Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Symmetries of null Geometry in Indefinite Kenmotsu Manifolds

by

Fortuné Massamba

Preprint no.: 39 2011

SYMMETRIES OF NULL GEOMETRY IN INDEFINITE KENMOTSU MANIFOLDS

FORTUNÉ MASSAMBA*

ABSTRACT. Null hypersurfaces have metrics with vanishing determinants and this degeneracy of these metrics leads to several difficulties. In this paper, null hypersurfaces of indefinite Kenmotsu space forms, tangent to the structure vector field, are studied with specific attention to locally symmetric, semi-symmetric and Ricci semi-symmetric null hypersurfaces. We show that locally symmetric and semi-symmetric null hypersurfaces are totally geodesic and parallel. These also hold for Ricci semi-symmetric null hypersurfaces, under a certain condition. We prove that, in null Einstein hypersurfaces of an indefinite Kenmotsu space form, tangent to the structure vector field, the local symmetry, semi-symmetry and Ricci semi-symmetry notions are equivalent. For totally contact umbilical null hypersurfaces, we show that there are η-"Weyl" connections adapted to the induced structure on the null hypersurface.

1. INTRODUCTION

A semi-Riemannian manifold is *locally symmetric* if its curvature tensor R is parallel, i.e. $\nabla R = 0$, where ∇ is the Levi-Civita connection on semi-Riemannian manifold extended to act on tensors as a derivation and R is the corresponding curvature tensor. This class of manifolds contains one of manifolds of constant curvature. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$, where again R is extended to act on tensors as a derivation. Manifolds which satisfy the latter condition are called *semi-symmetric* and have been classified by Szabó ([29] and [30], for details). A semi-Riemannian manifold is called Ricci semi-symmetric, if $R \cdot Ric = 0$. The set of all manifolds which are Ricci semi-symmetric contains the set of manifolds which are semi-symmetric. This means that every semi-symmetric manifold is Ricci semi-symmetric. The converse is not true in general.

We are again interested to answer the following question: "Are conditions $R \cdot R = 0$ and $R \cdot Ric = 0$ equivalent on null hypersurfaces of semi-Riemannian manifolds?" and also its extension to $\nabla R = 0$. These equivalences are not true in general. In [27], Ryan raised the following question for hypersurfaces of Euclidean spaces: Are conditions $R \cdot R = 0$ and $R \cdot Ric = 0$ equivalent for hypersurfaces of Euclidean spaces? However, there are many results which contributed to the solution of the above question in the affirmative under some conditions (see [6],

2000 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25, 53C50.

Key words and phrases. Indefinite Kenmostu manifold; Null hypersurface; locally symmetric Null hypersurface; semi-symmetric Null hypersurface; Ricci semi-symmetric Null hypersurface; η-conformal connection.

[7], [26] and references therein). In [5] a survey on Ricci semi-symmetric spaces and contributions to the solution of above problem are given. In virtue of results given by Günes *et al* ([11], Theorem 3.1) and Sahin ([28], Theorem 4.2), we see that the conditions $\nabla R = 0$ and $R \cdot R = 0$ are equivalent for null hypersurfaces of semi-Euclidean space under conditions $Ric(E, X) = 0$ and $A_N E$ a vector field non-null. In [17], the authors proved that the conditions $\nabla R = 0$, $R \cdot R = 0$ and $R \cdot Ric = 0$ are equivalent for lightlike hypersurfaces of an indefinite Sasakian space form, tangent to the structure vector field under some conditions (Theorem 8 and Theorem 12). In this paper we give an affirmative answer to the question above for Einstein null hypersurfaces of indefinite Kenmotsu space forms, tangent to the structure vector field (Theorem 4.14).

As is well known, the geometry of null (lightlike) submanifolds [3] is different because of the fact that their normal vector bundle intersects with the tangent bundle. Thus, the study becomes more difficult and strikingly different from the study of non-degenerate submanifolds. This means that one cannot use, in the usual way, the classical submanifold theory to define any induced object on a null submanifold. To deal with this anomaly, the null submanifolds were introduced and presented in a book by Duggal and Bejancu [3]. They introduced a non-degenerate screen distribution to construct a non-intersecting null transversal vector bundle of the tangent bundle. Several authors have studied null hypersurfaces of semi-Riemannian manifolds ([10] and many more references therein). Concerning the null contact geometry, some specific discussions can be found in [17], [18], [19], [20], [21], [22], [23], [24] and [25].

In the present paper, we investigate some symmetries of null hypersurfaces in indefinite Kenmotsu manifolds, tangent to the structure vector field, by particularly paying attention to the locally symmetry, semi-symmetry and Ricci semi-symmetry, as well as their relationships with induced connections studied, for instance in [3], with the choice of screen distribution. By defining an η-Weyl connection, we remark that an induced connection cannot be an η-Weyl connection.

The paper is organized as follows. In Section 2, we give basic definition on indefinite Kenmotsu manifolds and null hypersurfaces of semi-Riemannian manifolds. In section 3, we give the decomposition of almost contact metrics of null hypersurfaces in indefinite Kenmotsu manifolds, tangent to the structure vector field, supported by an example. In Section 4, we consider a null hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$ and study local symmetry, semi-symmetry and Ricci semi-symmetry conditions on this hypersurface. We prove, among other results, that in an null Einstein hypersurface of an indefinite Kenmotsu space forms, tangent to the structure vector field, the mentioned three symmetries are equivalent. We also prove that local symmetry property of a screen integrable null hypersurface of an indefinite Kenmotsu space form is related with local symmetry property of leaves of its screen distribution (Theorem 4.8). Finally in Section 5, we introduce a concept of η-Weyl connection (η-semi-conformal connection) and we give a total contact umbilicity criterion which shows that there are η-Weyl connections for any totally contact umbilical null hypersurfaces.

2. PRELIMINARIES

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0 \quad \text{and} \quad \overline{\phi}\xi = 0. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$ [2]

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \quad \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}). \tag{2.2}$$

If, moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\,\overline{X}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an indefinite Kenmotsu manifold (see [13] for details). Here, without loss of generality, the vector field ξ is assumed to be spacelike, that is, $\overline{g}(\xi, \xi) = 1$.

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-sectional curvature. If a Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-sectional curvature c, then, by virtue of the Proposition 12 in [14], the curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\begin{aligned}\overline{R}(\overline{X}, \overline{Y})\overline{Z} &= \frac{c-3}{4}\left\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\right\} + \frac{c+1}{4}\left\{\eta(\overline{X})\eta(\overline{Z})\overline{Y}\right. \\ &-\eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X}, \overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y}, \overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y}, \overline{Z})\overline{\phi}\,\overline{X} \\ &\left. - \overline{g}(\overline{\phi}\,\overline{X}, \overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})\overline{\phi}\,\overline{Z}\right\}, \quad \overline{X},\ \overline{Y},\ \overline{Z} \in \Gamma(T\overline{M}).\end{aligned} \tag{2.3}$$

A Kenmotsu manifold $\overline{M}$ of constant $\overline{\phi}$-sectional curvature c will be called *Kenmotsu space form* and denoted $\overline{M}(c)$.

If a $(2n+1)$-dimensional Kenmotsu manifold $\overline{M}$ has a constant $\overline{\phi}$-sectional curvature c, then the Ricci tensor $\overline{Ric}$ and the scalar curvature $\overline{r}$ are given by [14]

$$\overline{Ric} = \frac{1}{2}\{n(c-3) + c + 1\}\,\overline{g} - \frac{1}{2}(n+1)(c+1)\eta \otimes \eta, \tag{2.4}$$

$$\overline{r} = \frac{1}{2}\{n(2n+1)(c-3) - n(c+1)\}. \tag{2.5}$$

This means that $\overline{M}$ is η-Einstein. Since $\overline{M}$ is Kenmotsu and η-Einstein, by Corollary 9 in [14], $\overline{M}$ is an Einstein one and consequently, $c+1=0$, that is, $c=-1$. So, the Ricci tensor (2.4) becomes $\overline{Ric} = -2n\overline{g}$ and the scalar curvature is given by $\overline{r} = -2n(2n+1)$.

Thus, *if a Kenmotsu manifold $\overline{M}$ is a space form, then it is Einstein and $c = -1$.* This means that, it is a space of constant curvature -1, so, in the Riemannian case $\overline{M}(c=-1)$ is locally isometric to the hyperbolic $\mathbb{H}^{2n+1}(-1)$ and in the proper semi-Riemannian case $\overline{M}(c=-1)$ is locally isometric to the pseudo hyperbolic space $\mathbb{H}_s^{2n+1}(-1)$, s being the index of the metric $\overline{g}$.

Example 2.1. We consider the 7-dimensional manifold

$$\overline{M}^7 = \{(x_1, x_2, ..., x_7) \in \mathbb{R}^7 : x_7 > 0\},$$

where $x = (x_1, x_2, ..., x_7)$ are the standard coordinates in $\mathbb{R}^7$. The vector fields,

$$e_p = x_7 \frac{\partial}{\partial x_p}, \quad e_q = -x_7 \frac{\partial}{\partial x_q},$$

for any $p = 1, 2, 3, 4$, $q = 5, 6, 7$, are linearly independent at each point of $\overline{M}^7$. Let $\overline{g}$ be the semi-Riemannian metric defined by

$$\overline{g}(e_i, e_j) = 0, \ \forall\, i \neq j, \ i, j = 1, 2, ..., 7,$$
$$\overline{g}(e_l, e_l) = 1, \ \forall\, l = 1, 2, 3, 4, 7 \quad \text{and} \quad \overline{g}(e_m, e_m) = -1, \forall\, m = 5, 6.$$

Its tensor product form is given by

$$\overline{g} = \frac{1}{(x_7)^2}\{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 + dx_7^2\}.$$

Let η be the 1-form defined by $\eta(\overline{X}) = \overline{g}(\overline{X}, e_7)$, for any $\overline{X} \in \Gamma(T\overline{M}^7)$. Let $\overline{\phi}$ be the $(1,\,1)$ tensor field defined by, for any $r = 1, 2, ..., 6$, $\overline{\phi} e_{2r-1} = -e_{2r}$, $\overline{\phi} e_{2r} = e_{2r-1}$ and $\overline{\phi} e_7 = 0$. Then using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\overline{\phi}^2 \overline{X} = -\overline{X} + \eta(\overline{X}) e_7$, $\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$, for any $\overline{X}, \overline{Y} \in \Gamma(T\overline{M}^7)$. Thus, for $e_7 = \xi$, $(\overline{\phi},\, \xi,\, \eta,\, \overline{g})$ defines an almost contact metric structure on $\overline{M}^7$.

Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric $\overline{g}$. Then, we have $[e_r,\, e_7] = e_i$, $\forall\, r = 1, 2, ..., 6$ and $[e_r,\, e_s] = 0$, $\forall\, r \neq s,\ r,\, s = 1, 2, ..., 6$. The metric connection $\overline{\nabla}$ of the metric $\overline{g}$ is given by

$$2\overline{g}(\overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z}) = \overline{X}.\overline{g}(\overline{Y}, \overline{Z}) + \overline{Y}.\overline{g}(\overline{Z}, \overline{X}) - \overline{Z}.\overline{g}(\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}])$$
$$- \overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]),$$

which is known as Koszul's formula. Using this formula, the non-vanishing covariant derivatives are given by, for any $p = 1, 2, 3, 4$, $m = 5, 6$, $r = 1, 2, 3, ..., 6$, $\overline{\nabla}_{e_p} e_p = -e_7$, $\overline{\nabla}_{e_m} e_m = e_7$, $\overline{\nabla}_{e_r} e_7 = e_r$. From these relations, it follows that the manifold $\overline{M}$ satisfies $(\overline{\nabla}_{\overline{X}} \overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}) - \eta(\overline{Y})\overline{\phi}\,\overline{X}$. Hence, $\overline{M}$ is indefinite Kenmotsu manifold.

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is said to be a null (lightlike) hypersurface of $\overline{M}$ if g is of constant rank $(2n-1)$ and the orthogonal complement $TM^\perp$ of tangent space TM, defined as

$$TM^\perp = \bigcup_{x \in M} \left\{ Y_x \in T_x\overline{M} : \overline{g}_x(X_x, Y_x) = 0,\ \forall\, X_x \in T_x M \right\}, \qquad (2.6)$$

is a distribution of rank 1 on M [3]: $TM^\perp \subset TM$ and then coincides with the radical distribution $\text{Rad}\, TM = TM \cap TM^\perp$. A complementary bundle of $TM^\perp$ in TM is a rank $(2n-1)$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. Existence of $S(TM)$ is secured provided M is paracompact. However, in general, $S(TM)$ is not canonical (thus it is not unique) and the null geometry depends on its choice but it is canonically isomorphic to the vector bundle $TM/\mathrm{Rad}\,TM$ [16].

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a null hypersurface.

Theorem 2.2. [3] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, E) = 1 \quad \textit{and} \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.7}$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\perp$ and $\oplus$ the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.2 we may write down the following decompositions

$$TM = S(TM) \perp TM^{\perp}, \tag{2.8}$$

$$T\overline{M} = TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \tag{2.9}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, then by using decomposition of (2.9) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.2, we have the Gauss and Weingarten formulae in the form,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.10}$$

$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.11}$$

for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$, $V \in \Gamma(N(TM))$, where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is an induced symmetric linear connection on M, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$, h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.2. Then (2.10) takes the following form,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)\, N \tag{2.12}$$

$$\text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X) N, \tag{2.13}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and the induced linear torsion-free connection, respectively, on $TM|_{\mathcal{U}}$.

It is important to mention that the second fundamental form B of M is independent of the choice of screen distribution and $B(\cdot, E) = 0$. In fact, from (2.12), we obtain, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$, $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$ and

$$\tau(X) = \overline{g}(\nabla^{\perp}_X N, E). \tag{2.14}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have,

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \tag{2.15}$$

$$\text{and} \quad \nabla_X E = -A^*_E X - \tau(X)E, \tag{2.16}$$

for any X, $Y \in \Gamma(TM|_{\mathcal{U}})$, $E \in \Gamma(TM^{\perp})$, where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced linear metric connection, respectively, on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.17}$$

where θ is a differential 1-form locally defined on M by

$$\theta(\cdot) := \overline{g}(N, \cdot). \tag{2.18}$$

The local second fundamental forms B and C, respectively, of M and on $S(TM)$ are related to their shape operators by

$$g(A^*_E X, PY) = B(X, PY), \quad g(A^*_E X, N) = 0, \tag{2.19}$$

$$g(A_N X, PY) = C(X, PY), \quad g(A_N X, N) = 0. \tag{2.20}$$

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{M}$ and M, respectively. From Gauss-Codazzi equations, we have, for any X, Y, $Z \in \Gamma(TM_{|\mathcal{U}})$,

$$\begin{aligned}\overline{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X + \{(\nabla_X B)(Y, Z) \\ &- (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N, \end{aligned}\tag{2.21}$$

$$\overline{g}(\overline{R}(X, Y)Z, N) = \overline{g}(R(X, Y)Z, N), \tag{2.22}$$

$$\begin{aligned}\overline{g}(\overline{R}(X, Y)PZ, N) &= \nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) \\ &- \tau(X)C(Y, PZ), \end{aligned}\tag{2.23}$$

$$\overline{g}(\overline{R}(X, Y)E, N) = C(Y, \overset{*}{A}_\xi X) - C(X, \overset{*}{A}_\xi Y) - 2d\tau(X, Y). \tag{2.24}$$

3. Null hypersurfaces of indefinite Kenmotsu manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and (M, g) be a null hypersurface of $(\overline{M}, \overline{g})$, *tangent to the structure vector field ξ ($\xi \in TM$).*

If E is a local section of $TM^{\perp}$, it is easy to check that $\overline{\phi}E \neq 0$ and $\overline{g}(\overline{\phi}E, E) = 0$, then $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. In fact, if $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} \neq \{0\}$, there exists a non-zero smooth real valued function f such that $\overline{\phi}E = \mu E$. Applying $\overline{\phi}$ to this and using (2.1), we obtain $(\mu^2 + 1)E = 0$, which implies $\mu^2 + 1 = 0$. It is an impossible case for real submanifold M. Therefore, we have $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $N(TM)$, we have $\overline{\phi}N \neq 0$. Since $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$, we deduce that $\overline{\phi}E$ belongs to $S(TM)$ and $\overline{\phi}N$ is also tangent to M. At the same time, since $\overline{g}(\overline{\phi}N, N) = 0$, we see that the component of $\overline{\phi}N$, with respect to E, vanishes.

Thus $\overline{\phi}\, N \in \Gamma(S(TM))$, that is, $\overline{\phi}(N(TM))$ is also a vector subbundle of $S(TM)$ of rank 1. We have

Lemma 3.1. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$. Then, the distributions $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are vector subbundles of $S(TM)$ of rank* 1*.*

From (2.1), we have $\overline{g}(\overline{\phi}\, N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2.

If M is tangent to the structure vector field ξ, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp < \xi >, \tag{3.1}$$

where $\langle\xi\rangle$ is the distribution spanned by ξ. The distribution D_0 is invariant under $\overline{\phi}$, i.e. $\overline{\phi}(D_0) = D_0$. Moreover, from (2.8) and (3.1) we obtain the decompositions

$$TM = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp < \xi > \perp TM^{\perp}, \tag{3.2}$$

$$T\overline{M} = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp < \xi > \perp (TM^{\perp} \oplus N(TM)). \tag{3.3}$$

Example 3.2. Let M be a hypersurface of $(\overline{M}^7, \overline{\phi}, \xi, \eta, \overline{g})$, indefinite Kenmotsu manifold defined in the Example 2.1, given by

$$x_5 = \sqrt{2}(x_2 + x_3),$$

where $(x_1, x_2, ..., x_7)$ is a local coordinate system for $\mathbb{R}^7$. Thus, the tangent space TM is spanned by $\{U_i\}_{1 \le i \le 6}$, where $U_1 = e_1$, $U_2 = e_2 - e_3$, $U_3 = \frac{1}{\sqrt{2}}(e_2 + e_3) - e_5$, $U_4 = e_4$, $U_5 = e_6$, $U_6 = \xi$ and the 1-dimensional distribution $TM^{\perp}$ of rank 1 is spanned by E, where $E = \frac{1}{\sqrt{2}}(e_2 + e_3) - e_5$. It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional null hypersurface of $\overline{M}^7$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}\{\frac{1}{\sqrt{2}}(e_2 + e_3) + e_5\}$. On the other hand, by using the almost contact structure of $\overline{M}^7$ and also by taking into account the decomposition (3.1), the distribution D_0 is spanned by $\left\{F, \overline{\phi}F\right\}$, where $F = U_2$, $\overline{\phi}F = U_1 + U_4$ and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by ξ, $\overline{\phi}E = \frac{1}{\sqrt{2}}(U_1 - U_4) + U_5$ and $\overline{\phi}N = \frac{1}{2}\{\frac{1}{\sqrt{2}}(U_1 - U_4) - U_5\}$. Hence, M is a null hypersurface of $\overline{M}^7$.

Now, we consider the distributions on M, $D := TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0$, $D' := \overline{\phi}(N(TM))$. Then, D is invariant under $\overline{\phi}$ and

$$TM = (D \oplus D') \perp \langle\xi\rangle. \tag{3.4}$$

Let us consider the local null vector fields $U := -\overline{\phi}N$, $V := -\overline{\phi}E$. Then, from (3.4), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(X) := g(V, X), \quad \forall X \in \Gamma(TM)$. Applying $\overline{\phi}$ to X and (2.1), one obtains

$$\overline{\phi} X = \phi X + u(X) N, \tag{3.5}$$

where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi X := \overline{\phi} RX$. In addition, we obtain, $\phi^2 X = -X + \eta(X)\xi + u(X)U$ and $\nabla_X \xi = X - \eta(X)\xi$. Using (2.1), we derive $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y)$, where v is a differential 1-form locally defined on M by $v(\cdot) = g(U, \cdot)$. We have the following identities, for any $X \in \Gamma(TM)$, $\nabla_X \xi = X - \eta(X)\xi$ and

$$B(X, \xi) = 0, \tag{3.6}$$

$$C(X, \xi) = \theta(X), \tag{3.7}$$

$$B(X, U) = C(X, V). \tag{3.8}$$

A section $X \in \Gamma(TM)$ is said to be an η-conformal Killing vector field if

$$L_X g = \Omega(g - \eta \otimes \eta), \tag{3.9}$$

where Ω is a smooth function on $\mathcal{U} \subset M$.

Lemma 3.3. *Let M be a null hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then ξ is an η-conformal Killing vector field on M, that is, $L_\xi g = \Omega(g - \eta \otimes \eta)$, with $\Omega = 2$.*

Proof. The proof follows by direct calculation. □

Although the use of a non-degenerate screen distribution $S(TM)$ has been helpful in defining induced objects on the null spaces, because of the degenerate metric, $S(TM)$ is not unique. Therefore, a lot of induced geometric objects depend on the choice of a screen, which creates a problem. For this reason, it is desirable to look for a unique or canonical screen distribution so that the induced objects on M are well-defined. To clarify this point, we first present a brief review of the dependence on the choice of a screen distribution.

By Theorem 2.2 and relation (2.8), we say that there exists a quasi-orthonormal basis of $\overline{M}$ along M, given by

$$\{E,\, N,\, W_i\}, \quad i \in \{1, ..., 2n-1\}, \tag{3.10}$$

where $\{E\}$, $\{N\}$ and $\{W_i\}$ are the null basis of $TM^\perp$, $N(TM)$ and the orthonormal basis of $S(TM)$, respectively. Consider two quasi-orthonormal frames fields $\{E,\, N,\, W_i\}$ and $\{E,\, N',\, W_i'\}$ induced on $\mathcal{U} \subset M$ by $\{S(TM),\, N(TM)\}$ and $\{S(TM)',\, N(TM)'\}$, respectively for the same E. Using (2.7) and (2.9), we obtain

$$W_i' = \sum_{j=1}^{2n-1} W_i^j \left(W_j - \epsilon_j \mathbf{f}_j E\right), \tag{3.11}$$

$$N' = N + \mathbf{f} E + \sum_{i=1}^{2n-1} \mathbf{f}_i W_i, \tag{3.12}$$

where ϵ_i are signature of the orthonormal basis $\{W_i\}$ and W_i^j, $\mathbf{f}$ and $\mathbf{f}_i$ are smooth functions on $\mathcal{U}$ such that $\{W_i^j\}$ are $(2n-1)\times(2n-1)$ semi-orthogonal matrices. Computing $\overline{g}(N', N') = 0$ by (2.7) and $\overline{g}(W_i, W_i) = 1$ we get

$$2\mathbf{f} + \sum_{i=1}^{2n-1} \epsilon_i (\mathbf{f}_i)^2 = 0.$$

Using this in the second relation of the above two equations, we have

$$W_i' = \sum_{j=1}^{2n-1} W_i^j \left(W_j - \epsilon_j \mathbf{f}_j E\right), \tag{3.13}$$

$$N' = N - \frac{1}{2}\{\sum_{i=1}^{2n-1} \epsilon_i (\mathbf{f}_i)^2\}E + \sum_{i=1}^{2n-1} \mathbf{f}_i W_i. \tag{3.14}$$

The above two relations are used to investigate the transformation of the induced objects when one changes the pair $\{S(TM),\ N(TM)\}$ with respect to a change in the basis. Using (2.12) for both screens we have

$$B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E) = B'(X,Y), \quad \forall\ X,\ Y \in \Gamma(TM|_{\mathcal{U}}). \tag{3.15}$$

Thus, $B = B'$ on $\mathcal{U}$. Take $\overline{E} = \alpha E$, for some positive smooth function α on M. Then, it follows that $\overline{N} = (1/\alpha)N$. From (2.12) and (2.13), the associated local fundamental form $\overline{B}$ and 1-form $\overline{\tau}$ are related to B and τ, respectively, by

$$\overline{B} = \alpha B, \tag{3.16}$$

$$\tau(X) = \overline{\tau}(X) + X(\ln \alpha), \tag{3.17}$$

for any $X \in \Gamma(TM|_{\mathcal{U}})$, which proves that B and τ depend on the section E on $\mathcal{U}$. Finally, taking the exterior derivative d on both sides of (3.17) we get $d\tau = d\overline{\tau}$ on $\mathcal{U}$, that is, $d\tau$ is independent of the section E.

Define the Ricci tensor $\overline{Ric}$ of $\overline{M}$ and induced Ricci type tensor $R^{(0,2)}$ of M, respectively, as

$$\overline{Ric}(\overline{X}, \overline{Y}) = \mathrm{trace}(\overline{Z} \longrightarrow \overline{R}(\overline{Z}, \overline{X})\overline{Y}), \forall\, \overline{X},\ \overline{Y} \in \Gamma(TM), \tag{3.18}$$

$$R^{(0,2)}(X,Y) = \mathrm{trace}(Z \longrightarrow R(Z,X)Y), \forall\, X,\ Y \in \Gamma(TM). \tag{3.19}$$

Since the induced connection ∇ on M is not a Levi-Civita connection, in general, $R^{(0,2)}$ is not symmetric. Therefore, in general, it is just a tensor quantity and has no geometric or physical meaning similar to the symmetric Ricci tensor of $\overline{M}$.

Let consider a local quasi-orthogonal frame field $\{X_0, N, X_i\}_{i=1,\ldots,2n-1}$ on $\overline{M}$ where $\{X_0, X_i\}$ is a local frame field on M with respect to the decomposition (3.3) with N, the unique section of transversal bundle $N(TM)$ satisfying (2.7), and $E = X_0$. It is easy to obtain from (3.19) the following local expression for the Ricci tensor

$$R^{(0,2)}(X,Y) = g^{ij} g(R(X_i, X)Y, X_j) + g(R(X_0, X)Y, N). \tag{3.20}$$

From this we obtain

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = g^{ij}\{C(Y,X_j)B(X,X_i) - C(X,X_j)B(Y,X_i)\} - \overline{g}(\overline{R}(X,Y)X_0,N). \quad (3.21)$$

Put $R^{(0,2)}_{ls} := R^{(0,2)}(X_s,X_l)$ and $R^{(0,2)}_{0k} := R^{(0,2)}(X_k,X_0)$. Using the frame field $\{X_0,N,X_i\}$ and replacing X and Y by X_s and X_l respectively, a direct calculation gives locally

$$R^{(0,2)}_{ls} - R^{(0,2)}_{sl} = A^i_s B_{il} - A^i_l B_{is} + \overline{R}^0_{0ls} = 2d\tau(X_l,X_s) \quad (3.22)$$

$$\text{and} \quad R^{(0,2)}_{0k} - R^{(0,2)}_{k0} = -A^i_0 B_{ik} + \overline{R}^0_{00k} = 2d\tau(X_0,X_k), \quad (3.23)$$

where $\overline{R}^0_{ijk} = \overline{g}(\overline{R}(X_k,X_j)X_i,N)$. The Gauss-Codazzi equations are expressed locally by using coefficients of ∇ and local components of h, A_N and τ and they are given by ([3])

$$\overline{R}^0_{00s} = A^i_0 B_{is} + 2d\tau(X_0,X_s), \quad (3.24)$$

$$\text{and} \quad \overline{R}^0_{0js} = R^0_{0js} = A^i_j B_{is} - A^i_s B_{ij} + 2d\tau(X_j,X_s). \quad (3.25)$$

Putting (3.24) and (3.25) into (3.22) and (3.23), respectively, we have

$$R^{(0,2)}_{ls} - R^{(0,2)}_{sl} = 2d\tau(X_l,X_s) \quad \text{and} \quad R^{(0,2)}_{0k} - R^{(0,2)}_{k0} = 2d\tau(X_0,X_k). \quad (3.26)$$

This means that $R^{(0,2)}$ is symmetric on M if and only if $d\tau = 0$ on $\mathcal{U} \subset M$, that is τ is closed. Suppose $R^{(0,2)}$ is a symmetric Ricci tensor Ric. Then, the 1-form τ is closed. Thus there exists a smooth function f on $\mathcal{U}$ such that

$$\tau = df. \quad (3.27)$$

Consequently we get $\tau(X) = X(f)$. This relation, using (3.17), for $\alpha = \exp(f)$, yields

$$\tau(X) = \overline{\tau}(X) + X(\ln\alpha) = \overline{\tau}(X) + \tau(X),$$

therefore $\overline{\tau}(X) = 0$, for any $X \in \Gamma(TM|_{\mathcal{U}})$. Then, by taking $\overline{E} = \exp(f)E$, one obtains $\overline{\tau} = 0$ on $\mathcal{U}$. The corresponding $\overline{N}$ is $\overline{N} = (1/\exp(f))N$. We call the pair $\{\xi,N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ vanishes the *canonical null pair of* M.

As it is mentioned above, we observe that the existence of a symmetric Ricci tensor on M is equivalent to $d\tau = 0$, on any $\mathcal{U} \subset M$ and τ need not vanish. Therefore, only vanishing of $d\tau$ is needed to get a symmetric Ricci tensor for M.

If $\overline{M}$ is an indefinite Kenmotsu space form $(\overline{M}(c),\overline{g})$, then, the relation (2.3) becomes, for any X, Y, $Z \in \Gamma(TM)$,

$$\overline{R}(X,Y)Z = g(X,Z)Y - g(Y,Z)X. \quad (3.28)$$

Using (2.21), a direct calculation gives

$$R^{(0,2)}(X,Y) = -(2n-1)g(X,Y) + B(X,Y)trA_N - B(A_NX,Y), \quad (3.29)$$

where trace tr is written with respect to g restricted to $S(TM)$. Note that the Ricci tensor does not depend on the choice of the vector field E of the distribution $TM^{\perp}$. From (3.29), we have

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = B(A_N X, Y) - B(A_N Y, X). \tag{3.30}$$

The tensor field $R^{(0,2)}$ of a null hypersurface M of an indefinite Kenmotsu manifold $\overline{M}$ is called induced Ricci tensor if it is symmetric [8] .

4. Symmetries of null hypersurfaces in indefinite Kenmotsu manifolds

This section deals with locally symmetric, semi-symmetric and Ricci semi-symmetric null hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field ξ. Let (M, g) be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (Theorem 2.2). The relation (3.28) implies that, for any X, Y, $Z \in \Gamma(TM)$,

$$\overline{g}(\overline{R}(X,Y)Z, E) = 0, \tag{4.1}$$

$$\text{and} \quad \overline{g}(\overline{R}(X,Y)Z, N) = \overline{g}(X,Z)\theta(Y) - \overline{g}(Y,Z)\theta(X). \tag{4.2}$$

From (2.21) and (3.5) and comparing the tangential and transversal parts, we obtain

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X + B(Y,Z)A_N X - B(X,Z)A_N Y, \tag{4.3}$$

and

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z). \tag{4.4}$$

Using (4.3), then, for any X, $Y \in \Gamma(TM)$,

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{4.5}$$

Proposition 4.1. *A lightlike hypersurface (M, g) of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$, cannot be flat.*

Proof. Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. If M is flat, then from (4.5), we obtain, for any $Z \in \Gamma(TM)$,

$$\eta(X)g(Y,Z) = \eta(Y)g(X,Z),$$

from which we obtain $\overline{g}(\overline{\phi}Y, \overline{\phi}Z) = 0$, a contradiction. □

The Theorem 4.1 shows that the curvature tensor R and Ricci tensor Ric of a lightlike hypersuface M of an indefinite Kenmotsu space form $\overline{M}(c)$ are not vanishing. This guarantees the fact that we are dealing with non-trivial curvature and Ricci tensors.

In the sequel, we need the following identities. For any X, $Y \in \Gamma(TM)$ and $Z \in \Gamma(S(TM))$,

$$g((\nabla_X A_E^*)Y, Z) = (\nabla_X B)(Y,Z), \quad g((\nabla_X A_N)Y, Z) = (\nabla_X C)(Y,Z). \tag{4.6}$$

A null hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be locally symmetric if the curvature tensor R of M satisfies ([17])

$$g((\nabla_W R)(X,Y)Z, PT) = 0 \quad \text{and} \quad g((\nabla_W R)(X,Y)Z, N) = 0, \tag{4.7}$$

for any X, Y, Z, W, $T \in \Gamma(TM)$ and $N \in \Gamma(N(TM))$. From this definition, we have $(\nabla_W R)(X,Y)Z = 0, \forall\ X,\ Y,\ Z \in \Gamma(TM)$.

Note that, using the relation (3.28), it is easy to see that an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ is locally symmetric.

Proposition 4.2. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then,*

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,T) &= B(Y,Z)(\nabla_W C)(X,T) - B(X,Z)(\nabla_W C)(Y,T) \\ &+ \{B(W,X)g(Y,T) - B(W,Y)g(X,T)\}\theta(Z) + (\nabla_W B)(Y,Z)C(X,T) \\ &- (\nabla_W B)(X,Z)C(Y,T) + B(W,Z)\{\theta(X)g(Y,T) - \theta(Y)g(X,T)\}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,N) &= B(Y,Z)C(X,A_N W) - B(X,Z)C(Y,A_N W) \\ &+ \{B(W,X)\theta(Y) - B(W,Y)\theta(X)\}\theta(Z), \end{aligned} \tag{4.9}$$

for any X, Y, Z, $W \in \Gamma(TM)$, $T \in \Gamma(S(TM))$ and $N \in \Gamma(N(TM))$.

Proof. Using (4.3), the covariant derivative of R gives, for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \nabla_W R(X,Y)Z - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z \\ &- R(X,Y)\nabla_W Z \\ &= B(Y,Z)(\nabla_W A_N)X - B(X,Z)(\nabla_W A_N)Y + (\nabla_W g)(X,Z)Y \\ &+ (\nabla_W B)(Y,Z)A_N X - (\nabla_W B)(X,Z)A_N Y - (\nabla_W g)(Y,Z)X, \end{aligned} \tag{4.10}$$

which implies, for any $T \in \Gamma(S(TM))$ and $N \in \Gamma(N(TM))$,

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,T) &= B(Y,Z)(\nabla_W C)(X,T) - B(X,Z)(\nabla_W C)(Y,T) \\ &+ \{B(W,X)g(Y,T) - B(W,Y)g(X,T)\}\theta(Z) + B(W,Z)\{\theta(X)g(Y,T) \\ &- \theta(Y)g(X,T)\} + (\nabla_W B)(Y,Z)C(X,T) - (\nabla_W B)(X,Z)C(Y,T), \end{aligned}$$

and

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,N) &= B(Y,Z)C(X,A_N W) - B(X,Z)C(Y,A_N W) \\ &+ \{B(W,X)\theta(Y) - B(W,Y)\theta(X)\}\theta(Z), \end{aligned}$$

which completes the proof. □

Theorem 4.3. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, M is locally symmetric if and only if it is totally geodesic.*

Proof. Let (M, g) be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Suppose that M is locally symmetric. Then, for any W, Y, $Z \in \Gamma(TM)$, $(\nabla_W R)(X,Y)Z = 0$. Taking $Y = E$ and $Z = \xi$ in (4.8), one obtains $0 = g((\nabla_W R)(X, E)\xi, N) = B(W, X)$, which implies that M is totally geodesic. The converse is obvious. □

Let $\overline{M}(c)$ be an indefinite Kenmotsu space form and M be a null hypersurface of $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (Theorem 2.2) and by using (2.21), we obtain

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \quad (4.11)$$

Lemma 4.4. *Let M be a null hypersurface of an indefinite Kenmotsu space form $\overline{M}$ with $\xi \in TM$. Then, the Lie derivative of the local second fundamental form B with respect to ξ is given by*

$$(L_\xi B)(X, Y) = (1 - \tau(\xi))B(X, Y), \; \forall\, X,\, Y \in \Gamma(TM). \quad (4.12)$$

Proof. Using (2.21), we obtain

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) - 2B(X, Y). \quad (4.13)$$

Likewise, using again (2.21), we have

$$(\nabla_X B)(\xi, Y) = -B(X, Y). \quad (4.14)$$

Subtracting (4.13) and (4.14), we obtain

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = (L_\xi B)(X, Y) - B(X, Y). \quad (4.15)$$

From (4.11) and after calculations, the left hand side of (4.15) becomes

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = -\tau(\xi)B(X, Y). \quad (4.16)$$

The relations (4.15) and (4.16) imply $(L_\xi B)(X, Y) = (1 - \tau(\xi))B(X, Y)$. □

Next, we give characterization on parallel null hypersuface of an indefinite Kenmotsu manifold. In fact, it shows that there do not exist non-totally geodesic totally umbilical null hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field ξ.

The second fundamental form h of M is said to be parallel if $(\nabla_X h)(Y, Z) = 0,\; \forall\, X,\, Y,\, Z \in \Gamma(TM)$. That is,

$$(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z). \quad (4.17)$$

In [28], Sahin characterizes lightlike hypersurfaces with parallel second fundamental form in Lorentzian manifold. He showed that there do not exist non-totally geodesic parallel lightlike hypersurfaces in a Lorentzian manifold.

Theorem 4.5. *Let M be a null hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. If the second fundamental form h of M is parallel, then M is totally geodesic.*

Proof. Suppose that the second fundamental form h of M is parallel. Then (4.17) is satisfied. Using (4.17), we obtain

$$(\nabla_\xi B)(X, Y) = -\tau(\xi)B(X, Y). \quad (4.18)$$

From (2.21) and using (4.12), the left hand side of (4.18) becomes

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) - 2B(X, Y) = -(1 + \tau(\xi))B(X, Y). \quad (4.19)$$

From the expressions (4.18) and (4.19) we complete the proof. □

To study the dependence of the induced objects $\{\tau,\, \nabla\}$ on the screen distribution $S(TM)$, let $\{\tau',\, \nabla'\}$ be another set of induced objects with respect to another screen distribution $S(TM)'$ and its transversal $N(TM)'$. Consider two quasi-orthonormal frames fields $\{E,\, N,\, W_i\}$ and $\{E,\, N',\, W_i'\}$ induced on $\mathcal{U} \subset M$ by $\{S(TM),\; N(TM)\}$ and $\{S(TM)',\, N(TM)'\}$, respectively. Using the transformation equations (3.15) and (3.17), we obtain relationship between the geometrical objects induced by the Gauss-Weingarten equations with respect to $S(TM)$ and $S(TM)'$ as follows:

$$\tau'(X) = \tau(X) + B(X, N' - N), \tag{4.20}$$

$$\nabla'_X Y = \nabla_X Y + B(X,Y)\{\frac{1}{2} g(W,W)E - W\}, \tag{4.21}$$

for any $X,\, Y \in \Gamma(TM|_{\mathcal{U}})$, where $W = \sum_{i=1}^{2n-1} \mathbf{f}_i W_i$ is the characteristic vector field of the screen change.

The covariant derivative of the second fundamental form h depends on ∇, N and τ which depend on the choice of the screen vector bundle. The covariant derivatives ∇ of $h = B \otimes N$ and ∇' of $h' = B \otimes N'$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: for any $X, Y, Z \in \Gamma(TM)$,

$$\overline{g}((\nabla'_X h')(Y,Z), E) = \overline{g}((\nabla_X h)(Y,Z), E) + \mathcal{L}_{(X,Y)} Z,$$

with $\mathcal{L}_{(X,Y)} Z = B(X,Y)B(Z,W) + B(X,Z)B(Y,W) + B(Y,Z)B(X,W)$. It is easy to check that the parallelism of h is independent of the screen distribution $S(TM)$ ($\nabla' h' \equiv \nabla h$) if and only the second fundamental form B of M vanishes identically on M.

In virtue of Theorem 4.3, we have the following result.

Theorem 4.6. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, M is locally symmetric if and only if it is parallel.*

It is known that null submanifolds whose screen distribution is integrable have interesting properties. Referring to the decomposition (3.2) and for any $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp \langle\xi\rangle)$, we have

$$\nabla_X Y = \nabla_X^{D\perp\langle\xi\rangle} Y + h^{D\perp\langle\xi\rangle}(X,Y), \tag{4.22}$$

where $\nabla^{D\perp\langle\xi\rangle}$ is a linear connection on $D \perp \langle\xi\rangle$ and $h^{D\perp\langle\xi\rangle} : \Gamma(TM) \times \Gamma(D \perp \langle\xi\rangle) \longrightarrow D'$ is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U} \subset M$ be a coordinate neighborhood as fixed in Theorem 2.2. Then, using (3.2), (4.22) can be rewritten locally as,

$$\nabla_X Y = \nabla_X^{D\perp\langle\xi\rangle} Y + \overline{g}(\overline{\nabla}_X Y, V)U = \nabla_X^{D\perp\langle\xi\rangle} Y + B(X, \phi Y)U, \tag{4.23}$$

for any $X \in \Gamma(TM)$, $Y \in \Gamma(D \perp \langle\xi\rangle)$ and the local expression of $h^{D\perp\langle\xi\rangle}$ is

$$h^{D\perp\langle\xi\rangle}(X,Y) = B(X, \phi Y)U. \tag{4.24}$$

This means that $h^{D\perp\langle\xi\rangle}$ is symmetric on $D \perp \langle\xi\rangle$ if and only if $B(X,\phi Y) = B(\phi X, Y)$, $\forall\; X,\; Y \in \Gamma(D \perp \langle\xi\rangle)$. Since $u([X,Y]) = B(X,\phi Y) - B(\phi X, Y)$, we deduce that $h^{D\perp\langle\xi\rangle}$ is symmetric on $D \perp \langle\xi\rangle$ if and only if $D \perp \langle\xi\rangle$ is integrable. Moreover the integrability of $D \perp \langle\xi\rangle$ implies that $\nabla^{D\perp\langle\xi\rangle}$ is a linear symmetric connection on the integral manifolds.

In the following this property is considered.

Definition 4.7. Let $(M, g, S(TM))$ be a screen integrable null hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. A leaf M' of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold is said to be locally symmetric if the induced curvature R^* of Levi-Civita connection ∇^* satisfies

$$(\nabla^*_W R^*)(X, Y)Z = 0, \quad \forall\, W, X, Y, Z \in \Gamma(TM'). \tag{4.25}$$

In the following theorem, we show that local symmetry property of a screen integrable null hypersurface of an indefinite Kenmotsu space form is closely related to the local symmetry property of leaves of its screen distribution. First of all, we note that when null hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$ is locally symmetric, then the following identities

$$R(E, Y, Z, T) = 0, \quad R(X, E, Z, T) = 0, \quad R(X, Y, E, T) = 0, \tag{4.26}$$

for any X, Y, Z, $T \in \Gamma(TM)$, hold.

Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Using Gauss and Weingarten equations, we have,

$$\begin{aligned} R(X, Y)Z = &R^*(X, Y)Z + C(X, Z)A^*_E Y - C(Y, Z)A^*_E X \\ &+ \{(\nabla_X C)(Y, Z) - (\nabla_Y C)(X, Z) + \tau(Y)C(X, Z) \\ &- \tau(X)C(Y, Z)\}E, \quad \forall X, Y, Z \in \Gamma(TM'), \end{aligned} \tag{4.27}$$

where $(\nabla_X C)(Y, Z) = X.C(Y, Z) - C(\nabla^*_X Y, Z) - C(Y, \nabla^*_X Z)$. By covariant derivative, we have for any W, X, Y, $Z \in \Gamma(TM')$,

$$\begin{aligned} &(\nabla_W R)(X, Y)Z = (\nabla^*_W R^*)(X, Y)Z + (\nabla_W C)(X, Z)A^*Y \\ &- (\nabla_W C)(Y, Z)A^*X + C(X, Z)(\nabla_W A^*)Y - C(Y, Z)(\nabla_W A^*)X \\ &- \big\{(\nabla_X C)(Y, Z) - (\nabla_Y C)(X, Z) + \tau(Y)C(X, Z) \\ &- \tau(X)C(Y, Z)\big\}A^*W + \big\{(\nabla_W \nabla_X C)(Y, Z) - (\nabla_W \nabla_Y C)(X, Z) \\ &+ C(X, Z)(\nabla_W \tau)Y - C(Y, Z)(\nabla_W \tau)X + \tau(Y)(\nabla_W C)(X, Z) \\ &- \tau(X)(\nabla_W C)(Y, Z) + \tau(W)(\nabla_Y C)(X, Z) - \tau(W)(\nabla_X C)(Y, Z) \\ &+ \tau(W)\tau(X)C(Y, Z) - \tau(W)\tau(Y)C(X, Z) + C(X, Z)C(W, A^*Y) \\ &- C(Y, Z)C(W, A^*X) + (\nabla_{\nabla^*_W Y} C)(X, Z) - (\nabla_{\nabla^*_W X} C)(Y, Z) \\ &+ C(W, R^*(X, Y)Z)\big\}E - R(C(W, X)E, Y)Z - R(X, C(W, Y)E)Z \\ &- R(X, Y)C(W, Z)E. \end{aligned} \tag{4.28}$$

So, for any $W, X, Y, Z, T \in \Gamma(TM')$, we have,

$$g((\nabla_W R)(X,Y)Z,T) = g((\nabla^*_W R^*)(X,Y)Z,T) + B(Y,T)(\nabla_W C)(X,Z)$$
$$- B(X,T)(\nabla_W C)(Y,Z) + C(X,Z)g((\nabla_W A^*)Y,T)$$
$$- C(Y,Z)g((\nabla_W A^*)X,T) + B(W,T)(\nabla_Y C)(X,Z)$$
$$- B(W,T)(\nabla_X C)(Y,Z) + B(W,T)\tau(X)C(Y,Z)$$
$$- B(W,T)\tau(Y)C(X,Z) - C(W,X)R(E,Y,Z,T)$$
$$- C(W,Y)R(X,E,Z,T) - C(W,Z)R(X,Y,E,T). \tag{4.29}$$

By virtue of the relation (4.6), we have

$$g((\nabla_W A^*)Y,T) = (\nabla_W B)(Y,T).$$

If M is locally symmetric, then, using Theorem 4.3, $B = 0$. By relations in (4.26), $g((\nabla^*_W R^*)(X,Y)Z,T) = 0$, that is M' is locally symmetric in $\overline{M}$. Therefore,

Theorem 4.8. *Let $(M, g, S(TM))$ be a screen integrable null hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. If M is locally symmetric, then any leaf M' of $S(TM)$ immersed in $\overline{M}$ as a non-degenerate submanifold is locally symmetric.*

Note that the locally symmetry has an integrability condition, namely, the semi-symmetry. Now, we deal with semi-symmetric null hypersurfaces of indefinite Kenmotsu spaces form, tangent to the structure vector field ξ.

A null hypersurface M of a semi-Riemannian manifold $\overline{M}$ is said to be semi-symmetric if the following condition is satisfied ([10])

$$(R(W_1, W_2) \cdot R)(X, Y, Z, T) = 0, \ \ \forall\, W_1, W_2, X, Y, Z, T \in \Gamma(TM), \tag{4.30}$$

where R is the induced Riemann curvature on M.

This is equivalent to

$$-R(R(W_1, W_2)X, Y, Z, T) - ... - R(X, Y, Z, R(W_1, W_2)T) = 0.$$

In general the condition (4.30) is not equivalent to $(R(W_1, W_2) \cdot R)(X, Y)Z = 0$ like in the non-degenerate case.

Next, we investigate the effect of semi-symmetry condition on geometry of null hypersurfaces in an indefinite Kenmotsu space form.

Theorem 4.9. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$, with $\xi \in TM$. Then M is semi-symmetric if and only if it is totally geodesic.*

Proof. From (4.30), we have, for any X, Y, Z, $T \in \Gamma(TM)$,

$$(R(E,X) \cdot R)(E,Y,Z,T) = -B(X,Y)B(A_N E,Z)g(A_N E,T)$$
$$- B(Y,A_N E)B(X,Z)g(A_N E,T) - B(Y,Z)B(X,T)g(A_N E,A_N E). \tag{4.31}$$

If M is semi-symmetric, the left hand side of (4.31) vanishes and we have,

$$0 = B(X,Y)B(A_N E,Z)g(A_N E,T) + B(Y,A_N E)B(X,Z)g(A_N E,T)$$
$$+ B(Y,Z)B(X,T)g(A_N E,A_N E).$$

which leads, by taking $T = \xi$ and using (3.7), to

$$\begin{aligned} 0 &= B(A_N E, B(X,Y)Z + B(X,Z)Y) \\ &= g(A_E^* A_N E, B(X,Y)Z + B(X,Z)Y). \end{aligned} \tag{4.32}$$

This means that

$$\begin{aligned} B(X,Y)Z + B(X,Z)Y = \{B(X,Y)\theta(Z) + B(X,Z)\theta(Y)\}E \\ + \{B(X,Y)\eta(Z) + B(X,Z)\eta(Y)\}\xi, \end{aligned}$$

that is,

$$B(X,Y)\{Z - \theta(Z)E - \eta(Z)\xi\} = B(X,Z)\{Y - \theta(Y)E - \eta(Y)\xi\}.$$

Taking $PZ' = Z - \theta(Z)E - \eta(Z)\xi$ and $PY' = Y - \theta(Y)E - \eta(Y)\xi$, we have

$$B(X,Y)PZ' = B(X,Z)PY'. \tag{4.33}$$

Now suppose that there exists a vector field Y_0' on some neighborhood of M such that $B(X, Y_0) \neq 0$, for any $X \in \Gamma(TM)$, at some point p in the neighborhood. Then, from (4.33) it follows that all vectors of the fibre $(S(TM) - \langle\xi\rangle)_p := \left(\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM)) \perp D_0\right)_p \subset S(TM)_p$ are collinear with $(PY_0')_p$. This contradicts $\dim(S(TM) - \langle\xi\rangle)_p > 1$. This implies that $B(X,Y) = 0$. The converse is obvious. □

In virtue of Theorem 4.3 and Theorem 4.9, we have the following result.

Theorem 4.10. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then M is locally symmetric if and only if it is semi-symmetric.*

Next, we study Ricci semi-Symmetric null hypersurfaces of an indefinite Kenmotsu spaces form, tangent to the structure vector field ξ. We prove that Ricci semi-Symmetric null hypersurfaces are totally geodesic under some condition.

A null submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be Ricci semi-symmetric if the following condition is satisfied ([5])

$$(R(W_1, W_2) \cdot Ric)(X,Y) = 0, \;\; \forall\, W_1, W_2, X, Y \in \Gamma(TM), \tag{4.34}$$

where R and Ric are induced Riemannian curvature and Ricci tensor on M, respectively. The latter condition is equivalent to

$$-Ric(R(W_1, W_2)X, Y) - Ric(X, R(W_1, W_2)Y) = 0.$$

In the following result we show the effect of Ricci semi-symmetric condition on the geometry of null hypersurfaces of an indefinite Kenmotsu space form.

Theorem 4.11. *Let $(M, g, S(TM))$ be a Ricci semi-symmetric null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then either M is totally geodesic or $Ric(E, A_N E) = 0$.*

The proof of this theorem is similar to the one of Theorem 10 in [17] in case of indefinite Sasakian space form of constant curvature 1.

Let us consider the following distribution

$$\widehat{D} = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \tag{4.35}$$

so that the tangent space of M is written

$$TM = \widehat{D} \perp \langle\xi\rangle \perp TM^{\perp}. \tag{4.36}$$

Let $\widehat{P}$ be the morphism of $S(TM)$ on $\widehat{D}$ with respect to the orthogonal decomposition of $S(TM)$ such that

$$\widehat{P}X = PX - \eta(X)\xi, \tag{4.37}$$

for any $X \in \Gamma(TM)$. Using (4.37), one obtains

$$\widehat{P}^2X = \widehat{P}(PX - \eta(X)\xi = P^2X - \eta(X)\xi = \widehat{P}X.$$

This means that the morphism $\widehat{P}$ is a projection.

A submanifold M is said to be Einstein if its induced Ricci tensor Ric satisfies

$$Ric = ag, \tag{4.38}$$

where the non-zero function a is not necessarily constant on M.

Note that when the null hypersurface M is totally geodesic, by relation (3.29), M is Einstein. This also occurs when M is parallel or totally umbilical (see [23] for details). Below we have another characterization of Einstein null hypersurface and this is related with the shape operator concerning the normal vector field N.

If the null hypersurface M of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$, is Einstein, then, using (3.29), the function a satisfies

$$a = -(2n-1). \tag{4.39}$$

Due to the symmetry of the induced degenerate metric g, the induced Ricci type tensor $R^{(0,2)}$ is symmetric, and the notion of Einstein manifold does not depend on the choice of the screen distribution $S(TM)$.

In this case, using (3.29) and (4.38), we have, for any $X, Y \in \Gamma(TM)$,

$$-(2n-1)g(X,Y) + B(X,Y)trA_N - B(A_NX,Y) = ag(X,Y). \tag{4.40}$$

This implies that

$$B(X,Y)trA_N - B(A_NX,Y) = 0, \tag{4.41}$$

which is equivalent to $g((trA_N)X - A_NX, A^*_EY) = 0$. Therefore, $(trA_N)X - A_NX = \alpha_1E + \alpha_2\xi$. It is easy to check that $\alpha_1 = (trA_N)\theta(X)$ and $\alpha_2 = (trA_N)\eta(X) - \theta(X)$. Hence

$$A_NX = (trA_N)\widehat{P}X + \theta(X)\xi. \tag{4.42}$$

From the relation (4.42), we have

Lemma 4.12. *The screen distribution $S(TM)$ of any null Einstein Einstein hypersurface M of an indefinite Kenmotsu space form with $\xi \in TM$, is totally contact umbilical in the direction of the Kernel of the differential 1-form θ in (2.18).*

Using (4.3), the left hand-side of (4.34) is deduced as, for any W_1, W_2, X, $Y \in \Gamma(TM)$,

$$(R(W_1, W_2) \cdot Ric)(X, Y) = -aB(W_2, X)C(W_1, Y) + aB(W_1, X)C(W_2, Y) - aB(W_2, Y)C(W_1, X) + aB(W_1, Y)C(W_2, X). \quad (4.43)$$

Taking $W_1 = E$ and $Y = \xi$ in (4.43), we have

$$(R(E, W_2) \cdot Ric)(X, \xi) = (2n - 1)B(W_2, X). \quad (4.44)$$

Theorem 4.13. *Let $(M, g, S(TM))$ be an null Einstein hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then M is Ricci semi-symmetric if and only if it is totally geodesic.*

By Theorems 4.10 and 4.13, we have

Theorem 4.14. *Let $(M, g, S(TM))$ be an null Einstein hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, the following assertions are equivalent:*

(i) *M is locally symmetric.*
(ii) *M is semi-symmetric.*
(iii) *M is Ricci semi-symmetric.*

As an example to this Theorem 4.14, we have

Example 4.15. Let M be a hypersurface of $\overline{M}^7$, of Example 3.2, given by

$$x_5 = \sqrt{2}\,(x_2 + x_3)\,,$$

where $(x_1, ..., x_7)$ is a local coordinate system for $\overline{M}^7$. As explained in Example 3.2, M is a null hypersurface of $\overline{M}^7$ having a local quasi-orthogonal field of frames $\{U_1 = e_1,\ U_2 = e_2 - e_3,\ U_3 = E = \frac{1}{\sqrt{2}}(e_2 + e_3) - e_5,\ U_4 = e_4,\ U_5 = e_6,\ U_6 = \xi,\ N = \frac{1}{2}\{\frac{1}{\sqrt{2}}(e_2 + e_3) + e_5\}\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}^7$. Then, we obtain $\overline{\nabla}_{U_3} N = -\xi$ and $\overline{\nabla}_X N = 0$, $\forall X \in \Gamma(TM)$, $X \neq U_3$. Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulas we have

$$A_N U_3 = \xi,\ \ A_N X = 0,\ \ \forall X \in \Gamma(TM),\ X \neq U_3, \quad (4.45)$$

$$A^*_E X = 0,\ \ \nabla_X E = 0,\ \ \forall X \in \Gamma(TM). \quad (4.46)$$

From (4.45) and (4.46), $C(U_3, \xi) = 1$, $trA_N = 0$ and $trA^*_E = 0$, i.e. the shape operators A_N and A^*_E are trace-free. Then, the null hypersurface M is totally geodesic and its screen distribution is not parallel. The non-zero components of the curvature tensor are given by

$$R(e_i, e_j)e_i = e_j,\ \forall\, i,\ j,\ i \neq j, \quad R(e_i, e_m)e_m = e_i,\ \forall\, i,\ m = 5, 6,$$
$$R(e_i, e_l)e_l = -e_i,\ \forall\, i \neq l, l = 1, 2, 3, 4, 7,$$

and those for the Ricci tensor are

$$Ric(e_l, e_l) = -5,\ \forall\, l = 1, 2, 3, 4, 7, \quad Ric(e_m, e_m) = 5,\ \forall\, m = 5, 6.$$

Using these relations, it is easy to check that M is Einstein with $a = -5$, and the components of ∇R, $R \cdot R$ and $R \cdot Ric$ vanish, that is,

$$(\nabla_{e_r} R)(e_p, e_q)e_s = 0, \quad (R(\cdot,\cdot)\cdot R)(e_p, e_q, e_r, e_s) = 0,$$
$$(R(\cdot,\cdot)\cdot Ric)(e_p, e_q) = 0, \quad \forall\, p,\ q,\ r,\ s.$$

This means M is locally symmetric, semi-symmetric and Ricci semi-symmetric.

5. Total contact umbilicity of null hypersurfaces in indefinite Kenmotsu manifolds

In this section, we deal with the totally contact umbilical null hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$ by introducing a new concept. First of all, a submanifold M is said to be a totally umbilical null hypersurface of a semi-Riemannian manifold $\overline{M}$ if its local second fundamental form B satisfies

$$B(X,Y) = \rho g(X,Y), \ \ \forall X,\ Y \in \Gamma(TM) \tag{5.1}$$

where ρ is a smooth function on $\mathcal{U} \subset M$. If we assume that M is a totally umbilical null hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$, using (3.6), we have $0 = B(\xi,\xi) = \rho$. Hence M is totally geodesic. Therefore we have

Proposition 5.1. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. If M is totally umbilical, then M is totally geodesic.*

It follows from the Proposition 5.1 that a Kenmotsu $\overline{M}(c)$ does not admit any non-totally geodesic, totally umbilical null hypersurface. From this point of view, Bejancu [1] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [15]. We follow Bejancu [1] definition of totally contact umbilical submanifolds and state the following definition for totally null hypersurfaces.

A submanifold M is said to be totally contact umbilical if its second fundamental form $h = B \otimes N$ satisfies ([1], [19])

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\, H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \tag{5.2}$$

for any $X, Y \in \Gamma(TM)$, where H is a normal vector field to M, that is, $H = \lambda N$, λ is a smooth function on $\mathcal{U} \subset M$. Using (3.6), it is easy to check that a totally contact umbilical null hypersurface of an indefinite Kenmotsu manifold is *η-totally umbilical*.

Using the projection morphism $\widehat{P}$, we have the following identity,

$$B(X, PY) = B(X, \widehat{P}Y), \tag{5.3}$$

for any $X, Y \in \Gamma(TM)$.

The relation (5.2) is now equivalent to

$$A^*_E X = \lambda \widehat{P} X, \quad \forall\, X \in \Gamma(TM|\mathcal{U}). \tag{5.4}$$

If the function λ is nowhere vanishing on M, then the latter is said to be proper totally contact umbilical. It is easy to check that this is an intrinsic notion that is independent, on $\mathcal{U}$, of the choice of a screen distribution, E, and hence N as in Theorem 2.2.

Note that totally contact umbilicity is the nearest situation from being totally geodesic (λ is identically zero). The null hypersurface M is totally umbilical in the direction of $\widehat{D}$. Then, taking into account (5.3), the relation (5.4) may be rewritten for a given E in $TM^{\perp}$ as

$$\overline{g}(\overline{\nabla}_X \widehat{P}Y, E) = \varphi(E) g(X, \widehat{P}Y), \tag{5.5}$$

with φ a 1-form on $TM^{\perp}$ which coincides with the function λ of normal vector $H = \lambda N$ in (5.2), that is, $\varphi(E) = \lambda$ on M. Therefore, the map

$$(X, Y) \longmapsto \overline{g}(\overline{\nabla}_X E, \widehat{P}Y) = -\varphi(E) g(X, \widehat{P}Y), \tag{5.6}$$

is a bilinear symmetric form on TM. Since $\overline{g}(\overline{\nabla}_X E, \widehat{P}Y) = \overline{g}(\overline{\nabla}_X E, Y)$ and $g(X, \widehat{P}Y) = g(X, Y) - \eta(X)\eta(Y)$, the map (5.6) redefined as

$$(X, Y) \longmapsto \overline{g}(\overline{\nabla}_X E, Y) = -\varphi(E)\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{5.7}$$

Lemma 5.2. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, M is totally contact umbilical if and only if $TM^{\perp}$ is an η-conformal Killing distribution, that is, there exists a 1-form φ on $TM^{\perp}$ such that, for any section E of $TM^{\perp}$,*

$$L_E g = -2\varphi(E)\{g - \eta \otimes \eta\}. \tag{5.8}$$

Proof. The proof follows from the relation $(L_E g)(X, Y) = -2B(X, Y)$, for any $X, Y \in \Gamma(TM)$, obtained by using (2.16). ☐

In a (pseudo-) Riemannian setting, manifolds M with conformal structure $[g]$ and torsion-free connection D, such that parallel translation induces conformal transformations, are called Weyl manifolds. If D is locally the Levi-Civita connection of a compatible metric in $[g]$, the structure is said to be closed, and the D-compatible metric is locally Einstein [12].

A conformal change of the metric leads to a metric which is no more compatible with the almost contact structure. This can be corrected by a convenient change of the structure vector field ξ and the 1-form η, which implies rather strong restrictions. Therefore, in case there is an integral manifold, of an integrable distribution of M, which has an indefinite Kenmotsu structure, we may consider a change of the form

$$\phi' = \phi, \quad \xi' = e^{\rho}\xi, \quad \eta' = e^{-\rho}\eta, \quad g' = e^{-2\rho}g, \tag{5.9}$$

where ρ is a differential function on considered integral manifold, to preserve the relations given by the Kenmotsu structure. To support this statement, we have the following example of integral manifolds. Suppose the distribution $D_0 \perp \langle\xi\rangle$ is integrable. Let M_0 be a leaf of $D_0 \perp \langle\xi\rangle$, then for any $p \in M_0$, we have $T_pM_0 = (D_0 \perp \langle\xi\rangle)_p$ and $\dim M_0 = 2n - 3$. If $X_0 = X_0' + \eta(X_0)\xi \in \Gamma(TM_0)$, $\phi X_0 = \overline{\phi} R X_0' = \overline{\phi} X_0' = \overline{\phi} X_0$, where $R : \Gamma(TM) \longrightarrow \Gamma(D \perp \langle\xi\rangle)$ being the projection morphism and $D = TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0 \perp \langle\xi\rangle$. We put $\overset{\circ}{\phi} = \phi|_{D_0 \perp \langle\xi\rangle}$ and $\overset{\circ}{\eta} = \eta|_{D_0 \perp \langle\xi\rangle}$, so ϕ^* defines an $(1,1)$-type tensor field on M_0

because $\overline{\phi}(D \perp \langle\xi\rangle) \subset D$. Now we consider $(M_0, \phi_0, \xi, \eta_0, g)$ and check that this is an indefinite Kenmotsu structure. We know that $\phi^2 X = -X + \eta(X)\xi + u(X)U$, for any $X \in \Gamma(TM)$, and that $u(X) = 0$ for any $X \in \Gamma(D)$, so we deduce $(\overset{\circ}{\phi})^2 X_0 = -X_0 + \eta(X_0)\xi$, for any $X_0 \in \Gamma(TM_0)$. Then $\overset{\circ}{\eta}(\xi) = 1$ and (ϕ_0, ξ, η_0) is an almost contact structure. Now, we prove the compatibility between the (ϕ_0, ξ, η_0)-structure and the metric g on M^*. By relation (3.5), for any X_0, $Y_0 \in \Gamma(TM_0)$, we have $g(\overset{\circ}{\phi} X_0, \overset{\circ}{\phi} Y_0) = g(X_0, Y_0) - \eta(X_0)\eta(Y_0)$. Let $\overset{\circ}{\nabla}$ be a linear connection on the bundle $D_0 \perp \langle\xi\rangle$. For any $X \in \Gamma(TM)$, $Y_0 \in \Gamma(D_0 \perp \langle\xi\rangle)$, we have

$$\nabla_X Y_0 = \overset{\circ}{\nabla}_X Y_0 + \overset{\circ}{h}(X, Y_0), \tag{5.10}$$

where $\overset{\circ}{h}$: $\Gamma(TM) \times \Gamma(D_0 \perp \langle\xi\rangle) \longrightarrow \Gamma(\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\} \perp TM^\perp)$ is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U} \subset M$ be a coordinate neighborhood as fixed in Theorem 2.2. Then, for any X_0, $Y_0 \in \Gamma(D_0 \perp \langle\xi\rangle)$

$$\nabla_{X_0} Y_0 = \overset{\circ}{\nabla}_{X_0} Y_0 + C(X_0, \phi Y_0)V + B(X_0, \phi Y_0)U + C(X_0, Y_0)E, \tag{5.11}$$

and the local expression of $\overset{\circ}{h}$ is $\overset{\circ}{h}(X_0, Y_0) = C(X_0, \phi Y_0)V + B(X_0, \phi Y_0)U + C(X_0, Y_0)E$. Since $D_0 \perp \langle\xi\rangle$ is integrable, $\overset{\circ}{h}$ is symmetric, that is $C(X_0, \phi Y_0) = C(Y_0, \phi X_0)$, $B(X_0, \phi Y_0) = B(Y_0, \phi X_0)$ and $C(X_0, Y_0) = C(Y_0, X_0)$. The Levi-Civita conncection $\overline{\nabla}$ on $\overline{M}$ and the induced connection $\overset{\circ}{\nabla}$ are related as

$$\overline{\nabla}_{X_0} Y_0 = \overset{\circ}{\nabla}_{X_0} Y_0 + B(X_0, Y_0)N + \overset{\circ}{h}(X_0, Y_0). \tag{5.12}$$

It easy to check that $\overset{\circ}{\nabla}_{X_0} Y_0 \in \Gamma(D_0 \perp \langle\xi\rangle)$, for any X_0, $Y_0 \in \Gamma(D_0 \perp \langle\xi\rangle)$, that is, the distribution $D_0 \perp \langle\xi\rangle$ defines a totally geodesic foliation. Hence M_0 is a totally geodesic leaf in both M and $\overline{M}$. Moreover, $\overset{\circ}{\nabla}$ is the Levi-Civita connection on M_0. In fact, using (5.10) and (5.12), and since $D_0 \perp \langle\xi\rangle \subset S(TM)$, for any X_0, Y_0, $Z_0 \in \Gamma(D_0 \perp \langle\xi\rangle)$, we have

$$(\overset{\circ}{\nabla}_{X_0} g)(Y_0, Z_0) = (\nabla_{X_0} g)(Y_0, Z_0) = 0, \tag{5.13}$$

$$(\overset{\circ}{\nabla}_{X_0} \overset{\circ}{\phi})Y_0 = g(\overset{\circ}{\phi} X_0, Y_0)\xi - \overset{\circ}{\eta}(Y_0)\overset{\circ}{\phi} X_0. \tag{5.14}$$

Therefore, $(M_0, \phi_0, \xi, \eta_0, g)$ has an indefinite Kenmotsu structure.

Note that being null is invariant under conformal change of the metric. In this respect we define the following.

A connection ∇ on a null hypersurface (M, g) is said to be *η-conformal* if the covariant derivative of g is proportional to $g - \eta \otimes \eta$, that is, there exists a differential 1-form β such that the following

$$\nabla g = -\beta \otimes \{g - \eta \otimes \eta\}, \tag{5.15}$$

holds. If in addition, ∇ is torsion-free, it is said to be *Weyl-connection* [12] in the direction of the distribution $Ker(\eta)$. But on M, such a connection will be called *η-Weyl connection.*

Suppose now that ∇ is an η-conformal connection on the null hypersurface (M, g) of an indefinite Kenmotsu space form $(\overline{M}, \overline{g})$ with $\xi \in TM$. From (5.15), we have

$$\begin{aligned} 0 &= (\nabla_\xi g)(X, Y) + \beta(\xi)\{g(X, Y) - \eta(X)\eta(Y)\} \\ &= (L_\xi g)(X, Y) + (\beta(\xi) - 2)\{g(X, Y) - \eta(X)\eta(Y)\}. \end{aligned} \tag{5.16}$$

This relation leads to $(L_\xi g)(X, Y) = (2-\beta(\xi))\{g(X, Y)-\eta(X)\eta(Y)\}$ and using Lemma 3.3, we have $\beta(\xi) = 0$. This means that the differential 1-form β vanishes in the direction of the distribution $\langle\xi\rangle$.

If we denote by X^T the component of X orthogonal to ξ, then, using (2.17) we have for arbitrary vector fields X, Y, Z on M,

$$\begin{aligned} 0 &= (\nabla_X g)(Y, Z) + \beta(X)\{g(X - \eta(X)\xi, Y - \eta(Y)\xi)\} \\ &= B(X^T, Y^T)\theta(Z^T) + B(X^T, Z^T)\theta(Y) + \beta(X^T)g(Y^T, Z^T) \\ &= (\nabla_{X^T} g)(Y^T, Z^T) + \beta(X^T)g(Y^T, Z^T). \end{aligned}$$

That is, the relation (5.15) is equivalent to

$$(\nabla_{X^T} g)(Y^T, Z^T) = -\beta(X^T)g(Y^T, Z^T). \tag{5.17}$$

Assume that the induced connection ∇ satisfies (5.15) for some smooth 1-form β on M. Then, from (2.7) we get

$$\begin{aligned} 0 &= (\nabla_X g)(Y, Z) + \beta(X)\{g(Y, Z) - \eta(Y)\eta(Z)\} \\ &= B(X, Y)\theta(Z) + B(X, Z)\theta(Y) + \beta(X)\{g(Y, Z) - \eta(Y)\eta(Z)\}, \end{aligned} \tag{5.18}$$

for any X, Y and Z in $\Gamma(TM)$. Taking $Y = E$ in (5.18), we have $B(X, Z) = 0$, for all vector fields X and Z in TM, which is equivalent to saying that (M, g) is totally geodesic. If M is a proper totally contact umbilical null hypersurface, the local fundamental form B is nowhere vanishing. Therefore, we have

Theorem 5.3. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, the induced connection is an η-Weyl connection if and only if M is totally geodesic. Moreover, the induced connection on a proper totally contact umbilical null hypersurface is never an η-Weyl connection.*

The Theorem 5.3 shows that the connection associated to a screen distribution on M is never a η-Weyl connection unless M is totally geodesic. Note that, for a totally geodesic null hypersurface (M, g) , not all metrics in the conformal class of g guarantee the geometric condition of geodesibility. Therefore, only an appropriate conformal structure on a given totally geodesic submanifold should be considered.

The Theorem 4.14 can be extended to the one above as follows.

Theorem 5.4. *Let $(M, g, S(TM))$ be an null Einstein hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, the following assertions are equivalent:*

(i) *M is totally geodesic.*

(ii) *M is locally symmetric.*
(iii) *M is semi-symmetric.*
(iv) *The induced connection ∇ is torsion-free and η-conformal.*
(v) *M is Ricci semi-symmetric.*

Now, we shall show that, indeed, there always exists η-Weyl connections on a proper totally contact umbilical null hypersurfaces.

Suppose there exists an η-Weyl connection ∇ on (M, g), that is, ∇ is torsion-free and there exists on M a smooth 1-form β such that $\nabla g = -\beta \otimes \{g - \eta \otimes \eta\}$. Using thus relation and for any X and Y in TM, we obtain

$$(\nabla_E g)(X, Y) = -\beta(E)\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{5.19}$$

or equivalently, since ∇ is torsion-free,

$$\begin{aligned}(L_E g)(X, Y) = -\beta(E)\{g(X, Y) - \eta(X)\eta(Y)\} + g(\nabla_X E, Y)\\ + g(X, \nabla_Y E).\end{aligned} \tag{5.20}$$

Since $\nabla g = -\beta \otimes \{g - \eta \otimes \eta\}$, for any $X, Y \in \Gamma(TM)$ and $E \in \Gamma(TM^\perp)$,

$$0 = -\beta(X)\{g(E, Y) - \eta(E)\eta(Y)\} = (\nabla_X g)(E, Y) = -g(\nabla_X E, Y). \tag{5.21}$$

Then, $g(\nabla_X E, Y) = 0$, $\forall\ X,\ Y \in \Gamma(TM)$ and the relation (5.20) becomes $(L_E g)(X, Y) = -\beta(E)\{g(X, Y) - \eta(X)\eta(Y)\}$ which from Lemma 5.2 means that M is totally contact umbilical with $\varphi(E) = \frac{1}{2}\beta(E)$.

Now, assume that (M, g) is a proper totally contact umbilical null hypersurface. The 1-form φ in (5.7) is related to the means curvature H of M as $H = \lambda N = \varphi(E)N$. Hence, the 1-form φ is a section of $(TM^\perp)^*$ and the latter is canonically isomorphic to $(T\overline{M}|_M)/TM$. Since the projection

$$T\overline{M}|_M \longrightarrow (T\overline{M}|_M)/TM, \tag{5.22}$$

has contractible fibres, then there exists a section $\varphi^\sharp$ of $T\overline{M}|_M$ such that $\overline{g}(\varphi^\sharp, E) = \varphi(E), \quad \forall\, E \in \Gamma(TM^\perp)$. In fact, the section $\varphi^\sharp$ is the metrical dual vector of φ. We also observe that two sections $\varphi^\sharp$ differ by exactly one section of TM. Let β be the differential 1-form on M, locally defined by

$$\beta(X) = 2\overline{g}(\varphi^\sharp, X), \tag{5.23}$$

and we define ∇^β as

$$\nabla^\beta_X Y = \overline{D}_X Y + \sigma_X Y, \tag{5.24}$$

where σ is the symmetric $(1, 2)$-tensor field on M, locally defined by

$$\begin{aligned}\sigma_X Y = \frac{1}{2}\beta(X)\{Y - \eta(Y)\xi\} + \frac{1}{2}\beta(Y)\{X - \eta(X)\xi\}\\ - \{g(X, Y) - \eta(X)\eta(Y)\}\varphi^\sharp - 2\varphi(E)\theta(X)\theta(Y)E,\end{aligned} \tag{5.25}$$

and $\overline{D}$ is the Levi-Civita connection on the ambient manifold $(\overline{M}, \overline{g})$. Note that, for any $X, Y \in \Gamma(TM)$, we have

$$\overline{g}(\sigma_X Y, E) = -\varphi(E)\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{5.26}$$

Lemma 5.5. *Let $(M, g, S(TM))$ be a proper totally contact umbilical null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Then, ∇^β in (5.24) is a torsion-free connection on M and for any X, $Y \in \Gamma(TM)$,*

$$\nabla^\beta_X Y \in \Gamma(TM).$$

Proof. ∇^β is clearly a torsion-free connection on M. Using the relation (5.7) and for any $X, \in \Gamma(TM)$, we have

$$\begin{aligned}\overline{g}(\nabla^\beta_X Y, E) = \overline{g}(\overline{D}_X Y, E) + \overline{g}(\sigma_X Y, E) = \varphi(E)\{g(X, Y) - \eta(X)\eta(Y)\} \\ - \varphi(E)\{g(X, Y) - \eta(X)\eta(Y)\} = 0,\end{aligned}$$

which completes the proof. □

Finally, we show that ∇^β is η-conformal. Using Lemma 5.5 and (5.24), and let X, Y and Z be tangent vector fields to M. We have

$$\begin{aligned}(\nabla^\beta_X g)(Y, Z) &= X(g(Y, Z)) - g(\nabla^\beta_X Y, Z) - g(Y, \nabla^\beta_X Z) \\ &= \overline{g}(\overline{D}_X Y, Z) + \overline{g}(Y, \overline{D}_X Z) - \overline{g}(\overline{D}_X Y, Z) \\ &\quad - \overline{g}(\sigma_X Y, Z) - \overline{g}(Y, \overline{D}_X Z) - \overline{g}(Y, \sigma_X Z) \\ &= -\overline{g}(\sigma_X Y, Z) - \overline{g}(Y, \sigma_X Z). \end{aligned} \tag{5.27}$$

From (5.25), we have

$$\begin{aligned}\overline{g}(\sigma_X Y, Z) = \frac{1}{2}\beta(X)\{g(Y, Z) - \eta(Y)\eta(Z)\} + \frac{1}{2}\beta(Y)\{g(X, Z) - \eta(X)\eta(Z)\} \\ - \overline{g}(\varphi^\sharp, Z)\{g(X, Y) - \eta(X)\eta(Y)\},\end{aligned} \tag{5.28}$$

and

$$\begin{aligned}\overline{g}(Y, \sigma_X Z) = \frac{1}{2}\beta(X)\{g(Y, Z) - \eta(Y)\eta(Z)\} + \frac{1}{2}\beta(Z)\{g(X, Y) - \eta(X)\eta(Y)\} \\ - \overline{g}(\varphi^\sharp, Y)\{g(X, Z) - \eta(X)\eta(Z)\}.\end{aligned} \tag{5.29}$$

Putting these relations together into (5.27), one obtains

$$\begin{aligned}(\nabla^\beta_X g)(Y, Z) = -\beta(X)\{g(Y, Z) - \eta(Y)\eta(Z)\} \\ + \{\overline{g}(\varphi^\sharp, Y) - \frac{1}{2}\beta(Y)\}\{g(X, Z) - \eta(X)\eta(Z)\} \\ + \{\overline{g}(\varphi^\sharp, Z) - \frac{1}{2}\beta(Z)\}\{g(X, Y) - \eta(X)\eta(Y)\}.\end{aligned} \tag{5.30}$$

By (5.23), the last two terms in (5.27) are zero. Therefore

$$(\nabla^\beta_X g)(Y, Z) = -\beta(X)\{g(Y, Z) - \eta(Y)\eta(Z)\}, \tag{5.31}$$

for any tangent vector fields X, Y and Z in M. We have

Theorem 5.6. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite Kenmotsu space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. For M to be proper totally contact umbilical, it is necessary and sufficient that it admits an η-Weyl connection.*

Let K be a $(0,3)$-tensor, locally defined by

$$K(X,Y,Z) := (\nabla_{X^T} g)(Y^T, Z^T) + \beta(X^T) g(Y^T, Z^T). \tag{5.32}$$

Here $K(X,Y,Z) = K(X^T, Y^T, Z^T)$. We replace g and the 1-form β by

$$(g', \beta') = (e^{-2\rho} g, \beta + 2d\rho). \tag{5.33}$$

We call the replacement a *conformal gauge change*. Under the conformal gauge change, we have

$$\begin{aligned} K'(X,Y,Z) &= (\nabla_{X^T} g')(Y^T, Z^T) + \beta'(X^T) g'(Y^T, Z^T) \\ &= e^{-2\rho}\{(\nabla_{X^T} g)(Y^T, Z^T) + \beta(X^T) g(Y^T, Z^T)\} \\ &= e^{-2\rho} K(X,Y,Z). \end{aligned} \tag{5.34}$$

This formula implies that the tensor K is not a conformal gauge invariant. Then, the definition of η-Weyl connection (i.e. $K \equiv 0$) is invariant under conformal gauge change (5.33).

If the $(0,3)$-tensor K satisfies, for any X, Y, $Z \in \Gamma(TM)$,

$$K(X,Y,Z) = K(Y,X,Z), \tag{5.35}$$

then the connection ∇ is said to be η-semi-conformal.

Suppose the induced connection ∇ on the null hypersurface M is η-semi-conformal. Then, K is symmetric for all variables and the definition of η-semi-conformal is invariant under conformal gauge change (5.33). If the 1-form β vanishes, the induced connection ∇ will be called *statistical connection* (see [26] for details). This connection appears in, for instance, all totally umbilical, parallel, totally geodesic null hypersurfaces of indefinite Kenmotsu manifolds.

Theorem 5.7. *Let (M, g) be a null hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. If the induced connection ∇ on M is an η-semi-conformal connection, then M is totally contact umbilical.*

Proof. Suppose that the induced connection is an η-semi-conformal connection. Then, for any X, Y and Z tangent vector fields to M, we have

$$(\nabla_X g)(Y,Z) + \beta(X) g(Y, \widehat{P}Z) = (\nabla_Y g)(X,Z) + \beta(Y) g(X, \widehat{P}Z).$$

This relation leads, by using (2.17), to

$$\begin{aligned} &B(X,Z)\theta(Y) + \beta(X)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &\qquad = B(Y,Z)\theta(X) + \beta(Y)\{g(X,Z) - \eta(X)\eta(Z)\}. \end{aligned}$$

Taking $Y = E$ in this, one obtains $B(X,Z) = \beta(E)\{g(X,Z) - \eta(X)\eta(Z)\}$, that is, M is totally contact umbilical and this completes the proof. □

By Theorems 5.6 we note that a null hypersurface of an indefinite Kenmotsu space form, tangent to the structure vector field ξ and endowed with an η-semi-conformal structure admits an η-Weyl connection.

Now taking into account all studied aspects, we remark that if the section $\varphi^\sharp \in T\overline{M}|_M$ is tangent to M, then $\varphi(E)$ is identically zero and the null hypersurface

M becomes totally geodesic. Otherwise, $\varphi^\sharp$ is nowhere tangent to M and consequently, $\varphi(E)$ is everywhere zero and M is proper totally contact umbilical.

In [25], the author showed that screen conformal null hypersurfaces M of indefinite Kenmotsu space forms $\overline{M}(c)$ with $\xi \in TM$ are proper totally contact umbilical as well as the leaves of its screen distributions. This result guarantees the existence of proper totally contact umbilical in null hypersurfaces of indefinite Kenmotsu space forms, tangent to the structure vector field ξ.

It is well known that the second fundamental form and the shape operator of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.10)-(2.16) that in the case of null hypersurfaces, there are interrelations between these geometric objects and those of its screen distributions. So, the geometry of null hypersurfaces depends on the vector bundles $S(TM)$, $S(TM^{\perp})$ and $N(TM)$. In this case, it is known that the local second fundamental form of M on $\mathcal{U}$ is independent of the choice of the above vector bundles. This means that all results of this paper which depend only on B are stable with respect to any change of those vector bundles.

Denote by ω is the dual 1-form of $W = \sum_{i=1}^{2n-1} \mathbf{f}_i W_i$, characteristic vector field of the screen change, with respect to the induced metric g of M, that is $\omega(X) = g(X, W)$, $\forall\ X \in \Gamma(TM)$. Using (3.14), it is easy to check that $\theta'(X) = \theta(X) + \omega(X)$, for any X, $Y \in \Gamma(TM)$. The relationship between the symmetric $(1, 2)$-tensor fields σ and σ' of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by

$$\sigma'_X Y = \sigma_X Y - \beta(E)\{\theta(X)\omega(Y) + \theta(Y)\omega(X) + \omega(X)\omega(Y)\}. \qquad (5.36)$$

The symmetric $(1, 2)$-tensor field σ is independent of the screen distribution $S(TM)$ if and only if ω vanishes identically on M.

ACKNOWLEDGMENTS

The author wishes to thank the Max Planck Institute for Mathematics in the Sciences at Leipzig (Germany) for its support and hospitality.

REFERENCES

[1] A. Bejancu, Umbilical Semi-invariant submanifolds of a Sasakian manifold, Tensor N. S. 37 (1982), 203-213.

[2] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics 203. Birkhauser Boston, Inc., Boston, MA 2002.

[3] A. Bejancu and K. L. Duggal, Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and Its Applications. Kluwer Publishers, 1996.

[4] M. Dajczer et al., Submanifolds and isometric immersions. Mathematics lecture series 13. Publish or Perish, Inc., Houston, Texas, 1990.

[5] F. Defever, Ricci semi-symmetric hypersurfaces, Balkan J. of Geometry and Its Appl. 5, 1 (2000), 81-91.

[6] F. Defever, R. Descz, D. Z. Senturk, L. Verstraelen and S.Yaprak, On problem of P. J. Ryan, Kyngpook Math. J. 37 (1997), 371-376.

[7] F. Defever, R. Descz, D. Z. Senturk, L. Verstraelen and S. Yaprak, P. J. Ryan's problem in semi-Riemannian space forms, Glasgow Math. J. 41 (1999), 271-281.

[8] K. L. Duggal, On scalar curvature in lightlike geometry, J. Geom. Phys. 57 (2007), no.2, 473-481.
[9] K. L. Duggal and D. H. Jin, Null curves and hypersurfaces of semi-Riemannian manifolds, World Scientific Publishing Co. Pte. Ltd, 2007.
[10] K. L. Duggal and B. Sahin, Differential Geometry of Lightlike Submanifolds, Frontiers in Mathematics, 2010.
[11] R. Günes, B. Sahin and E. Kiliç, On lightlike hypersurfaces of semi-Riemannian space form, Turk. J. Math. 27 (2003), 283-297.
[12] S. Ivanov, Einstein-Weyl structures on compact conformal manifolds, Quart. J. Math. Oxford Ser. (2) 50 (1999), 457462.
[13] D. Janssens, L. Vanhecke, Almost contact structures and curvature tensors, Kodai Math. J., 4 (1981), 1-27.
[14] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J., 24 (1972), 93-103.
[15] M. Kon, Remarks on anti-invariant submanifold of a Sasakian manifold, Tensor, N. S., 30 (1976), 239-246.
[16] D. N. Kupeli, Singular semi-invariant geometry, vol. 366 of Mathematics and Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
[17] O. Lungiambudila, F. Massamba and J. Tossa, Symmetry properties of lightlike hypersurface in indefinite Sasakian manifolds, SUT J. Math. 46, no. 2 (2010), 177-204.
[18] F. Massamba, Lightlike hypersurfaces of indefinite Sasakian manifolds with parallel symmetric bilinear forms, Differ. Geom. Dyn. Syst., 10 (2008), 226-234.
[19] F. Massamba, Totally contact umbilical lightlike hypersurfaces of indefinite Sasakian manifolds, Kodai Math. J., 31 (2008), 338-358.
[20] F. Massamba, Screen integrable lightlike hypersurfaces of indefinite Sasakian manifolds, Mediterr. J. Math, 6 (2009), 27-46.
[21] F. Massamba, On semi-parallel lightlike hypersurfaces of indefinite Kenmotsu manifolds, J. Geom., 95 (2009), 73-89.
[22] F. Massamba, Relative nullity foliations and lightlike hypersurfaces in indefinite Kenmotsu manifolds, Turk. J. Math., 35 (2011), 129-149
[23] F. Massamba, On lightlike geometry in indefinite Kenmotsu manifolds, in press, Mathematica Slovaca.
[24] F. Massamba, Lightlike geometry of leaves in indefinite Kenmotsu manifolds, in press, Cubo A Mathematical Journal.
[25] F. Massamba, Screen conformal lightlike geometry in indefinite Kenmotsu space forms, preprint.
[26] H. Matsuzoe, Geometry of semi-Weyl manifolds and Weyl manifolds. Kyushu J. Math. 55 (2001), 107117.
[27] P. J. Ryan, A class of complex hypersurfaces, Colloquium Math. 26 (1972), 175-182.
[28] B. Sahin, Lightlike hypersurfaces of semi-Euclidean spaces satisfying curvature conditions of semi-symmetry, Turk. J. Math. 31 (2007) 139-162.
[29] Z. I. Szabo, Structure theorem on Riemannian spaces satisfying $R(X,Y).R = 0$, I: The local version, J. Differential Geom. 17 (1982), 531-582.
[30] Z. I.Szabo, Structure theorem on Riemannian spaces satisfying $R(X,Y).R = 0$, II: The global version, Geom. Dedicata 19 (1985), 65-108.

* DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BOTSWANA
PRIVATE BAG 0022
GABORONE
BOTSWANA
E-mail address: massfort@yahoo.fr, massambaf@mopipi.ub.bw