

## SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS

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**ABSTRACT.** We establish a link between the study of completely integrable systems of partial differential equations and the study of generic submanifolds in  $\mathbb{C}^n$ . Using the recent developments of Cauchy-Riemann geometry we provide the set of symmetries of such a system with a Lie group structure. Finally we determine the precise upper bound of the dimension of this Lie group for some specific systems of partial differential equations.

### 1. Introduction

To study the geometry of a real analytic Levi nondegenerate hypersurface  $M$  in  $\mathbb{C}^2$ , one of the principal ideas of H. Poincaré, of B. Segre and of É. Cartan in the fundamental memoirs [20], [21], [22], [3] was to associate to  $M$  a system  $(\mathcal{E}_M)$  of (partial) differential equations, in order to solve the so-called equivalence problem. Establishing a natural correspondence between the local holomorphic automorphisms of  $M$  and the Lie symmetries of  $(\mathcal{E}_M)$  they could use the classification results on differential equations achieved by S. Lie in [5] and pursued by A. Tresse in [28].

Starting with such a correspondence, we shall establish a general link between the study of a real analytic generic submanifold of codimension  $m$  in  $\mathbb{C}^{n+m}$  and the study of completely integrable systems of analytic partial differential equations. We shall observe that the recent theories in Cauchy-Riemann (CR) geometry may be transposed to the setting of partial differential equations, providing some new information on their Lie symmetries.

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Indeed consider for  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  a  $\mathbb{K}$ -analytic system  $(\mathcal{E})$  of the following general form:

$$(\mathcal{E}) \quad u_{x^\alpha}^j(x) = F_\alpha^j \left( x, u(x), (u_{x^{\beta(q)}}^{j(q)}(x))_{1 \leq q \leq p} \right).$$

Here  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ ,  $u = (u^1, \dots, u^m) \in \mathbb{K}^m$ , the integers  $j(1), \dots, j(p)$  satisfy  $1 \leq j(q) \leq m$  for  $q = 1, \dots, p$ , and  $\alpha$  and the multiindices  $\beta(1), \dots, \beta(p) \in \mathbb{N}^n$  satisfy  $|\alpha|, |\beta(q)| \geq 1$ . We also require  $(j, \alpha) \neq (j(1), \beta(1)), \dots, (j(p), \beta(p))$ . For  $j = 1, \dots, m$  and  $\alpha \in \mathbb{N}^n$ , we denote by  $u_{x^\alpha}^j$  the partial derivative  $\partial^{|\alpha|} u^j / \partial x^\alpha$ . We assume that the system  $(\mathcal{E})$  is *completely integrable*, namely that the Pfaffian system naturally associated in the jet space is involutive in the sense of Frobenius. We note that in that case  $(\mathcal{E})$  is *locally solvable*, meaning that through every point  $(x^*, u^*, u_\beta^*, u_\alpha^*)$  in the jet space, satisfying  $u_\alpha^* = F_\alpha(x^*, u^*, u_\beta^*)$  (written in a condensed form), there exists a local  $\mathbb{K}$ -analytic solution  $u = u(x)$  of  $(\mathcal{E})$  satisfying  $u(x^*) = u^*$  and  $u_{x^\beta}(x^*) = u_\beta^*$ . Consequently the Lie theory ([18]) may be applied to such systems. We shall associate with  $(\mathcal{E})$  the *submanifold of solutions*  $\mathcal{M}$  in  $\mathbb{K}^{n+2m+p}$  given by  $\mathbb{K}$ -analytic equations of the form

$$(1) \quad u^j = \Omega_j(x, \nu, \chi), \quad j = 1, \dots, m,$$

where  $\nu \in \mathbb{K}^m$  and where  $\chi \in \mathbb{K}^p$ . Moreover the integer  $m+p$  is the number of initial conditions for the general solution  $u(x) := \Omega(x, \nu, \chi)$  of  $(\mathcal{E})$ , whose existence and uniqueness follow from complete integrability. Precisely, the parameters  $\nu, \chi$  correspond to the data  $u(0), (u_{x^{\beta(q)}}^{j(q)}(0))_{1 \leq q \leq p}$ . In the special case where the system  $(\mathcal{E})$  is constructed from a generic submanifold  $M$  as in [21], [24] (see also Subsection 2.2 below), the corresponding submanifold of solutions is exactly the extrinsic complexification of  $M$ .

A pointwise  $\mathbb{K}$ -analytic transformation  $(x', u') = \Phi(x, u)$  defined in a neighborhood of the origin and sufficiently close to the identity mapping is called a *Lie symmetry of  $(\mathcal{E})$*  if it transforms the graph of every solution to the graph of another local solution. A vector field  $X = \sum_{l=1}^n Q^l(x, u) \partial / \partial x_l + \sum_{j=1}^m R^j(x, u) \partial / \partial u^j$  is called an *infinitesimal symmetry of  $(\mathcal{E})$*  if for every  $s$  close to zero in  $\mathbb{K}$  the local diffeomorphism  $(x, u) \mapsto \exp(sX)(x, u)$  associated to the flow of  $X$  is a Lie symmetry of  $\mathcal{E}$ . According to [18] (Chapter 2) the infinitesimal symmetries of  $(\mathcal{E})$  form a Lie algebra of vector fields defined in a neighborhood of the origin in  $\mathbb{K}^n \times \mathbb{K}^m$ , denoted by  $\mathfrak{Sym}(\mathcal{E})$ . Inspired by recent developments in CR geometry we shall provide in Section 2 nondegeneracy conditions on  $\mathcal{M}$  insuring firstly that  $\mathfrak{Sym}(\mathcal{E})$  may be identified with

the Lie algebra  $\mathfrak{Sym}(\mathcal{M})$  of vector fields of the form

$$(2) \sum_{l=1}^n Q^l(x, u) \frac{\partial}{\partial x_l} + \sum_{j=1}^m R^j(x, u) \frac{\partial}{\partial u^j} + \sum_{j=1}^m \Pi^j(\nu, \chi) \frac{\partial}{\partial \nu^j} + \sum_{q=1}^p \Lambda^q(\nu, \chi) \frac{\partial}{\partial \chi_q},$$

which are tangent to  $\mathcal{M}$ , and secondly that  $\mathfrak{Sym}(\mathcal{M}) \cong \mathfrak{Sym}(\mathcal{E})$  is finite dimensional. The strength of this identification is to provide some (non-optimal) bound on the dimension of  $\mathfrak{Sym}(\mathcal{E})$  for arbitrary systems of partial differential equations with an arbitrary number of variables, see Theorem 3.

In the second part of the paper (Sections 3, 4 and 5), using the classical Lie theory (cf. [5], [18], [19] and [2]), we provide an optimal upper bound on the dimension of  $\mathfrak{Sym}(\mathcal{E})$  for a completely integrable  $\mathbb{K}$ -analytic system ( $\mathcal{E}$ ) of the following form:

$$(\mathcal{E}) \quad u_{x^\alpha}^j = F_\alpha^j(x, u(x), (u_{x^\beta}(x))_{1 \leq |\beta| \leq \kappa-1}), \quad \alpha \in \mathbb{N}^n, \quad |\alpha| = \kappa, \quad j = 1, \dots, m.$$

This system is a special case of the system studied in Section 2. For instance the *homogeneous* system  $(\mathcal{E}_0) : u_{x_{k_1} \dots x_{k_\kappa}}^j(x) = 0$  is completely integrable. The solutions of  $(\mathcal{E}_0)$  are the polynomials of the form  $u^j(x) = \sum_{\beta \in \mathbb{N}^n, |\beta| \leq \kappa-1} \lambda_\beta^j x^\beta$ ,  $j = 1, \dots, m$ , where  $\lambda_\beta^j \in \mathbb{K}$  and a Lie symmetry of  $(\mathcal{E}_0)$  is a transformation stabilizing the graphs of polynomials of degree  $\leq \kappa - 1$ . We prove the following Theorem:

**THEOREM 1.** *Let  $(\mathcal{E})$  be the  $\mathbb{K}$ -analytic system of partial differential equations of order  $\kappa \geq 2$ , with  $n$  independent variables and  $m$  dependent variables, defined just above. Assume that  $(\mathcal{E})$  is completely integrable. Then the Lie algebra  $\mathfrak{Sym}(\mathcal{E})$  of its infinitesimal symmetries satisfies the following estimates:*

$$(3) \quad \begin{aligned} \dim_{\mathbb{K}}(\mathfrak{Sym}(\mathcal{E})) &\leq (n+m+2)(n+m), & \text{if } \kappa = 2, \\ \dim_{\mathbb{K}}(\mathfrak{Sym}(\mathcal{E})) &\leq n^2 + 2n + m^2 + m C_{n+\kappa-1}^{\kappa-1}, & \text{if } \kappa \geq 3, \end{aligned}$$

where we denote  $C_{n+\kappa-1}^{\kappa-1} := \frac{(n+\kappa-1)!}{n! (\kappa-1)!}$ . Moreover the inequalities (3) become equalities for the homogeneous system  $(\mathcal{E}_0)$ .

We remark that there is no combinatorial formula interpolating these two estimates. Theorem 1 is a generalization of the following results. For  $n = m = 1$ , S. Lie proved that the dimension of the Lie algebra  $\mathfrak{Sym}(\mathcal{E})$  is less than or equal to 8 if  $\kappa = 2$  and is less than or equal to  $\kappa + 4$  if  $\kappa \geq 3$ , these bounds being reached for the homogeneous system (cf. [5]). For  $n = 1$ ,  $m \geq 1$  and  $\kappa = 2$ , F. González-Gascón and A. González-López proved in [11] that the dimension of  $\mathfrak{Sym}(\mathcal{E})$  is less than or equal to  $(m+3)(m+1)$ . For  $n = 1$ ,  $m \geq 1$  and  $\kappa = 2$ , using the equivalence

method due to É. Cartan, M. Fels [6] proved that the dimension of  $\mathfrak{Sym}(\mathcal{E})$  is less than or equal to  $m^2 + 4m + 3$ , with equality if and only if the system  $(\mathcal{E})$  is equivalent to the system  $u_{x_2}^j = 0$ ,  $j = 1, \dots, m$ . He also generalized this result to the case  $n = 1$ ,  $m \geq 1$ ,  $\kappa = 3$ . For  $n \geq 1$ ,  $m \geq 1$  and  $\kappa = 2$ , A. Sukhov proved in [24] that the dimension of  $\mathfrak{Sym}(\mathcal{E})$  is less than or equal to  $(n + m + 2)(n + m)$  (the first inequality in Theorem 1), with equality for the homogeneous system  $u_{x_{k_1} x_{k_2}}^j = 0$ .

Consequently, for the case  $\kappa = 2$ , we will only give the general form of the Lie symmetries of the homogeneous system  $(\mathcal{E}_0)$  (see Subsection 5.2). We will prove Theorem 1 for the case  $\kappa \geq 3$ . The formulas obtained in Sections 3, 4 and 5 were checked with the help of MAPLE release 6.

## 2. Submanifold of solutions

### 2.1. Preliminary

Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $n \geq 1$  and let  $x = (x_1, \dots, x_n) \in \mathbb{N}$ . We denote by  $\mathbb{K}\{x\}$  the local ring of  $\mathbb{K}$ -analytic functions  $\varphi = \varphi(x)$  defined in some neighborhood of the origin in  $\mathbb{K}^n$ . If  $\varphi \in \mathbb{K}\{x\}$  we denote by  $\bar{\varphi}$  the function in  $\mathbb{K}\{x\}$  satisfying  $\overline{\varphi(x)} \equiv \bar{\varphi}(\bar{x})$ . Recall that a  $\mathbb{K}$ -analytic function  $\varphi$  defined in a domain  $U \subset \mathbb{K}^n$  is called  $\mathbb{K}$ -algebraic (in the sense of Nash) if there exists a nonzero polynomial  $P = P(X_1, \dots, X_n, \Phi) \in \mathbb{K}[X_1, \dots, X_n, \Phi]$  such that  $P(x, \varphi(x)) \equiv 0$  on  $U$ . All the considerations in this paper will be local: functions, submanifolds and mappings will always be defined in a small connected neighborhood of some point (most often the origin) in  $\mathbb{K}^n$ .

### 2.2. System of partial differential equations associated to a generic submanifold of $\mathbb{C}^{n+m}$

Let  $M$  be a real algebraic or analytic local submanifold of codimension  $m$  in  $\mathbb{C}^{n+m}$ , passing through the origin. We assume that  $M$  is *generic*, namely  $T_0M + iT_0M = T_0\mathbb{C}^{n+m}$ . Classically (cf. [1]) there exists a choice of complex linear coordinates  $t = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m$  centered at the origin such that  $T_0M = \{\text{Im } w = 0\}$  and such that there exist  $m$  complex algebraic or analytic defining equations representing  $M$  as the set of  $(z, w)$  in a neighborhood of the origin in  $\mathbb{C}^{n+m}$  which satisfy

$$(4) \quad w_1 = \Theta_1(z, \bar{z}, \bar{w}), \dots, w_m = \Theta_m(z, \bar{z}, \bar{w}).$$

Furthermore, the mapping  $\Theta = (\Theta_1, \dots, \Theta_m)$  satisfies the functional equation

$$(5) \quad w \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)),$$

which reflects the reality of the generic submanifold  $M$ . It follows in particular from (5) that the local holomorphic mapping  $\mathbb{C}^m \ni \bar{w} \mapsto (\Theta_j(0, 0, \bar{w}))_{1 \leq j \leq m} \in \mathbb{C}^m$  is of rank  $m$  at  $\bar{w} = 0$ .

Generalizing an idea due to B. Segre in [21] and [22], exploited by É. Cartan in [3] and more recently by A. Sukhov in [24], [25], [26], we shall associate to  $M$  a system of partial differential equations. For this, we need some general nondegeneracy condition, which generalizes Levi nondegeneracy. Let  $\ell_0 \in \mathbb{N}$  with  $\ell_0 \geq 1$ . We shall assume that  $M$  is  $\ell_0$ -finitely nondegenerate at the origin (cf. [1], [17], [8]). This means that there exist multiindices  $\beta(1), \dots, \beta(n) \in \mathbb{N}^n$  with  $|\beta(k)| \geq 1$  for  $k = 1, \dots, n$  and  $\max_{1 \leq k \leq n} |\beta(k)| = \ell_0$ , and integers  $j(1), \dots, j(n)$  with  $1 \leq j(k) \leq m$  for  $k = 1, \dots, n$  such that the local holomorphic mapping

$$(6) \quad \mathbb{C}^{n+m} \ni (\bar{z}, \bar{w}) \mapsto \left( (\Theta_j(0, \bar{z}, \bar{w}))_{1 \leq j \leq m}, (\Theta_{j(k), z^{\beta(k)}}(0, \bar{z}, \bar{w}))_{1 \leq k \leq n} \right) \in \mathbb{C}^{m+n}$$

is of rank equal to  $n + m$  at  $(\bar{z}, \bar{w}) = (0, 0)$ . Here, we denote the partial derivative  $\partial^{|\beta|} \Theta_j(0, \bar{z}, \bar{w}) / \partial z^\beta$  simply by  $\Theta_{j, z^\beta}(0, \bar{z}, \bar{w})$ . Then  $M$  is Levi nondegenerate at the origin if and only if  $\ell_0 = 1$ . By complexifying the variables  $\bar{z}$  and  $\bar{w}$ , we get new independent variables  $\zeta \in \mathbb{C}^n$  and  $\xi \in \mathbb{C}^m$  together with a complex algebraic or analytic  $m$ -codimensional submanifold  $\mathcal{M}$  in  $\mathbb{C}^{2(n+m)}$  of equations

$$(7) \quad w_j = \Theta_j(z, \zeta, \xi), \quad j = 1, \dots, m,$$

called the *extrinsic complexification of  $M$* . In the defining equations (7) of  $\mathcal{M}$ , following [21] and [24], we may consider the “dependent variables”  $w_1, \dots, w_m$  as algebraic or analytic functions of the “independent variables”  $z = (z_1, \dots, z_n)$ , with additional dependence on the extra “parameters”  $(\zeta, \xi) \in \mathbb{C}^{n+m}$ . Then by applying the differential operator  $\partial^{|\alpha|} / \partial z^\alpha$  to (7), we obtain  $w_{j, z^\alpha}(z) = \Theta_{j, z^\alpha}(z, \zeta, \xi)$ . Writing these equations for  $(j, \alpha) = (j(k), \beta(k))$  with  $k = 1, \dots, n$ , we obtain a system of  $m + n$  equations

$$(8) \quad \begin{aligned} w_j(z) &= \Theta_j(z, \zeta, \xi), & j &= 1, \dots, m, \\ w_{j(k), z^{\beta(k)}}(z) &= \Theta_{j(k), z^{\beta(k)}}(z, \zeta, \xi), & k &= 1, \dots, n. \end{aligned}$$

In this system (8), by the assumption of  $\ell_0$ -finite nondegeneracy (6), the algebraic or analytic implicit function theorem allows to solve the parameters  $(\zeta, \xi)$  in terms of the variables  $(z_k, w_j(z), w_{j(k), z^{\beta(k)}}(z))$ , providing a local algebraic or analytic  $\mathbb{C}^{n+m}$ -valued mapping  $R$  such that

$(\zeta, \xi) = R(z_k, w_j(z), w_{j(k), z^{\beta(k)}}(z))$ . Finally, for every pair  $(j, \alpha)$  different from  $(1, 0), \dots, (m, 0), (j(1), \beta(1)), \dots, (j(n), \beta(n))$ , we may replace  $(\zeta, \xi)$  by  $R$  in the differentiated expression  $w_{j, z^\alpha}(z) = \Theta_{j, z^\alpha}(z, \zeta, \xi)$ . This yields

$$(9) \quad \begin{aligned} w_{j, z^\alpha}(z) &= \Theta_{j, z^\alpha}\left(z, R(z_k, w_j(z), w_{j(k), z^{\beta(k)}}(z))\right) \\ &=: F_{j, \alpha}\left(z_k, w_j(z), w_{j(k), z^{\beta(k)}}(z)\right). \end{aligned}$$

This is the *system of partial differential equations associated with  $\mathcal{M}$* . As argued by B. Segre in [21], the geometric study of generic submanifolds of  $\mathbb{C}^n$  may gain much information from the study of their associated systems of partial differential equations (cf. [24], [25]). The next paragraphs are devoted to provide a *general one-to-one correspondence* between completely integrable systems of analytic partial differential equations and their associated “submanifolds of solutions” (to be defined precisely below) like  $\mathcal{M}$  above. Afterwards, we shall observe that conversely, the study of systems of analytic partial differential equations also gains much information from the direct study of their associated submanifolds of solutions.

### 2.3. Completely integrable systems of partial differential equations

Let now  $n, m, p \in \mathbb{N}$  with  $n, m, p \geq 1$ , let  $\kappa \in \mathbb{N}$  with  $\kappa \geq 2$  and let  $u = (u^1, \dots, u^m) \in \mathbb{K}^m$ . Consider a collection of  $p$  multi-indices  $\beta(1), \dots, \beta(p) \in \mathbb{N}^n$  with  $|\beta(q)| \geq 1$  for  $q = 1, \dots, p$  and  $\max_{1 \leq q \leq p} |\beta(q)| = \kappa - 1$ . Consider also  $p$  integers  $j(1), \dots, j(p)$  with  $1 \leq j(q) \leq m$  for  $q = 1, \dots, p$ . Inspired by (9), we consider a general system of partial differential equations of  $n$  independent variables  $(x_1, \dots, x_n)$  and  $m$  dependent variables  $(u^1, \dots, u^m)$  which is of the following form:

$$(E) \quad u_{x^\alpha}^j(x) = F_\alpha^j\left(x, u(x), (u_{x^{\beta(q)}}^{j(q)}(x))_{1 \leq q \leq p}\right),$$

where  $(j, \alpha) \neq (j(1), \beta(1)), \dots, (j(p), \beta(p))$  and  $j = 1, \dots, m, |\alpha| \leq \kappa$ . Here, we assume that  $u = 0$  is a local solution of the system (E) and that the functions  $F_\alpha^j$  are  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic in a neighborhood of the origin in  $\mathbb{K}^{n+m+p}$ . Among such systems are included ordinary differential equations of any order  $\kappa \geq 2$ , systems of second order partial differential equation as studied in [24], etc.

Throughout this article, we shall assume the system (E) completely integrable. By analyzing the application of the Frobenius theorem in

jet spaces, one can show (we will not develop this) that the general solution of the system  $(\mathcal{E})$  is given by  $u(x) := \Omega(x, \nu, \chi)$ , where the parameters  $\nu \in \mathbb{K}^n$  and  $\chi \in \mathbb{K}^n$  essentially correspond to the “initial conditions”  $u(0)$  and  $(u_{x^{\beta(q)}}^{j(q)}(0))_{1 \leq q \leq p}$ , and  $\Omega$  is a  $\mathbb{K}$ -analytic  $\mathbb{K}^n$ -valued mapping. In the case of a generic submanifold as in Subsection 2.2 above, we recover the mapping  $\Theta$ . In the sequel, we shall use the following terminology: the coordinates  $(x, u)$  will be called the *variables* and the coordinates  $(\nu, \chi)$  will be called the *parameters* or the *initial conditions*. In Subsection 2.5 below, we shall introduce a certain duality where the rôles between variables and parameters are exchanged.

#### 2.4. Associated submanifold of solutions

The existence of the function  $\Omega$  and the analogy with Subsection 2.2 leads us to introduce the *submanifold of solutions associated to the completely integrable system*  $(\mathcal{E})$ , which by definition is the  $m$ -codimensional  $\mathbb{K}$ -analytic submanifold of  $\mathbb{K}^{n+2m+p}$ , equipped with the coordinates  $(x, u, \nu, \chi)$ , defined by the Cartesian equations

$$(10) \quad u_j = \Omega_j(x, \nu, \chi), \quad j = 1, \dots, m.$$

Let us denote this submanifold by  $\mathcal{M}$ . We stress that in general such a submanifold cannot coincide with the complexification of a generic submanifold of  $\mathbb{C}^{m+n}$ , for instance because  $\mathbb{K}$  may be equal to  $\mathbb{R}$  or, if  $\mathbb{K} = \mathbb{C}$ , because the integer  $p$  is not necessarily equal to  $n$ . Also, even if  $\mathbb{K} = \mathbb{C}$  and  $n = p$ , the mapping  $\Omega$  does not satisfy a functional equation like (5). In fact, it may be easily established that the submanifold of solutions of a completely integrable system of partial differential equations like  $(\mathcal{E})$  coincides with the complexification of a generic submanifold if and only if  $\mathbb{K} = \mathbb{C}$ ,  $p = n$  and the mapping  $\Omega$  satisfies a functional equation like (5).

Let now  $\mathcal{M}$  be a submanifold of  $\mathbb{K}^{n+2n+p}$  of the form (10), but not necessarily constructed as the submanifold of solutions of a system  $(\mathcal{E})$ . We shall always assume that  $\Omega_j(0, \nu, \chi) \equiv \nu^j$  and  $\Omega_j(x, \nu, 0) \equiv \nu^j$ . We say that  $\mathcal{M}$  is *solvable with respect to the parameters* if there exist multi-indices  $\beta(1), \dots, \beta(p) \in \mathbb{N}^n$  with  $|\beta(q)| \geq 1$  for  $q = 1, \dots, p$  and integers  $j(1), \dots, j(p)$  with  $1 \leq j(q) \leq m$  for  $q = 1, \dots, p$  such that the local  $\mathbb{K}$ -analytic mapping

$$(11) \quad \mathbb{K}^{m+p} \ni (\nu, \chi) \longmapsto \left( (\Omega_j(0, \nu, \chi))_{1 \leq j \leq m}, (\Omega_{j(q).x^{\beta(q)}}(0, \nu, \chi))_{1 \leq q \leq p} \right) \in \mathbb{K}^{m+p}$$

is of rank equal to  $m+p$  at  $(\zeta, \chi) = (0, 0)$  (notice that since  $\Omega_j(0, \nu, \chi) \equiv \nu^j$ , then the  $m$  first components of the mapping (11) are already of rank

*m*). We remark that the submanifold of solutions of a system  $(\mathcal{E})$  is automatically solvable with respect to the variables, the multiindices  $\beta(q)$  and the integers  $j(q)$  being the same as in the arguments of the right hand side terms  $F_\alpha^j$  in  $(\mathcal{E})$ .

### 2.5. Dual system of defining equations

Since  $\Omega_j(0, \nu, \chi) \equiv \nu^j$ , we may solve the equations (10) with respect to  $\nu$  by means of the analytic implicit function theorem, getting an equivalent system of equations for  $\mathcal{M}$ :

$$(12) \quad \nu^j = \Omega_j^*(\chi, x, u), \quad j = 1, \dots, m.$$

We call this the *dual system of defining equations for  $\mathcal{M}$* . By construction, we have the functional equation

$$(13) \quad \nu \equiv \Omega^*(\chi, x, \Omega(x, \nu, \chi)),$$

implying the identities  $\nu \equiv \Omega^*(0, x, \Omega(x, \nu, 0)) \equiv \Omega^*(0, x, \nu)$ . We say that  $\mathcal{M}$  is *solvable with respect to the variables* if there exist multi-indices  $\delta(1), \dots, \delta(n) \in \mathbb{N}^p$  with  $|\delta(l)| \geq 1$  for  $l = 1, \dots, n$  and integers  $j(1), \dots, j(n)$  with  $1 \leq j(l) \leq m$  for  $l = 1, \dots, n$  such that the local  $\mathbb{K}$ -analytic mapping

$$(14) \quad \mathbb{K}^{n+m} \ni (x, u) \mapsto \left( (\Omega_j^*(0, x, u))_{1 \leq j \leq m}, \left( \Omega_{j(l)}^*(0, x, u) \right)_{1 \leq l \leq n} \right) \in \mathbb{K}^{m+n}$$

is of rank equal to  $n+m$  at  $(x, u) = (0, 0)$  (notice that since  $\Omega_j^*(0, x, u) \equiv u^j$ , the  $m$  first components of the mapping (14) are already of rank  $m$ ).

In the case where  $\mathcal{M}$  is the complexification of a generic submanifold then the solvability with respect to the parameters is equivalent to the solvability with respect to the variables since  $\Omega^* \equiv \bar{\Omega}$ . However we notice that a submanifold  $\mathcal{M}$  of solutions of a system  $(\mathcal{E})$  is not automatically solvable with respect to the variables, as shows the following trivial example.

**EXAMPLE 1.** Let  $n = 2, m = 1$  and let  $(\mathcal{E})$  denote the system  $u_{x_2} = 0, u_{x_1 x_1} = 0$ , whose general solutions are  $u(x) = \nu + x_1 \chi =: \Omega(x_1, x_2, \nu, \chi)$ . Notice that the variable  $x_2$  is absent from the dual equation  $\nu = u - x_1 \chi =: \Omega^*(\chi, x_1, x_2, u)$ . It follows that  $\mathcal{M}$  is not solvable with respect to the variables.



## 2.6. Symmetries of $(\mathcal{E})$ , their lift to the jet space and their lift to the parameter space

We denote by  $\mathcal{J}_{n,m}^\kappa$  the space of jets of order  $\kappa$  of  $\mathbb{K}$ -analytic mappings  $u = u(x)$  from  $\mathbb{K}^n$  to  $\mathbb{K}^m$ . Let

$$(15) \quad (x_l, u^j, U_{l_1}^{i_1}, U_{l_1, l_2}^{i_1, i_2}, \dots, U_{l_1, \dots, l_\lambda}^{i_1, \dots, i_\lambda}) \in \mathbb{K}^{n+m} C_{\kappa+n}^\kappa$$

denote the natural coordinates on  $\mathcal{J}_{n,m}^\kappa$ . Here, the superscripts  $j, i_1$  and the subscripts  $l, l_1, l_2, \dots, l_\lambda$  satisfy  $j, i_1 = 1, \dots, m$  and  $l, l_1, l_2, \dots, l_\lambda = 1, \dots, n$ . The independent coordinate  $U_{l_1, \dots, l_\lambda}^{i_1, \dots, i_\lambda}$  corresponds to the partial derivative  $u_{x_{l_1} \dots x_{l_\lambda}}^{i_1 \dots i_\lambda}$ . Finally, by symmetry of partial differentiation, we identify every coordinate  $U_{l_1, \dots, l_\lambda}^{i_1, \dots, i_\lambda}$  with the coordinates  $U_{\sigma(l_1), \dots, \sigma(l_\lambda)}^{i_1, \dots, i_\lambda}$ , where  $\sigma$  is an arbitrary permutation of the set  $\{1, \dots, \lambda\}$ . With these identifications, the  $\kappa$ -th order jet space  $\mathcal{J}_{n,m}^\kappa$  is of dimension  $n+m C_{\kappa+n}^\kappa$ , where  $C_p^q := \frac{p!}{q!(p-q)!}$  denotes the binomial coefficient. Also, we shall sometimes use an equivalent notation for coordinates on  $\mathcal{J}_{n,m}^\kappa$ :

$$(16) \quad (x_l, u^j, U_\beta^i) \in \mathbb{K}^{n+m} C_{\kappa+n}^\kappa,$$

where  $\beta \in \mathbb{N}^n$  satisfies  $|\beta| \leq \kappa$  and where the independent coordinate  $U_\beta^i$  corresponds to the partial derivative  $u_{x_\beta}^i$ . Associated to the system  $(\mathcal{E})$  is the so-called *skeleton*  $\Delta_{\mathcal{E}}$ , which is the  $\mathbb{K}$ -analytic submanifold of dimension  $n+m+p$  in  $\mathcal{J}_{n,m}^\kappa$  simply defined by replacing the partial derivatives of the dependent variables  $u^j$  by the independent jet variables in  $(\mathcal{E})$ :

$$(17) \quad U_\alpha^j = F_\alpha^j \left( x, u, (U_{\beta(q)}^{j(q)})_{1 \leq q \leq p} \right),$$

for  $(j, \alpha) \neq (j(1), \beta(1)), \dots, (j(p), \beta(p))$  and  $j = 1, \dots, m$ ,  $|\alpha| \leq \kappa$ . Clearly, the natural coordinates on the submanifold  $\Delta_{\mathcal{E}}$  of  $\mathcal{J}_{n,m}^\kappa$  are the  $n+m+p$  coordinates

$$(18) \quad \left( x, u, (U_{\beta(q)}^{j(q)})_{1 \leq q \leq p} \right).$$

Let  $h = h(x, u)$  be a local  $\mathbb{K}$ -analytic diffeomorphism of  $\mathbb{K}^{n+m}$  close to the identity mapping and let  $\pi_\kappa : \mathcal{J}_{n,m}^\kappa \rightarrow \mathbb{K}^{n+m}$  be the canonical projection. According to [18] (Chapter 2) there exists a unique lift  $h^{(\kappa)}$  of  $h$  to  $\mathcal{J}_{n,m}^\kappa$  such that  $\pi_\kappa \circ h^{(\kappa)} = h \circ \pi_\kappa$ . The components of  $h^{(\kappa)}$  may be computed by means of universal combinatorial formulas and they are rational functions of the jet variables (15), their coefficients being partial derivatives of the components of  $h$ , see for instance §3.3.5 of [2]. By definition,  $h$  is a local symmetry of  $(\mathcal{E})$  if  $h$  transforms the graph

of every local solution of  $(\mathcal{E})$  into the graph of another local solution of  $(\mathcal{E})$ . This definition seems to be rather uneasy to handle, because of the abstract quantification of “every local solution”, but we have the following concrete characterization for  $h$  to be a local symmetry of  $(\mathcal{E})$  (cf. Chapter 2 in [18]).

LEMMA 1. *The following conditions are equivalent:*

- (1) *the local transformation  $h$  is a local symmetry of  $(\mathcal{E})$ ,*
- (2) *its  $\kappa$ -th prolongation  $h^{(\kappa)}$  is a local self-transformation of the skeleton  $\Delta_{\mathcal{E}}$  of  $(\mathcal{E})$ .*

These considerations have an infinitesimal version. Indeed, let  $X = \sum_{l=1}^n Q^l(x, u) \partial/\partial x_l + \sum_{j=1}^m R^j(x, u) \partial/\partial u^j$  be a local vector field with  $\mathbb{K}$ -analytic coefficients which is defined in a neighborhood of the origin in  $\mathbb{K}^{n+m}$ . Let  $s \in \mathbb{K}$  and consider the flow of  $L$  as the one-parameter family  $h_s(x, u) := \exp(sX)(x, u)$  of local transformations. We recall that  $X$  is an *infinitesimal symmetry* of  $(\mathcal{E})$  if for every small  $s \in \mathbb{K}$ , the mapping  $h_s(x, u) := \exp(sX)(x, u)$  is a local symmetry of  $(\mathcal{E})$ . By differentiating with respect to  $s$  the  $\kappa$ -th prolongation  $(h_s)^{(\kappa)}$  of  $h_s$  at  $s = 0$ , we obtain a unique vector field  $X^{(\kappa)}$  on the  $\kappa$ -th jet space, called the  *$\kappa$ -th prolongation of  $X$*  and which satisfies  $(\pi_k)_*(X^{(\kappa)}) = X$ . In Subsections 3.1 and 3.2 below, we shall analyze the combinatorial formulas for the coefficients of  $X^{(\kappa)}$ , since they will be needed to prove Theorem 1.

Let  $X_{\mathcal{E}}$  be the projection to the restricted jet space  $\mathbb{K}^{m+n+p}$ , equipped with the coordinates (18), of the restriction of  $X^{(\kappa)}$  to  $\Delta_{\mathcal{E}}$ , namely

$$(19) \quad X_{\mathcal{E}} := (\pi_{\kappa,p})_*(X^{(\kappa)}|_{\Delta_{\mathcal{E}}}).$$

The following Lemma, called the *Lie criterion*, is the concrete characterization for  $X$  to be an infinitesimal symmetry of  $(\mathcal{E})$  and is a direct corollary of Lemma 1 (cf. Chapter 2 in [18]). This criterion will be central in the next Sections 3, 4 and 5.

LEMMA 2. *The following conditions are equivalent:*

- (1) *the vector field  $X$  is an infinitesimal symmetry of  $(\mathcal{E})$ ,*
- (2) *its  $\kappa$ -th prolongation  $X^{(\kappa)}$  is tangent to the skeleton  $\Delta_{\mathcal{E}}$ .*

We denote by  $\mathfrak{Sym}(\mathcal{E})$  the set of infinitesimal symmetries of  $(\mathcal{E})$ . Since it may be easily checked that  $(cX + dY)^{(\kappa)} = cX^{(\kappa)} + dY^{(\kappa)}$  and that  $[X^{(\kappa)}, Y^{(\kappa)}] = ([X, Y])^{(\kappa)}$  (see Theorem 2.39 in [18]), it follows from Lemma 2 (2) that  $\mathfrak{Sym}(\mathcal{E})$  is a Lie algebra of locally defined vector fields. Our main question in this section is the following: under which natural conditions is  $\mathfrak{Sym}(\mathcal{E})$  finite-dimensional ?

EXAMPLE 2. We observe that the Lie algebra  $\mathfrak{Sym}(\mathcal{E})$  of the system  $(\mathcal{E})$  presented in Example 1 is infinite-dimensional, since it includes all vector fields of the form  $X = Q^2(x_1, x_2, u) \partial/\partial x_2$ , as may be verified. As we will argue in Proposition 2 below, this phenomenon is typical, the main reason lying in the first order relation  $u_{x_2} = 0$ .

By analyzing the construction of the submanifold of solutions  $\mathcal{M}$  associated to the system  $(\mathcal{E})$ , we may establish the following correspondence (we shall not develop its proof).

PROPOSITION 1. *To every infinitesimal symmetry  $X = \sum_{l=1}^n Q^l(x, u) \partial/\partial x_l + \sum_{j=1}^m R^j(x, u) \partial/\partial u^j$  of  $(\mathcal{E})$ , there corresponds a unique vector field of the form*

$$(20) \quad \mathcal{X} = \sum_{j=1}^m \Pi^j(\nu, \chi) \frac{\partial}{\partial \nu^j} + \sum_{q=1}^p \Lambda^q(\nu, \chi) \frac{\partial}{\partial \chi_q},$$

whose coefficients depend only on the parameters  $(\nu, \chi)$ , such that  $X + \mathcal{X}$  is tangent to the submanifold of solutions  $\mathcal{M}$ .

This leads us to define the Lie algebra  $\mathfrak{Sym}(\mathcal{M})$  of vector fields of the form

$$(21) \quad \sum_{l=1}^n Q^l(x, u) \frac{\partial}{\partial x_l} + \sum_{j=1}^m R^j(x, u) \frac{\partial}{\partial u^j} + \sum_{j=1}^m \Pi^j(\nu, \chi) \frac{\partial}{\partial \nu^j} + \sum_{q=1}^p \Lambda^q(\nu, \chi) \frac{\partial}{\partial \chi_q}$$

which are tangent to  $\mathcal{M}$ . We shall say that the submanifold  $\mathcal{M}$  is *degenerate* if there exists a nonzero vector field of the form  $X = \sum_{l=1}^n Q^l(x, u) \partial/\partial x_l + \sum_{j=1}^m R^j(x, u) \partial/\partial u^j$  which is tangent to  $\mathcal{M}$ , which means that the corresponding  $\mathcal{X}$  part is zero. In this case, we claim that  $\mathfrak{Sym}(\mathcal{M})$  is infinite dimensional. Indeed there exists then a nonzero vector field  $T = \sum_{l=1}^n Q^l(x, u) \partial/\partial x_l + \sum_{j=1}^m R^j(x, u) \partial/\partial u^j$  tangent to  $\mathcal{M}$ . Consequently, for every  $\mathbb{K}$ -analytic function  $A(x, u)$ , the vector field  $A(x, u) T$  belongs to  $\mathfrak{Sym}(\mathcal{M})$ , hence  $\mathfrak{Sym}(\mathcal{M})$  is infinite dimensional.

By developing the dual defining functions of  $\mathcal{M}$  with respect to the powers of  $\chi$ , we may write

$$(22) \quad \nu^j = \Omega_{j,\gamma}^*(\chi, x, u) = \sum_{\gamma \in \mathbb{N}^p} \chi^\gamma \Omega_{j,\gamma}^*(x, u),$$

where the functions  $\Omega_{j,\gamma}^*(x, u)$  are  $\mathbb{K}$ -analytic in a neighborhood of the origin, we may formulate a criterion for  $\mathcal{M}$  to be non degenerate with respect to the variables (whose proof is skipped).

PROPOSITION 2. *The submanifold  $\mathcal{M}$  is not degenerate with respect to the variables if and only if there exists an integer  $k$  such that the generic rank of the local  $\mathbb{K}$ -analytic mapping*

$$(23) \quad (x, u) \longmapsto \left( \Omega_{j,\gamma}^*(x, u) \right)_{1 \leq j \leq m, \gamma \in \mathbb{N}^p, |\gamma| \leq k}$$

is equal to  $n + m$ .

Seeking for conditions which insure that  $\mathfrak{S}\eta\mathfrak{m}(\mathcal{M})$  is finite-dimensional, it is therefore natural to assume that the generic rank of the mapping (23) is equal to  $n + m$ . Furthermore, to simplify the presentation, we shall assume that the rank at  $(x, u) = (0, 0)$  (not only the generic rank) of the mapping (23) is equal to  $n + m$  for  $k$  large enough. This is a “Zariski-generic” assumption. Coming back to (14), we observe that this means exactly that  $\mathcal{M}$  is solvable with respect to the variables. Then we denote by  $\ell_0^*$  the smallest integer  $k$  such that the rank at  $(x, u) = (0, 0)$  of the mapping (23) is equal to  $n + m$  and we say that  $\mathcal{M}$  is  $\ell_0^*$ -solvable with respect to the variables. Also, we denote by  $\ell_0$  the integer  $\max_{1 \leq q \leq p} |\beta(q)|$  and we say that  $\mathcal{M}$  is  $\ell_0$ -solvable with respect to the parameters.

## 2.7. Fundamental isomorphism between $\mathfrak{S}\eta\mathfrak{m}(\mathcal{E})$ and $\mathfrak{S}\eta\mathfrak{m}(\mathcal{M})$

In the remainder of this Section 2, we shall assume that  $\mathcal{M}$  is  $\ell_0$ -solvable with respect to the parameters and  $\ell_0^*$ -solvable with respect to the variables. In this case, viewing the variables  $(\nu^1, \dots, \nu^m)$  in the dual equations  $\nu^j = \Omega_j^*(\chi, x, u)$  of  $\mathcal{M}$  as a mapping of  $\chi$  with (dual) “parameters”  $(x, u)$  and proceeding as in Subsection 2.2, we may construct a dual system of completely integrable partial differential equations of the form

$$(E^*) \quad \nu_{\chi^\gamma}^j(\chi) = G_\gamma^j \left( \chi, \nu(\chi), (\nu_{\chi^{\delta(l)}}^{j(l)}(\chi))_{1 \leq l \leq n} \right),$$

where  $(j, \gamma) \neq (j(1), \delta(1)), \dots, (j(n), \delta(n))$ . This system has its own infinitesimal symmetry Lie algebra  $\mathfrak{S}\eta\mathfrak{m}(E^*)$ .

THEOREM 2. *If  $\mathcal{M}$  is both solvable with respect to the parameters and solvable with respect to the variables, we have the following two isomorphisms:*

$$(24) \quad \mathfrak{S}\eta\mathfrak{m}(\mathcal{E}) \cong \mathfrak{S}\eta\mathfrak{m}(\mathcal{M}) \cong \mathfrak{S}\eta\mathfrak{m}(E^*),$$

namely  $X \longleftrightarrow X + \mathcal{X} \longleftrightarrow \mathcal{X}$ .

In Subsection 2.10 below, we shall introduce a second geometric condition which is in general necessary for  $\mathfrak{S}\eta\mathfrak{m}(\mathcal{M})$  to be finite-dimensional.

## 2.8. Local (pseudo) group $\text{Sym}(\mathcal{M})$ of point transformations of $\mathcal{M}$

We shall study the geometry of a local  $\mathbb{K}$ -analytic submanifold  $\mathcal{M}$  of  $\mathbb{K}^{n+2m+p}$  whose equations and dual equations are of the form

$$(25) \quad \begin{aligned} u^j &= \Omega_j(x, \nu, \chi), & j &= 1, \dots, m, \\ \nu^j &= \Omega_j^*(\chi, x, u), & j &= 1, \dots, m. \end{aligned}$$

Let  $t := (x, u) \in \mathbb{K}^{n+m}$  and  $\tau := (\nu, \chi) \in \mathbb{K}^{n+m}$ . We are interested in describing the set of local  $\mathbb{K}$ -analytic transformations of the space  $\mathbb{K}^{n+2m+p}$  which are of the specific form

$$(26) \quad (t, \tau) \longmapsto (h(t), \phi(\tau)),$$

and which stabilize  $\mathcal{M}$ , in a neighborhood of the origin. We denote the local Lie pseudogroup of such transformations (possibly infinite-dimensional) by  $\text{Sym}(\mathcal{M})$ . Importantly, each transformation of  $\text{Sym}(\mathcal{M})$  stabilizes both the sets  $\{t = ct.\}$  and the sets  $\{\tau = c\tau.\}$ . Of course, the Lie algebra of  $\text{Sym}(\mathcal{M})$  coincides with  $\mathfrak{Sym}(\mathcal{M})$  defined above.

## 2.9. Fundamental pair of foliations on $\mathcal{M}$

Let  $p_0 \in \mathbb{K}^{n+2m+p}$  be a fixed point of coordinates  $(t_{p_0}, \tau_{p_0})$ . Firstly, we observe that the intersection  $\mathcal{M} \cap \{\tau = \tau_{p_0}\}$  consists of the  $n$ -dimensional  $\mathbb{K}$ -analytic submanifold of equation  $u = \Omega(x, \tau_{p_0})$ . As  $\tau_{p_0}$  varies, we obtain a local  $\mathbb{K}$ -analytic foliation of  $\mathcal{M}$  by  $n$ -dimensional submanifolds. Let us denote this first foliation by  $\mathcal{F}_p$  and call it the *foliation of  $\mathcal{M}$  with respect to parameters*. Secondly, and dually, we observe that the intersection  $\mathcal{M} \cap \{t = t_{p_0}\}$  consists of the  $p$ -dimensional  $\mathbb{K}$ -analytic submanifold of equation  $\nu = \Omega^*(\chi, t_{p_0})$ . As  $t_{p_0}$  varies, we obtain a local  $\mathbb{K}$ -analytic foliation of  $\mathcal{M}$  by  $p$ -dimensional submanifolds. Let us denote this second foliation by  $\mathcal{F}_v$  and call it the *foliation of  $\mathcal{M}$  with respect to the variables*. We call  $(\mathcal{F}_p, \mathcal{F}_v)$  the *fundamental pair of foliations on  $\mathcal{M}$* .

## 2.10. Covering property of the fundamental pair of foliations

We wish to formulate a geometric condition which says that starting from the origin in  $\mathcal{M}$  and following alternately the leaves of  $\mathcal{F}_p$  and the leaves of  $\mathcal{F}_v$ , we cover a neighborhood of the origin in  $\mathcal{M}$ . Let us introduce two collections  $(\mathcal{L}_k)_{1 \leq k \leq n}$  and  $(\mathcal{L}_q^*)_{1 \leq q \leq p}$  of vector fields whose

integral manifolds coincide with the leaves of  $\mathcal{F}_p$  and  $\mathcal{F}_v$ :

$$\begin{aligned}
 \mathcal{L}_k &:= \frac{\partial}{\partial x_k} + \sum_{j=1}^m \frac{\partial \Omega_j}{\partial x_k}(x, \nu, \chi) \frac{\partial}{\partial \nu^j}, & k = 1, \dots, n, \\
 \mathcal{L}_q^* &:= \frac{\partial}{\partial \chi_q} + \sum_{j=1}^m \frac{\partial \Omega_j^*}{\partial \chi_q}(\chi, x, u) \frac{\partial}{\partial \nu^j}, & k = 1, \dots, n.
 \end{aligned}
 \tag{27}$$

Let  $p_0$  be a fixed point in  $\mathcal{M}$  of coordinates  $(x_{p_0}, u_{p_0}, \nu_{p_0}, \chi_{p_0}) \in \mathbb{K}^{n+2m+p}$ , let  $x_1 := (x_{1,1}, \dots, x_{1,n}) \in \mathbb{K}^n$  be a “multitime” parameter and define the multiple flow map

$$\begin{aligned}
 \mathcal{L}_{x_1}(x_{p_0}, u_{p_0}, \nu_{p_0}, \chi_{p_0}) &:= \exp(x_1 \mathcal{L})(p_0) \\
 &:= \exp(x_{1,n} \mathcal{L}_n(\dots(\exp(x_{1,1} \mathcal{L}_1(p_0)))) \dots) \\
 &:= (x_{p_0} + x_1, \Omega(x_{p_0} + x_1, \nu_{p_0}, \chi_{p_0}), \nu_{p_0}, \chi_{p_0}).
 \end{aligned}
 \tag{28}$$

Similarly, for  $\chi = (\chi_{1,1}, \dots, \chi_{1,p}) \in \mathbb{K}^p$ , define the multiple flow map

$$\mathcal{L}_{\chi_1}^*(x_{p_0}, u_{p_0}, \nu_{p_0}, \chi_{p_0}) := (x_{p_0}, u_{p_0}, \Omega^*(\chi_{p_0} + \chi_1, x_{p_0}, u_{p_0}), \chi_{p_0} + \chi_1).
 \tag{29}$$

We may define now the mappings which correspond to start from the origin and to move alternately along the two foliations  $\mathcal{F}_p$  and  $\mathcal{F}_v$ . If the first movement consists in moving along the foliation  $\mathcal{F}_v$ , we define

$$\begin{aligned}
 \Gamma_1(x_1) &:= \mathcal{L}_{x_1}(0), \\
 \Gamma_1(x_1, \chi_1) &:= \mathcal{L}_{\chi_1}^*(\mathcal{L}_{x_1}(0)), \\
 \Gamma_3(x_1, \chi_1, x_2) &:= \mathcal{L}_{x_2}(\mathcal{L}_{\chi_1}^*(\mathcal{L}_{x_1}(0))), \\
 \Gamma_4(x_1, \chi_1, x_2, \chi_2) &:= \mathcal{L}_{\chi_2}^*(\mathcal{L}_{x_2}(\mathcal{L}_{\chi_1}^*(\mathcal{L}_{x_1}(0)))).
 \end{aligned}
 \tag{30}$$

Generally, we may define the maps  $\Gamma_k([x\chi]_k)$ , where  $[x\chi]_k = (x_1, \chi_1, x_2, \chi_2, \dots)$  with exactly  $k$  terms and where each  $x_l$  belongs to  $\mathbb{K}^n$  and each  $\chi_l$  belongs to  $\mathbb{K}^p$ . On the other hand, if the first movement consists in moving along the foliation  $\mathcal{F}_p$ , we start with  $\Gamma_1^*(\chi_1) := \mathcal{L}_{\chi_1}^*(0)$ ,  $\Gamma_2^*(\chi_1, x_1) := \mathcal{L}_{x_1}(\mathcal{L}_{\chi_1}^*(0))$ , etc., and generally we may define the maps  $\Gamma_k^*([\chi x]_k)$ , where  $[\chi x]_k = (\chi_1, x_1, \chi_2, x_2, \dots)$ , with exactly  $k$  terms. The range of both maps  $\Gamma_k$  and  $\Gamma_k^*$  is contained in  $\mathcal{M}$ . We call  $\Gamma_k$  the  $k$ -th chain and  $\Gamma_k^*$  the  $k$ -th dual chain.

DEFINITION 1. The pair of foliations  $(\mathcal{F}_p, \mathcal{F}_v)$  is called *covering at the origin* if there exists an integer  $k$  such that the generic rank of  $\Gamma_k$  is (maximal possible) equal to  $\dim_{\mathbb{K}} \mathcal{M}$ . Since the dual  $(k + 1)$ -th chain  $\Gamma_{k+1}^*$  for  $\chi_1 = 0$  identifies with the  $k$ -th chain  $\Gamma_k$ , it follows that the same property holds for the dual chains.

In terms of Sussmann's approach [27], this means that the *local orbit* of the two systems of vector fields  $(\mathcal{L}_k)_{1 \leq k \leq n}$  and  $(\mathcal{L}_q^*)_{1 \leq q \leq p}$  is of maximal dimension. Reasoning as in [27] (using the so-called *backward trick* in Control Theory, see also [17]), it may be shown that there exists the smallest even integer  $2\mu_0$  such that the ranks of the two maps  $\Gamma_{2\mu_0}$  and  $\Gamma_{2\mu_0}^*$  at the origin (not only their generic rank) in  $\mathbb{K}^{n\mu_0+p\mu_0}$  are both equal to  $\dim_{\mathbb{K}} \mathcal{M}$ . This means that  $\Gamma_{2\mu_0}$  and  $\Gamma_{2\mu_0}^*$  are submersive onto a neighborhood of the origin in  $\mathcal{M}$ . We call  $\mu_0$  the *type of the pair of foliations*  $(\mathcal{F}_p, \mathcal{F}_v)$ . It may also be established that  $\mu_0 \leq m + 2$ .

EXAMPLE 3. We give an example of a submanifold which is both 1-solvable with respect to the parameters and with respect to the variables but whose pair of foliations is not covering: with  $n = 1$ ,  $m = 2$  and  $p = 1$ , this is given by the two equations  $u^1 = \nu^1$ ,  $u^2 = \nu^2 + x\chi_1$ . Then  $\mathfrak{Sym}(\mathcal{M})$  is infinite-dimensional since it contains the vector fields  $a(u^1) \partial/\partial u^1 + a(\nu^1) \partial/\partial \nu^1$ , where  $a$  is an arbitrary  $\mathbb{K}$ -analytic function. For this reason, we shall assume in the sequel that the pair of foliations  $(\mathcal{F}_p, \mathcal{F}_v)$  is covering at the origin.

### 2.11. Estimate on the dimension of the local symmetry group of the submanifold of solutions

We may now formulate the main theorem of this section, which shows that, under suitable nondegeneracy conditions,  $\text{Sym}(\mathcal{M})$  is a finite dimensional local Lie group of local transformations. If  $t \in \mathbb{K}^{n+m}$ , we denote by  $|t| := \max_{1 \leq k \leq n+m} |t_k|$ . If  $(h, \phi) \in \text{Sym}(\mathcal{M})$  we denote by  $J_t^k h(0)$  the  $k$ -th order jet of  $h$  at the origin and by  $J_\tau^k \phi(0)$  the  $k$ -th order jet of  $\phi$  at the origin. Also, we shall assume that  $\mathcal{M}$  is either  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic. Of course, the  $\mathbb{K}$ -algebraicity of the submanifold of solutions does not follow from the  $\mathbb{K}$ -algebraicity of the right hand sides  $F_\alpha^j$  of the system of partial differential equations  $(\mathcal{E})$ .

THEOREM 3. Assume that the  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic submanifold of solutions  $\mathcal{M}$  of the completely integrable system of partial differential equations  $(\mathcal{E})$  is both  $\ell_0$ -solvable with respect to the parameters and  $\ell_0^*$ -solvable with respect to the variables. Assume that the fundamental pair of foliations  $(\mathcal{F}_p, \mathcal{F}_v)$  is covering at the origin and let  $\mu_0$  be its type at the origin. Then there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon$  with  $0 < \varepsilon < \varepsilon_0$ , the following four properties hold:

- (a) The (pseudo)group  $\text{Sym}(\mathcal{M})$  of local  $\mathbb{K}$ -analytic diffeomorphisms defined for  $\{(t, \tau) \in \mathbb{K}^{n+2m+p} : |t| < \varepsilon, |\tau| < \varepsilon\}$  which are of

the form  $(t, \tau) \mapsto (h(t), \phi(\tau))$  and which stabilize  $\mathcal{M}$  is a local Lie pseudogroup of transformations of finite dimension  $d \in \mathbb{N}$ .

- (b) Let  $\kappa_0 := \mu_0(\ell_0 + \ell_0^*)$ . Then there exist two  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic mappings  $H_{\kappa_0}$  and  $\Phi_{\kappa_0}$  which depend only on  $\mathcal{M}$  and which may be constructed algorithmically by means of the defining equations of  $\mathcal{M}$  such that every element  $(h, \phi) \in \text{Sym}(\mathcal{M})$ , sufficiently close to the identity mapping, may be represented by

$$(31) \quad \begin{aligned} h(t) &= H_{\kappa_0}(t, J_t^{\kappa_0} h(0)), \\ \phi(\tau) &= \Phi_{\kappa_0}(\tau, J_\tau^{\kappa_0} \phi(0)). \end{aligned}$$

Consequently, every element of  $\text{Sym}(\mathcal{M})$  is uniquely determined by its  $\kappa_0$ -th jet at the origin and the dimension  $d$  of the Lie algebra  $\mathfrak{Sym}(\mathcal{M})$  is bounded by the number of components of the vector  $(J_t^{\kappa_0} h(0), J_\tau^{\kappa_0} \phi(0))$ , namely we have

$$(32) \quad \begin{aligned} \dim_{\mathbb{K}} \mathfrak{Sym}(\mathcal{E}) &= \dim_{\mathbb{K}} \mathfrak{Sym}(\mathcal{M}) \\ &\leq (n + m) C_{n+m+\kappa_0}^{\kappa_0} + (m + p) C_{m+p+\kappa_0}^{\kappa_0}. \end{aligned}$$

- (c) There exists  $\varepsilon'$  with  $0 < \varepsilon' < \varepsilon$  and a  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic mapping  $(H_{\mathcal{M}}, \Phi_{\mathcal{M}})$  which may be constructed algorithmically by means of the defining equations of  $\mathcal{M}$ , defined in a neighborhood of the origin in  $\mathbb{K}^{n+2m+p} \times \mathbb{K}^d$  with values in  $\mathbb{K}^{n+2m+p}$  and which satisfies  $(H_{\mathcal{M}}(t, 0), \Phi_{\mathcal{M}}(\tau, 0)) \equiv (t, \tau)$ , such that every element  $(h, \phi) \in \text{Sym}(\mathcal{M})$  defined on the set  $\{(t, \tau) \in \mathbb{K}^{n+2m+p} : |t| < \varepsilon', |\tau| < \varepsilon'\}$ , sufficiently close to the identity mapping and stabilizing  $\mathcal{M}$  may be represented as  $(h(t), \phi(\tau)) \equiv (H_{\mathcal{M}}(t, s_{h,\phi}), \Phi_{\mathcal{M}}(\tau, s_{h,\phi}))$  for a unique element  $s_{h,\phi} \in \mathbb{K}^d$  depending on the mapping  $(h, \phi)$ .
- (d) The mapping  $(t, \tau, s) \mapsto (H_{\mathcal{M}}(t, s), \Phi_{\mathcal{M}}(\tau, s))$  defines a local  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic Lie group of local  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic transformations stabilizing  $\mathcal{M}$ .

### 2.12. Applications

The proof of Theorem 3, which possesses strong similarities with the proof of Theorem 4.1 in [8], will not be presented. It seems that Theorem 3, together with the argumentation on the necessity of assumptions that  $\mathcal{M}$  be solvable with respect to the variables and that its fundamental pair of foliations be covering, is a new result about the finite-dimensionality of a completely integrable system of partial differential equations having an arbitrary number of independent and dependent



variables. The main interest lies in the fact that we obtain the algorithmically constructible representation formula (31) together with the local Lie group structure mapping  $(H_{\mathcal{M}}, \Phi_{\mathcal{M}})$ . In particular, we get as a corollary that every transformation  $(h(t), \phi(\tau))$  given by a formal power series (not necessarily convergent) is as smooth as the applications  $(H_{\kappa_0}, \Phi_{\kappa_0})$  are, namely every formal element of  $\text{Sym}(\mathcal{M})$  is necessarily  $\mathbb{K}$ -algebraic or  $\mathbb{K}$ -analytic. As a counterpart of its generality, Theorem 3 does not provide optimal bounds, as shows the following illustration.

**EXAMPLE 4.** Let  $n = m = 1$ , let  $\kappa \geq 3$  and let  $(\mathcal{E})$  denote the ordinary differential equation  $u_{x^\kappa}(x) = F(x, u(x), u_x(x), \dots, u_{x^{\kappa-1}}(x))$ . Then the submanifold of solutions  $\mathcal{M}$  is of the form  $u = \nu + x\chi_1 + \dots + x^{\kappa-1}\chi_{\kappa-1} + O(|x|^\kappa) + O(|x|^2)$ . It may be checked that  $\ell_0 = \kappa - 1$ ,  $\ell_0^* = 1$  and  $\mu_0 = 3$ , hence  $\kappa_0 = 3\kappa$ . Then the dimension estimate in (32) is:  $\dim_{\mathbb{K}} \mathfrak{Sym}(\mathcal{E}) \leq 2C_{2+3\kappa}^{3\kappa} + \kappa C_{4\kappa}^{3\kappa}$ . This bound is much larger than the optimal bound  $\dim_{\mathbb{K}} \mathfrak{Sym}(\mathcal{E}) \leq \kappa + 4$  due to S. Lie (cf. [5]; see also the case  $n = m = 1$  of Theorem 1).

Until now we focused on providing the set of Lie symmetries of a general system of partial differential equations with a local Lie group structure. As a byproduct we obtained the (non optimal) dimensional upper bound (32) of Theorem 3. In the next Sections 3, 4 and 5, using the classical Lie algorithm based on the Lie criterion (see Lemma 2), we provide an optimal bound for some specific systems of partial differential equations, answering an open problem raised in p.206 of [19].

### 3. Lie theory for partial differential equations

#### 3.1. Prolongation of vector fields to the jet spaces

Consider the following  $\mathbb{K}$ -analytic system  $(\mathcal{E})$  of non linear partial differential equations:

$$(33) \quad u_{x_{k_1} \dots x_{k_\kappa}}^j(x) = F_{k_1, \dots, k_\kappa}^j(x, u(x), u_{x_{l_1}}^{i_1}(x), \dots, u_{x_{l_1} \dots x_{l_{\kappa-1}}}^{i_{\kappa-1}}(x)),$$

where  $1 \leq k_1 \leq \dots \leq k_\kappa \leq n$ ,  $1 \leq j \leq m$ , and  $F_{k_1, \dots, k_\kappa}^j$  are analytic functions of  $n + m C_{n+\kappa-1}^{\kappa-1}$  variables, defined in a neighborhood of the origin. We assume that  $(\mathcal{E})$  is *completely integrable*. The Lie theory consists in studying the *infinitesimal symmetries*  $X = \sum_{l=1}^n Q^l(x, u) \partial / \partial x_l + \sum_{j=1}^m R^j(x, u) \partial / \partial w^j$  of  $(\mathcal{E})$ . Consider the *skeleton* of  $(\mathcal{E})$ , namely the complex subvariety  $\Delta_{\mathcal{E}}$  of codimension  $m C_{n+\kappa-1}^{\kappa}$  in the jet space  $\mathcal{J}_{n,m}^{\kappa}$ ,

defined by

$$(34) \quad U_{k_1, \dots, k_\kappa}^j = F_{k_1, \dots, k_\kappa}^j \left( x, u, U_{l_1}^{i_1}, \dots, U_{l_1, \dots, l_{\kappa-1}}^{i_1} \right),$$

where  $j, i_1 = 1, \dots, m$  and  $k_1, \dots, k_\kappa, l_1, \dots, l_{\kappa-1} = 1, \dots, n$ . For  $k = 1, \dots, n$  let  $D_k$  be the  $k$ -th operator of total differentiation, characterized by the property that for every integer  $\lambda \geq 2$  and for every analytic function  $P = P(x, u, U_{l_1}^{i_1}, \dots, U_{l_1, \dots, l_{\lambda-1}}^{i_1})$  defined in the jet space  $\mathcal{J}_{n,m}^{\lambda-1}$ , the operator  $D_k$  is the unique formal infinite differential operator satisfying the relation

$$(35) \quad \begin{aligned} & [D_k P] \left( x, u(x), u_{x_{l_1}}^{i_1}(x), \dots, u_{x_{l_1} \dots x_{l_{\lambda-1}}}^{i_1}(x) \right) \\ & \equiv \frac{\partial}{\partial x_k} \left[ P \left( x, u(x), u_{x_{l_1}}^{i_1}(x), \dots, u_{x_{l_1} \dots x_{l_{\lambda-1}}}^{i_1}(x) \right) \right]. \end{aligned}$$

Note that this identity involves only the truncature of  $D_k$  to order  $\lambda$ , denoted by  $D_k^\lambda$ , and defined by

$$(36) \quad \begin{aligned} D_k^\lambda := & \frac{\partial}{\partial x_k} + \sum_{i_1=1}^m U_k^{i_1} \frac{\partial}{\partial u^{i_1}} + \sum_{i_1=1}^m \sum_{l_1=1}^n U_{k, l_1}^{i_1} \frac{\partial}{\partial U_{l_1}^{i_1}} \\ & + \dots + \sum_{i_1=1}^m \sum_{l_1, \dots, l_{\lambda-1}=1}^n U_{k, l_1, \dots, l_{\lambda-1}}^{i_1} \frac{\partial}{\partial U_{l_1, \dots, l_{\lambda-1}}^{i_1}}. \end{aligned}$$

According to Theorem 2.36 of [18], the *prolongation of order  $\kappa$  of a vector field  $X = \sum_{l=1}^n Q^l(x, u) \partial/\partial x_l + \sum_{j=1}^m R^j(x, u) \partial/\partial u^j$* , denoted by  $X^{(\kappa)}$ , is the unique vector field on the space  $\mathcal{J}_{n,m}^\kappa$  of the form

$$(37) \quad \begin{aligned} X^{(\kappa)} = & X + \sum_{j=1}^m \sum_{k_1=1}^n \mathbf{R}_{k_1}^j \frac{\partial}{\partial U_{k_1}^j} + \sum_{j=1}^m \sum_{k_1, k_2=1}^n \mathbf{R}_{k_1, k_2}^j \frac{\partial}{\partial U_{k_1, k_2}^j} \\ & + \dots + \sum_{j=1}^m \sum_{k_1, \dots, k_\kappa=1}^n \mathbf{R}_{k_1, \dots, k_\kappa}^j \frac{\partial}{\partial U_{k_1, k_2, \dots, k_\kappa}^j}, \end{aligned}$$

corresponding to the infinitesimal action of the flow of  $X$  on the jets of order  $\kappa$  of the graphs of maps  $u = u(x)$ , and whose coefficients are

computed recursively by the formulas

$$\begin{aligned}
 \mathbf{R}_{k_1}^j &:= D_{k_1}^1(R^j) - \sum_{l_1=1}^n D_{k_1}^1(Q^{l_1})U_{l_1}^j, \\
 \mathbf{R}_{k_1, k_2}^j &:= D_{k_2}^2(\mathbf{R}_{k_1}^j) - \sum_{l_1=1}^n D_{k_2}^1(Q^{l_1})U_{k_1, l_1}^j, \\
 &\vdots \\
 \mathbf{R}_{k_1, k_2, \dots, k_\lambda}^j &:= D_{k_\lambda}^\lambda(\mathbf{R}_{k_1, \dots, k_{\lambda-1}}^j) - \sum_{l_1=1}^n D_{k_\lambda}^1(Q^{l_1})U_{k_1, \dots, k_{\lambda-1}, l_1}^j.
 \end{aligned}
 \tag{38}$$

For a better comprehension of the general computation, let us start by computing  $R^\kappa$  in the case  $n = m = 1$ .

### 3.2. Computation of $R^\kappa$ when $n = m = 1$

A direct application of the preceding formulas leads to the following classical expressions:

$$\begin{aligned}
 \mathbf{R}^1 &= R_x + [R_u - Q_x]U^1 + [-Q_u](U^1)^2, \\
 \mathbf{R}^2 &= R_{x^2} + [2R_{xu} - Q_{x^2}]U^1 + [R_{u^2} - 2Q_{xu}](U^1)^2 + [-Q_{u^2}](U^1)^3 \\
 &\quad + [R_u - 2Q_x]U^2 + [-3Q_u]U^1U^2.
 \end{aligned}
 \tag{39}$$

Observe that these expressions are polynomial in the jet variables, their coefficients being differential expressions involving a partial derivative of  $R$  (with a positive integer coefficient) and a partial derivative of  $Q$  (with a negative integer coefficient). We have also:

$$\begin{aligned}
 \mathbf{R}^3 &= R_{x^3} + [3R_{x^2u} - Q_{x^3}]U^1 + [3R_{xu^2} - 3Q_{x^2u}](U^1)^2 \\
 &\quad + [R_{u^3} - 3Q_{xu^2}](U^1)^3 + [-Q_{u^3}](U^1)^4 + [3R_{xu} - 3Q_{x^2}]U^2 \\
 &\quad + [3R_{u^2} - 9Q_{xu}]U^1U^2 + [-6Q_{u^2}](U^1)^2U^2 + [-3Q_u](U^2)^2 \\
 &\quad + [R_u - 3Q_x]U^3 + [-4Q_u]U^1U^3, \\
 \mathbf{R}^4 &= R_{x^4} + [4R_{x^3u} - Q_{x^4}]U^1 + [6R_{x^2u^2} - 4Q_{x^3u}](U^1)^2 \\
 &\quad + [4R_{xu^3} - 6Q_{x^2u^2}](U^1)^3 + [R_{u^4} - 4Q_{xu^3}](U^1)^4 + [-Q_{u^4}](U^1)^5 \\
 &\quad + [6R_{x^2u} - 4Q_{x^3}]U^2 + [12R_{xu^2} - 18Q_{x^2u}]U^1U^2
 \end{aligned}
 \tag{40}$$

$$\begin{aligned}
& + [6R_{u^3} - 24Q_{xu^2}] (U^1)^2 U^2 + [-10Q_{u^3}] (U^1)^3 U^2 \\
& + [3R_{u^2} - 12Q_{xu}] (U^2)^2 + [-15Q_{u^2}] U^1 (U^2)^2 + [4R_{xu} - 6Q_{x^2}] U^3 \\
& + [4R_{u^2} - 16Q_{xu}] U^1 U^3 + [-10Q_{u^2}] (U^1)^2 U^3 + [-10Q_u] U^2 U^3 \\
& + [R_u - 4Q_x] U^4 + [-5Q_u] U^1 U^4.
\end{aligned}$$

Remark that all the brackets involved in equations (40) are of the form  $[\lambda R_{x^a u^{b+1}} - \mu Q_{x^{a+1} u^b}]$ , where  $\lambda, \mu \in \mathbb{N}$  and  $a, b \in \mathbb{N}$ .

In what follows we will not need the complete form of  $R^\kappa$  but only the following partial form:

LEMMA 3. For  $\kappa \geq 4$ ,

$$\begin{aligned}
(41) \quad \mathbf{R}^\kappa &= R_{x^\kappa} + [C_\kappa^1 R_{x^{\kappa-1} u} - Q_{x^\kappa}] U^1 + [C_\kappa^2 R_{x^{\kappa-2} u} - C_\kappa^1 Q_{x^{\kappa-1}}] U^2 \\
&+ [C_\kappa^2 R_{x^2 u} - C_\kappa^3 Q_{x^3}] U^{\kappa-2} + [C_\kappa^1 R_{xu} - C_\kappa^2 Q_{x^2}] U^{\kappa-1} \\
&+ [C_\kappa^1 R_{u^2} - \kappa^2 Q_{xu}] U^1 U^{\kappa-1} + [-C_{\kappa+1}^2 Q_u] U^2 U^{\kappa-1} \\
&+ [R_u - C_\kappa^1 Q_x] U^\kappa + [-C_{\kappa+1}^1 Q_u] U^1 U^\kappa \\
&+ \text{Remainder},
\end{aligned}$$

where the term *Remainder* denotes the remaining terms in the expansion of  $R^\kappa$ .

We note that the formula (41) is valid for  $\kappa = 3$ , comparing with (40), with the convention that the terms  $U^{\kappa-2}$  and  $U^{\kappa-1}$  vanish (they coincide with  $U^1$  and  $U^2$ ), and replacing the coefficient  $-C_{\kappa+1}^2 Q_u = -C_4^2 Q_u = -6Q_u$  of the monomial  $U^2 U^{\kappa-1}$  by  $-3Q_u$ , as it appears in (40). The proof goes by a straightforward computation, applying the recursive definition of this partial formula.

### 3.3. Computation of $R^\kappa$ in the general case

Following the exact same scheme as in the case  $n = 1$  we give the general partial formula for  $R^\kappa$ . We start with the first three families of coefficients  $\mathbf{R}_{k_1}^j$ ,  $\mathbf{R}_{k_1, k_2}^j$  and  $\mathbf{R}_{k_1, k_2, k_3}^j$ . Let  $\delta_p^q$  be the Kronecker symbol, equal to 1 if  $p = q$  and to 0 if  $p \neq q$ . More generally, the *generalized Kronecker symbols* are defined by  $\delta_{p_1, \dots, p_k}^{q_1, \dots, q_k} := \delta_{p_1}^{q_1} \delta_{p_2}^{q_2} \cdots \delta_{p_k}^{q_k}$ .

By convention, the indices  $j, i_1, i_2, \dots, i_\lambda$  run in the set  $\{1, \dots, m\}$ , the indices  $k, k_1, k_2, \dots, k_\lambda$  and  $l, l_1, l_2, \dots, l_\lambda$  running in  $\{1, \dots, n\}$ . Hence we will write  $\sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_\lambda=1}^m$  as  $\sum_{i_1, \dots, i_\lambda}$  and  $\sum_{l_1=1}^n \sum_{l_2=1}^n \cdots \sum_{l_\lambda=1}^n$  as  $\sum_{l_1, \dots, l_\lambda}$ . The letters  $i_1, i_2, \dots, i_\lambda$  and  $l_1, l_2, \dots, l_\lambda$  will always be used for the summations in the development of  $\mathbf{R}_{k_1, k_2, \dots, k_\lambda}^j$ . We

will always use the indices  $j$  and  $k_1, k_2, \dots, k_\lambda$  to write the coefficient

$\mathbf{R}_{k_1, k_2, \dots, k_\lambda}^j$ .

We have:

$$(42) \quad \begin{aligned} \mathbf{R}_{k_1}^j &= R_{x_{k_1}}^j + \sum_{i_1} \sum_{l_1} \left[ \delta_{k_1}^{l_1} R_{u^{i_1}}^j - \delta_{i_1}^j Q_{x_{k_1}}^{l_1} \right] U_{l_1}^{i_1} \\ &+ \sum_{i_1, i_2} \sum_{l_1, l_2} \left[ -\delta_{i_2}^j \delta_{k_1}^{l_1} Q_{u^{i_1}}^{l_2} \right] U_{l_1}^{i_1} U_{l_2}^{i_2}. \end{aligned}$$

For  $\mathbf{R}_{k_1, k_2}^j$  we have:

$$(43) \quad \begin{aligned} \mathbf{R}_{k_1, k_2}^j &= R_{x_{k_1} x_{k_2}}^j + \sum_{i_1} \sum_{l_1} \left[ \delta_{k_2}^{l_1} R_{x_{k_1} u^{i_1}}^j + \delta_{k_1}^{l_1} R_{x_{k_2} u^{i_1}}^j - \delta_{i_1}^j Q_{x_{k_1} x_{k_2}}^{l_1} \right] U_{l_1}^{i_1} \\ &+ \sum_{i_1, i_2} \sum_{l_1, l_2} \left[ \delta_{k_1, k_2}^{l_1, l_2} R_{u^{i_1} u^{i_2}}^j - \delta_{i_2}^j \left( \delta_{k_1}^{l_1} Q_{x_{k_2} u^{i_1}}^{l_2} + \delta_{k_2}^{l_1} Q_{x_{k_1} u^{i_1}}^{l_2} \right) \right] U_{l_1}^{i_1} U_{l_2}^{i_2} \\ &+ \sum_{i_1, i_2, i_3} \sum_{l_1, l_2, l_3} \left[ -\delta_{i_3}^j \delta_{k_1, k_2}^{l_1, l_2} Q_{u^{i_1} u^{i_2}}^{l_3} \right] U_{l_1}^{i_1} U_{l_2}^{i_2} U_{l_3}^{i_3} \\ &+ \sum_{i_1} \sum_{l_1, l_2} \left[ \delta_{k_1, k_2}^{l_1, l_2} R_{u^{i_1}}^j - \delta_{i_1}^j \delta_{k_2}^{l_1} Q_{x_{k_1}}^{l_2} - \delta_{i_1}^j \delta_{k_1}^{l_1} Q_{x_{k_2}}^{l_2} \right] U_{l_1, l_2}^{i_1} \\ &+ \sum_{i_1, i_2} \sum_{l_1, l_2, l_3} \left[ -\delta_{i_2}^j \delta_{k_1, k_2}^{l_1, l_2} Q_{u^{i_1}}^{l_3} - \delta_{i_2}^j \delta_{k_1, k_2}^{l_3, l_1} Q_{u^{i_1}}^{l_2} - \delta_{i_1}^j \delta_{k_1, k_2}^{l_2, l_3} Q_{u^{i_2}}^{l_1} \right] U_{l_1}^{i_1} U_{l_2, l_3}^{i_2}. \end{aligned}$$

Since we also treat systems of order  $\kappa \geq 3$ , it is necessary to compute

$\mathbf{R}_{k_1, k_2, k_3}^j$ . We write this as follows:

$$(44) \quad \mathbf{R}_{k_1, k_2, k_3}^j = \text{I} + \text{II} + \text{III},$$

where the first term I involves only polynomials in  $U_{l_1}^{i_1}$ :

$$(45) \quad \begin{aligned} \text{I} &= R_{x_{k_1} x_{k_2} x_{k_3}}^j + \sum_{i_1} \sum_{l_1} \left[ \delta_{k_1}^{l_1} R_{x_{k_2} x_{k_3} u^{i_1}}^j + \delta_{k_2}^{l_1} R_{x_{k_1} x_{k_3} u^{i_1}}^j + \delta_{k_3}^{l_1} R_{x_{k_1} x_{k_2} u^{i_1}}^j \right. \\ &\quad \left. - \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{l_1} \right] U_{l_1}^{i_1} + \sum_{i_1, i_2} \sum_{l_1, l_2} \left[ \delta_{k_1, k_2}^{l_1, l_2} R_{x_{k_3} u^{i_1} u^{i_2}}^j + \delta_{k_3, k_1}^{l_1, l_2} R_{x_{k_2} u^{i_1} u^{i_2}}^j \right. \end{aligned}$$

$$\begin{aligned}
 & + \delta_{k_2, k_3}^{l_1, l_2} R_{x_{k_1} u^{i_1} u^{i_2}}^j - \delta_{i_2}^j \delta_{k_1}^{l_1} Q_{x_{k_2} x_{k_3} u^{i_1}}^{l_2} - \delta_{i_2}^j \delta_{k_2}^{l_1} Q_{x_{k_1} x_{k_3} u^{i_1}}^{l_2} \\
 & - \delta_{i_2}^j \delta_{k_3}^{l_1} Q_{x_{k_1} x_{k_2} u^{i_1}}^{l_2} \Big] U_{l_1}^{i_1} U_{l_2}^{i_2} + \sum_{i_1, i_2, i_3} \sum_{l_1, l_2, l_3} \left[ \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} R_{u^{i_1} u^{i_2} u^{i_3}}^j \right. \\
 & - \delta_{i_3}^j \delta_{k_1, k_2}^{l_1, l_2} Q_{x_{k_3} u^{i_1} u^{i_2}}^{l_3} - \delta_{i_3}^j \delta_{k_2, k_3}^{l_1, l_2} Q_{x_{k_1} u^{i_1} u^{i_2}}^{l_3} \\
 & \left. - \delta_{i_3}^j \delta_{k_1, k_3}^{l_1, l_2} Q_{x_{k_2} u^{i_1} u^{i_2}}^{l_3} \right] U_{l_1}^{i_1} U_{l_2}^{i_2} U_{l_3}^{i_3} \\
 & + \sum_{i_1, i_2, i_3, i_4} \sum_{l_1, l_2, l_3, l_4} \left[ -\delta_{i_4}^j \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} Q_{u^{i_1} u^{i_2} u^{i_3}}^{l_4} \right] U_{l_1}^{i_1} U_{l_2}^{i_2} U_{l_3}^{i_3} U_{l_4}^{i_4},
 \end{aligned}$$

the second term II involves at least once the monomial  $U_{l_1, l_2}^{i_1}$ :

(46)

$$\begin{aligned}
 \text{II} = & \sum_{i_1} \sum_{l_1, l_2} \left[ \delta_{k_1, k_2}^{l_1, l_2} R_{x_{k_3} u^{i_1}}^j + \delta_{k_3, k_1}^{l_1, l_2} R_{x_{k_2} u^{i_1}}^j + \delta_{k_2, k_3}^{l_1, l_2} R_{x_{k_1} u^{i_1}}^j \right. \\
 & \left. - \delta_{i_1}^j \left( \delta_{k_1}^{l_1} Q_{x_{k_2} x_{k_3}}^{l_2} + \delta_{k_2}^{l_1} Q_{x_{k_1} x_{k_3}}^{l_2} + \delta_{k_3}^{l_1} Q_{x_{k_1} x_{k_2}}^{l_2} \right) \right] U_{l_1, l_2}^{i_1} \\
 & + \sum_{i_1, i_2} \sum_{l_1, l_2, l_3} \left[ \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} R_{u^{i_1} u^{i_2}}^j + \delta_{k_1, k_2, k_3}^{l_3, l_1, l_2} R_{u^{i_1} u^{i_2}}^j + \delta_{k_1, k_2, k_3}^{l_2, l_3, l_1} R_{u^{i_1} u^{i_2}}^j \right. \\
 & - \delta_{i_1}^j \left( \delta_{k_1, k_2}^{l_2, l_3} Q_{x_{k_3} u^{i_2}}^{l_1} + \delta_{k_3, k_1}^{l_2, l_3} Q_{x_{k_2} u^{i_2}}^{l_1} + \delta_{k_2, k_3}^{l_2, l_3} Q_{x_{k_1} u^{i_2}}^{l_1} \right) \\
 & - \delta_{i_2}^j \left( \delta_{k_1, k_2}^{l_1, l_2} Q_{x_{k_3} u^{i_1}}^{l_3} + \delta_{k_3, k_1}^{l_1, l_2} Q_{x_{k_2} u^{i_1}}^{l_3} + \delta_{k_2, k_3}^{l_1, l_2} Q_{x_{k_1} u^{i_1}}^{l_3} \right. \\
 & \left. + \delta_{k_1, k_2}^{l_3, l_1} Q_{x_{k_3} u^{i_1}}^{l_2} + \delta_{k_3, k_1}^{l_3, l_1} Q_{x_{k_2} u^{i_1}}^{l_2} + \delta_{k_2, k_3}^{l_3, l_1} Q_{x_{k_1} u^{i_1}}^{l_2} \right) \Big] U_{l_1}^{i_1} U_{l_2, l_3}^{i_2} \\
 & + \sum_{i_1, i_2, i_3} \sum_{l_1, l_2, l_3, l_4} \left[ -\delta_{i_3}^j \left( \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} Q_{u^{i_1} u^{i_2}}^{l_4} + \delta_{k_1, k_2, k_3}^{l_1, l_4, l_2} Q_{u^{i_1} u^{i_2}}^{l_3} \right. \right. \\
 & \left. \left. + \delta_{k_1, k_2, k_3}^{l_3, l_1, l_2} Q_{u^{i_1} u^{i_2}}^{l_4} \right) - \delta_{i_1}^j \left( \delta_{k_1, k_2, k_3}^{l_3, l_2, l_4} Q_{u^{i_2} u^{i_3}}^{l_1} + \delta_{k_1, k_2, k_3}^{l_4, l_3, l_2} Q_{u^{i_2} u^{i_3}}^{l_1} \right. \right. \\
 & \left. \left. + \delta_{k_1, k_2, k_3}^{l_2, l_3, l_4} Q_{u^{i_1} u^{i_2}}^{l_1} \right) \right] U_{l_1}^{i_1} U_{l_2}^{i_2} U_{l_3, l_4}^{i_3} \\
 & + \sum_{i_1, i_2} \sum_{l_1, l_2, l_3, l_4} \left[ -\delta_{i_2}^j \left( \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} Q_{u^{i_1}}^{l_4} + \delta_{k_1, k_2, k_3}^{l_3, l_1, l_2} Q_{u^{i_1}}^{l_4} \right. \right. \\
 & \left. \left. + \delta_{k_1, k_2, k_3}^{l_2, l_3, l_1} Q_{u^{i_1}}^{l_4} \right) \right] U_{l_1, l_2}^{i_1} U_{l_3, l_4}^{i_2}
 \end{aligned}$$

and the third term III involves at least once the monomial  $U_{l_1, l_2, l_3}^{i_1}$  (note that there is no term involving simultaneously  $U_{l_1, l_2}^{i_1}$  and  $U_{l_1, l_2, l_3}^{i_1}$ ):

$$\begin{aligned}
(47) \quad \text{III} = & \sum_{i_1} \sum_{l_1, l_2, l_3} \left[ \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} R_{u^{i_1}}^j - \delta_{i_1}^j \left( \delta_{k_2, k_3}^{l_1, l_2} Q_{x_{k_1}}^{l_3} + \delta_{k_3, k_1}^{l_1, l_2} Q_{x_{k_2}}^{l_3} \right. \right. \\
& \left. \left. + \delta_{k_1, k_2}^{l_1, l_2} Q_{x_{k_3}}^{l_3} \right) \right] U_{l_1, l_2, l_3}^{i_1} + \sum_{i_1, i_2} \sum_{l_1, l_2, l_3, l_4} \left[ -\delta_{i_1}^j \delta_{k_1, k_2, k_3}^{l_2, l_3, l_4} Q_{u^{i_2}}^{l_1} \right. \\
& \left. - \delta_{i_2}^j \left( \delta_{k_1, k_2, k_3}^{l_1, l_2, l_3} Q_{u^{i_1}}^{l_4} + \delta_{k_1, k_2, k_3}^{l_4, l_1, l_2} Q_{u^{i_1}}^{l_3} + \delta_{k_1, k_2, k_3}^{l_3, l_4, l_1} Q_{u^{i_1}}^{l_2} \right) \right] U_{l_1}^{i_1} U_{l_2, l_3, l_4}^{i_2}.
\end{aligned}$$

Before giving the partial expression of  $R^\kappa$  we introduce some notations. For  $p \in \mathbb{N}$  with  $p \geq 1$ , let  $\mathfrak{S}_p$  be the group of permutations of  $\{1, 2, \dots, p\}$ . For  $q \in \mathbb{N}$  with  $1 \leq q \leq p-1$ , let  $\mathfrak{S}_p^q$  be the set of permutations  $\sigma \in \mathfrak{S}_p$  such that  $\sigma(1) < \sigma(2) < \dots < \sigma(q)$  and  $\sigma(q+1) < \sigma(q+2) < \dots < \sigma(p)$ . Its cardinal is  $C_p^q$ . Let  $\mathfrak{C}_p$  be the group of cyclic permutations of  $\{1, 2, \dots, p\}$ . Reasoning recursively from the formula of  $\mathbf{R}_{k_1, k_2, k_3}^j$  given by (44), we may generalize Lemma 3:

LEMMA 4. For every  $\kappa \geq 4$  and for every  $j = 1, \dots, m$ ,  $k_1, \dots, k_\kappa = 1, \dots, n$ , we have:

$$(48) \quad \mathbf{R}_{k_1, k_2, \dots, k_\kappa}^j = I_1 + \dots + I_9 + \text{Remainder}$$

where

$$\begin{aligned}
I_1 &= R_{x_{k_1} x_{k_2} \dots x_{k_\kappa}}^j, \\
I_2 &= \sum_{i_1} \sum_{l_1} \left[ \sum_{\sigma \in \mathfrak{S}_\kappa^1} \delta_{k_{\sigma(1)}}^{l_1} R_{x_{k_{\sigma(2)}} \dots x_{k_{\sigma(\kappa)}}}^j u^{i_1} - \delta_{i_1}^j Q_{x_{k_1} \dots x_{k_\kappa}}^{l_1} \right] U_{l_1}^{i_1}, \\
I_3 &= \sum_{i_1} \sum_{l_1, l_2} \left[ \sum_{\sigma \in \mathfrak{S}_\kappa^2} \delta_{k_{\sigma(1)}, k_{\sigma(2)}}^{l_1, l_2} R_{x_{k_{\sigma(3)}} \dots x_{k_{\sigma(\kappa)}}}^j u^{i_1} \right. \\
&\quad \left. - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^1} \delta_{k_{\sigma(1)}}^{l_1} Q_{x_{k_{\sigma(2)}} \dots x_{k_{\sigma(\kappa)}}}^{l_2} \right) \right] U_{l_1, l_2}^{i_1}, \\
I_4 &= \sum_{i_1} \sum_{l_1, \dots, l_{\kappa-2}} \left[ \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-2}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-2)}}^{l_1, \dots, l_{\kappa-2}} R_{x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^j u^{i_1} \right. \\
&\quad \left. - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-3}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-3)}}^{l_1, \dots, l_{\kappa-3}} Q_{x_{k_{\sigma(\kappa-2)}} x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^{l_{\kappa-2}} \right) \right] U_{l_1, \dots, l_{\kappa-2}}^{i_1},
\end{aligned}$$

$$\begin{aligned}
I_5 &= \sum_{i_1} \sum_{l_1, \dots, l_{\kappa-1}} \left[ \sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l_1, \dots, l_{\kappa-1}} R_{x_{k_{\sigma(\kappa)}}}^j u^{i_1} \right. \\
&\quad \left. - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-2}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-2)}}^{l_1, \dots, l_{\kappa-2}} x_{k_{\sigma(\kappa-1)}}^{l_{\kappa-1}} x_{k_{\sigma(\kappa)}} \right) \right] U_{l_1, \dots, l_{\kappa-1}}^{i_1}, \\
I_6 &= \sum_{i_1, i_2} \sum_{l_1, \dots, l_{\kappa}} \left[ \sum_{\tau \in \mathfrak{C}_{\kappa}} \delta_{k_1, \dots, k_{\kappa}}^{l_{\tau(1)}, \dots, l_{\tau(\kappa)}} R_{u^{i_1} u^{i_2}}^j \right. \\
&\quad - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l_1, \dots, l_{\kappa}} Q_{x_{k_{\sigma(\kappa)}}}^{l_1} u^{i_2} \right) \\
&\quad - \delta_{i_2}^j \left( \sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \left( \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l_1, \dots, l_{\kappa-1}} Q_{x_{k_{\sigma(\kappa)}}}^{l_{\kappa}} u^{i_1} \right. \right. \\
&\quad \left. \left. + \dots + \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l_3, \dots, l_1} Q_{x_{k_{\sigma(\kappa)}}}^{l_2} u^{i_2} \right) \right) \left. \right] \times U_{l_1}^{i_1} U_{l_2, \dots, l_{\kappa}}^{i_2}, \\
I_7 &= \sum_{i_1, i_2} \sum_{l_3, \dots, l_{\kappa+1}} \left[ -\delta_{i_1}^j \left( \delta_{k_1, \dots, k_{\kappa}}^{l_2, \dots, l_{\kappa+1}} Q_{u^{i_2}}^{l_1} + \dots + \delta_{k_1, \dots, k_{\kappa}}^{l_{\kappa+1}, \dots, l_2} Q_{u^{i_2}}^{l_1} \right) \right. \\
&\quad \left. - \delta_{i_2}^j \left( \sum_{\tau \in \mathfrak{S}_{\kappa}^2} \delta_{k_1, \dots, k_{\kappa}}^{l_{\tau(1)}, \dots, l_{\tau(\kappa)}} Q_{u^{i_1}}^{l_{\kappa+1}} \right) \right] U_{l_1, l_2}^{i_1} U_{l_3, \dots, l_{\kappa+1}}^{i_2}, \\
I_8 &= \sum_{i_1} \sum_{l_1, \dots, l_{\kappa}} \left[ \delta_{k_1, \dots, k_{\kappa}}^{l_1, \dots, l_{\kappa}} R_{u^{i_1}}^j \right. \\
&\quad \left. - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l_1, \dots, l_{\kappa-1}} Q_{x_{k_{\sigma(\kappa)}}}^{l_{\kappa}} \right) \right] U_{l_1, \dots, l_{\kappa}}^{i_1}, \\
I_9 &= \sum_{i_1, i_2} \sum_{l_1, \dots, l_{\kappa+1}} \left[ -\delta_{i_1}^j \delta_{k_1, \dots, k_{\kappa}}^{l_2, \dots, l_{\kappa+1}} Q_{u^{i_2}}^{l_1} \right. \\
&\quad \left. - \delta_{i_2}^j \left( \delta_{k_1, \dots, k_{\kappa}}^{l_1, \dots, l_{\kappa}} Q_{u^{i_1}}^{l_{\kappa+1}} + \dots + \delta_{k_1, \dots, k_{\kappa}}^{l_3, \dots, l_1} Q_{u^{i_1}}^{l_2} \right) \right] \times U_{l_1}^{i_1} U_{l_2, \dots, l_{\kappa+1}}^{i_2}
\end{aligned}$$

and where the term *Remainder* denotes the remaining terms in the expansion of  $R_{k_1, k_2, \dots, k_{\kappa}}^j$ .



In  $I_6$  the summation on the upper indices  $(l_1, \dots, l_\kappa)$  gets on all the circular permutations of  $\{1, 2, \dots, \kappa\}$  except the identity. In  $I_7$  the summation gets on all the circular permutations of  $\{2, 3, \dots, \kappa+1\}$ . In  $I_9$  the summation gets on all the circular permutations of  $\{1, 2, \dots, \kappa+1\}$  except the one transforming  $(l_1, l_2, \dots, l_{\kappa+1})$  into  $(l_2, l_3, \dots, l_1)$ . For  $\kappa = 3$ , comparing with (44), we see that the formula remains valid, with the same conventions as in the case  $n = 1$ .

**3.4. Lie criterion and defining equations of  $\mathfrak{Sym}(\mathcal{E})$**

We recall the Lie criterion, presented in Subsection 2.6 (see Theorem 2.71 of [18]):

A vector field  $X$  is an infinitesimal symmetry of the completely integrable system  $(\mathcal{E})$  if and only if its prolongation  $X^{(\kappa)}$  of order  $\kappa$  is tangent to the skeleton  $\Delta_{\mathcal{E}}$  in the jet space  $\mathcal{J}_{n,m}^\kappa$ .

The set of infinitesimal symmetries of  $(\mathcal{E})$  forms a Lie algebra, since we have the relation  $[X, X']^{(\kappa)} = [X^{(\kappa)}, X'^{(\kappa)}]$  (cf. [18]). We will denote by  $\mathfrak{Sym}(\mathcal{E})$  this Lie algebra. The aim of the foregoing Section is to obtain precise bounds on the dimension of the Lie algebra  $\mathfrak{Sym}(\mathcal{E})$  of infinitesimal symmetries of  $(\mathcal{E})$ . For simplicity we start with the case  $n = m = 1$ .

**4. Optimal upper bound on  $\dim_{\mathbb{K}} \mathfrak{Sym}(\mathcal{E})$  when  $n = m = 1$**

**4.1. Defining equations for  $\mathfrak{Sym}(\mathcal{E})$**

Applying the Lie criterion, the tangency condition of  $X^{(\kappa)}$  to  $\Delta_{\mathcal{E}}$  is equivalent to the identity:

$$(49) \quad \mathbf{R}^\kappa - \left[ Q \frac{\partial F}{\partial x} + R \frac{\partial F}{\partial u} + \mathbf{R}^1 \frac{\partial F}{\partial U^1} + \mathbf{R}^2 \frac{\partial F}{\partial U^2} + \dots + \mathbf{R}^{\kappa-1} \frac{\partial F}{\partial U^{\kappa-1}} \right] \equiv 0,$$

on the subvariety  $\Delta_{\mathcal{E}}$ , that is to a formal identity in  $\mathbb{K}\{x, u, U^1, \dots, U^{\kappa-1}\}$ , in which we replace the variable  $U^\kappa$  by  $F(x, u, U^1, \dots, U^{\kappa-1})$  in the two monomials  $U^\kappa$  and  $U^1 U^\kappa$  of  $\mathbf{R}^\kappa$ , cf. Lemma 3. Expanding  $F$  and its partial derivatives in power series of the variables  $(U^1, \dots, U^{\kappa-1})$  with analytic coefficients in  $(x, u)$ , we may rewrite (49) as follows:

$$(50) \quad \sum_{\mu_1, \dots, \mu_{\kappa-1} \geq 0} \left[ \Phi_{\mu_1, \dots, \mu_{\kappa-1}}(x, u, (Q_{x^k u^l})_{k+l \leq \kappa}, (R_{x^k u^l})_{k+l \leq \kappa}) \right] \\ \times (U^1)^{\mu_1} \dots (U^{\kappa-1})^{\mu_{\kappa-1}} \equiv 0,$$

where the expressions

$$(51) \quad \Phi_{\mu_1, \dots, \mu_{\kappa-1}}(x, u, (Q_{x^k u^l})_{k+l \leq \kappa}, (R_{x^k u^l})_{k+l \leq \kappa})$$

are *linear* with respect to the partial derivatives  $((Q_{x^k u^l})_{k+l \leq \kappa}, (R_{x^k u^l})_{k+l \leq \kappa})$ , with analytic coefficients in  $(x, u)$ . By construction these coefficients essentially depend on the expansion of  $F$ . The tangency condition (50) is equivalent to the following infinite linear system of partial differential equations, called *defining equations of  $\mathfrak{S}\eta\mathfrak{m}(\mathcal{E})$* :

$$(52) \quad \Phi_{\mu_1, \dots, \mu_{\kappa-1}}(x, u, (Q_{x^k u^l}(x, u))_{k+l \leq \kappa}, (R_{x^k u^l}(x, u))_{k+l \leq \kappa}) = 0,$$

satisfied by  $(Q(x, u), R(x, u))$ . The Lie method consists in studying the solutions of this linear system of partial differential equations.

## 4.2. Homogeneous system

As mentioned in the introduction, we focus our attention on the case  $\kappa \geq 3$ . Denote by  $(\mathcal{E}_0)$  the homogeneous equation  $u_{x^\kappa} = 0$  of order  $\kappa$ . The general solution  $u = \sum_{l=0}^{\kappa-1} \lambda_l x^l$  consists of polynomials of degree  $\leq \kappa - 1$  and the defining equation (49) reduces to  $\mathbf{R}^\kappa = 0$ . Using the expression (41), expanding (50), (51) and considering only the coefficients of the five monomials  $ct., U^{\kappa-2}, U^{\kappa-1}, U^1 U^{\kappa-1}$  and  $U^2 U^{\kappa-1}$ , we obtain the five following partial differential equations, which are sufficient to determine  $\mathfrak{S}\eta\mathfrak{m}(\mathcal{E}_0)$ :

$$(53) \quad \begin{aligned} R_{x^\kappa} &= 0, \\ R_{x^2 u} - \frac{(\kappa-2)}{3} Q_{x^3} &= 0, \\ R_{xu} - \frac{(\kappa-1)}{2} Q_{x^2} &= 0, \\ R_{u^2} - \kappa Q_{xu} &= 0, \\ Q_u &= 0. \end{aligned}$$

The general solution of this system is evidently:

$$(54) \quad \begin{aligned} Q &= A + Bx + Cx^2, \\ R &= (\kappa-1)Cxu + Du + E^0 + E^1x + \dots + E^{\kappa-1}x^{\kappa-1}, \end{aligned}$$

where the  $(\kappa+4)$  constants  $A, B, C, D, E^0, E^1, \dots, E^{\kappa-1}$  are arbitrary. Computing explicitly the flows of the  $(\kappa+4)$  generators  $\partial/\partial x, x\partial/\partial x, x^2\partial/\partial x + (\kappa-1)xu\partial/\partial u, u\partial/\partial u, \partial/\partial u, x\partial/\partial u, \dots, x^{\kappa-1}\partial/\partial u$ , we check easily that they stabilize the graphs of polynomials of degree  $\leq$

$\kappa - 1$ . Moreover they span a Lie algebra of dimension  $(\kappa + 4)$  and the general form of a Lie symmetry is:

$$(55) \quad (x, u) \mapsto \left( \frac{\alpha_0 + \alpha_1 x}{1 + \varepsilon x}, \frac{\beta u + \gamma_0 + \gamma_1 x + \cdots + \gamma_{\kappa-1} x^{\kappa-1}}{(1 + \varepsilon x)^{\kappa-1}} \right).$$

### 4.3. Nonhomogeneous system

Consider for  $\kappa \geq 3$  the equation (49) after replacing the variable  $U^\kappa$  by  $F$ . Let  $\Phi(U^\lambda)$  denote an arbitrary term of the form  $\phi(x, u) U^\lambda$ , where  $\phi(x, u)$  is an analytic function. We consider the five following terms  $\Phi(\text{ct.})$ ,  $\Phi(U^{\kappa-2})$ ,  $\Phi(U^{\kappa-1})$ ,  $\Phi(U^1 U^{\kappa-1})$  and  $\Phi(U^2 U^{\kappa-1})$ . Since some multiplications of monomials appear in the expression (49), we must be aware of the fact that  $\Phi(U^1 U^{\kappa-1}) \equiv \Phi(U^1) \Phi(U^{\kappa-1})$  and  $\Phi(U^2 U^{\kappa-1}) \equiv \Phi(U^2) \Phi(U^{\kappa-1})$ . Consequently in the expansion of (49) we must take into account the seven types of monomials  $\Phi(\text{ct.})$ ,  $\Phi(U^1)$ ,  $\Phi(U^2)$ ,  $\Phi(U^{\kappa-2})$ ,  $\Phi(U^{\kappa-1})$ ,  $\Phi(U^1 U^{\kappa-1})$  and  $\Phi(U^2 U^{\kappa-1})$ . The  $(\kappa + 1)$  derivatives  $\partial F/\partial x$ ,  $\partial F/\partial u$ ,  $\partial F/\partial U^1, \dots, \partial F/\partial U^{\kappa-1}$  appearing in the brackets of (49), and the term  $F$  appearing in the expression of  $\mathbf{R}^\kappa$  after replacing  $U^\kappa$  by  $F$  (cf. the last two monomials  $U^\kappa$  and  $U^1 U^\kappa$  in (41)) may all contain the seven monomials  $\text{ct.}$ ,  $U^1$ ,  $U^2$ ,  $U^{\kappa-2}$ ,  $U^{\kappa-1}$ ,  $U^1 U^{\kappa-1}$  and  $U^2 U^{\kappa-1}$ . For  $F$  and its  $(\kappa + 1)$  first derivatives we use the generic simplified notation

$$(56) \quad \Phi(\text{ct.}) + \Phi(U^1) + \Phi(U^2) + \Phi(U^{\kappa-2}) + \Phi(U^{\kappa-1}) + \Phi(U^1 U^{\kappa-1}) + \Phi(U^2 U^{\kappa-1}),$$

to name the seven monomials appearing a priori. Hence, expanding (49), picking up the only terms which may contain the five monomials we are interested in, and using the formula of Lemma 3 for  $\mathbf{R}^\lambda$  ( $1 \leq \lambda \leq \kappa$ ), we obtain the following expression:

$$(57) \quad \begin{aligned} & R_{x^\kappa} + [C_\kappa^2 R_{x^2 u} - C_\kappa^3 Q_{x^3}] U^{\kappa-2} + [C_\kappa^1 R_{xu} - C_\kappa^2 Q_{x^2}] U^{\kappa-1} \\ & + [C_\kappa^1 R_{u^2} - \kappa^2 Q_{xu}] U^1 U^{\kappa-1} + [-C_{\kappa+1}^2 Q_u] U^2 U^{\kappa-1} \\ & + \{R_u - C_\kappa^1 Q_x + [-C_{\kappa+1}^1 Q_u] U^1\} \\ & \times \{\Phi(\text{ct.}) + \Phi(U^1) + \Phi(U^2) + \Phi(U^{\kappa-2}) + \Phi(U^{\kappa-1}) + \Phi(U^1 U^{\kappa-1}) \\ & + \Phi(U^2 U^{\kappa-1})\} \end{aligned}$$

$$\begin{aligned}
& - \{Q + R + R_x + [R_u - Q_x] U^1 + R_{x^2} + [2R_{xu} - Q_{x^2}] U^1 \\
& + [R_u - 2Q_x] U^2 + \dots + R_{x^{\kappa-3}} + [C_{\kappa-3}^1 R_{x^{\kappa-4}u} - Q_{x^{\kappa-3}}] U^1 \\
& + [C_{\kappa-3}^2 R_{x^{\kappa-5}u} - C_{\kappa-3}^1 Q_{x^{\kappa-4}}] U^2 + R_{x^{\kappa-2}} \\
& + [C_{\kappa-2}^1 R_{x^{\kappa-3}u} - Q_{x^{\kappa-2}}] U^1 + [C_{\kappa-2}^2 R_{x^{\kappa-4}u} - C_{\kappa-2}^1 Q_{x^{\kappa-3}}] U^2 \\
& + [R_u - C_{\kappa-2}^1 Q_x] U^{\kappa-2} + [-C_{\kappa-1}^1 Q_u] U^1 U^{\kappa-2} \\
& + R_{x^{\kappa-1}} + [C_{\kappa-1}^1 R_{x^{\kappa-2}u} - Q_{x^{\kappa-1}}] U^1 \\
& + [C_{\kappa-1}^2 R_{x^{\kappa-3}u} - C_{\kappa-1}^1 Q_{x^{\kappa-2}}] U^2 + [C_{\kappa-1}^1 R_{xu} - C_{\kappa-1}^2 Q_{x^2}] U^{\kappa-2} \\
& + [C_{\kappa-1}^1 R_{u^2} - (\kappa-1)^2 Q_{xu}] U^1 U^{\kappa-2} + [R_u - C_{\kappa-1}^1 Q_x] U^{\kappa-1} \\
& + [-C_{\kappa}^1 Q_u] U^1 U^{\kappa-1} \} \\
& \times \{ \Phi(\text{ct.}) + \Phi(U^1) + \Phi(U^2) + \Phi(U^{\kappa-2}) + \Phi(U^{\kappa-1}) + \Phi(U^1 U^{\kappa-1}) \\
& + \Phi(U^2 U^{\kappa-1}) \} \\
& + \text{Remainder} \equiv 0.
\end{aligned}$$

Here the term **Remainder** consists of the monomials, in the jet variables, different from the five ones we are concerned with. The first four lines before the sign “-” develop  $\mathbf{R}^\kappa$  and the fourth line consists of the factor  $F$  replaced by (56). In the last line (note that this is multiplied by the nine preceding lines) we replaced the  $(\kappa+1)$  first partial derivatives of  $F$  appearing in (49) by the term (56) which we factorized.

By expanding the product appearing in this expression (57), and equating to zero the coefficients of the five monomials  $\text{ct.}$ ,  $U^{\kappa-2}$ ,  $U^{\kappa-1}$ ,  $U^1 U^{\kappa-1}$  and  $U^2 U^{\kappa-1}$ , we obtain the five following partial differential equations

(58)

$$\begin{aligned}
R_{x^\kappa} &= \Pi(x, u, Q, Q_x, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\
C_\kappa^2 R_{x^2u} - C_\kappa^3 Q_{x^3} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u, R_{xu}), \\
C_\kappa^1 R_{xu} - C_\kappa^2 Q_{x^2} &= \Pi(x, u, Q, Q_x, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\
C_\kappa^1 R_{u^2} - \kappa^2 Q_{xu} &= \Pi(x, u, Q, Q_x, \dots, Q_{x^{\kappa-1}}, Q_u, R, R_x, \dots, R_{x^{\kappa-1}}, \\
& \quad R_u, R_{xu}, \dots, R_{x^{\kappa-2}u}), \\
-C_{\kappa+2}^2 Q_u &= \Pi(x, u, Q, Q_x, \dots, Q_{x^{\kappa-2}}, R, R_x, \dots, R_{x^{\kappa-1}}, \\
& \quad R_u, R_{xu}, \dots, R_{x^{\kappa-3}u}).
\end{aligned}$$

Here by convention  $\Pi$  denotes any linear quantity in  $Q$ ,  $R$  and some of their derivatives, of the form

$$(59) \quad \begin{aligned} & \Pi(x, u, Q_{x^{a_1}u^{b_1}}, \dots, Q_{x^{a_p}u^{b_p}}, R_{x^{c_1}u^{d_1}}, \dots, R_{x^{c_q}u^{d_q}}) \\ &= \sum_{i=1}^p \phi_i(x, u) Q_{x^{a_i}u^{b_i}}(x, u) + \sum_{j=1}^q \psi_j(x, u) R_{x^{c_j}u^{d_j}}(x, u), \end{aligned}$$

where  $\phi_i$  and  $\psi_j$  are analytic in  $(x, u)$ . For instance, the differentiation of  $\Pi(x, u, Q, R, R_u)$  with respect to  $x$  gives the expression  $\Pi(x, u, Q, Q_x, R, R_x, R_{xu})$ . Let us introduce the following collection of  $(\kappa + 4)$  partial derivatives of  $(Q, R)$  defined by  $J := (Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u)$ . The aim is now to make linear substitutions on the system (58) to obtain the system (68) where the five second members depend only on the collection  $J$ . The desired estimate  $\dim_{\mathbb{K}} \mathfrak{Sym}(\mathcal{E}) \leq \kappa + 4$  will follow from (68).

Let us differentiate the third equation of (58) with respect to  $x$ . Dividing by  $C_{\kappa}^1$  we obtain:

$$(60) \quad R_{x^2u} - \frac{(\kappa - 1)}{2} Q_{x^3} = \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa}}, R_u, R_{xu}).$$

Solving  $R_{x^2u}$  and  $Q_{x^3}$  by the second equality in (58) and by (4.3) we find

$$(61) \quad \begin{aligned} Q_{x^3} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa}}, R_u, R_{xu}), \\ R_{x^2u} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa}}, R_u, R_{xu}). \end{aligned}$$

Replacing  $R_{x^{\kappa}}$  by its value given by the first equality in (58) we obtain for  $Q_{x^3}$ :

$$(62) \quad Q_{x^3} = \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u, R_{xu}).$$

If we write the third equality in (58) as

$$(63) \quad R_{xu} = \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u),$$

we may replace  $R_{xu}$  in (62). This gives the desired dependence of  $Q_{x^3}$  on the collection  $J$ :

$$(64) \quad Q_{x^3} = \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u).$$

We may now differentiate the equalities (63) and (64) with respect to  $x$  up to the order  $l$ . At each differentiation we replace  $Q_{x^3}$ ,  $R_{xu}$  and  $R_{x^{\kappa}}$  by their values in (63), in (64) and in the first equality in (58) respectively. We obtain for  $l \in \mathbb{N}$ :

$$(65) \quad \begin{aligned} Q_{x^{l+3}} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\ R_{x^{l+1}u} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u). \end{aligned}$$

Replacing these values in the fifth equality of (58), we obtain

$$(66) \quad Q_u = \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u).$$

By replacing the fourth equality of (58) we obtain finally

$$(67) \quad R_{u^2} = \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u).$$

To summarize, using the first equality of (58), using (66), (67), (63) and (64), we obtained the desired system:

$$(68) \quad \begin{aligned} R_{x^\kappa} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\ Q_u &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\ R_{u^2} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\ R_{xu} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u), \\ Q_{x^3} &= \Pi(x, u, Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u). \end{aligned}$$

We recall that the terms  $\Pi$  are linear expressions of the form (59). Let us differentiate every equation of system (68) with respect to  $x$  at an arbitrary order and let us replace in the right hand side the terms  $R_{x^\kappa}$ ,  $R_{xu}$  and  $Q_{x^3}$  that may appear at each step by their value in (68), and then differentiate with respect to  $u$  at an arbitrary order. We deduce that all the partial derivatives of the five functions  $R_{x^\kappa}$ ,  $Q_u$ ,  $R_{u^2}$ ,  $R_{xu}$  and  $Q_{x^3}$  are also linear functions of the  $(\kappa + 4)$  partial derivatives  $(Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u)$ . Thus the analytic functions  $Q$  and  $R$  are determined uniquely by the value at the origin of the  $(\kappa + 4)$  partial derivatives  $(Q, Q_x, Q_{x^2}, R, R_x, \dots, R_{x^{\kappa-1}}, R_u)$ . This ends the proof of the inequality  $\dim_{\mathbb{K}} \mathfrak{S}\eta\mathfrak{m}(\mathcal{E}) \leq \kappa + 4$ .  $\square$

## 5. Optimal upper bound on $\dim_{\mathbb{K}} \mathfrak{S}\eta\mathfrak{m}(\mathcal{E})$ in the general dimensional case

### 5.1. Defining equations for $\mathfrak{S}\eta\mathfrak{m}(\mathcal{E})$

In the general dimensional case, the tangency condition of the prolongation  $X^\kappa$  of  $X$  to the skeleton gives the following equations for  $j = 1, \dots, m$  and  $k_1, \dots, k_\kappa = 1, \dots, n$ :

$$(69) \quad \begin{aligned} \mathbf{R}_{k_1, \dots, k_\kappa}^j &- \left[ \sum_{l=1}^n Q^l \frac{\partial F_{k_1, \dots, k_\kappa}^j}{\partial x_l} + \sum_{i=1}^m R^i \frac{\partial F_{k_1, \dots, k_\kappa}^j}{\partial u^i} \right. \\ &\left. + \sum_{i_1} \sum_{l_1} \mathbf{R}_{l_1}^{i_1} \frac{\partial F_{k_1, \dots, k_\kappa}^j}{\partial U_{l_1}^{i_1}} + \dots + \sum_{i_1} \sum_{l_1, \dots, l_{\kappa-1}} \mathbf{R}_{l_1, \dots, l_{\kappa-1}}^{i_1} \frac{\partial F_{k_1, \dots, k_\kappa}^j}{\partial U_{l_1, \dots, l_{\kappa-1}}^{i_1}} \right] \equiv 0, \end{aligned}$$

on  $\Delta_{\mathcal{E}}$ , by replacing the variables  $U_{l_1, \dots, l_{\kappa}}^{i_1}$  by  $F_{l_1, \dots, l_{\kappa}}^{i_1}$  wherever they appear. Let us expand  $F_{k_1, \dots, k_{\kappa}}^j$  and their partial derivatives and use the fact that  $\mathbf{R}_{k_1, \dots, k_{\lambda}}^j$  are polynomial expressions of the jets variables  $(U_{l_1}^{i_1}, \dots, U_{l_1, \dots, l_{\lambda}}^{i_1})$ , with coefficients being linear expressions of the partial derivatives of order  $\leq \lambda + 1$  of  $Q^l$  and  $R^j$ . We obtain for  $j = 1, \dots, m$  and  $k_1, \dots, k_{\kappa} = 1, \dots, n$  some identities of the form

$$(70) \quad \sum_{i_1, \dots, l_1, \dots} \Phi_{k_1, \dots, k_{\kappa}; l_1, \dots}^{j; i_1, \dots} (x, u, (Q_{x^{\alpha} u^{\beta}}^l)_{1 \leq l \leq n, |\alpha| + |\beta| \leq \kappa + 1}, (R_{x^{\alpha} u^{\beta}}^j)_{1 \leq j \leq m, |\alpha| + |\beta| \leq \kappa + 1}) \\ \times U_{l_1}^{i_1} \dots U_{l_{\mu_1}}^{i_{\mu_1}} \times U_{l_{\mu_1} + 1, l_{\mu_1} + 2}^{i_{\mu_1} + 1} \dots U_{l_{\mu_1} + 2\mu_2 - 1}^{i_{\mu_1} + \mu_2 - 1} U_{l_{\mu_1} + 2\mu_2}^{i_{\mu_1} + \mu_2} \times \dots \equiv 0,$$

satisfied if and only if the functions  $Q^l$  and  $R^j$  are solutions of the following system of partial differential equations

$$(71) \quad \Phi_{k_1, \dots, k_{\kappa}; l_1, \dots}^{j; i_1, \dots} (x, u, (Q_{x^{\alpha} u^{\beta}}^l)_{1 \leq l \leq n, |\alpha| + |\beta| \leq \kappa + 1}, (R_{x^{\alpha} u^{\beta}}^j)_{1 \leq j \leq m, |\alpha| + |\beta| \leq \kappa + 1}) = 0.$$

## 5.2. Homogeneous system

We start by giving the general form of the symmetries of the homogeneous system in the case  $\kappa = 2$ . Then we prove the equality  $\dim_{\mathbb{K}}(\mathfrak{Sym}(\mathcal{E}_0)) = n^2 + 2n + m^2 + m C_{n+\kappa-1}^{\kappa-1}$  in the case  $\kappa \geq 3$ .

In the case  $\kappa = 2$  we obtain:

$$(72) \quad Q^l(x, u) = A^l + \sum_{k_1=1}^n B_{k_1}^l x_{k_1} + \sum_{i_1=1}^m C_{i_1}^l u^{i_1} \\ + \sum_{k_1=1}^n D_{k_1} x_l x_{k_1} + \sum_{i_1=1}^m E_{i_1} x_l u^{i_1}, \\ R^j(x, u) = F^j + \sum_{k_1=1}^n G_{k_1}^j x_{k_1} + \sum_{i_1=1}^m H_{i_1}^j u^{i_1} \\ + \sum_{k_1=1}^n D_{k_1} x_{k_1} u^j + \sum_{i_1=1}^m E_{i_1} u^{i_1} u^j.$$

Here the  $(n+m)(n+m+2)$  constants  $A^l, B_{k_1}^l, C_{i_1}^l, D_{k_1}, E_{i_1}, F^j, G_{k_1}^j, H_{i_1}^j \in \mathbb{K}$  are arbitrary. Moreover one can check that the vector space

spanned by the  $(n + m)(n + m + 2)$  vector fields

$$(73) \quad \begin{aligned} & \frac{\partial}{\partial x_{k_1}}, \quad x_{k_1} \frac{\partial}{\partial x_{k_2}}, \quad u^{i_1} \frac{\partial}{\partial x_{k_1}}, \\ & x_{k_1} \left( x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} + u^1 \frac{\partial}{\partial u^1} + \cdots + u^m \frac{\partial}{\partial u^m} \right), \\ & u^{i_1} \left( x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} + u^1 \frac{\partial}{\partial u^1} + \cdots + u^m \frac{\partial}{\partial u^m} \right), \\ & \frac{\partial}{\partial u^{i_1}}, \quad x_{k_1} \frac{\partial}{\partial u^{i_1}}, \quad u^{i_1} \frac{\partial}{\partial u^{i_2}} \end{aligned}$$

is stable under the Lie bracket action and that the flow of each of these generators is a Lie symmetry of the system  $(\mathcal{E}_0)$ . This proves that  $\mathfrak{Sym}(\mathcal{E}_0)$  is indeed a Lie algebra with dimension  $(n + m)(n + m + 2)$ . Finally the corresponding transformations close to the identity mapping are projective, represented by the formula:

$$(74) \quad (x, u) \mapsto \left( \left( \frac{\alpha_{l,0} + \sum_{k=1}^n \alpha_{l,k} x_k + \sum_{i=1}^m \alpha_{l,n+i} u^i}{1 + \sum_{k=1}^n \gamma_k x_k + \sum_{i=1}^m \gamma_{n+i} u^i} \right)_{1 \leq l \leq n}, \right. \\ \left. \left( \frac{\beta_{j,0} + \sum_{k=1}^n \beta_{j,k} x_k + \sum_{i=1}^m \beta_{j,n+i} u^i}{1 + \sum_{k=1}^n \gamma_k x_k + \sum_{i=1}^m \gamma_{n+i} u^i} \right)_{1 \leq j \leq m} \right).$$

It is clear that these transformations preserve all the solutions of  $(\mathcal{E}_0)$  :  $u_{x_{k_1} x_{k_2}}^j = 0$ , the graphs of affine maps from  $\mathbb{K}^n$  to  $\mathbb{K}^m$ .

In the case  $\kappa \geq 3$  we consider the homogeneous system  $(\mathcal{E}_0)$  in which the second members  $F_{k_1, \dots, k_\kappa}^j$  vanish identically. Its solutions are the graphs of polynomial maps of degree  $\leq (\kappa - 1)$  from  $\mathbb{K}^n$  to  $\mathbb{K}^m$ . The defining equations of its Lie algebra of infinitesimal symmetries are  $\mathbf{R}_{k_1, \dots, k_\kappa}^j = 0$ , after having replaced the variables  $U_{l_1, \dots, l_\kappa}^{i_1}$  by  $0 = F_{l_1, \dots, l_\kappa}^{i_1}$  in  $I_8$  and  $I_9$  in (48). We will keep in this system the only equations coming from the vanishing of the coefficients of the five families of monomials *ct.*,  $U_{l_1, \dots, l_{\kappa-2}}^{i_1}$ ,  $U_{l_1, \dots, l_{\kappa-1}}^{i_1}$ ,  $U_{l_1}^{i_1} U_{l_2, \dots, l_\kappa}^{i_2}$  and  $U_{l_1, l_2}^{i_1} U_{l_3, \dots, l_{\kappa+1}}^{i_2}$  (this is inspired from the computations in Subsection 4.2). The coefficients of these five monomials families already appear in the expression (48). Moreover we fix  $l_1 = l_2 = \cdots = l_{\kappa+1} = l$  and  $i_1 = i_2$ , except for the coefficient of the monomial  $U_l^{i_1} U_{l, \dots, l}^{i_2}$ , where we fix first  $i_1 = i_2$  and then  $i_1 \neq i_2$ . This



provides the six partial differential linear equations:

$$\begin{aligned}
(75) \quad & 0 = R_{x_{k_1} x_{k_2} \dots x_{k_\kappa}}^j, \\
& 0 = \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-2}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-2)}}^{l, \dots, l} R_{x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^j u^{i_1} \\
& \quad - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-3}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-3)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa-2)}} x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^l \right), \\
& 0 = \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} R_{x_{k_{\sigma(\kappa)}}}^j u^{i_1} \\
& \quad - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-2}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-2)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^l \right), \\
& 0 = \kappa \delta_{k_1, \dots, k_\kappa}^{l, \dots, l} R_{u^{i_1} u^{i_1}}^j - \kappa \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa)}}}^l u^{i_1} \right), \\
& 0 = 2\kappa \delta_{k_1, \dots, k_\kappa}^{l, \dots, l} R_{u^{i_1} u^{i_2}}^j - \kappa \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa)}}}^l u^{i_2} \right) \\
& \quad - \kappa \delta_{i_2}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa)}}}^l u^{i_1} \right), \quad i_1 \neq i_2, \\
& 0 = -C_{\kappa+1}^2 \delta_{i_1}^j \delta_{k_1, \dots, k_\kappa}^{l, \dots, l} Q_{u^{i_1}}^l.
\end{aligned}$$

To solve the system (75) we fix the indices  $k_1 = \dots = k_\kappa = l$  and  $j = i_1$  in the sixth equation, implying  $Q_{u^{i_1}}^l = 0$ . Hence the terms following  $\delta_{i_1}^j$  and  $\delta_{i_2}^j$  in the fourth and in the fifth equations vanish identically. Let us choose the indices  $k_1 = \dots = k_\kappa$  in the fourth and the fifth equations (this last equation is satisfied only for  $i_1 \neq i_2$ ). We obtain first three simple equations, without any restriction on the indices:

$$\begin{aligned}
(76) \quad & 0 = R_{x_{k_1} x_{k_2} \dots x_{k_\kappa}}^j, \\
& 0 = Q_{u^{i_1}}^l, \\
& 0 = R_{u^{i_1} u^{i_2}}^j.
\end{aligned}$$

Finally we specify the indices in the third equation of (75) as follows:  $l = k_\kappa = \dots = k_3 = k_2 = k_1$ ; then  $l = k_\kappa = \dots = k_3 = k_2 \neq k_1$ ; finally

$l = k_\kappa = \dots = k_3$ ,  $k_3 \neq k_2$ ,  $k_3 \neq k_1$ . This gives the three following equations:

$$(77) \quad \begin{aligned} 0 &= C_\kappa^1 R_{x_{k_1} u^{i_1}}^j - C_\kappa^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_1}}^{k_1}, \\ 0 &= R_{x_{k_1} u^{i_1}}^j - C_{\kappa-1}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\ 0 &= -\delta_{i_1}^j Q_{x_{k_1} x_{k_2}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2. \end{aligned}$$

We specify the indices in the second equation of (75) as follows:  $l = k_\kappa = \dots = k_3 = k_2 = k_1$ ; then  $l = k_\kappa = \dots = k_3 = k_2 \neq k_1$ ; then  $l = k_\kappa = \dots = k_3$ ,  $k_3 \neq k_2$ ,  $k_3 \neq k_1$ ; finally  $l = k_\kappa = \dots = k_4$ ,  $l \neq k_1$ ,  $l \neq k_2$ ,  $l \neq k_3$ . This gives the four following equalities:

$$(78) \quad \begin{aligned} 0 &= C_\kappa^2 R_{x_{k_1} x_{k_1} u^{i_1}}^j - C_\kappa^3 \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\ 0 &= C_{\kappa-1}^1 R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-1}^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\ 0 &= R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-2}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2, \\ 0 &= -\delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^l, \quad l \neq k_1, \quad l \neq k_2, \quad l \neq k_3. \end{aligned}$$

Let us differentiate now the equations (77) with respect to the variables  $x_l$  as follows: we differentiate (77)<sub>1</sub> with respect to  $x_{k_1}$ ; then we differentiate (77)<sub>2</sub> with respect to  $x_{k_2}$ ; finally we differentiate (77)<sub>3</sub> with respect to  $x_{k_3}$ . This gives the three following equations:

$$(79) \quad \begin{aligned} 0 &= C_\kappa^1 R_{x_{k_1} x_{k_1} u^{i_1}}^j - C_\kappa^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\ 0 &= R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-1}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\ 0 &= -\delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2. \end{aligned}$$

The seven equations given by the systems (78) and (79) may be considered as three systems of two equations (of two variables) with a nonzero determinant, to which we add the last equation (78)<sub>4</sub>. We get immediately:

$$(80) \quad \begin{aligned} 0 &= R_{x_{k_1} x_{k_1} u^{i_1}}^j = \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\ 0 &= R_{x_{k_1} x_{k_2} u^{i_1}}^j = \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\ 0 &= R_{x_{k_1} x_{k_2} u^{i_1}}^j = \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2, \\ 0 &= \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^l, \quad l \neq k_1, \quad l \neq k_2, \quad l \neq k_3. \end{aligned}$$

It follows from these relations and from the relations  $Q_{u^{i_1}}^l = R_{u^{i_1}u^{i_2}}^j = 0$  obtained in (76) that all the third order partial derivatives of  $Q^l$  vanish identically, this being also satisfied by the third order partial derivatives of  $R^j$  containing at least one partial derivative with respect to  $u^{i_1}$ :

$$(81) \quad \begin{aligned} 0 &= Q_{x_{k_1}x_{k_2}x_{k_3}}^l = Q_{x_{k_1}x_{k_2}u^{i_1}}^l = Q_{x_{k_1}u^{i_1}u^{i_2}}^l = Q_{u^{i_1}u^{i_2}u^{i_3}}^l, \\ 0 &= R_{x_{k_1}x_{k_2}u^{i_1}}^j = R_{x_{k_1}u^{i_1}u^{i_2}}^j = R_{u^{i_1}u^{i_2}u^{i_3}}^j. \end{aligned}$$

It follows from the equations (76) and (81) that all the functions  $Q^l$  are polynomials of degree  $\leq 2$  with respect to the variables  $x_{k_1}$  and all the functions  $R^j$  are a sum of a polynomial of degree  $\leq (\kappa - 1)$  in the variables  $x_{k_1}$  and of monomials of the form  $u^{i_1}$  and  $x_{k_1}u^{i_1}$ . Let us develop now the relations (77) separately for  $j = i_1$  and  $j \neq i_1$ . We obtain the five equations:

$$(82) \quad \begin{aligned} 0 &= C_{\kappa}^1 R_{x_{k_1}u^{i_1}}^{i_1} - C_{\kappa}^2 Q_{x_{k_1}x_{k_1}}^{k_1}, \\ 0 &= C_{\kappa}^1 R_{x_{k_1}u^{i_1}}^j, \quad j \neq i_1, \\ 0 &= R_{x_{k_1}u^{i_1}}^{i_1} - C_{\kappa-1}^1 Q_{x_{k_1}x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\ 0 &= R_{x_{k_1}u^{i_1}}^j, \quad j \neq i_1, \\ 0 &= -Q_{x_{k_1}x_{k_2}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2. \end{aligned}$$

According to the equations (76), (81), (82), we have the following form of the general solution:

$$(83) \quad \begin{aligned} Q^l(x, u) &= A^l + \sum_{k_1=1}^n B_{k_1}^l x_{k_1} + \sum_{k_1=1}^n C_{k_1} x_{k_1} x_l, \\ R^j(x, u) &= \sum_{k_1=1}^n (\kappa - 1) C_{k_1} x_{k_1} u^j + \sum_{i_1=1}^m D_{i_1}^j u^{i_1} + E^{j,0} + \sum_{k_1=1}^n E_{k_1}^{j,1} x_{k_1} \\ &\quad + \cdots + \sum_{1 \leq k_1 \leq \cdots \leq k_{\kappa-1} \leq n} E_{k_1, \dots, k_{\kappa-1}}^{j, \kappa-1} x_{k_1} \cdots x_{k_{\kappa-1}}. \end{aligned}$$

Here the  $n + n^2 + n + m^2 + m C_{n+\kappa-1}^{\kappa-1}$  constants  $A^l, B_{k_1}^l, C_{k_1}, D_{i_1}^j, E^{j,0}, E_{k_1}^{j,1}, \dots, E_{k_1, \dots, k_{\kappa-1}}^{j, \kappa-1} \in \mathbb{K}$  are arbitrary. Moreover one can check that the

vector space spanned by the vector fields

$$(84) \quad \begin{aligned} & \frac{\partial}{\partial x_{k_1}}, \quad x_{k_1} \frac{\partial}{\partial x_{k_2}}, \\ & x_{k_1} \left( x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} + (\kappa - 1) \left( u^1 \frac{\partial}{\partial u^1} + \cdots + u^m \frac{\partial}{\partial u^m} \right) \right), \\ & u^{i_1} \frac{\partial}{\partial u^{i_2}}, \quad \frac{\partial}{\partial u^{i_1}}, \quad x_{k_1} \frac{\partial}{\partial u^{i_1}}, \quad \dots, \quad x_{k_1} \cdots x_{k_{\kappa-1}} \frac{\partial}{\partial u^{i_1}}, \end{aligned}$$

is stable under the Lie bracket action and that the flow of each of these generators is indeed a Lie symmetry of the system  $(\mathcal{E}_0)$ . Finally the Lie symmetries of  $(\mathcal{E}_0)$  have the following form:

$$(85) \quad (x, u) \mapsto \left( \left( \frac{\alpha_{l,0} + \sum_{k=1}^n \alpha_{l,k} x_k}{1 + \sum_{k=1}^n \varepsilon_k x_k} \right)_{1 \leq l \leq n}, \left( \frac{\omega_j}{[1 + \sum_{k=1}^n \varepsilon_k x_k]^{\kappa-1}} \right)_{1 \leq j \leq m} \right),$$

$$\begin{aligned} \text{where } \omega_j = & \sum_{i_1=1}^m \beta_{i_1}^j u_{i_1} + \gamma^{0,j} + \sum_{k_1=1}^n \gamma_{k_1}^{1,j} x_{k_1} \\ & + \cdots + \sum_{k_1 \leq \dots \leq k_{\kappa-1}} \gamma_{k_1, \dots, k_{\kappa-1}}^{\kappa-1, j} x_{k_1} \cdots x_{k_{\kappa-1}}. \end{aligned}$$

We note again that these transformations preserve the solutions of  $(\mathcal{E}_0)$  :  $u_{x_{k_1} \cdots x_{k_\kappa}}^j = 0$ , namely the graphs of polynomial maps of degree  $\leq (\kappa - 1)$  from  $\mathbb{K}^n$  to  $\mathbb{K}^m$ .

### 5.3. Nonhomogeneous system

Let  $\kappa \geq 3$ . Let us expand the defining equations (69) as done in (70). We will write only the coefficients of the five monomial families ct.,  $U_{l_1, \dots, l_{\kappa-2}}^{i_1}$ ,  $U_{l_1, \dots, l_{\kappa-1}}^{i_1}$ ,  $U_{l_1}^{i_1} U_{l_2, \dots, l_\kappa}^{i_2}$  and  $U_{l_1, l_2}^{i_1} U_{l_3, \dots, l_{\kappa+1}}^{i_2}$ . Moreover, we fix always  $l_1 = l_2 = \dots = l_\kappa = l_{\kappa+1} = l$  and  $i_1 = i_2$ , except for the fourth family of monomials where we distinguish the two cases  $i_1 = i_2$  and  $i_1 \neq i_2$ . Thus we obtain six linear equations of partial derivatives, the members on the left side (coming from the expression of  $\mathbf{R}_{k_1, \dots, k_\kappa}^{j_1}$  given by Lemma 4) coincide with the members on the right hand side of (75). Furthermore, the members on the right hand side are exactly the same as those obtained in (58), with more indices! We use the letters  $l', k'_1, \dots, k'_\kappa = 1, \dots, n$  and  $j', i'_1 = 1, \dots, m$  for the indices of the

arguments of the expressions  $\Pi$ , obtaining the six following equations, which generalize the equations (58):

(86)

$$[1]: R_{x_{k_1} x_{k_2} \dots x_{k_\kappa}}^j = \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'} \right),$$

$$[2]: \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-2}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-2)}}^{l, \dots, l} R_{x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^j u^{i_1} \\ - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-3}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-3)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa-2)}} x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^l \right) \\ = \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, Q_{x_{k'_1} x_{k'_2}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'}, R_{x_{k'_1} u^{i'_1}}^{j'} \right),$$

$$[3]: \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} R_{x_{k_{\sigma(\kappa)}}}^j u^{i_1} \\ - \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-2}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-2)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^l \right) \\ = \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'} \right),$$

$$[4]: \kappa \delta_{k_1, \dots, k_\kappa}^{l, \dots, l} R_{u^{i_1} u^{i_2}}^j - \kappa \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa)}}}^l u^{i_1} \right) \\ = \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, \dots, Q_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{l'}, Q_{u^{i'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, \right. \\ \left. R_{u^{i'_1}}^{j'}, R_{x_{k'_1} u^{i'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-2}} u^{i'_1}}^{j'} \right),$$

$$[5]: 2\kappa \delta_{k_1, \dots, k_\kappa}^{l, \dots, l} R_{u^{i_1} u^{i_2}}^j - \kappa \delta_{i_1}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa)}}}^l u^{i_2} \right) \\ - \kappa \delta_{i_2}^j \left( \sum_{\sigma \in \mathfrak{S}_\kappa^{\kappa-1}} \delta_{k_{\sigma(1)}, \dots, k_{\sigma(\kappa-1)}}^{l, \dots, l} Q_{x_{k_{\sigma(\kappa)}}}^l u^{i_1} \right) \\ = \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, \dots, Q_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{l'}, Q_{u^{i'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, \right.$$

$$\begin{aligned}
 & \left. R_{u^{i_1}'}^{j'}, R_{x_{k_1}'}^{j'}, \dots, R_{x_{k_1}' \dots x_{k_{\kappa-2}}'}^{j'} u^{i_1} \right), \quad i_1 \neq i_2, \\
 [6] : & -C_{\kappa+1}^2 \delta_{i_1}^j \delta_{k_1, \dots, k_{\kappa}}^{l, \dots, l} Q_{u^{i_1}}^l \\
 & = \Pi \left( x, u, Q', Q_{x_{k_1}'}', \dots, Q_{x_{k_1}' \dots x_{k_{\kappa-2}}'}', R^{j'}, R_{x_{k_1}'}^{j'}, \dots, R_{x_{k_1}' \dots x_{k_{\kappa-1}}'}^{j'}, \right. \\
 & \quad \left. R_{u^{i_1}'}^{j'}, R_{x_{k_1}'}^{j'} u^{i_1}, \dots, R_{x_{k_1}' \dots x_{k_{\kappa-3}}'}^{j'} u^{i_1} \right).
 \end{aligned}$$

Then we get the following Lemma:

LEMMA 5. *Let  $J$  denote the collection of  $n + n^2 + n + m C_{n+\kappa-1}^{\kappa-1} + m^2$  partial derivatives*

$$(87) \quad J := \left( Q', Q_{x_{k_1}'}', Q_{x_{k_1}' x_{k_1}'}^{k_1'}, R^{j'}, R_{x_{k_1}'}^{j'}, \dots, R_{x_{k_1}' \dots x_{k_{\kappa-1}}'}^{j'}, R_{u^{i_1}'}^{j'} \right).$$

After linear combinations on the system (86) we obtain the following equations:

$$\begin{aligned}
 (88) \quad & \Pi(x, u, J) = R_{x_{k_1}' \dots x_{k_{\kappa}}'}^j, \\
 & \Pi(x, u, J) = Q_{u^{i_1}}^l, \\
 & \Pi(x, u, J) = R_{u^{i_1} u^{i_2}}^j, \\
 & \Pi(x, u, J) = Q_{x_{k_1}' x_{k_2}' x_{k_3}'}^l, \\
 & \Pi(x, u, J) = R_{x_{k_1}'}^j u^{i_1}, \\
 & \Pi(x, u, J) = Q_{x_{k_1}' x_{k_2}'}^{k_1}, \quad k_1 \neq k_2, \\
 & \Pi(x, u, J) = Q_{x_{k_1}' x_{k_2}'}^l, \quad l \neq k_1, \quad l \neq k_2.
 \end{aligned}$$

Moreover all the partial derivatives (with respect to  $x_l$  and  $u^i$ ) up to order three of the coefficients  $Q^l$  and  $R^j$  of the vector field  $X \in \mathfrak{S}\eta\mathfrak{m}(\mathcal{E})$  are of the form  $\Pi(x, u, J)$ . Hence every function  $Q^l$  and  $R^j$  is uniquely determined by the values at the origin of the  $n + n^2 + n + m C_{n+\kappa-1}^{\kappa-1} + m^2$  partial derivatives (87). This implies that  $\dim_{\mathbb{K}} \mathfrak{S}\eta\mathfrak{m}(\mathcal{E}) \leq n^2 + 2n + m^2 + m C_{n+\kappa-1}^{\kappa-1}$ .

*Proof.* Since the second part of Lemma 5 is immediate let us establish only the identities (88). We first specify the indices in the equation (86)<sub>[3]</sub> as follows:  $l = k_{\kappa} = \dots = k_3 = k_2 = k_1$ ; then  $l = k_{\kappa} = \dots = k_3 = k_2 \neq k_1$ ; and finally  $l = k_{\kappa} = \dots = k_3, k_3 \neq k_2, k_3 \neq k_1$ . This gives three equations whose members on the right hand

side are the same as those in the equation (77) and whose members on the left hand side are the same as those in the equation (86)<sub>[3]</sub>:

$$\begin{aligned}
 & \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'} \right) \\
 &= C_{\kappa}^1 R_{x_{k_1} u^{i_1}}^j - C_{\kappa}^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_1}}^{k_1}, \\
 (89) \quad & \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'} \right) \\
 &= R_{x_{k_1} u^{i_1}}^j - C_{\kappa-1}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\
 & \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'} \right) \\
 &= -\delta_{i_1}^j Q_{x_{k_1} x_{k_2}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2.
 \end{aligned}$$

We remark that these three equations (after specialization of  $j = i_1$  or of  $j \neq i_1$  and after some easy linear combinations) provide directly the fifth, sixth and seventh equations of (88). In particular we may replace the values of the partial derivatives  $R_{x_{j'_1} u^{i'_1}}^{j'}$  and  $Q_{x_{k'_1} x_{k'_2}}^{l'}$  with  $k'_1 \neq k'_2$

or  $l' \neq k'_1, l' \neq k'_2$  appearing in the expressions  $\Pi$  of the second member of (86)<sub>[1]</sub> by their values just obtained from the fifth, the sixth and the seventh equations of (88). This gives the first equation of (88).

Then we specify the indices in (86)<sub>[2]</sub> as follows:  $l = k_{\kappa} = \dots = k_3 = k_2 = k_1$ ; then  $l = k_{\kappa} = \dots = k_3 = k_2 \neq k_1$ ; then  $l = k_{\kappa} = \dots = k_3, k_3 \neq k_2, k_3 \neq k_1$ ; and finally  $l = k_{\kappa} = \dots = k_4, l \neq k_1, l \neq k_2, l \neq k_3$ . This gives four equations, whose members on the right hand side are the same as those in (78) and the members on the left hand side are the same as those in (86)<sub>[2]</sub>:

$$\begin{aligned}
 (90) \quad & \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, Q_{x_{k'_1} x_{k'_2}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'}, R_{x_{k'_1} u^{i'_1}}^{j'} \right) \\
 &= C_{\kappa}^2 R_{x_{k_1} x_{k_1} u^{i_1}}^j - C_{\kappa}^3 \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\
 & \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, Q_{x_{k'_1} x_{k'_2}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, R_{u^{i'_1}}^{j'}, R_{x_{k'_1} u^{i'_1}}^{j'} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= C_{\kappa-1}^1 R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-1}^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\
 &\quad \Pi \left( x, u, Q^{l'}, Q_{x_{k_1}'}^{l'}, Q_{x_{k_1} x_{k_2}'}^{l'}, R^{j'}, R_{x_{k_1}'}^{j'}, \dots, R_{x_{k_1} \dots x_{k_{\kappa-1}}'}^{j'}, R_{u^{i_1}'}^{j'}, R_{x_{k_1} u^{i_1}'}^{j'} \right) \\
 &= R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-2}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2, \\
 &\quad \Pi \left( x, u, Q^{l'}, Q_{x_{k_1}'}^{l'}, Q_{x_{k_1} x_{k_2}'}^{l'}, R^{j'}, R_{x_{k_1}'}^{j'}, \dots, R_{x_{k_1} \dots x_{k_{\kappa-1}}'}^{j'}, R_{u^{i_1}'}^{j'}, R_{x_{k_1} u^{i_1}'}^{j'} \right) \\
 &= -\delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^l, \quad l \neq k_1, \quad l \neq k_2, \quad l \neq k_3.
 \end{aligned}$$

Using the fifth, the sixth and the seventh equations of (88) just obtained, we may replace the partial derivatives  $R_{x_{j_1} u^{i_1}}^{j'}$  and  $Q_{x_{k_1} x_{k_2}'}^{l'}$  with  $k_1' \neq k_2'$  or  $l' \neq k_1'$ ,  $l' \neq k_2'$  appearing in the expressions  $\Pi$  of (90), providing four new equations in which the arguments of  $\Pi$  are the desired ones:  $(x, u, J)$ , where  $J$  is defined in (87):

$$\begin{aligned}
 (91) \quad &\Pi(x, u, J) = C_{\kappa}^2 R_{x_{k_1} x_{k_1} u^{i_1}}^j - C_{\kappa}^3 \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\
 &\Pi(x, u, J) = C_{\kappa-1}^1 R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-1}^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\
 &\Pi(x, u, J) = R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-2}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2, \\
 &\Pi(x, u, J) = -\delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^l, \quad l \neq k_1, \quad l \neq k_2, \quad l \neq k_3.
 \end{aligned}$$

Let us differentiate now the equations (89) with respect to the variables  $x_l$  as follows: first we differentiate (89)<sub>1</sub> with respect to  $x_{k_1}$ ; then we differentiate (89)<sub>2</sub> with respect to  $x_{k_2}$ ; finally we differentiate (89)<sub>3</sub> with respect to  $x_{k_3}$ . The arguments in the expressions  $\Pi$  in the equation (89) contain now the terms  $R_{x_{k_1} \dots x_{k_{\kappa}}'}^{j'}$ ; we replace them by their value given in the first equation of (88) already obtained. The arguments also contain the terms  $R_{x_{j_1} u^{i_1}}^{j'}$  and  $Q_{x_{k_1} x_{k_2}'}^{l'}$  with  $k_1' \neq k_2'$  or  $l' \neq k_1'$ ,  $l' \neq k_2'$ . We replace them by their value given by the fifth, the sixth and the seventh equations of (88). We obtain three new equations in which the arguments of the expressions  $\Pi$  are the desired ones:  $(x, u, J)$ , where  $J$  is defined



in (87):

$$\begin{aligned}
 \Pi(x, u, J) &= C_\kappa^1 R_{x_{k_1} x_{k_1} u^{i_1}}^j - C_\kappa^2 \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\
 (92) \quad \Pi(x, u, J) &= R_{x_{k_1} x_{k_2} u^{i_1}}^j - C_{\kappa-1}^1 \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\
 \Pi(x, u, J) &= -\delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2.
 \end{aligned}$$

The seven equations (91) and (92) may be considered as three systems of two linear equations of two variables with a nonzero determinant, the seventh equation being the last equation in (91). We immediately obtain:

$$\begin{aligned}
 \Pi(x, u, J) &= R_{x_{k_1} x_{k_1} u^{i_1}}^j = \delta_{i_1}^j Q_{x_{k_1} x_{k_1} x_{k_1}}^{k_1}, \\
 \Pi(x, u, J) &= R_{x_{k_1} x_{k_2} u^{i_1}}^j = \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_2}}^{k_2}, \quad k_2 \neq k_1, \\
 (93) \quad \Pi(x, u, J) &= R_{x_{k_1} x_{k_2} u^{i_1}}^j = \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^{k_3}, \quad k_3 \neq k_1, \quad k_3 \neq k_2, \\
 \Pi(x, u, J) &= \delta_{i_1}^j Q_{x_{k_1} x_{k_2} x_{k_3}}^l, \quad l \neq k_1, \quad l \neq k_2, \quad l \neq k_3,
 \end{aligned}$$

giving the fourth equation in (88).

It remains now to obtain the second and the third equations in (88). Let us write firstly equation (86)<sub>[6]</sub> with the choice of the indices  $j = i_1$ ,  $l = k_1 = \dots = k_\kappa$ . This gives the equation:

$$\begin{aligned}
 (94) \quad Q_{u^{i_1}}^l &= \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, \dots, Q_{x_{k'_1} \dots x_{k'_{\kappa-2}}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, \right. \\
 &\quad \left. R_{u^{i'_1}}^{j'}, R_{x_{k'_1} u^{i'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-3}} u^{i'_1}}^{j'} \right).
 \end{aligned}$$

We observe first that the differentiation with respect to the variables  $x_l$  of one of the expressions  $\Pi(x, u, J)$  remains an expression  $\Pi(x, u, J)$ . Indeed we see from (87) that there appears, in the partial derivative  $J_{x_l}$ , derivatives  $Q_{x_{k'_1} x_{k'_2}}^{l'}$  with  $k'_1 \neq k'_2$  or  $l' \neq k'_1$ ,  $l' \neq k'_2$ . We may replace them by their value obtained in the sixth and the seventh equations of (88). It also appears some derivatives  $Q_{x_{k'_1} x_{k'_2} x_{k'_3}}^{l'}$  (we replace them by their value obtained in the fourth equation of (88)), some derivatives  $R_{x_{k'_1} \dots x_{k'_\kappa}}^{j'}$  (we replace them by their value obtained in the first equation of (88)) and some derivatives  $R_{x_{k'_1} u^{i'_1}}^{j'}$  (we replace them by their value

obtained in the fifth equation of (88)). Consequently we may write:

$$(95) \quad [\Pi(x, u, J)]_{x_l} = \Pi(x, u, J).$$

It follows that any derivative with respect to  $x_l$  (to any order) of the fourth and the fifth equations of (88) provides expressions of the form  $\Pi(x, u, J)$ . In other words for any integer  $\lambda \geq 3$  and any integer  $\mu \geq 1$  we have

$$(96) \quad \begin{aligned} \Pi(x, u, J) &= Q_{x_{k_1} x_{k_2} x_{k_3} \dots x_{k_\lambda}}^l, \\ \Pi(x, u, J) &= R_{x_{k_1} \dots x_{k_\mu} u^{i_1}}^j. \end{aligned}$$

We may replace then these values in the equation (94), replacing also the derivatives  $Q_{x_{k'_1} x_{k'_2}}^{l'}$  with  $k'_1 \neq k'_2$  or  $l' \neq k'_1, l' \neq k'_2$  by their values obtained in the sixth and the seventh equations of (88). This gives the second equation of (88).

We also remark that by a differentiation with respect to the variables  $x_l$ , the second equation  $Q_{u^{i_1}}^l = \Pi(x, u, J)$  just obtained implies, using (95):

$$(97) \quad \Pi(x, u, J) = Q_{x_{k_1} u^{i_1}}^l.$$

It remains finally to write (86)<sub>[4]</sub> first with the choice of indices  $l = k_1 = \dots = k_\kappa, j = i_1$  then with the choice of indices  $l = k_1 = \dots = k_\kappa, j \neq i_1$ . We also write (86)<sub>[5]</sub> first with the choice of indices  $l = k_1 = \dots = k_\kappa, j = i_2$  then with the choice of indices  $l = k_1 = \dots = k_\kappa, j \neq i_1, j \neq i_2$ . We obtain four new equations:

$$(98) \quad \begin{aligned} &R_{u^{i_1} u^{i_1}}^{i_1} - \kappa Q_{x_{k_1} u^{i_1}}^{k_1} \\ &= \Pi \left( x, u, Q^{l'}, Q_{x_{k'_1}}^{l'}, \dots, Q_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{l'}, Q_{u^{i'_1}}^{l'}, R^{j'}, R_{x_{k'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-1}}}^{j'}, \right. \\ &\quad \left. R_{u^{i'_1}}^{j'}, R_{x_{k'_1} u^{i'_1}}^{j'}, \dots, R_{x_{k'_1} \dots x_{k'_{\kappa-2}} u^{i'_1}}^{j'} \right), \end{aligned}$$

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