathrm{D}_{SS}|\,|r^\mathrm{D}_{SP}| = |2\,\mathrm{Im}\,t^\mathrm{D}_{SP}|$$

together with a number of similar relations.

6.5 TURNING POINT FOR *P* WAVES

If the only radical which is imaginary is $\nu_{\alpha n}$ (Fig. 2c) the roles of **J** and $\bar{\mathbf{J}}$ in (74) are reversed

$\mathbf{J} = \mathrm{diag}\{1, 1, 0, 1\}, \quad \bar{\mathbf{J}} = \mathrm{diag}\{0, 0, 1, 0\}$

and now equation (66) yields that the matrix

$$\mathbf{U} = \begin{pmatrix} r^{\mathrm{D}}_{PP} & r^{\mathrm{D}}_{PS} & t^{\mathrm{U}}_{PS} \\ r^{\mathrm{D}}_{SP} & r^{\mathrm{D}}_{SS} & t^{\mathrm{U}}_{SS} \\ t^{\mathrm{D}}_{SP} & t^{\mathrm{D}}_{SS} & r^{\mathrm{U}}_{SS} \end{pmatrix} \tag{77}$$

is unitary, and thus

$$|r^{\mathrm{D}}_{PP}|^2 + |r^{\mathrm{D}}_{SP}|^2 + |t^{\mathrm{D}}_{SP}|^2 = 1$$

together with similar results. The $\bar{\mathbf{J}}$ projector now reduces the right-hand side of (67) to a single element and so

$$|t^{\mathrm{D}}_{PP}|^2 + |t^{\mathrm{D}}_{PS}|^2 + |r^{\mathrm{U}}_{PS}|^2 = 2\,\mathrm{Im}\,(r^{\mathrm{U}}_{PP}). \tag{78}$$

We may extend this approach to all the other possible cases and obtain a number of similar results for relationships between the reflection and transmission coefficients for perfectly elastic media.

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Appendix: the effect of alternative normalizations on the reflection and transmission coefficients

The decomposition of the stress–displacement field **B** into up and downgoing wave components (5) is dependent on the eigenvector matrix **T** (12, 13) and as we have noted we have a free choice of scaling parameters $\epsilon_\alpha^{U,D}$, $\epsilon_\beta^{U,D}$. With the choice of values appropriate to energy normalization we were able to arrive at rather simple forms for the symmetry relations.

Suppose, however, that we choose

$$\tilde{\epsilon}_\alpha^{U,D} = \left(\frac{2}{\rho\omega^3\nu_\alpha}\right)^{1/2} \tilde{\eta}_\alpha^{U,D}, \quad \tilde{\epsilon}_\beta^{U,D} = \left(\frac{2}{\rho\omega^3\nu_\beta}\right)^{1/2} \tilde{\eta}_\beta^{U,D} \tag{A1}$$

where throughout this section a tilde will indicate that a quantity is not energy normalized. The energy normalized results can be recovered by setting $\tilde{\eta}_{\alpha,\beta}^{U,D} = 1$. We define the 2×2 matrices $\tilde{\mathrm{H}}^U$, $\tilde{\mathrm{H}}^D$

$$\tilde{\mathrm{H}}^D = \begin{pmatrix} \tilde{\eta}_\alpha^D & 0 \\ 0 & \tilde{\eta}_\beta^D \end{pmatrix}, \qquad \tilde{\mathrm{H}}^U = \begin{pmatrix} \tilde{\eta}_\alpha^U & 0 \\ 0 & \tilde{\eta}_\beta^U \end{pmatrix} \tag{A2}$$

and then any alternative eigenvector matrix $\tilde{\mathbf{T}}$ takes the form

$$\tilde{\mathbf{T}} = \mathbf{T}\begin{pmatrix} \tilde{\mathrm{H}}^U & 0 \\ 0 & \tilde{\mathrm{H}}^D \end{pmatrix}. \tag{A3}$$

Since the propagator matrix $\mathbf{P}(k, z_1, z_n)$ is the solution of the differential equation (2) it will be independent of the normalization and thus the matrix $\mathbf{Q}$ defined in equation (36a) transforms as

$$\tilde{\mathbf{Q}} = \begin{pmatrix} (\tilde{\mathrm{H}}_0^U)^{-1} & 0 \\ 0 & (\tilde{\mathrm{H}}_0^D)^{-1} \end{pmatrix} \mathbf{Q} \begin{pmatrix} \tilde{\mathrm{H}}_n^U & 0 \\ 0 & \tilde{\mathrm{H}}_n^D \end{pmatrix}. \tag{A4}$$

Thus using the equations (38) which define the reflection and transmission coefficients in terms of the subpartitions of we find

$$\begin{aligned} \tilde{\boldsymbol{T}}_D &= (\tilde{\mathrm{H}}_n^D)^{-1} \boldsymbol{T}_D \tilde{\mathrm{H}}_0^D \\ \tilde{\boldsymbol{R}}_D &= (\tilde{\mathrm{H}}_0^U)^{-1} \boldsymbol{R}_D \tilde{\mathrm{H}}_0^D \\ \tilde{\boldsymbol{T}}_U &= (\tilde{\mathrm{H}}_0^U)^{-1} \boldsymbol{T}_U \tilde{\mathrm{H}}_n^U \\ \tilde{\boldsymbol{R}}_U &= (\tilde{\mathrm{H}}_n^D)^{-1} \boldsymbol{R}_U \tilde{\mathrm{H}}_n^U \end{aligned} \tag{A5}$$

and the individual coefficients can be obtained from similar formulae, e.g.

$$\tilde{r}_{PS}^D = (\tilde{\eta}_{0\alpha}^U)^{-1} r_{PS}^D \tilde{\eta}_{0\beta}^D.$$

If we consider the first of the general symmetry relations for layered media (43)

$$\begin{aligned} \tilde{r}_{PS}^D &= \frac{\tilde{\eta}_{0\beta}^D}{\tilde{\eta}_{0\alpha}^U} r_{PS}^D = \frac{\tilde{\eta}_{0\beta}^D}{\tilde{\eta}_{0\alpha}^U} r_{SP}^D \\ &= \frac{\tilde{\eta}_{0\beta}^D \tilde{\eta}_{0\beta}^U}{\tilde{\eta}_{0\alpha}^U \tilde{\eta}_{0\alpha}^D} \tilde{r}_{SP}^D \end{aligned} \tag{A6}$$

thus the simple results of (43) do not hold in general for arbitrary normalizations. For example the results of Lapwood & Hudson (1975) can be reproduced by setting

$$\tilde{\eta}_\alpha^U = \tilde{\eta}_\alpha^D = \tilde{\eta}_\beta^U = 1, \qquad \tilde{\eta}_\beta^D = -1$$

so that

$$\begin{aligned}\tilde{r}^{D}_{PS} &= -\tilde{r}^{D}_{SP}\\ \tilde{r}^{U}_{PS} &= -\tilde{r}^{U}_{SP}\\ \tilde{t}^{D}_{PS} &= -\tilde{t}^{U}_{PS}\\ \tilde{t}^{D}_{SP} &= -\tilde{t}^{U}_{SP}.\end{aligned} \tag{A7}$$

If we wish to preserve some of the symmetry relations discussed earlier in this paper there are two classes of normalization. Firstly if we seek to maintain the overall symmetries (43) for a stack of layers, we require, e.g.

$$\tilde{\boldsymbol{R}}_{D} = \tilde{\boldsymbol{R}}_{D}^{T}$$

and thus from (A6) since $\boldsymbol{R}_D$ is symmetric we need

$$\boldsymbol{R}_{D}(\tilde{H}^{U}_{0}\tilde{H}^{D}_{0}) = (\tilde{H}^{U}_{0}\tilde{H}^{D}_{0})\boldsymbol{R}_{D} \tag{A8a}$$

since the H matrices are diagonal and therefore commute. From the other two symmetries in (43) we require

$$\boldsymbol{R}_{U}(\tilde{H}^{U}_{n}\tilde{H}^{D}_{n}) = (\tilde{H}^{U}_{n}\tilde{H}^{D}_{n})\boldsymbol{R}_{U} \tag{A8b}$$

$$\boldsymbol{T}_{D}(\tilde{H}^{U}_{0}\tilde{H}^{D}_{0}) = (\tilde{H}^{U}_{n}\tilde{H}^{D}_{n})\boldsymbol{T}_{D}. \tag{A8c}$$

For all three equations (A8) to be compatible

$$\tilde{H}^{U}_{0}\tilde{H}^{D}_{0} = \tilde{H}^{U}_{n}\tilde{H}^{D}_{n} = \lambda(\omega, k)\boldsymbol{I} \tag{A9}$$

where the factor λ is independent of the velocity structure. Alternatively if we wish to preserve the relationships (52) for a single interface we now require

$$\begin{aligned}\boldsymbol{R}_{U}\tilde{H}^{U}_{n}(\tilde{H}^{D}_{n})^{-1} + \boldsymbol{T}_{D}\tilde{H}^{D}_{0}(\tilde{H}^{U}_{0})^{-1}\boldsymbol{R}_{D}\boldsymbol{T}_{D}^{-1} &= 0\\ \boldsymbol{T}_{U}\tilde{H}^{U}_{n}(\tilde{H}^{D}_{n})^{-1}\boldsymbol{T}_{D} + \boldsymbol{R}_{D}\tilde{H}^{D}_{0}(\tilde{H}^{U}_{0})^{-1}\boldsymbol{R}_{D} &= \tilde{H}^{U}_{0}(\tilde{H}^{D}_{0})^{-1}\end{aligned} \tag{A10}$$

where now the indices 0 and n refer to the two sides of an interface. For equations (A10) to be compatible

$$\tilde{H}^{D}_{0}(\tilde{H}^{U}_{0})^{-1} = \lambda_0\boldsymbol{I}, \quad \tilde{H}^{D}_{n}(\tilde{H}^{U}_{n})^{-1} = \lambda_n\boldsymbol{I}$$

with

$$\lambda_0\lambda_n = 1$$

and to have the simple symmetries for the interface, from (A9) we need

$$\lambda_0 = \lambda_n = 1$$

and thus

$$\tilde{H}^{U} = \tilde{H}^{D}. \tag{A11}$$