SYMMETRIES OF LINKS

BY
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In this paper certain properties called *symmetries* are defined for links, and the problem of determining those links *admitting* a particular symmetry is attacked. The problems of symmetry are the generalization to links of the problems of amphichaerality and invertibility of knots which, because of Trotter's proof that there are noninvertible knots [13], are now fairly well under control. A link of two components is called *interchangeable* if it possesses a special type of symmetry, and certain invariants of interchangeability of a link are given and examined.

By a *link* of μ components we shall understand the union of μ oriented, ordered, and disjoint knots $K_1, \ldots, K_\mu$ tamely imbedded in the oriented 3-sphere S. Two links, L and L', are of the same (oriented) *type* if and only if there is an orientation-preserving autohomeomorphism ϕ of S which takes L onto L' such that $\phi(+K_\alpha) = +K'_\alpha$ for each α; that is, the orientation of $\phi(+K_\alpha)$ is to match that of K'_α for each α.

The problems of amphichaerality and invertibility of knots (see either [1] or [13]) have been generalized to links by R. H. Fox as follows. Let L be a link of μ components, S_μ the symmetric group of degree μ, and $Z_2^{\mu+1}$ the direct product of $\mu+1$ copies of the multiplicative group $Z_2=\{-1, 1\}$. Define Γ_μ as a split extension $1 \to Z_2^{\mu+1} \to \Gamma_\mu \to S_\mu \to 1$ with isomorphism W taking S_μ into $\mathrm{Aut}\,(Z_2^{\mu+1})$ by $W(p) = W_p$ for each p in S_μ, where W_p is given by

$$W_p(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu) = (\varepsilon_0, \varepsilon_{p(1)}, \ldots, \varepsilon_{p(\mu)})$$

for each $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu)$ in $Z_2^{\mu+1}$; that is, S_μ permutes the last μ factors of $Z_2^{\mu+1}$. $\gamma=(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu, p)$, where $\varepsilon_i = \pm 1$ and p is a permutation on $\{1, 2, \ldots, \mu\}$, is an element of Γ_μ. We shall say that L *admits* γ if $L^\gamma = \varepsilon_1 K_{p(1)} \cup \cdots \cup \varepsilon_\mu K_{p(\mu)}$ is of the same type as $L = +K_1 \cup \cdots \cup +K_\mu$; that is, if there is an autohomeomorphism ψ of S such that $\psi(+S) = \varepsilon_0 S$, and $\psi(+K_\alpha) = \varepsilon_\alpha K_{p(\alpha)}$ for each α.

The elements γ admitted by L form a subgroup $\Sigma(L)$ of Γ_μ, the *group of symmetries* of L. $\Sigma(L)$ is unaltered by change of orientation of S or of any K_α, but change in the orientation of any K_α or in the order of $K_1, \ldots, K_\mu$ induces an inner automorphism of Γ_μ which replaces $\Sigma(L)$ by a conjugate subgroup of Γ_μ.

Three questions which now arise are:

(1) *What links admit a given γ?*

(2) *What links have a given subgroup Σ of Γ_μ as their group of symmetries?*

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(3) *In particular, for what subgroups* Σ *of* Γ_μ *does there exist a link L for which* $\Sigma(L)=\Sigma$?

For $\mu=1$, the five (proper and improper) subgroups of $\Gamma_1=Z_2\times Z_2$ describe five possible properties of a knot: 1. nonamphichaeral and noninvertible; 2. nonamphichaeral and invertible; 3. +amphichaeral and noninvertible; 4. −amphichaeral and noninvertible; 5. amphichaeral and invertible. For $\mu=2$, a link is called *interchangeable* (see problem 11 of [2]) if and only if it admits any γ of the form $(\varepsilon_0, \varepsilon_1, \varepsilon_2, (12))$.

In §1 of this paper, the problem (1) above is reduced to the (oriented) type problem and the special case of problem (1) in which the link L is unsplittable and prime. As a corollary to Theorem 2 of §1, necessary and sufficient conditions are obtained for an unsplittable link of two components to be interchangeable. These conditions can be used on occasion to show that certain links are not interchangeable.

In §2, the cyclic covering spaces of S branched over a link are brought into play. If M_g denotes the g-fold cyclic covering space of S branched over $L=K_1 \cup \cdots \cup K_\mu$ and $\tilde{L}=\tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu \subset M_g$ lies over L with $\tilde{K}_i$ lying over K_i, $(i=1,\ldots,\mu)$, then $H_1((M_g-\tilde{L}) \cup \tilde{K}_i)$ and $H_1((M_g-\tilde{L}) \cup \tilde{K}_j)$ are shown to be isomorphic. As an immediate corollary to the proof of this fact, $H_1(M_g-\tilde{L})$ must be isomorphic to the direct sum of the infinite cyclic group and $H_1((M_g-\tilde{L}) \cup \tilde{K}_i)$. For interchangeability invariants of $L=K_1 \cup K_2$, we go to the various unbranched cyclic covering spaces of $M_g-\tilde{K}_1$ and $M_g-\tilde{K}_2$.

Examples are given in §3.

I wish to express my appreciation to Professor R. H. Fox of Princeton University for his generous assistance. Also, I would like to thank Professor J. W. Milnor for a helpful suggestion.

1. **Reduction of problem (1).**

THEOREM 1. *If L is a splittable link of* μ *components and its unsplittable parts are* $L_i=K_{\nu_{i-1}+1} \cup \cdots \cup K_{\nu_i}$, $(i=1,\ldots,m)$, *then in order that L admit* $\gamma=(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu, p)$ *it is necessary and sufficient that p induce a permutation*

$$\begin{pmatrix} T_1 \cdots T_m \\ T_{q(1)} \cdots T_{q(m)} \end{pmatrix}$$

of the sets $T_i=\{\nu_{i-1}+1,\ldots,\nu_i\}$ *such that for each i,* $L_{q(i)}$ *and* $L_i^{\gamma_i}$ *are equal as point sets, and* $L_i^{\gamma_i}$ *be of the same type as* L_i, *where*

$$\gamma_i = (\varepsilon_0, \varepsilon_{\nu_{i-1}+1}, \ldots, \varepsilon_{\nu_i}, p_i)$$

and

$$p_i = \begin{pmatrix} \nu_{i-1}+1 \cdots \nu_i \\ p(\nu_{i-1}+1) \cdots p(\nu_i) \end{pmatrix}.$$

In particular, for those i for which $q(i)=i$, *the requirement is that* L_i *admit the element* γ_i *of* $\Gamma_{\nu_i-\nu_{i-1}}$.

Proof. The proof is quite simple. Since the necessity is obvious, it only remains to prove that the condition is sufficient. All maps are to be autohomeomorphisms of S.

Since L is splittable with unsplittable parts $L_1, \ldots, L_m$, there exist (tame) 3-cells $Q_1, \ldots, Q_m$ in S with $L_i \subset \text{Int}(Q_i)$ and $Q_i \cap Q_j = \varnothing$ for $i \neq j$. There is a map ψ_1 such that $\psi_1(+S) = \varepsilon_0 S$ and $\psi_1(Q_i) = Q_{q(i)}$ for each $i = 1, \ldots, m$. Finally, since $L_i^{\gamma_i}$ is of the same type as L_i for each i, there exists a map ψ_2 which acts as the identity on $S - (\text{Int}(Q_1) \cup \cdots \cup \text{Int}(Q_m))$ such that for the mapping $\psi = \psi_2\psi_1$, $\psi(+S) = \varepsilon_0 S$ and $\psi(+K_\alpha) = \varepsilon_\alpha K_{p(\alpha)}$ for each $\alpha = 1, \ldots, \mu$.

COROLLARY. *A splittable link of two components is interchangeable if and only if its components are of the same knot type.*

Links L and L' will be called *equivalent* (denoted by $L \approx L'$), if there is an orientation-preserving autohomeomorphism θ taking L onto L' such that θ restricted to each component of L is also orientation-preserving. θ is not required to preserve the order of the components of L.

REMARK. All 3-cells and 2-spheres mentioned are to be considered as tamely imbedded in S.

Let L be an unsplittable link and let Q be a 3-cell whose boundary 2-sphere Bd (Q) intersects L in exactly two points a and b of some component, say K_α, of L (that is, $L \cap \text{Bd}(Q) = \{a, b\} \subset K_\alpha$) in such a way that points of L belong to both the interior and exterior of Q. If q is a simple arc (to be called a *connection arc*) on Bd (Q) whose endpoints are a and b, then for appropriate orientations of q, $L_1 = [L \cap (S - Q)] \cup q$ and $L_2 = [L \cap \text{Int}(Q)] \cup q$ are links which are uniquely determined by L and Bd (Q) up to equivalence. We set $k_1 = [K_\alpha \cap (S - Q)] \cup q$ and $k_2 = [K_\alpha \cap \text{Int}(Q)] \cup q$, noting that k_1 and k_2 are components of the links L_1 and L_2, respectively, and that $K_\alpha = k_1 \# k_2$ (see p. 139 of [1]).

After Schubert (see p. 142 of [11]), we say that L and Bd (Q) form a *product representation* of L with respect to K_α, and that L_1 and L_2 are *factors* of L. L is called the *product* of L_1 and L_2 with respect to k_1 and k_2, and we write $L = L_1 \cdot L_2$. L is determined up to equivalence by L_1, L_2, k_1, and k_2.

The link L shall be called *prime* if, whenever $L = L_1 \cdot L_2$, one but not both of L_1 and L_2 is a trivial knot. A trivial knot will be interpreted as the empty product of prime links, while a prime link is treated as a product of prime links with one factor. It has been proved [7] that each unsplittable link L is the product of a finite collection of prime links which are uniquely determined by L up to equivalence. Conversely, L is determined up to equivalence by its prime factors and an algorithm for the multiplication of these factors.

THEOREM 2. *If an unsplittable link L is the product of prime factors $L_j = K_{j,\alpha(j,1)} \cup \cdots \cup K_{j,\alpha(j,l_j)}$, $(j = 1, \ldots, n)$, where the indices $\alpha(j, k)$ are inherited from the* *indexing* α *of the components of* $L=\bigcup K_\alpha$, *then in order that* L *admit* $\gamma=(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu, p)$ *it is necessary and sufficient that* p *induce a permutation*

$$\begin{pmatrix} V_1 \cdots V_n \\ V_{r(1)} \cdots V_{r(n)} \end{pmatrix}$$

of the sets $V_j=\{(j, \alpha(j, 1)), \ldots, (j, \alpha(j, l_j))\}$, $(j=1, \ldots, n)$, *such that for each* j,

$$L_j^{\gamma_j} = \varepsilon_{\alpha(j,1)} K_{r(j),p_j(\alpha(j,1))} \cup \cdots \cup \varepsilon_{\alpha(j,l_j)} K_{r(j),p_j(\alpha(j,l_j))}$$

be of the same type as L_j, *where* $\gamma_j=(\varepsilon_0, \varepsilon_{\alpha(j,1)}, \ldots, \varepsilon_{\alpha(j,l_j)}, p_j)$ *and*

$$p_j = \begin{pmatrix} \alpha(j, 1) \cdots \alpha(j, l_j) \\ p(\alpha(j, 1)) \cdots p(\alpha(j, l_j)) \end{pmatrix}.$$

In particular, for those j *for which* $r(j)=j$, *the requirement is that* L_j *admit the element* γ_j *of* Γ_{l_j}.

Proof. *Necessity.* There is an autohomeomorphism ψ of S such that $\psi(S)=\varepsilon_0 S$ and $\psi(+K_\alpha)=\varepsilon_\alpha K_{p(\alpha)}$ for each $\alpha=1, \ldots, \mu$. Since L is the product of prime factors $L_1, \ldots, L_n$, (the prime factors of L may be fixed by considering a *decomposition system* of L (see p. 286 of [7])), then L^γ is the product of prime factors $\psi(L_1), \ldots, \psi(L_n)$. A second prime factorization $L'_1, \ldots, L'_n$ of L^γ may clearly be obtained from $L_1, \ldots, L_n$ by properly orienting all components of each L_i. By the unique factorization theorem for links [7] there is a one-to-one correspondence between $\{\psi(L_1), \ldots, \psi(L_n)\}$ and $\{L'_1, \ldots, L'_n\}$ such that corresponding prime factors are equivalent.

Clearly, $\psi(L_j)$ may be written as

$$\psi(L_j) = K^*_{j,p(\alpha(j,1))} \cup \cdots \cup K^*_{j,p(\alpha(j,l_j))},$$

where the indices $p(\alpha(j, k))$ are inherited from the indexing $p(\alpha)$ of the components of $L^\gamma=\bigcup_{\alpha=1}^{\mu} \varepsilon_\alpha K_{p(\alpha)}$, and $K^*_{j,p(\alpha(j,k))}=\psi(K_{j,\alpha(j,k)})$. (Note that no two distinct components of L_j inherit α-indexing from the same component of L.) Now given $\psi(L_j)$, the combined proofs of Lemma 3 and the unique factorization theorem of [7] construct (by use of the well-known methods of Schubert [10]) an orientation-preserving autohomeomorphism θ_j of S taking $\psi(L_j)$ onto one of the links $L'_1, \ldots, L'_n$ in such a way that the α-indexing of $\theta_j(K^*_{j,p(\alpha(j,k))})$ is precisely $p(\alpha(j, k))$ for each $k=1, \ldots, l_j$. If then $\theta_j\psi(L_j)=L'_{r(j)}$, it follows from the construction of θ_j that $\theta_j(K^*_{j,p(\alpha(j,k))})=\varepsilon_{\alpha(j,k)} K_{r(j),p(\alpha(j,k))}$, $(k=1, \ldots, l_j)$. Again from the construction of the θ_j, we have $\theta_i\psi(L_i)=\theta_j\psi(L_j)$ only if $i=j$, and the necessity of the condition is proved.

In particular, in those cases for which $r(j)=j$, each $p(\alpha(j, k))$ must clearly belong to $\{\alpha(j, 1), \ldots, \alpha(j, l_j)\}$, so that L_j admits γ_j of Γ_{l_j}.

Sufficiency. The proof proceeds by induction on n, the number of prime factors of L. For $n=1$ there is nothing to prove.

If $n=2$, L is the product of prime factors L_1 and L_2. There is a 3-cell Q such that L and Bd (Q) form a product representation of L with respect to some component

K_α of L, and such that $L_1=[L \cap \operatorname{Int}(Q)] \cup q$ and $L_2=[L \cap (S-Q)] \cup q$. q is a connection arc on Bd (Q) appropriately oriented in each case.

Let ψ_1 be an autohomeomorphism of S such that $\psi_1(S)=\varepsilon_0 S$. By hypothesis there is an orientation-preserving autohomeomorphism ψ_2 of S taking $\psi_1(L_1)$ onto $L_1^{\gamma_1}$. It may be arranged (see the proof of Theorem 3′ of [10]) that ψ_2 take $\psi_1(\operatorname{Bd}(Q))$ onto Bd (Q). Again by hypothesis, there is an orientation-preserving autohomeomorphism ψ_3 of S which is the identity on either Q or $S-\operatorname{Int}(Q)$, and which takes $\psi_2\psi_1(L_2)$ onto $L_2^{\gamma_2}$. Evidently, ψ_2 and ψ_3 may be so chosen that $\psi_3\psi_2\psi_1(K_\alpha)=\varepsilon_\alpha K_{p(\alpha)}$, $(\alpha=1,\dots,\mu)$. Hence, L admits γ.

Now assume that L is the product of a link K and a prime link L_{m+1}, where K is the product of prime factors $L_1,\dots,L_m$. It is not difficult to see that L^γ is the product of a link denoted by K^γ and the prime link $L_{m+1}^{\gamma_{m+1}}$, where K^γ is the product of prime factors $L_1^{\gamma_1},\dots,L_m^{\gamma_m}$. Then there are 3-cells Q and Q^γ such that L and Bd (Q) form a product representation of L with respect to some component K_α of L, L^γ and Bd (Q^γ) form a product representation of L^γ with respect to the component $\varepsilon_\alpha K_{p(\alpha)}$ of L^γ, and such that $K\approx[L \cap \operatorname{Int}(Q)] \cup q$, $L_{m+1}\approx[L\cap(S-Q)]\cup q$, $K^\gamma\approx[L^\gamma\cap \operatorname{Int}(Q^\gamma)]\cup q^\gamma$, and $L_{m+1}^{\gamma_{m+1}}\approx[L^\gamma\cap(S-Q^\gamma)]\cup q^\gamma$, where q and q^γ are connecting arcs on Bd (Q) and Bd(Q^γ), respectively. It is convenient to denote $[L\cap \operatorname{Int}(Q)]\cup q$ by its equivalent K, $[L\cap(S-Q)]\cup q$ by its equivalent L_{m+1}, etc.

If now ψ_1 is an autohomeomorphism of S such that $\psi_1(S)=\varepsilon_0 S$, then by the induction hypothesis $\psi_1(K)$ is equivalent to K^γ, and there is, therefore, an orientation-preserving autohomeomorphism ψ_2 of S taking $\psi_1(K)$ onto K^γ and $\psi_1(Q)$ onto Q^γ. Since L_{m+1} and $L_{m+1}^{\gamma_{m+1}}$ are of the same type, there is an orientation-preserving autohomeomorphism ψ_3 of S taking $\psi_2\psi_1(L_{m+1})$ onto $L_{m+1}^{\gamma_{m+1}}$ which acts as the identity on Q^γ. Finally, by the induction hypothesis and the hypothesis of the theorem, ψ_2 and ψ_3 can certainly be chosen so that $\psi_3\psi_2\psi_1(K_\alpha)=\varepsilon_\alpha K_{p(\alpha)}$, $(\alpha=1,\dots,\mu)$; thus, L admits γ.

Corollary. *If $L=K_1\cup K_2$ is an unsplittable link, then its prime factors consist of a prime link $L^*=K_1^*\cup K_2^*$ and a collection $K_{11},\dots,K_{1m}$ and $K_{21},\dots,K_{2n}$ of prime knots, where K_1^* and $K_{1\beta}$ are factors of K_1 and K_2^* and $K_{2\beta}$ are factors of K_2. In order that L be interchangeable it is necessary and sufficient that*

(1) *L^* admit an element $\gamma\in\Gamma_2$ of the form $\gamma=(\varepsilon_0,\varepsilon_1,\varepsilon_2,(12))$.*

(2) *$m=n$, and there are two permutations q and r on $\{1,\dots,m\}$ together with two autohomeomorphisms ϕ_β and θ_β for each $\beta=1,\dots,m=n$ such that $\phi_\beta(S)=\varepsilon_0 S$, $\theta_\beta(S)=\varepsilon_0 S$, $\phi_\beta(K_{1\beta})=\varepsilon_1 K_{2q(\beta)}$, and $\theta_\beta(K_{2\beta})=\varepsilon_2 K_{1r}(\beta)$.*

Remark. The prime link L^* of the above corollary will be called the *hub* of the link L.

2. **Covering spaces.** Let M_g denote the gth cyclic covering space of S branched over the link $L=K_1\cup K_2$. Let $\tilde{L}=\tilde{K}_1\cup\tilde{K}_2\subset M_g$ lie over L, where $\tilde{K}_i$ lies over K_i.

Theorem 3. *The first homology group of $M_g-\tilde{K}_1$ is isomorphic to that of $M_g-\tilde{K}_2$.*

Proof. The proof is patterned after that of Theorem (6.1) of [5]. Let

$$(x_0, \ldots, x_{n_1}, y_0, \ldots, y_{n_2}: r_0, \ldots, r_{n_1+n_2+1})^{\phi}$$

be an over presentation [12] for the group $G=\pi_1(S-L)$, where the abelianizing homomorphism ψ of G given by $x_i^{\psi\phi}=t_1$ and $y_j^{\psi\phi}=t_2$ maps G onto the free abelian group $G/[G, G]$ of rank 2 generated by elements denoted by t_1 and t_2, t_m being represented by a loop whose linking number with K_n is δ_{mn}. We change from the above presentation to

$$(x, a_1, \ldots, a_{n_1}, y, b_1, \ldots, b_{n_2}: s_0, \ldots, s_{n_1+n_2+1})^{\phi}$$

by introducing new generators $a_j=x_jx_0^{-1}$, $b_i=y_iy_0^{-1}$, $x=x_0$, $y=y_0$, writing

$$s_k(x, a_1, \ldots, a_{n_1}, y, b_1, \ldots, b_{n_2})$$

for

$$r_k(x_0, a_1x_0, \ldots, a_{n_1}x_0, y_0, b_1y_0, \ldots, b_{n_2}y_0),$$

and ϕ again for the canonical homomorphism. Then $x^{\psi\phi}=t_1$, $a_j^{\psi\phi}=1$, $y^{\psi\phi}=t_2$, and $b_i^{\psi\phi}=1$. By the fundamental formula of the free calculus [3, (2.3)],

$$(t_1-1)(\partial s_k/\partial x)^{\psi\phi}+(t_2-1)(\partial s_k/\partial y)^{\psi\phi} = 0.$$

Let Z denote an infinite cyclic group with generator t. Let $\tau: G/[G, G]\to Z$ by $t_1^{\tau}=t_2^{\tau}=t$ and extend to the group rings. It follows immediately that

$$(*) \qquad (\partial s_k/\partial x)^{\tau\psi\phi}+(\partial s_k/\partial y)^{\tau\psi\phi} = 0.$$

Now let $\sigma: t\to(0\,1\ldots(g-1))$ be the representation of Z upon Z_g, (the cyclic subgroup of S_g generated by $(0\,1\ldots(g-1))$, and let θ be the regular representation of Z_g on the group of $g\times g$ matrices. $\rho=\sigma\tau\psi$ is then a transitive representation of G upon Z_g corresponding to the unbranched covering space $M_g-\tilde{L}$. Now extend ρ and θ to the group rings.

Then a relation matrix for $H_1(M_g-\tilde{K}_2)\oplus A_{g-1}$ ([5] or p. 148 of [1]), where A_{g-1} denotes the free abelian group of rank $g-1$, is the integral matrix

$$\left\| \begin{array}{c} \left(\dfrac{\partial(s_0, s_1, \ldots, s_{n_1+n_2+1})}{\partial(x, y, a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2})}\right)^{\theta\rho\phi} \\ \hline 1\ldots 1 \mid 0\ldots\ldots\ldots\ldots 0 \\ (g \text{ times}) \end{array} \right\|.$$

A relation matrix for $H_1(M_g-\tilde{K}_1)\oplus A_{g-1}$ is

$$\left\| \begin{array}{c} \left(\dfrac{\partial(s_0, s_1, \ldots, s_{n_1+n_2+1})}{\partial(x, y, a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2})}\right)^{\theta\rho\phi} \\ \hline 0\ldots 0 \mid 1\ldots 1 \mid 0\ldots\ldots 0 \\ (g \text{ times}) \quad (g \text{ times}) \end{array} \right\|.$$

By use of equation (*), it is clear that each of these matrices is equivalent to

$$\left\| \begin{array}{c|c} 0 & \left(\dfrac{\partial(s_0, s_1, \ldots, s_{n_1+n_2+1})}{\partial(y, a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2})}\right)^{\theta\rho\phi} \\ \hline \underset{(g \text{ times})}{0 \ldots 0\,1} & 0 \ldots\ldots\ldots\ldots\ldots\ldots 0 \end{array} \right\|$$

and the theorem follows.

Now consider a link $L = K_1 \cup \cdots \cup K_\mu$, let M_g denote the gth cyclic covering space of S branched over L, and let $\tilde{L} = \tilde{K}_1 \cup \cdots \cup \tilde{K}_\mu \subset M_g$ lie over L, where as before $\tilde{K}_i$ lies over K_i, $(i = 1, \ldots, \mu)$. Finally, let K_i and K_j be any two components of L.

THEOREM 4. *The first homology group of* $(M_g - \tilde{L}) \cup \tilde{K}_i$ *is isomorphic to that of* $(M_g - \tilde{L}) \cup \tilde{K}_j$.

A proof of Theorem 4 may be constructed from that of Theorem 3 with the obvious changes.

Immediately we have an important

COROLLARY. *The group* $H_1(M_g - \tilde{L})$ *is isomorphic to the direct sum of* $H_1((M_g - \tilde{L}) \cup \tilde{K}_i)$ *and the infinite cyclic group for any* $i = 1, \ldots, \mu$.

If now $L = K_1 \cup K_2$ is interchangeable, then $M_g - \tilde{K}_1$ is homeomorphic to $M_g - \tilde{K}_2$, which implies the existence of a one-to-one correspondence between the first homology groups of the r-fold unbranched cyclic covering spaces of $M_g - \tilde{K}_1$ and those of $M_g - \tilde{K}_2$ such that corresponding groups are isomorphic. These r-fold unbranched cyclic covering spaces of $M_g - \tilde{K}_i$ (see Figure 1) are rg-fold branched covering spaces of $S - K_i$.

The hubs of the links given in Figures 2(a) and 3(a) are illustrated in Figures 2(b) and 3(b), respectively. Right(left)-handed trefoil knots are denoted by + (−) signs as shown.

If a link of two components is interchangeable, it follows from the corollary to Theorem 2 that the knots $k_1 = K_{11} \# \cdots \# K_{1m}$ and $k_2 = K_{21} \# \cdots \# K_{2m}$ are of the same knot type. The link of Figure 2(a) is not interchangeable, since for it, k_1 and k_2 are of different types (see, for example, [6]). Notice that the components of this link are, however, of the same knot type.

Figure 3(a) shows a link whose hub is interchangeable and for which $k_1 = K_{11}$ and $k_2 = K_{21}$ are of the same knot type. This link does not, however, satisfy condition (2) of the corollary to Theorem 2. Since the trefoil is nonamphichaeral, any autohomeomorphism of S taking K_{11} onto K_{21} must reverse the orientation of S. On the other hand, since the hub of the link is also nonamphichaeral [8], any autohomeomorphism of S which interchanges the hub must preserve the orientation of S. Hence, the link cannot be interchanged.

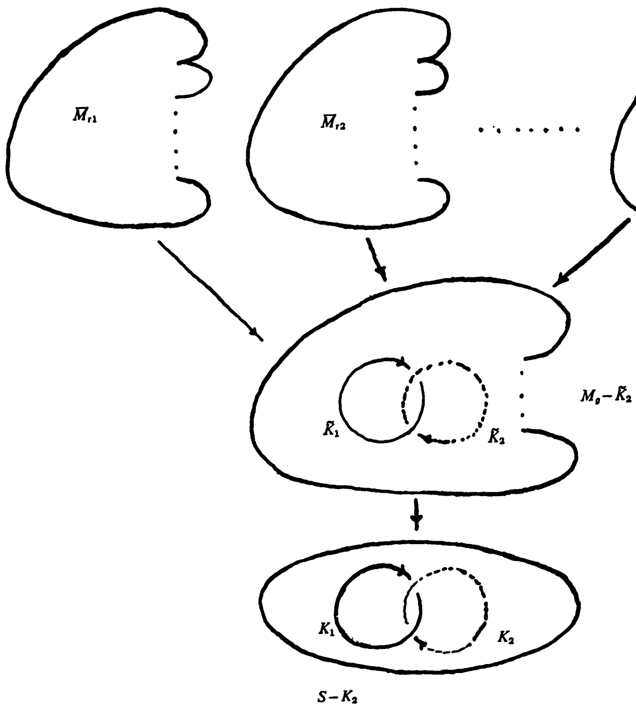

FIGURE 1

3. Examples.

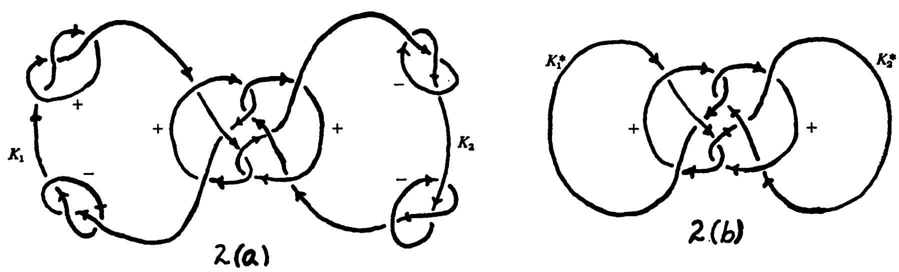

FIGURE 2

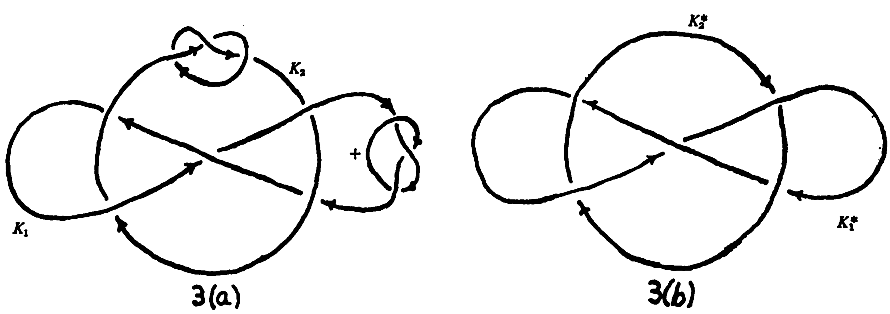

FIGURE 3

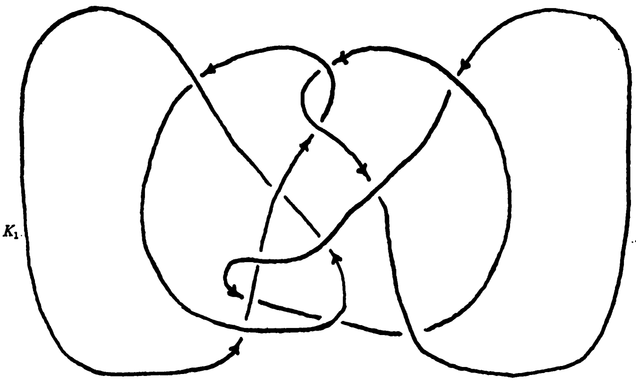

FIGURE 4

Now, consider the link shown in Figure 4. For each of $i=1, 2$, there is exactly one two-fold and one three-fold unbranched cyclic covering space $M_i^*(2)$ and $M_i^*(3)$, respectively, of $M_2-\tilde{K}_i$. I calculate that

$$H_1(M_1^*(2)) \approx Z \oplus Z_3 \oplus Z_{51} \approx H_1(M_2^*(2));$$

however, $H_1(M_1^*(3)) \approx Z \oplus Z_8 \oplus Z_{136}$ and

$$H_1(M_2^*(3)) \approx Z \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_{34},$$

which shows that our final example is not interchangeable.

Let $\Delta(x, y)$ denote the (normalized) Alexander polynomial [4] of a link L of two components. If L is interchangeable, then $\Delta(x, y)$ must be an associate of at least one of the four L-polynomials, $\Delta(y^{\varepsilon_1}, x^{\varepsilon_2})$, where each of ε_1 and ε_2 is either $+1$ or -1. The link of Figure 4 has polynomial

$$\Delta(x, y) = y(x^4-2x^3+x^2-2x+1)+y^2(x^3-x^2+2x-1)+(-x^4+2x^3-x^2+x),$$

which does not have the necessary property.

More generally, if $\Delta(x_1, \ldots, x_\mu)$ denotes the (normalized) Alexander polynomial of $L=+K_1 \cup \cdots \cup +K_\mu$, where x_i is represented by a loop whose linking number with $+K_j$ is δ_{ij}, then *in order that L admit* $\gamma=(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mu, p)$, *it is necessary that* $\Delta(x_1, \ldots, x_\mu)$ *be an associate of* $\Delta(x_{p(1)}^{\varepsilon_1}, \ldots, x_{p(\mu)}^{\varepsilon_\mu})$. Using this condition it is easy to see that for the link L of Figure 4, $\Sigma(L)$ is either trivial or isomorphic to Z_2.

The nonsymmetric homotopy linking numbers (see p. 633 of [9]) of a link are clearly interchangeability invariants. It is not difficult to see that one of these numbers for the link of Figure 4 is $+1$, while experimentation seems to indicate that the other is $+3$.

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