



## Symmetries of Schrödinger Operators with Point Interactions

S. ALBEVERIO<sup>1</sup>, L. DĄBROWSKI<sup>2</sup> and P. KURASOV<sup>3</sup>

Fakultät für Mathematik, Universität Bonn, D-53315 Bonn, Germany

<sup>1</sup>Institute of Mathematics, Ruhr-Universität, D-44780 Bochum; SFB 237; BiBoS; Cerfim (Locarno); Acc.Arch, USI (Mendrisio)

<sup>2</sup>SISSA, 34014 Trieste, Italy

<sup>3</sup>Alexander von Humboldt fellow, Institute of Mathematics, Ruhr-Universität, Bochum Department of Mathematics, Stockholm University; Department of Mathematics, Luleå University; Department of Mathematical and Computational Physics, St. Petersburg University.

(Received: 27 January 1998)

**Abstract.** The transformations of all the Schrödinger operators with point interactions in dimension one under space reflection  $P$ , time reversal  $T$  and (Weyl) scaling  $W_\lambda$  are presented. In particular, those operators which are invariant (possibly up to a scale) are selected. Some recent papers on related topics are commented upon.

**Mathematics Subject Classifications (1991):** 34L40, 47A55, 81Q10.

**Key words:** Schrödinger operators, symmetries, extension theory, point interactions, exactly solvable models.

### 1. Introduction

Differential operators with point interactions are used to obtain exactly solvable models in quantum mechanics, in the study of wave propagation in electrodynamics and more generally in some models of theoretical physics [3, 7, 12, 24]. Let us recall the definition of point interactions. Let  $A$  and  $A_1$  be two (different) selfadjoint operators acting in the Hilbert space  $L_2(\mathbb{R}^n)$ ,  $n \in \mathbf{N}$ . Let  $\Gamma$  be a set of points from  $\mathbb{R}^n: \Gamma \subset \mathbb{R}^n$ . Consider the restrictions  $A^0$  and  $A_1^0$  of the operators  $A$  and  $A_1$  to the sets of functions from the domains of the operators  $A$  and  $A_1$ , respectively, and vanishing (at least) in an arbitrary open neighborhood of the set  $\Gamma$ . Then the operator  $A_1$  is a singular perturbation with the support on  $\Gamma$  of the operator  $A$  if and only if the restricted operators  $A^0$  and  $A_1^0$  coincide. If the set  $\Gamma$  contains only one point, then such an operator  $A_1$  is called *a point perturbation* of the operator  $A$ . In other words, the operator  $A_1$  is equal to the operator  $A$  plus a certain point interaction. An interaction defined in this way generally cannot be written explicitly as an operator sum.

We are going to discuss here the point interactions for the Schrödinger operator in dimension one. It is known that the most general family of point interactions for these operators are described by unitary  $2 \times 2$  matrices using the extension theory for symmetric operators. It is astonishing that the truly whole family of point interactions has been investigated very little up to now, although many authors study various lower-dimensional subfamilies, most often a particular three parameter subfamily [8–10, 28]. The four-parameter family of selfadjoint extensions has been first studied for a dense domain in  $U(2)$  in [26], the full family was studied in [2], [19], however without a detailed description of the symmetry properties which is the main subject of this Letter.

We have already mentioned one important problem related to point interactions: how to describe the point interactions explicitly by certain linear operators which are singular and characterized by their singular set  $\Gamma$  so that they can be defined in a generalized sense? Every function with the support separated from the singular set  $\Gamma$  belongs to the kernel of such an operator. Such operators are finite rank singular perturbations for the original operator. The corresponding abstract problem has been studied recently using the extension theory for symmetric operators with finite deficiency indices [5, 6, 16–18, 29].

Let us now concentrate on the problem of one-dimensional Schrödinger operators with point interactions. Let us start our discussion by considering the following family of operators with singular interactions:

$$L_{x_1 x_2} = -\frac{d^2}{dx^2} + x_1 \delta + x_2 \delta^{(1)},$$

where  $\delta$  and  $\delta^{(1)}$  denote the Dirac delta function and its derivative. The term  $x_1 \delta$  occurring in such a singular perturbation of the second derivative operator cannot be defined as an operator in the Hilbert space  $L_2(\mathbf{R})$ . But the corresponding quadratic form  $\langle x_1 \delta \psi, \psi \rangle = x_1 |\psi(0)|^2$  is a form bounded perturbation of the operator  $-(d^2/dx^2)$  with the relative bound zero. Therefore, the operator  $-(d^2/dx^2) + x_1 \delta$  can be defined using the KLMN theorem [14, 25]. The quadratic form corresponding to the derivative of the delta function is not form bounded with the relative bound zero and therefore the operator  $-(d^2/dx^2) + x_2 \delta^{(1)}$  cannot be defined using standard perturbation theory. There were several attempts to define the corresponding selfadjoint operator using certain particular interpretations of the meaning of the interaction terms, leading to different one parameter families of selfadjoint operators [4, 13, 15, 21, 27]. In [19, 20], it was suggested using distribution theory with discontinuous test functions as a proper tool to define such operators in general. According to this approach the domain of a selfadjoint operator corresponding to the formal expression  $-(d^2/dx^2) + x_2 \delta^{(1)}$  should contain functions with a jump discontinuity at the origin. Let  $\psi$  be such a function having a nontrivial jump discontinuity at the origin. Then the product  $\delta^{(1)} \psi$  can be defined only within the framework of the distribution theory with discontinuous test functions. This theory developed in [19] also gives a possibility of describing

the most general four parameter family of singular operators related to the above Schrödinger operators. We comment on this result in Section 4. In particular, we discuss the Schrödinger operator with a  $\delta^{(1)}$  potential from this point of view.

Symmetries are well known for playing an important role in quantum mechanics. As far as symmetric (Hermitian) operators are concerned, they are capable of restricting the nonuniqueness of the selfadjoint extensions and selecting some of them (if there exists some selfadjoint extension which has the required symmetry properties). In that respect, the fundamental symmetries of Schrödinger operators with point interactions have not yet been analyzed in a systematic way. The time reversal symmetry has been discussed in the recent paper [10]. Unfortunately, this paper ignores completely the rich mathematical literature on point interactions in one dimension. In this Letter, we study the full four-parameter family of selfadjoint operators with the point interactions mentioned above and its behavior under fundamental symmetry transformations (at the same time putting the results of [10] in the right perspective by extending the study of symmetries of this particular case, and relating [10] to the mathematical literature).

We first discuss how the whole family of one-dimensional Schrödinger operators with point interactions can be described in terms of boundary conditions. Next, the complete classification of the point interactions with a singular set at the origin (or at a single point) is given. The basic symmetries are investigated in Section 5, where we study the parity, time reversal and scaling transformations. In Section 6, it is shown how the symmetry results can be obtained from the described classification of the point interactions.

## 2. Four Parameter Family of Point Interactions

Consider the second derivative operator  $A = -(d^2/dx^2)$  with the domain  $\text{Dom}(A) = W_2^2(\mathbb{R})$ . Every second derivative operator with the point interaction at the origin coincides with a certain selfadjoint extension of the second derivative operator  $-(d^2/dx^2)$  restricted to the set  $C_0^\infty(\mathbb{R} \setminus \{0\})$  of all  $C_0^\infty(\mathbb{R})$  functions having compact support separated from the origin. The closure of this operator is defined on the domain  $D_0$  of all functions vanishing at the origin together with the first derivative

$$D_0 = \{\psi \in W_2^2(\mathbb{R}) : \psi(0) = \psi'(0) = 0\}.$$

Let us denote the restricted symmetric operator by  $A^0$ . The operator  $A^0$  has deficiency indices (2,2). Let  $\mathfrak{F}\lambda > 0$  (with  $\mathfrak{F}$  denoting the imaginary part), then we choose the orthogonal basis in the deficiency subspace  $\text{Ker}(A^{0*} - \lambda)$  as follows:

$$g_-(\lambda, x) = \Theta(-x) \exp(-ikx), \quad g_+(\lambda, x) = \Theta(x) \exp(ikx), \quad (1)$$

where  $k = \sqrt{\lambda}$ ,  $\mathfrak{F}k > 0$  and  $\Theta$  is the Heaviside function

$$\Theta(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases}$$

The domain of the adjoint operator  $A^{0*}$  coincides with the space  $W_2^2(\mathbb{R} \setminus \{0\})$ . The operator  $A^0$  is closed, therefore every function from the domain of the adjoint operator  $A^{0*}$  possesses the following representation:

$$\psi = \tilde{\psi} + a_+ g_+(\lambda) + a_- g_-(\lambda) + b_+ g_+(\bar{\lambda}) + b_- g_-(\bar{\lambda}), \quad (2)$$

where  $\tilde{\psi} \in D_0$ ,  $a_{\pm}, b_{\pm} \in \mathbb{C}$  and  $\bar{\phantom{x}}$  means complex conjugation.

The deficiency elements (1) have equal norm and are orthogonal. Therefore, the selfadjoint extensions of the operator  $A^0$  can be parametrized by a  $2 \times 2$  unitary matrix using von Neumann formulas. Let

$$V = \begin{pmatrix} v_{++} & v_{+-} \\ v_{-+} & v_{--} \end{pmatrix}$$

be a unitary matrix. Then the restriction of the operator  $A^{0*}$  to the set

$$D_V = \left\{ \psi = \tilde{\psi} + a_+ g_+(\lambda) + a_- g_-(\lambda) + b_+ g_+(\bar{\lambda}) + b_- g_-(\bar{\lambda}) \in \text{Dom}(A^{0*}) : \begin{pmatrix} b_+(\psi) \\ b_-(\psi) \end{pmatrix} = -V \begin{pmatrix} a_+(\psi) \\ a_-(\psi) \end{pmatrix} \right\} \quad (3)$$

is a selfadjoint extension of the operator  $A^0$ . Let us denote this selfadjoint operator by  $A_V$ . In particular, the unitary matrix

$$V = \frac{-1}{\bar{k}} \begin{pmatrix} i\mathfrak{I}k & \Re k \\ \Re k & i\mathfrak{I}k \end{pmatrix}$$

(with  $\Re$  denoting real part) corresponds to the original operator  $A$ .

### 3. Boundary Conditions

An important characterization of the selfadjoint extensions of the operator  $A^0$  is provided by using the boundary conditions at the origin. Consider the boundary form of the adjoint operator  $A^{0*}$  calculated on the functions  $\psi, \varphi \in \text{Dom}(A^{0*})$  given by

$$\begin{aligned} \langle A^{0*} \psi, \varphi \rangle - \langle \psi, A^{0*} \varphi \rangle \\ = \psi'(+0) \overline{\varphi(+0)} - \psi(+0) \overline{\varphi'(+0)} - \psi'(-0) \overline{\varphi(-0)} + \psi(-0) \overline{\varphi'(-0)}, \end{aligned} \quad (4)$$

(with  $\langle \cdot, \cdot \rangle$  meaning the scalar product in  $L_2(\mathbb{R})$ ). The boundary form (4) determines a symplectic structure on the domain of the adjoint operator  $A^{0*}$ . Therefore, the family of selfadjoint extensions of the operator  $A^0$  is isomorphic to the set of

linear subspaces of  $\text{Dom}(A^{0*})$  that are Lagrangian with respect to the symplectic form (4). Every such subspace can be described by a certain boundary condition at the origin. The following Theorem describes the family of such boundary conditions.

**THEOREM 1.** *Every selfadjoint extension of the operator  $A^0$  coincides with the operator  $A^{0*}$ , restricted to the set of functions which satisfy the boundary conditions at the origin of one and only one of the following types*

$$1. \quad \begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \Lambda \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}, \quad (5)$$

where the matrix  $\Lambda$  is equal to

$$\Lambda = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (6)$$

with the real parameters  $\theta \in [0, \pi)$ ,  $a, b, c, d \in \mathbb{R}$  fulfilling the condition  $ad - bc = 1$ ;

$$2. \quad \psi'(+0) = h^+ \psi(+0) \quad \psi'(-0) = h^- \psi(-0), \quad (7)$$

with the parameters  $h^\pm \in \mathbb{R} \cup \{\infty\}$ . If  $h^+ = \infty$ , then the first equation (7) reads  $\psi(+0) = 0$ . Similarly for  $h^- = \infty$ .

*Remark.* The theorem establishes a one-to-one correspondence between the set of selfadjoint extensions of the operator  $A^0$  and the two families (5) and (7) of boundary conditions. The boundary conditions from the first family (5) ('boundary conditions of the first type') are generic and in terms of them those from the second family (7) ('boundary conditions of the second type') can in fact be obtained as limit cases. Only the boundary conditions (7) were used by P. Seba [26] and P. Chernoff and R. Hughes [9], since these conditions link together the two half axes. Our description gives all possible boundary conditions (as it was described in [2] and [19]).

The second family of boundary conditions (7) can be split into four subfamilies described by finite parameters. We combine them together, noting that they define selfadjoint operators which are equal to the orthogonal sum of two second derivative operators on the half axes with certain boundary conditions at the origin. Such extensions of the operator  $A^0$  can be called *separated*. Let us denote the corresponding separated selfadjoint operator by  $A_H^s$ , where  $H = (h^-, h^+) \in (\mathbb{R} \cup \infty)^2$ . In particular, if  $h^\pm = 0$  or  $\infty$ , we have various combinations of Dirichlet and Neumann boundary conditions.

Instead, the boundary conditions (5) of the first type determine operators which do not possess such a decomposition and therefore will be called *nonseparated*. Let us denote the corresponding selfadjoint operator by  $A_\Lambda^a$ .

*Proof of Theorem 1.* Every selfadjoint extension  $A_V$  of the operator  $A^0$  is described by the von Neumann conditions (3)

$$\begin{pmatrix} b_+(\psi) \\ b_-(\psi) \end{pmatrix} = -V \begin{pmatrix} a_+(\psi) \\ a_-(\psi) \end{pmatrix}, \quad (8)$$

where  $V$  is an unitary matrix. The boundary values of the function  $\psi$  on the left and right-hand sides of the origin are given by

$$\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \begin{pmatrix} 1 - v_{++} & -v_{+-} \\ ik + i\bar{k}v_{++} & i\bar{k}v_{+-} \end{pmatrix} \begin{pmatrix} a_+(\psi) \\ a_-(\psi) \end{pmatrix}; \quad (9)$$

$$\begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix} = \begin{pmatrix} -v_{-+} & 1 - v_{--} \\ -i\bar{k}v_{-+} & -ik - i\bar{k}v_{--} \end{pmatrix} \begin{pmatrix} a_+(\psi) \\ a_-(\psi) \end{pmatrix}. \quad (10)$$

Conditions (8) can be written in terms of the boundary values of the function  $\psi$  only. There are two possibilities to write these boundary conditions. If the matrix  $V$  is diagonal, then equality (9) can be simplified

$$\begin{aligned} \psi(+0) &= (1 - v_{++})a_+(\psi) \\ \psi'(+0) &= (ik + i\bar{k}v_{++})a_+(\psi) \end{aligned} \Rightarrow (1 - v_{++})\psi'(+0) = (ik + i\bar{k}v_{++})\psi(+0);$$

Multiplying the latter equation by  $(1 - \bar{v}_{++})$  we get a boundary condition of the same type with the real parameters

$$(1 - \bar{v}_{++})(1 - v_{++})\psi'(+0) = (i(k - \bar{k}) + i\bar{k}v_{++} - ik\bar{v}_{++})\psi(+0). \quad (11)$$

The second condition can be derived in a similar way using (10)

$$\begin{aligned} \psi(-0) &= (1 - v_{--})a_-(\psi) \\ \psi'(-0) &= -(ik + i\bar{k}v_{--})a_-(\psi) \\ &\Rightarrow (1 - v_{--})\psi'(-0) = -(ik + i\bar{k}v_{--})\psi(-0) \\ &\Rightarrow (1 - \bar{v}_{--})(1 - v_{--})\psi'(-0) = (i(k - \bar{k}) + i\bar{k}v_{--} - ik\bar{v}_{--})\psi(-0). \end{aligned} \quad (12)$$

Equations (11, 12) are boundary conditions of the second type (7).

If the matrix  $V$  is not diagonal, then the determinant of the linear system (10) is equal to  $iv_{-+}(k + \bar{k}) \neq 0$ . The second item does not vanish because  $k$  has nonzero real part. It follows that Equation (8) can be written in the form

$$\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \Lambda \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}$$

with the matrix  $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$  satisfying the following conditions:

$$\lambda_{11}\bar{\lambda}_{22} - \lambda_{21}\bar{\lambda}_{12} = 1; \quad \lambda_{11}\bar{\lambda}_{21} \in \mathbb{R}; \quad \lambda_{12}\bar{\lambda}_{22} \in \mathbb{R}.$$

Every such matrix  $\Lambda$  can be written in the form (4) with the real parameters  $\theta, a, b, c, d$ . The boundary form of the operator (4) vanishes on the subset of functions defined by the boundary conditions of the first and second types. It follows that the operator  $A^{0*}$  restricted to the linear set defined by these boundary conditions is selfadjoint and the theorem is proven.  $\square$

In geometric terms, the nonseparated boundary conditions provide a local parameterization of a dense part of  $U(2)$  by  $\theta, a, b, c, d \in \mathbb{R}$ , satisfying  $\theta \in [0, \pi)$  and  $ad - bc = 1$  (altogether there are four independent real coordinates, as  $\dim_{\mathbb{R}} U(2) = 4$ ). The remaining part of  $U(2)$ , which has dimension two and corresponds to separated boundary conditions, is (locally) parametrized by a pair of (real) projective coordinates.

#### 4. Classification of the Point Interactions

The relations between the four-parameter families of the selfadjoint extensions of the operator  $A^0$  and the second derivative operators with singular interactions having support at the origin have already been discussed in [19, 20]. In these references it was suggested using the distribution theory with discontinuous test functions to determine the selfadjoint operator corresponding to the following formally symmetric differential expression

$$L_X = -\frac{d^2}{dx^2}(1 + x_4\delta) + i\frac{d}{dx}(2x_3\delta - ix_4\delta^{(1)}) + x_1\delta + (x_2 - ix_3)\delta^{(1)}, \quad (13)$$

where  $X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ . Let  $u, v \in W_2^2(\mathbb{R})$ , then  $\langle L_X u, v \rangle = \langle u, L_X v \rangle$ . In fact, one has

$$\begin{aligned} L_X u &= -u'' + x_1 u(0)\delta + x_2 (u(0)\delta^{(1)} - u'(0)\delta) + \\ &\quad + ix_3 (u(0)\delta^{(1)} + u'(0)\delta) - x_4 u'(0)\delta^{(1)}, \end{aligned}$$

hence

$$\begin{aligned} \langle L_X u, v \rangle &= \langle u', v' \rangle + x_1 u(0)\overline{v(0)} + x_2 (-u(0)\overline{v'(0)} - u'(0)\overline{v(0)}) \\ &\quad + ix_3 (-u(0)\overline{v'(0)} + u'(0)\overline{v(0)}) + x_4 u'(0)\overline{v'(0)} \\ &= \langle u, L_X v \rangle. \end{aligned}$$

To define a selfadjoint operator with point interactions, one has to extend the latter sesquilinear form to a sesquilinear form on  $W_2^2(\mathbb{R} \setminus \{0\}) \oplus W_2^2(\mathbb{R} \setminus \{0\})$ . Such extensions have been classified in [19, 20] and it was shown that a unique extended

sesquilinear form exists possessing the same scaling properties as the original one. The latter problem has been solved using the following extension of the distribution  $\delta^{(n)}$  to the set of test functions with a jump discontinuity at the origin

$$\delta^{(n)}\psi = (-1)^n \frac{\psi^{(n)}(+0) + \psi^{(n)}(-0)}{2}. \quad (14)$$

It was proven that the selfadjoint operator corresponding to the extended sesquilinear form coincides with the restriction of the operator  $A^{0*}$  to the domain of functions from  $W_2^2(\mathbb{R} \setminus \{0\})$  satisfying the following boundary conditions at the origin

$$\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \begin{pmatrix} \frac{(2+x_2)^2 - x_1x_4 + x_3^2}{(2-ix_3)^2 + x_1x_4 - x_2^2} & \frac{-4x_4}{(2-ix_3)^2 + x_1x_4 - x_2^2} \\ \frac{4x_1}{(2-ix_3)^2 + x_1x_4 - x_2^2} & \frac{(2-x_2)^2 - x_1x_4 + x_3^2}{(2-ix_3)^2 + x_1x_4 - x_2^2} \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix} \quad (15)$$

if  $(2-ix_3)^2 + x_1x_4 - x_2^2 \neq 0$ . The exceptional case  $(2-ix_3)^2 + x_1x_4 - x_2^2 = 0$  and infinite values of the parameters  $x_1, x_2, x_3, x_4$  corresponds to the separated boundary conditions of the second type (see [19] for more details). In [19] it also has been proven that every point interaction at the origin can be described by a certain pseudo-operator with singular interaction.

The following one two parameter and two one-parameter subfamilies of operators from the general four parameter family (13) are described by the matrices  $\Lambda$

- the Schrödinger operator with a ‘generalized potential’

$$L_{x_1x_2} = -\frac{d^2}{dx^2} + x_1\delta + x_2\delta^{(1)} \Rightarrow \Lambda = \begin{pmatrix} \frac{2+x_2}{2-x_2} & 0 \\ \frac{4x_1}{4-x_2^2} & \frac{2-x_2}{2+x_2} \end{pmatrix};$$

- the regularized Schrödinger operator with a ‘singular gauge field’

$$L_{x_3} = -\frac{d^2}{dx^2} + ix_3 \left( 2\frac{d}{dx}\delta - \delta^{(1)} \right) \Rightarrow \Lambda = \begin{pmatrix} \frac{2+ix_3}{2-ix_3} & 0 \\ 0 & \frac{2+ix_3}{2-ix_3} \end{pmatrix}; \quad (16)$$



– the Schrödinger operator with a ‘singular density’

$$L_{x_4} = -\frac{d^2}{dx^2}(1 + x_4\delta) + x_4\frac{d}{dx}\delta^{(1)} \Rightarrow \Lambda = \begin{pmatrix} 1 & -x_4 \\ 0 & 1 \end{pmatrix}. \quad (17)$$

In particular, we see that the selfadjoint operator corresponding to the formal expression

$$L_{x_2} = -\frac{d^2}{dx^2} + x_3\delta^{(1)} \quad (18)$$

is determined by following diagonal matrix

$$\Lambda = \begin{pmatrix} \frac{2+x_2}{2-x_2} & 0 \\ 0 & \frac{2-x_2}{2+x_2} \end{pmatrix}. \quad (19)$$

According to the frequent terminology in the literature by a ‘ $\delta'$ -interaction’ one denotes the interactions determined by boundary conditions of the form (17) (see [3]). However, as it was shown by P. Seba [27], this ‘ $\delta'$ -interaction’ does not describe a Schrödinger operator perturbed by the derivative of a  $\delta$  potential. There were several attempts to define the latter interaction. Even one-dimensional families of nonselfadjoint operators have been suggested (see [4] for references). N. Elander and P. Kurasov [21] and D. Griffiths [15] independently derived the boundary conditions (19) using the extension (14). The selfadjoint operator described by the boundary conditions (19) possesses the same symmetry properties as the original formal expression (see Section 6). Let us remark that such a definition for the  $\delta^{(1)}$  potential interaction has been already used to study several physical problems [11, 22].

## 5. Symmetries

Consider the unitary operators  $P$ ,  $T$  and  $W_\lambda$ , with  $\lambda > 0$ , defined by

$$(P\psi)(x) = \psi(-x), \quad (T\psi)(x) = \bar{\psi}(x), \quad (W_\lambda\psi)(x) = \lambda^{1/2}\psi(\lambda x) \quad (20)$$

on the Hilbert space  $L_2(\mathbb{R}, dx)$ . The corresponding transformation of a selfadjoint operator  $A$  defined by  $U = P, T, W_\lambda$  is given by

$$A \mapsto UAU^{-1}. \quad (21)$$

The operators  $P$  and  $T$  map each of the second derivative operators with point interactions (as listed in Theorem 1) to the operator from the same class with the interaction defined by different parameters

$$A_{H^U}^s = UA_H^sU^{-1}; \quad A_{\Lambda^U}^n = UA_\Lambda^nU^{-1}, \quad (22)$$

where

$$\begin{aligned} H^P &= -H; & H^T &= H; \\ \Lambda^P &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda^T = \bar{\Lambda}. \end{aligned} \quad (23)$$

LEMMA 1. *Let  $\Omega_P$  be the set of selfadjoint extensions of the operator  $A^0$  which are invariant with respect to the symmetry transformation  $P$  i.e.  $B \in \Omega_P \Rightarrow B^P = B$ . Then the family  $\Omega_P^c$  of nonseparated extensions from  $\Omega_P$  is a two-parameter family and it is defined by all matrices  $\Lambda = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$  with the real parameters  $a, b, c \in \mathbb{R}$  fulfilling the condition  $a^2 - bc = 1$ . The family  $\Omega_P^s$  of separated extensions from  $\Omega_P$  is a one-parameter family and is described by all  $H$  such that  $h^+ = -h^-$ .*

*Proof.* Consider the family of nonseparated selfadjoint extensions of the operator  $A^0$ . Every such extension is described by a certain matrix  $\Lambda$  possessing the representation (6). If the extended operator is invariant with respect to the symmetry transformation  $P$ , then the matrices  $\Lambda^P$  and  $\Lambda$  coincide, i.e. the following equality holds:

$$e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-i\theta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The latter equality implies that

$$e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{-i\theta} \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Since the coefficients  $a, b, c, d$  are real and  $\theta \in [0, \pi)$ , the latter equality is satisfied if and only if  $\theta = 0$  and  $a = d$ . The lemma is thus proven for nonseparated extensions. The proof for separated extensions is similar.  $\square$

LEMMA 2. *Let  $\Omega_T$  be the set of selfadjoint extensions of the operator  $A^0$  which are invariant with respect to the symmetry transformation  $T$ , i.e.  $B \in \Omega_T \Rightarrow B^T = B$ . Then the family  $\Omega_T^c$  of nonseparated extensions from  $\Omega_P$  is a three-parameter family and it is defined by all matrices  $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with the real parameters  $a, b, c, d \in \mathbb{R}$  fulfilling the condition  $ad - bc = 1$ . Every separated extension of the operator  $A^0$  is invariant with respect to the symmetry operator  $T$ .*

*Proof.* The proof follows the same lines as the proof of Lemma 1. Considering the nonseparated extensions invariant with respect to the operator  $T$ , we arrive to the following equation for the matrix  $\Lambda$

$$e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{-i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The latter equality is satisfied if and only if  $\theta = 0$ . The parameters  $a, b, c, d$  are arbitrary real numbers constrained only by  $ad - bc = 1$ . The lemma has thus been proven for nonseparated extensions. But every separated extension is also invariant with respect to the symmetry operator  $T$ , since the coefficients  $h^\pm$  are real, which completes the proof of the lemma.  $\square$

The following lemma can be proven using the same method:

**LEMMA 3.** *Let  $\Omega_{PT}$  be the set of selfadjoint extensions of the operator  $A^0$  which are invariant with respect to the (composed) symmetry transformation  $PT$ , i.e.  $B \in \Omega_{PT} \Rightarrow B^{PT} = B$ . Then the family  $\Omega_{PT}^c$  of nonseparated extensions from  $\Omega_{PT}$  is a three-parameter family and it is defined by all matrices  $\Lambda = e^{i\theta} \begin{pmatrix} a & b \\ c & a \end{pmatrix}$  with the real parameters  $\theta \in [0, \infty)$ ;  $a, b, c \in \mathbb{R}$  fulfilling the condition  $a^2 - bc = 1$ . The family  $\Omega_{PT}^s$  of separated extensions from  $\Omega_{PT}$  is a one-parameter family and is described by all  $H$  such that  $h^+ = -h^-$ . It coincides with the family  $\Omega_p^s$ .*

Consider now the scaling transformation  $W_\lambda$  in more details. The unitary transformation (21) maps the second derivative operator  $-(d^2/dx^2)$  to the operator  $-(1/\lambda^2)d^2/dx^2$ . Therefore, the transformed operator can only be proportional to the original operator with point interactions. Again the unitary transformation  $W_\lambda$  does not change the class of boundary conditions

$$\begin{aligned} A_{HW}^s &= \frac{1}{\lambda^2} W_\lambda A_H^s W_\lambda^{-1}; \\ A_{\Lambda W}^n &= \frac{1}{\lambda^2} W_\lambda A_\Lambda^n W_\lambda^{-1}, \end{aligned} \quad (24)$$

where

$$\Lambda^W = \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} \Lambda \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad H^\Lambda = H/\lambda. \quad (25)$$

The following lemma describes the set of boundary conditions that are invariant with respect to the scaling transformation.

**LEMMA 4.** *The family of nonseparated boundary conditions invariant with respect to all the scaling transformations  $W_\lambda, \lambda > 0$  is described by all matrices  $\Lambda$  of the form*

$$\Lambda = e^{i\theta} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad (26)$$

with  $\theta \in [0, \infty)$ ,  $a \in \mathbb{R} \setminus \{0\}$ . The family of separated boundary conditions invariant with respect to the scaling transformation consists of four elements  $(0, 0)$ ,  $(0, \infty)$ ,  $(\infty, 0)$  and  $(\infty, \infty)$ .

*Proof.* Every matrix  $\Lambda$  invariant with respect to the scaling transformation for all  $\lambda > 0$  satisfies the following equation:

$$e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix},$$

which implies that  $b = c = 0$  for  $\lambda \neq 1$ . Thus such a  $\Lambda$  is given by (26).

Consider now separated boundary conditions. The equation  $h^\pm = h^\pm/\lambda, \forall \lambda > 0$ , has only two solutions  $h_\pm = 0, \infty$ . Therefore, all separated invariant boundary conditions are given by  $H = (0, 0), (0, \infty), (\infty, 0), (\infty, \infty)$ . The lemma is proven.  $\square$

## 6. Symmetries and Pseudo-Operators

Let us study the symmetry properties of the point interactions using the classification described in Section 4.

The following two parameter subfamily of differential pseudo-operators (13) is invariant with respect to the inversion  $P$

$$L_{x_1x_4} = -\frac{d^2}{dx^2}(1 + x_4\delta) + i\frac{d}{dx}(-ix_4\delta^{(1)}) + x_1\delta \quad (27)$$

and it is described by the nonseparated boundary conditions with the matrix

$$\Lambda = \begin{pmatrix} \frac{4 - x_1x_4}{4 + x_1x_4} & \frac{-4x_4}{4 + x_1x_4} \\ \frac{4x_1}{4 + x_1x_4} & \frac{4 - x_1x_4}{4 + x_1x_4} \end{pmatrix}.$$

The family of matrices obtained in this way coincides with the family of matrices  $\Lambda$  described by Lemma 1.

The parameter  $x_3$  in (13) describes the singular gauge field with the support at the origin as given by (16). The operator with gauge field is not invariant with respect to the time reversal symmetry. Therefore, the symmetric differential expressions (13) invariant with respect to the complex conjugation  $T$  form the three parameter subfamily

$$L_{x_1x_2x_4} = -\frac{d^2}{dx^2}(1 + x_4\delta) + \frac{d}{dx}(x_4\delta^{(1)}) + x_1\delta + x_2\delta^{(1)} \quad (28)$$

defined by the nonseparated boundary conditions with the matrix

$$\Lambda = \begin{pmatrix} \frac{(2+x_2)^2 - x_1x_4}{4+x_1x_4-x_2^2} & \frac{-4x_4}{4+x_1x_4-x_2^2} \\ \frac{4x_1}{4+x_1x_4-x_2^2} & \frac{(2-x_2)^2 - x_1x_4}{4+x_1x_4-x_2^2} \end{pmatrix}.$$

We get all matrices  $\Lambda$  which describe nonseparated boundary conditions given by Lemma 2.

The two parameter family

$$L_{x_2x_3} = -\frac{d^2}{dx^2} + 2i\frac{d}{dx}x_3\delta + x_2\delta^{(1)} \quad (29)$$

is selfsimilar with respect to the scaling  $W_\lambda$  and the corresponding operator is defined by the nonseparated boundary conditions with the matrix

$$\Lambda = \begin{pmatrix} \frac{(2+x_2)^2 + x_3^2}{(2-ix_3)^2 - x_2^2} & 0 \\ 0 & \frac{(2-x_2)^2 - x_3^2}{(2-ix_3)^2 - x_2^2} \end{pmatrix}.$$

All nonseparated boundary conditions given by Lemma 4 can be obtained in this way. In particular, the Schrödinger operator with the  $\delta^{(1)}$  potential belongs to the family. Therefore, the function from its domain satisfy the boundary conditions with the diagonal matrix  $\Lambda$ .

We have seen that the approach developed using the distribution theory with discontinuous test functions gives a possibility to separate the selfadjoint perturbations possessing certain symmetry properties. This approach can be generalized to include separated extensions. Infinite values of the parameters  $x_1, x_2, x_3, x_4$  should be considered. A more detailed analysis can be carried out using the four-dimensional projective coordinates, see [19].

### Acknowledgement

The financial support by the Alexander von Humboldt Foundation is gratefully acknowledged.

### References

1. Albeverio, S., Brzeźniak, Z. and Dąbrowski, L.: Time dependent propagator for point interactions *J. Phys. A* **27** (1994), 4933–4943.
2. Albeverio, S., Brzeźniak, Z. and Dąbrowski, L.: Fundamental solution of the heat and Schrödinger equation with point interactions, *J. Funct. Anal.* **128** (1995), 220–254.
3. Albeverio, S., Gesztesy, F., Høegh-Krohn, R. and Holden, H.: *Solvable Models in Quantum Mechanics*, Springer-Verlag, Berlin, 1988.

4. Albeverio, S., Gesztesy, F. and Holden, H.: Comment on a recent note on the Schrödinger equation with a  $\delta'$ -interaction, *J. Phys. A: Math. Gen.* **26** (1993), 3903–3904.
5. Albeverio, S. and Kurasov, P.: Rank one perturbations, approximations and selfadjoint extensions, *J. Funct. Anal.* **148** (1997), 152–169.
6. Albeverio, S. and Kurasov, P.: Rank one perturbations of not semibounded operators, *Integral Equations Operator Theory* **27** (1997), 349–400.
7. Berezin, F. A. and Faddeev, L. D.: A remark on Schrödinger equation with a singular potential, *Soviet Math. Dokl.* **2** 1961, 372–375 (*Math. USSR Dokl.*, **137** (1961), 1011–1014).
8. Cheon, T. and Shigehara, T.: Realizing discontinuous wave function with renormalized short-range potentials, quant-ph/9709035.
9. Chernoff, P. and Hughes, R.: A new class of point interactions in one dimension, *J. Funct. Anal.* **111** (1993), 97–117.
10. Coutinho, F. A. B., Nogami, Y. and Fernando Perez, J.: Generalized point interactions in one-dimensional quantum mechanics, *J. Phys. A: Math. Gen.* **30** (1997), 3937–3945.
11. Demkov, Yu. N., Kurasov, P. B. and Ostrovsky, V. N.: Doubly periodical in time and energy exactly soluble system with two interacting systems of states, *J. Phys. A: Math. Gen.* **28** (1995), 4361–4380.
12. Demkov, Yu. N. and Ostrovskii, V. N.: *Zero-range Potentials and their Applications in Atomic Physics*, Leningrad Univ. Press, Leningrad, 1975; English translation: Plenum Press, New York-London, 1988.
13. Gesztesy, F. and Holden, H.: A new class of solvable models in quantum mechanics describing point interactions on the line, *J. Phys. A: Math. Gen.* **20** (1987), 5157.
14. Gesztesy, F. and Kirsch, W.: One-dimensional Schrödinger operators with interactions singular on a discrete set, *J. Reine Angew. Math.* **362** (1985), 28–50.
15. Griffiths, D. J.: Boundary conditions at the derivative of a delta function, *J. Phys. A: Math. Gen.* **26** (1993), 2265–2267.
16. Hassi, S., Langer, H. and de Snoo, H.: Selfadjoint extensions for a class of symmetric operators with defect numbers (1,1), *15th OT Conference Proc.*, 1995, pp. 115–145.
17. Hassi, S. and de Snoo, H.: On rank one perturbations of selfadjoint operators, *Integral Equations Operator Theory* **29** (1997), 288–300.
18. Kiselev, A. and Simon, B.: Rank one perturbations with infinitesimal coupling, *J. Funct. Anal.* **130** (1995), 345–356.
19. Kurasov, P.: Distribution theory for the discontinuous test functions and differential operators with the generalized coefficients, *J. Math. Anal. Appl.* **201** (1996), 297–323.
20. Kurasov, P. and Boman, J.: Finite rank singular perturbations and distributions with discontinuous test functions, accepted for publication in *Proc. Amer. Math. Soc.*
21. Kurasov, P. and Elander, N.: On the  $\delta'$ -potential in one dimension, Preprint MSI 93-7, ISSN-1100-214X (1993).
22. Kurasov, P., Scrinzi, A. and Elander, N.: On the  $\delta'$ -potential arising in exterior complex scaling, *Phys. Rev. A* **49** (1994) 5095–5097.
23. Krein, M. G.: On the resolvents of a Hermitian operator with deficiency indices  $(m, m)$ , *Dokl. Akad. Nauk SSSR* **52** (1946), 657–660 (in Russian).
24. Pavlov, B. S.: The theory of extensions and explicitly-soluble models, *Russian Math. Surveys* **42**(6) (1987), 127–168.
25. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics*, vol. IV, Academic Press, New York, 1975.
26. Seba, P.: The generalized point interaction in one dimension, *Czech. J. Phys. B* **36** (1986), 667–673.
27. Seba, P.: Some remarks on the  $\delta'$ -interaction in one dimension, *Rep. Math. Phys.* **24** (1986), 111–120.

28. Shubin Christ, C. and Stolz, G.: Spectral theory of one-dimensional Schrödinger operators with point interactions, *J. Math. Anal. Appl.* **184** (1994), 491–516.
29. Simon, B.: Spectral analysis of rank one perturbations and applications, in: *CRM Proc. Lectures Notes* 8, Amer. Math. Soc., Providence, 1995, pp. 109–149.