ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

522

SYMMETRIZATION OF CONDENSERS IN *n*-SPACE

BY

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HELSINKI 1972
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1973.522

ISBN 951-41-0063-8

Communicated 12 May 1972 by Jussi Väisälä

KESKUSKIRJAPAINO
HELSINKI 1972

Preface

I wish to express my sincere gratitude to Professor JUSSI VÄISÄLÄ for suggesting this subject and for his kind interest and advice. My thanks are also due to Professor OLLI MARTIO and Dr. JOSEPH HESSE, who read the manuscript and made several valuable suggestions.

This work has been financially supported by a research grant from the Finnish Science Research Council.

Djursholm, April 1972

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Contents

1. Introduction

A pair (A, C) of sets in euclidean n-space R^n, $n \geqq 2$, is called a *condenser* if A is open and C is a compact non-empty subset of A. A condenser (A, C) is called *ringlike* if $A \setminus C = \{x \in A \mid x \notin C\}$ is connected and its complement in $\bar{R}^n$ has exactly two components, where $\bar{R}^n = R^n \cup \{\infty\}$ is the one point compactification of R^n.

A function $u : R^n \to R^1$ is called absolutely continuous on lines, abbreviated ACL, if u is continuous and, if for every closed cube $Q \subset R^n$, u is absolutely continuous on almost all lines in Q parallel to the coordinate axes. An ACL function u has a gradient ∇u almost everywhere in R^n.

For every condenser (A, C) and $p > 0$ we define the *p-capacity* of (A, C) to be the real number

$$\operatorname{cap}_p(A, C) = \inf_{u \in W(A,C)} \int_{R^n} |\nabla u|^p dm, \tag{1.1}$$

where $W(A, C)$ is the set of all ACL functions $u : R^n \to R^1$ such that $0 \leqq u(x) \leqq 1$ for $x \in R^n$, $u(x) = 0$ for $x \in C$, and the closure of $\{x \in R^n \mid 0 \leqq u(x) < 1\}$ is a compact subset of A. For $p = n$, $\operatorname{cap}_n(A, C)$ is the conformal capacity of the condenser (A, C) [6, p. 24].

Symmetrizations are geometric transformations usually defined for open and closed sets in R^n. They were first introduced by J. Steiner and were subsequently studied by others, especially Polya and Szegö [9]. Let Sym be some symmetrization in R^n and let Sym (A) denote the symmetrization of $A \subset R^n$ under Sym. Given a condenser (A, C) and $p > 0$ we will consider the validity of the following *capacity inequality*

$$\operatorname{cap}_p(A, C) \geqq \operatorname{cap}_p(\operatorname{Sym}(A), \operatorname{Sym}(C)), \tag{1.2}$$

whenever (Sym (A), Sym (C)) is also a condenser. This capacity inequality has many important applications in classical potential and function theory and has recently been used in the theory of spatial quasiconformal and quasiregular mappings. For $p = n$ it has been proved in the following cases: Hayman [4] for spherical symmetrizations in R^2, Gehring [2] for spherical and point symmetrizations in R^3, Mostow [7] for spherical symmetrizations in R^n, Anderson [1] for one and two dimensional Steiner symmetrizations in R^3, and Pfaltzgraff [8] for radial symmetrizations in R^n. In fact, all these proofs can be easily modified for the more general case $p \geqq 1$. Since $\operatorname{cap}_p (A, C) = 0$ for $0 < p < 1$ (see Lemma 5.6), this case has no interest for us.

Hayman's proof involves only elementary methods and Pfaltzgraff uses a similar technique for radial symmetrizations in R^n. Anderson's and Mostow's proofs are modifications of Gehring's proof which uses the Brunn-Minkowski inequalities and the co-area formula of Federer and Young, both of which are rather deep results.

The above symmetrizations, except the radial one, belong to two categories of symmetrizations in R^n: the k-dimensional Steiner symmetrizations, $k = 1, 2, \ldots, n$, and the k-dimensional cap symmetrizations, $k = 1, 2, \ldots, n-1$. In this paper we consider only Steiner and cap symmetrizations and our main result is:

Let Sym *be any Steiner or cap symmetrization and* (A, C) *a condenser. If* Sym *is a Steiner symmetrization we also assume* A *is bounded. Then* $(\operatorname{Sym}(A), \operatorname{Sym}(C))$ *is also a condenser and the capacity inequality* (1.2) *is valid for every* $p > 0$. *If, in addition,* (A, C) *is ringlike, then* $(\operatorname{Sym}(A), \operatorname{Sym}(C))$ *is also ringlike.*

This result is obtained in the following way. First we study Steiner and cap symmetrizations in detail and establish the following results: For $k \geqq 2$ in $R^n, n \geqq 3$, every k-dimensional Steiner or cap symmetrization can be approximated by successive $(k-1)$-dimensional symmetrizations of the same type (Theorems 4.29 and 4.32). Furthermore, in R^n, $n \geq 2$, every 1-dimensional Steiner symmetrization can be approximated by 1-dimensional cap symmetrizations (Lemma 4.19). Then we prove (Theorem 6.12) that every 1-dimensional cap symmetrization satisfies the capacity inequality (1.2). We do this by generalizing to R^n the methods used by Hayman [4]. See also Pfaltzgraff [8]. Finally, using only the above approximation results, we prove by induction that every Steiner and cap symmetrization satisfies the capacity inequality (1.2) (Theorem 7.5).

Thus using elementary methods we prove the capacity inequality for Steiner and cap symmetrizations in a unified fashion. In particular, we do not need the co-area formula of Federer and Young nor do we assume the Brunn-Minkowski inequalities; in fact, we get the latter as a corollary from our geometric considerations, see Remark 4.34.

1.3. *Notation and terminology.* Let R^1 denote the real number system and $R^n, n \geqq 2$, euclidean n-space. For $x \in R^n$ we write $x =$

$x_1 e_1 + \ldots + x_n e_n$, where $e_1, \ldots, e_n$ are the unit coordinate vectors of R^n . For $x, y \in R^n$ let $x \cdot y = \sum_{i=1}^{n} x_i y_i$ denote the scalar product of x and y and $|x| = |x \cdot x|^{1/2}$ the norm of x . Also, for non-zero $x, y \in R^n$ let angle $(x, y) = \arccos\left(\frac{x \cdot y}{|x||y|}\right) \in [0, \pi]$ denote the angle between the vectors x and y .

For $A \subset R^n$ let $\complement A$, cl$A = \bar{A}$, int A and ∂A denote the complement, the closure, the interior and the boundary of A , all taken with respect to R^n . Also, let $d(A)$ denote the diameter of A .

For $A, B \subset R^n$ let $d(A, B)$ be the distance between A and B , let $A + B = \{a + b \mid a \in A, b \in B\}$ and, for $r \in R^1$, let $rA = \{rx \mid x \in A\}$.

For $x \in R^n$ and $r > 0$ we define $B^n(x, r)$ to be the open ball $\{y \in R^n \mid |x - y| < r\}$ and $S^{n-1}(x, r) = \partial B^n(x, r)$. We will also use the notation $B^n(r) = B^n(0, r)$ and $B^n = B^n(0, 1)$.

A domain in R^n is an open connected non-empty set.

Let $\mathcal{F}$ denote the collection of non-empty compact sets in R^n .

We write $m_n(A)$ or $m(A)$ for the Lebesgue measure of a measurable set $A \subset R^n$. The measure m_n is also defined for sets in an n-dimensional linear submanifold or in an n-dimensional sphere in $R^{n'}$, $n' > n$. For the Lebesgue integral of a function f over a set $A \subset R^n$ we write

$$\int_A f \, dm_n \quad \text{or} \quad \int_A f(x) \, dm(x) .$$

Let N denote the set of positive integers.

If $a_1, \ldots, a_k$ are linearly independent vectors in R^n , then $E^k(a_1, \ldots, a_k)$ denotes the linear subspace of R^n generated by $a_1, \ldots, a_k$.

We call the linear submanifolds of R^n *planes*. So a k-dimensional plane T through a point $x \in R^n$, $1 \leqq k \leqq n$, is always of the form $T = \{x\} + E^k(a_1, \ldots, a_k)$ for some linearly independent vectors $a_1, \ldots, a_k$ in R^n . We also call a point $x \in R^n$ a 0-dimensional plane through x . Closed half-planes of a plane T are the sets $\{y \in T \mid (y - x) \cdot e \geqq 0\}$ where $x \in T$ and e is a unit vector so that $x + e \in T$.

We say that two planes $T \subset R^n$ and $L \subset R^n$ are *perpendicular* to each other if, for any $a, x \in T$ and $b, y \in L$ we have $(a - x) \cdot (b - y) = 0$. Similarly, a vector $x \in R^n$ is perpendicular to the plane T if, for any $a, y \in T$, $x \cdot (a - y) = 0$.

1.4. *Hausdorff metric.* We define the function $d_H : \mathcal{F} \times \mathcal{F} \to R^1$ by setting $d_H(A, B) = \inf\{r > 0 \mid A \subset B + r\bar{B}^n, B \subset A + r\bar{B}^n\}$ for all

$A, B \in \mathcal{F}$. It is not difficult to prove that d_H is a metric on $\mathcal{F}$, [3, p. 151—2]. This metric is often called the *Hausdorff metric.* We define, by the metric d_H, the convergence of a sequence of sets $\{F_i\} \subset \mathcal{F}$ to a set $F \in \mathcal{F}$ by $\lim_i F_i = F$ if and only if $\lim_i d_H(F_i, F) = 0$.
In this paper we consider the convergence of a sequence of sets only in the above sense. The following lemma is easy to prove [3, p. 152].

1.5. **Lemma.** *For every sequence* $\{F_i\} \subset \mathcal{F}$ *such that* $F_1 \supset F_2 \supset \ldots$,

$$\lim_i F_i = \bigcap_i F_i .$$

Furthermore, see [3, p. 154];

1.6. **Theorem.** *Let* $\{F_i\}$ *be a sequence in* $\mathcal{F}$ *and* $A \subset R^n$ *a bounded set such that every* $F_i \subset A$. *Then there exists a subsequence* $\{F_{i_j}\}$ *and* $F \in \mathcal{F}$ *such that* $\lim_j F_{i_j} = F$.

1.7. **Corollary.** *Let* $\{F_i\}$ *be a sequence in* $\mathcal{F}$ *and* $A \subset R^n$ *a bounded set such that* $F_i \subset A$ *for every* $i \in N$. *If every convergent subsequence of* $\{F_i\}$ *converges to a fixed set* $F \in \mathcal{F}$, *then* $\lim_i F_i = F$.

2. Set transformations. Steiner and cap symmetrizations

In this chapter we first introduce the concept of a set transformation and then list some of its properties which will be used when symmetrizations are considered as set transformations. We then define Steiner and cap symmetrizations and derive some of their elementary properties. Throughout the rest of this paper a symmetrization means a Steiner or cap symmetrization.

2.1. *Set transformations.* A function $f : \mathcal{A} \to \mathcal{R}$ is called *a set transformation* if $\mathcal{R}$ is the collection of all subsets of R^n and $\mathcal{A}$ is some nonempty subcollection of $\mathcal{R}$. We write Dom (f) for $\mathcal{A}$ and Im (f) for the family of image sets $f(A)$, $A \in \mathcal{A}$. If $f : \mathcal{A} \to \mathcal{R}$ is a set transformation such that Im $(f) \subset$ Dom (f), then f^2 denotes the composition $f \circ f$ and, in general, f^{i+1} is defined by $f^{i+1} = f \circ f^i$, $i = 2, 3, \ldots$.

The set transformation $f : \mathcal{A} \to \mathcal{R}$ is called monotone if $f(A) \subset f(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$. We say $f : \mathcal{A} \to \mathcal{R}$ is *open, closed, compact* or f *preserves bounded sets,* if $f(A)$ is open, closed, compact or bounded whenever $A \in \mathcal{A}$ and A is of the same kind, respectively.

We call $f : \mathscr{A} \to \mathcal{R}$ *continuous from the inside* if for every increasing sequence $\{G_i\}$ of open sets in $\mathscr{A}$ such that $\bigcup_i G_i$ is in $\mathscr{A}$, we have $f(\bigcup_i G_i) = \bigcup_i f(G_i)$. Similarly, we call $f : \mathscr{A} \to \mathcal{R}$ *continuous from the outside* if for every decreasing sequence $\{F_i\}$ of closed sets in $\mathscr{A}$ such that $\bigcap_i F_i$ is in $\mathscr{A}$, we have $f(\bigcap_i F_i) = \bigcap_i f(F_i)$.

Finally, we call $f : \mathscr{A} \to \mathcal{R}$ *smoothing*, if for every closed set $F \in \mathscr{A}$ and for every $r > 0$ such that $F + r\bar{B}^n \in \mathscr{A}$, we have $f(F + r\bar{B}^n) \supset f(F) + r\bar{B}^n$.

Clearly every function $g : A \to R^n$, $A \subset R^n$, induces a set transformation $\bar{g} : \mathscr{A} \to \mathcal{R}$ such that $\bar{g}(B) = g(B)$ for every $B \in \mathscr{A} = \{C \subset R^n \mid C \subset A\}$. We often make no notational difference between the function and the induced set transformation.

2.2. *Example.* Let $g : R^n \to R^n$ be a continuous function and let $h : \mathcal{R} \to \mathcal{R}$ be the set transformation defined by $h(A) = g^{-1}(A)$ for every $A \subset R^n$. Then h is monotone, open, closed and continuous from the inside and from the outside. In addition, h is smoothing if and only if $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in R^n$.

2.3. **Lemma.** *Let $f : \mathscr{A} \to \mathcal{R}$ be monotone, continuous from the inside and smoothing. If, for every open set $G \in \mathscr{A}$ all open and compact subsets of G are included in $\mathscr{A}$, then f is open.*

Proof. Let $G \in \mathscr{A}$ be open with $f(G) \neq \emptyset$. Choose an increasing sequence $\{G_i\}$ of open bounded sets such that $G = \bigcup_i G_i$ and $\bar{G}_i \subset G$. By the continuity from the inside, $f(G) = \bigcup_i f(G_i)$. Choose $x \in f(G)$. Then there exists a G_i and an $r > 0$ such that $x \in f(G_i)$ and $\bar{G}_i + r\bar{B}^n \subset G$. The smoothing property and monotonicity of f imply $f(G) \supset f(\bar{G}_i + r\bar{B}^n) \supset f(\bar{G}_i) + r\bar{B}^n \supset \{x\} + r\bar{B}^n$. Thus $f(G)$ is open.

2.4. **Lemma.** *Let f and f_i, $i = 1, 2, \ldots$, be set transformations such that $\mathcal{F} \subset \operatorname{Dom}(f) \cap \operatorname{Dom}(f_i)$, $i = 1, 2, \ldots$, and, for all $F \in \mathcal{F}$, every $f_i(F) \in \mathcal{F}$, $f(F) \in \mathcal{F}$ and $\lim_i f_i(F) = f(F)$. Suppose also that every f_i is smoothing. Then $f \mid \mathcal{F}$ is smoothing.*

Proof. Let $F \in \mathcal{F}$ and $r > 0$. Since $F + r\bar{B}^n \in \mathcal{F}$, $\lim f_i(F + r\bar{B}^n) = f(F + r\bar{B}^n)$. On the other hand, $f_i(F) + r\bar{B}^n \subset f_i(F + r\bar{B}^n)$, hence $f(F) + r\bar{B}^n = \lim (f_i(F) + r\bar{B}^n) \subset \lim f_i(F + r\bar{B}^n) = f(F + r\bar{B}^n)$.

2.5. *Steiner symmetrizations.* These symmetrizations are set transformations defined in the family of all bounded open and closed sets in R^n. Every $(n-k)$-dimensional plane $T \subset R^n$, $1 \leqq k \leqq n$, defines a *k-dimensional Steiner symmetrization* Sym as follows. Let A be a bounded open or closed set in R^n. For every $x \in T$ let $L(x)$ denote the k-dimensional plane through x and perpendicular to T. Now if $m_k(A \cap L(x)) = 0$, then $\mathrm{Sym}(A) \cap L(x)$ is empty or the point $\{x\}$ according as $A \cap L(x)$ is empty or non-empty. If $m_k(A \cap L(x)) > 0$, then

$$\mathrm{Sym}(A) \cap L(x) = \begin{cases} B^n(x, r) \cap L(x), & \text{if } A \text{ is open,} \\ \bar{B}^n(x, r) \cap L(x), & \text{if } A \text{ is closed,} \end{cases}$$

where $r > 0$ is defined by $m_k(B^n(x, r) \cap L(x)) = m_k(A \cap L(x))$.

The plane T is called *the symmetry plane* of Sym and the planes $L(x), x \in T$, *the symmetrizing planes* of Sym. Some authors call only 1-dimensional symmetrizations of the above type Steiner symmetrizations, while the $(n-1)$ and n-dimensional ones are called Schwarz and point symmetrizations, respectively.

2.6. *Sphere and its cap.* For every sphere $S^{n-1}(x, r)$, $x \in R^n$, $r > 0$ and every $(k+1)$-dimensional plane $T \subset R^n$ through x, $1 \leqq k \leqq n-1$, we call the intersection $K = S^{n-1}(x, r) \cap T$ a *k-dimensional sphere* in R^n with centre x and radius r. For every $y \in K$, *the open caps* of K with centre y are sets of the form $B^n(y, r') \cap K$ for some $r' > 0$, and *the closed caps* with the centre $y \in K$ are the closures of the corresponding open caps and also the point $\{y\}$.

2.7. *Cap symmetrizations.* Every cap symmetrization is a set transformation defined in the family of all open and closed sets in R^n. The definition of the cap symmetrization is analogous to that of the Steiner symmetrization, but instead of symmetrizing planes we now use spheres.

Consider an $(n-k)$-dimensional plane $T \subset R^n$, $1 \leqq k \leqq n-1$, and a closed half-plane T_s of T. The half-plane T_s defines a *k-dimensional cap symmetrization,* Sym, as follows. Let J be the boundary of T_s with respect to T and for every $z \in J$ and $r \geqq 0$ we define

$$K(z, r) = \begin{cases} \{z\}, & \text{if } r = 0, \\ S^{n-1}(z, r) \cap M(z) & \text{otherwise,} \end{cases} \tag{2.8}$$

where $M(z)$ is the $(k+1)$-dimensional plane through the point z and perpendicular to the plane J. Now for every open or closed $A \subset R^n$ we define $\mathrm{Sym}(A)$ by the conditions:

$$K(z, r) \cap \mathrm{Sym}(A) = \begin{cases} \emptyset & \text{if and only if } K(z, r) \cap A = \emptyset, \\ K(z, r) & \text{if and only if } K(z, r) \subset A, \end{cases}$$

and otherwise $K(z, r) \cap \mathrm{Sym}(A)$ is the cap of the sphere $K(z, r)$ such that

(i) the centre of the cap is the point $K(z, r) \cap T_s$,
(ii) $m_k[K(z, r) \cap \mathrm{Sym}(A)] = m_k[K(z, r) \cap A]$ and
(iii) the cap $K(z, r) \cap \mathrm{Sym}(A)$ is open or closed according as A is open or closed.

We call the plane T *the symmetry plane* of Sym and the $(n-k-1)$-dimensional plane J *the symmetrizing axis*. The spheres (2.8.) are called *the symmetrizing spheres* of Sym. An $(n-1)$-dimensional cap symmetrization in R^n is often called a spherical symmetrization.

2.9. Let Sym be a Steiner or cap symmetrization and T the symmetry plane of Sym and A a set in Dom(Sym). Clearly Sym(A) is symmetric in the plane T, i.e. if $x \in \mathrm{Sym}(A)$ and x' is the orthogonal projection of x on the plane T, then $x' - (x - x') \in \mathrm{Sym}(A)$. Later we shall prove that symmetrizations are open and closed set transformations. Then Sym(A) is measurable and, in particular, $m(A) = m(\mathrm{Sym}(A))$, which is easily verified by integrating along the symmetrizing planes or spheres and using Fubini's theorem. Clearly Sym is monotone and preserves bounded sets. Further, $\mathrm{Sym}(A) = \emptyset$ if and only if $A = \emptyset$.

2.10. **Lemma.** *Steiner and cap symmetrizations are continuous from the inside and from the outside.*

Proof. Let Sym be a k-dimensional cap symmetrization, $1 \leq k \leq n-1$, and let $G = \bigcup_i G_i$ where $\{G_i\}$ is an increasing sequence of open sets. Then $\mathrm{Sym}(\bigcup_i G_i) = \bigcup_i \mathrm{Sym}(G_i)$, if

$$K \cap \mathrm{Sym}(\bigcup_i G_i) = K \cap [\bigcup_i \mathrm{Sym}(G_i)] \tag{2.11}$$

for every symmetrizing sphere K of Sym. We may assume that $K \cap G \neq \emptyset$ and $K \not\subset G$. Hence, $K \cap \mathrm{Sym}(G) \neq K$ is an open cap and every $K \cap \mathrm{Sym}(G_i)$ is a concentric open subcap. Thus $K \cap [\bigcup_i \mathrm{Sym}(G_i)]$ is also an open concentric subcap $K \cap \mathrm{Sym}(G)$. Moreover,

$$m_k(K \cap \mathrm{Sym}(G)) = m_k(K \cap G) = \lim_i m_k(K \cap G_i) = \lim_i m_k(K \cap \mathrm{Sym}(G_i)) =$$
$$= m_k[K \cap (\bigcup_i \mathrm{Sym}(G_i))].$$

Thus we get (2.11). Arguing similarly, we see that Sym is also continuous from the outside. We prove the lemma for Steiner symmetrizations in the same way by considering symmetrizing planes instead of symmetrizing spheres.

2.12. For every set transformation $f : \mathcal{A} \to \mathcal{R}$ we define the complement transformation $f_c : \mathcal{A}' \to \mathcal{R}$ by the formula

$$f_c(A) = \mathbf{C}f(\mathbf{C}A) \quad \text{for every} \quad A \in \mathcal{A}' = \{\mathbf{C}B \mid B \in \mathcal{A}\}.$$

Observe the following useful property of cap symmetrizations: If Sym is a k-dimensional cap symmetrization, $1 \leqq k \leqq n-1$, T the symmetry plane of Sym and T_s the half-plane that defines Sym, then the k-dimensional cap symmetrization Sym_c defined by the half-plane $T_s' = cl(T \setminus T_s)$ is the complement transformation of Sym, or in other words,

$$\mathrm{Sym}\,(A) = \mathbf{C}\mathrm{Sym}_c\,(\mathbf{C}A) \tag{2.13}$$

for every $A \in \mathrm{Dom}\,(\mathrm{Sym})$. We also call Sym_c the complement symmetrization of Sym. Using (2.13) we immediately get from Lemmas 2.10 and 2.3:

2.14. **Lemma.** *If for some* $k \in N$, $1 \leqq k \leqq n-1$, *all k-dimensional cap symmetrizations in R^n are smoothing, then they are also open and closed.*

Next we consider the preserving of connectedness under a symmetrization.

2.15. **Lemma.** *Let* Sym *be a Steiner or cap symmetrization and* $A \in \mathrm{Dom}\,(\mathrm{Sym})$. *If* A *or* $\bar{R}^n \setminus A$ *is connected, then* $\mathrm{Sym}\,(A)$ *or* $\bar{R}^n \setminus \mathrm{Sym}(A)$ *is connected, respectively.*

Proof. Let Sym be a cap symmetrization defined by the half-plane T_s. For every $x \in R^n$ let $K(x)$ denote the symmetrizing sphere of Sym through x, and denote by $f : R^n \to T_s$ the continuous mapping for which $\{f(x)\} = K(x) \cap T_s$ for every $x \in R^n$. Put $S_a = [K(a) \cap \mathrm{Sym}\,(A)] \cup f(A)$ for every $a \in A$. Assume now that A is connected. Then $f(A)$ is connected, which implies that S_a is also connected, since $K(a) \cap \mathrm{Sym}\,(A)$ is connected and $[K(a) \cap \mathrm{Sym}\,(A)] \cap f(A) = \{f(a)\} \neq \emptyset$. Now $\mathrm{Sym}\,(A) = \bigcup_{a \in A} S_a$ is connected, since it is the union of connected sets whose intersection $\bigcap_{a \in A} S_a = f(A)$ is not empty.

Next, assume that $\bar{R}^n \setminus A$ is connected. We first observe that

$$\bar{R}^n \setminus \mathrm{Sym}\,(A) = \{\infty\} \cup \mathbf{C}\mathrm{Sym}\,(A) = \{\infty\} \cup \mathrm{Sym}_c\,(\mathbf{C}A),$$

where Sym_c is the complement symmetrization of Sym. Let T'_s be the half-plane defining Sym_c and let $K(x)$ for $x \in R^n$ be defined as above. We define a mapping $f_c : \bar{R}^n \to \bar{R}^n$ by setting

$$f_c(x) = \begin{cases} \text{the point } K(x) \cap T'_s, & \text{if } x \in R^n, \\ \infty, \text{ if } x = \infty . \end{cases}$$

Clearly f_c is continuous, whence $f_c(\bar{R}^n \setminus A) = \{\infty\} \cup f_c(\complement A)$ is connected. Hence, we get

$$\bar{R}^n \setminus \mathrm{Sym}(A) = \bigcup_{a \in \complement A} S'_a ,$$

where $S'_a = [K(a) \cap \mathrm{Sym}_c(\complement A)] \cup f_c(\bar{R}^n \setminus A)$ for every $a \in \complement A$. Now, arguing as above, we verify the connectedness of $\bar{R}^n \setminus \mathrm{Sym}(A)$.

Next we assume that Sym is a Steiner symmetrization with the symmetry plane T. For every $x \in R^n$ let $L(x)$ denote the symmetrizing plane of Sym through x. If A is connected, we verify the connectedness of $\mathrm{Sym}(A)$ just as we did in the case of a cap symmetrization; we need only replace T_s and $K(x)$ by T and $L(x)$.

On the other hand, $\bar{R}^n \setminus \mathrm{Sym}(A)$ is always connected whenever $A \in \mathrm{Dom}(\mathrm{Sym})$. To prove this, we first observe that $\mathrm{Sym}(A)$ is bounded and thus, for every $x \in R^n$ the set $V(x) = [L(x) \cup \{\infty\}] \setminus [L(x) \cap \mathrm{Sym}(A)]$ is connected. Then $\bar{R}^n \setminus \mathrm{Sym}(A)$ is connected because

$$\bar{R}^n \setminus \mathrm{Sym}(A) = \bigcup_{x \in R^n} V(x) \quad \text{and} \quad \bigcap_{x \in R^n} V(x) = \{\infty\} \neq \emptyset ,$$

and the proof of the lemma is complete.

3. Convergent set transformations

3.1. We call a set transformation $f : \mathcal{A} \to \mathcal{R}$ *regular*, if f is monotone, $\mathcal{F} \cup \{\emptyset\} \subset \mathcal{A}, f(\mathcal{F}) \subset \mathcal{F}$ and $f(\emptyset) = \emptyset$. A regular set transformation $f : \mathcal{A} \to \mathcal{R}$ is called *convergent*, if $\lim f^i(F)$ exists for every $F \in \mathcal{F}$.

In this chapter we first give sufficient conditions for a regular set transformation to be convergent. We then prove that if Sym is a k-dimensional cap or Steiner symmetrization, $k \geqq 2$, we can choose two $(k-1)$-dimensional symmetrizations $\mathrm{Sym}_j, j = 1, 2$, of the same type, such that $\mathrm{Sym}_2 \circ \mathrm{Sym}_1$ is convergent and, in addition,

$$\mathrm{Sym}(F) = \lim_i (\mathrm{Sym}_2 \circ \mathrm{Sym}_1)^i(F) \tag{3.2}$$

for every $F \in \mathcal{F}$. This result will be proved under the assumption that symmetrizations are regular and smoothing, which we shall show to be true in chapter 4.

The result (3.2) shows that every k-dimensional symmetrization, $k \geqq 2$, can be approximated in $\mathcal{F}$ by succesive $(k-1)$-dimensional symmetrizations of the same type. This result is essential for our inductive method.

3.3. Let $\mathcal{B}$ be a non-empty subcollection of $\mathcal{F}$ and $\mathcal{A}_{\mathcal{B}} = \{A \subset R^n \mid A \subset B$ for some $B \in \mathcal{B}\}$. Let $\beta : \mathcal{A}_{\mathcal{B}} \to \mathcal{F} \cup \{\emptyset\}$ be the set transformation

$$(3.4) \qquad \beta(A) = \bigcap_{A \subset B \in \mathcal{B}} B \,, \quad \text{for } A \in \mathcal{A}_{\mathcal{B}} \setminus \{\emptyset\} \,, \quad \text{and } \beta(\emptyset) = \emptyset \,.$$

Observe that $A \subset \beta(A)$ for every $A \in \mathcal{A}_{\mathcal{B}}$, and $\beta(B) = B$ for every $B \in \mathcal{B} \subset \mathcal{A}_{\mathcal{B}}$. We call the collection $\mathcal{B}$ *continuous* if $\beta : \mathcal{A}_{\mathcal{B}} \to \mathcal{F} \cup \{\emptyset\}$ in continuous from the outside and $\beta(\mathcal{A}_{\mathcal{B}} \setminus \{\emptyset\}) \subset \mathcal{B}$.

A regular set transformation $f : \mathcal{A} \to \mathcal{K}$ is called *rounding* with respect to a collection $\mathcal{B} \subset \mathcal{F}$ if the following two conditions hold:

(i) $f(B) \subset B$ for every $B \in \mathcal{B}$, and
(ii) if $F \in \mathcal{F}, F \subset B$ and $F \neq B$ for some $B \in \mathcal{B}$, then there exist $i \in N$ and $B' \in \mathcal{B}$ such that $f^i(F) \subset B' \subset B, B' \neq B$.

3.5. **Lemma.** *Let f be a regular set transformation which is continuous from the outside and rounding with respect to a continuous collection $\mathcal{B} \subset \mathcal{F}$. If $F \in \mathcal{F}$ is such that $F \subset B$ for some $B \in \mathcal{B}$, then*

$$(3.6) \qquad \lim_i f^i(F) = B^* \,,$$

where $B^* = \bigcap_i \beta[f^i(F)] \in \mathcal{B}$ and β is defined by $\mathcal{B}$ as in (3.4).

If, in addition, B belongs to some k-dimensional plane or sphere P in R^n, $m_k(f(A) \cap P) = m_k(A \cap P)$ for every $A \in \mathcal{F}$, and f is smoothing, then

$$(3.7) \qquad m_k(B^*) = m_k(F) \,.$$

Proof. Put $F_i = f^i(F)$ for every $i \in N$. Then $F_i = f^i(F) \subset f^i(B) \subset B$ for every $i \in N$. Hence to prove (3.6) we need, by Corollary 1.7, only to show that every convergent subsequence of $\{F_i\}$ converges to B^*.

In fact, assume that a subsequence $\{F_{i_j}\}$ converges to $A \in \mathcal{F}$. The sequence $B_i = \beta(F_i) = \beta(f^i(F))$, $i \in N$, is decreasing, since f is monotone, $f(B_i) \subset B_i$ by the rounding property of f and $f^i(F) \subset B_i \in \mathcal{B}$ for every $i \in N$. Thus $\bigcap_j B_{i_j} = \bigcap_i B_i = B^*$ and $\lim_j B_{i_j} = \bigcap_j B_{i_j}$

by Lemma 1.5. So $A = \lim_j F_{ij} \subset \lim_j B_{ij} = \bigcap_j B_{ij} = B^*$. On the other hand, $B^* \in \mathcal{B}$, since $B^* = \bigcap_i B_i = \bigcap_i \beta(B_i) = \beta(\bigcap_i B_i) \in \mathcal{B}$ by the continuity of the collection $\mathcal{B}$.

If $A \neq B^*$, then by the rounding property of f with respect to $\mathcal{B}$ there exist $i' \in N$ and $B' \in \mathcal{B}$ such that

$$f^{i'}(A) \subset B' \subset B^*, B' \neq B^* . \tag{3.8}$$

Since $\lim_j F_{ij} = A$ we can find, for any $m \in N$, an index i'' such that $F_{i''} \subset \left(A + \frac{1}{m}\bar{B}^n\right) \cap B$. Then $f^{i'+i''}(F) = f^{i'}(F_{i''}) \subset f^{i'}\left[\left(A + \frac{1}{m}\bar{B}^n\right) \cap B\right]$,

and thus $B^* \subset \beta[f^{i'+i''}(F)] \subset \beta\left[f^{i'}\left(\left(A + \frac{1}{m}\bar{B}^n\right) \cap B\right)\right]$. Now $\mathcal{B}$ is a continuous collection and f is continuous from the outside, whence $f^{i'}$ also has this property, and we get

$$B^* \subset \bigcap_{m=1}^{\infty} \beta\left[f^{i'}\left(\left(A + \frac{1}{m}\bar{B}^n\right) \cap B\right)\right] = \beta[f^{i'}(A)] \subset B' .$$

This contradicts (3.8). So $A = B^*$, which proves (3.6).

We still must show that, under the additional assumptions of the lemma, (3.7) holds.

Because $B^* = \bigcap B_i$, $B_1 \supset B_2 \supset \ldots$, $f^i(F) \subset B_i$ for every $i \in N$, and f is m_k-measure preserving with respect to sets in P, we get

$$m_k(F) = \lim_i m_k(f^i(F)) \leqq \lim_i m_k(B_i) = m_k(B^*) .$$

On the other hand, for every $\varepsilon > 0$ there exists $i \in N$ such that $B^* \subset f^i(F) + \varepsilon\bar{B}^n$. Since f is smoothing, so is f^i, and we get $B^* \subset (f^i(F) + \varepsilon\bar{B}^n) \cap P \subset f^i(F + \varepsilon\bar{B}^n) \cap P$, and thus, by the assumptions of the lemma

$$m_k(B^*) \leqq m_k(f^i(F + \varepsilon\bar{B}^n) \cap P) = m_k((F + \varepsilon\bar{B}^n) \cap P) .$$

Letting $\varepsilon \to 0$ we get $m_k(B^*) \leqq m_k(F \cap P) = m_k(F)$. The lemma follows.

For an index set I we call the system $(P_\alpha, \mathcal{B}_\alpha)$, $\alpha \in I$, a *continuous partition* of R^n if

(i) $\{P_\alpha \mid \alpha \in I\}$ is a collection of disjoint closed sets in R^n and $\mathcal{B}_\alpha \subset \mathcal{F}$ is a non-empty continuous collection for every $\alpha \in I$,

(ii) $R^n = \bigcup_{\alpha \in I} P_\alpha$, $P_\alpha = \bigcup_{B \in \mathcal{B}_\alpha} B$ for every $\alpha \in I$, and

(iii) for every $F \in \mathcal{F}$ the set $\bigcup_{\alpha \in I} \beta_\alpha(F \cap P_\alpha)$ is in $\mathcal{F}$, where β_α is the set transformation defined by $\mathcal{B}_\alpha$ as in (3.4).

3.9. **Lemma.** *Let f be a regular set transformation which is continuous from the outside and let the system $(P_\alpha, \mathcal{B}_\alpha)$, $\alpha \in I$, be a continuous partition of R^n such that*

(i) *for every $F \in \mathcal{F}$ and $\alpha \in I$, $f(F \cap P_\alpha) = f(F) \cap P_\alpha$, and*

(ii) *f is rounding with respect to every $\mathcal{B}_\alpha$, $\alpha \in I$.*

Then f is convergent and for every $F \in \mathcal{F}$,

$$\lim_i f^i(F) = \bigcup_{\alpha \in I} B_\alpha^* \in \mathcal{F},$$

where $B_\alpha^* = \bigcap_{i=1}^{\infty} A_{i\alpha}$, $A_{i\alpha} = \beta_\alpha[f^i(F) \cap P_\alpha]$, and $B_\alpha^* \in \mathcal{B}_\alpha$, unless $B_\alpha^* = \emptyset$.

Proof. Put $A_i = \bigcup_{\alpha \in I} A_{i\alpha}$, $i \in N$. Hence, $f^i(F) \subset A_i$ for every $i \in N$, and A_i is compact because $(P_\alpha, \mathcal{B}_\alpha)$, $\alpha \in I$, is a continuous partition of R^n. Furthermore, $A_1 \supset A_2 \supset \dots$ because $\beta_\alpha[f^i(F) \cap P_\alpha] \supset \beta_\alpha[f^{i+1}(F) \cap P_\alpha]$ for all $i \in N$, $\alpha \in I$. Hence $\bigcap_i A_i = \lim_i A_i$ by Lemma 1.5.

Since $A_{i\alpha} \subset P_\alpha$ for all $i \in N$, $\alpha \in I$, and P_α, $\alpha \in I$, are mutually disjoint, we have $\bigcap_i A_i = \bigcap_i (\bigcup_{\alpha \in I} A_{i\alpha}) = \bigcup_{\alpha \in I} (\bigcap_i A_{i\alpha}) = \bigcup_{\alpha \in I} B_\alpha^*$. Hence

$$\bigcup_{\alpha \in I} B_\alpha^* = \lim_i A_i \in \mathcal{F}. \tag{3.10}$$

Now assume that $\{f^{i_j}(F)\}$ is any convergent subsequence of $\{f^i(F)\}$ and $\lim_j f^{i_j}(F) = E$. Put $F^* = \bigcup_{\alpha \in I} B_\alpha^*$. To prove that $\lim_i f^i(F) = F^*$, we need, by Corollary 1.7, only to show that $E = F^*$, since every $f^i(F)$, $i \in N$, is included in the compact set A_1.

Since every $f^{i_j}(F) \subset A_{i_j}$, we get $E \subset F^*$ by (3.10). On the other hand, choose any B_α^*, $\alpha \in I$. If $F \cap P_\alpha = \emptyset$, then $\emptyset = B_\alpha^* \subset E$. If $F \cap P_\alpha \neq \emptyset$, then by Lemma 3.5, $\lim_i f^i(F \cap P_\alpha) = B_\alpha^* \in \mathcal{B}_\alpha$. Thus, for any $r > 0$ we can find an index i' such that

$$B_\alpha^* \subset f^{i'}(F \cap P_\alpha) + r\bar{B}^n \quad \text{and} \quad f^{i'}(F) \subset E + r\bar{B}^n.$$

Hence by the monotonicity of f,

$$B_\alpha^* \subset f^{i'}(F \cap P_\alpha) + r\bar{B}^n \subset f^{i'}(F) + r\bar{B}^n \subset (E + r\bar{B}^n) + r\bar{B}^n,$$

which implies $B_\alpha^* \subset E$, since $r > 0$ was arbitrary. Thus, $F^* = \bigcup_{\alpha \in I} B_\alpha^* \subset E$, which proves the lemma.

3.11. Let Sym be a k-dimensional cap symmetrization in R^n, $2 \leqq k \leqq n - 1$, and let T denote the symmetry plane of Sym, J the symmetrizing axis and T_s the half-plane which defines Sym. We associate with Sym two $(k - 1)$-dimensional cap symmetrizations Sym_i, $i = 1, 2$, as follows. Choose two mutually orthogonal unit vectors a_1, a_2 perpendicular to T. Put

$$T_{1s} = T_s + E^1(a_1)\,,\ T_{2s} = T + \{ta_2 \mid t \geqq 0\}\,,$$

and let Sym_i denote the $(k - 1)$-dimensional cap symmetrization defined by T_{is}, $i = 1, 2$.

For the following lemma let $K(x)$ denote the symmetrizing sphere of Sym through x for all $x \in T_s$, Let $\mathcal{B}_x$ denote the collection of all closed caps of $K(x)$ with centre x, and if $K(x) = \{x\}$, let $\{\{x\}\} = \mathcal{B}_x$.

3.12. **Lemma.** *If* Sym_1 *and* Sym_2 *are closed set transformations, then* $f = \mathrm{Sym}_2 \circ \mathrm{Sym}_1$ *is a regular set transformation, and for every* $x \in T_s$, f *is rounding with respect to* $\mathcal{B}_x$ *and*

$$f(F) \cap K(x) = f(F \cap K(x))\,, \tag{3.13}$$

$$m_k(f(F) \cap K(x)) = m_k(F \cap K(x)) \quad \textit{for every} \quad F \in \mathcal{F}\,. \tag{3.14}$$

Proof. The set transformations Sym_1 and Sym_2 are regular because they are closed and symmetrizations. Therefore, f is also regular.

To prove the other parts of the lemma let $a_0, a_3, a_4, \ldots, a_k$ be unit vectors such that $a_0, a_1, a_2, \ldots, a_k$ are mutually orthogonal and perpendicular to J, where a_1, a_2 are the unit vectors in 3.11, and a_0 is fixed by the condition $T_s = J + \{ta_0 \mid t \geqq 0\}$. Further, let T_i be the symmetry plane of Sym_i, $i = 1, 2$.

Now choose $x \in T_s$ and let $K = K(x)$ be the symmetrizing sphere of Sym through x. We can assume that $K \neq \{x\}$, for the case $K = \{x\}$ is trivial. Then there exists $z \in J$ and $r > 0$ such that

$$K = (\{z\} + E^{k+1}(a_0, a_1, \ldots, a_k)) \cap S^{n-1}(z, r)\,.$$

Put for every $t \in [-r, r]$

$$L_1(t) = \{z + ta_1\} + E^k(a_0, a_2, a_3, \ldots, a_k)\,,$$

$$L_2(t) = \{z + ta_0\} + E^k(a_1, a_2, \ldots, a_k)\,,$$

$$K_i(t) = L_i(t) \cap S^{n-1}(z, r)\,,\ i = 1, 2\,,$$

whence $K_i(t)$ is a symmetrizing sphere of Sym_i and, in particular,

$$K = \bigcup_{t\in[-r,r]} K_i(t)\,, i = 1\,, 2\,. \tag{3.15}$$

Hence from (3.15) we deduce that (3.13) is valid, and by Fubini's theorem we see that (3.15) implies (3.14).

It remains to prove that f is rounding with respect to the collection $\mathcal{B}_x$. Clearly by (3.15) $f(B) = B$ for every $B \in \mathcal{B}_x$. Let $F \in \mathcal{F}$ be such that $F \subset B\,, F \neq B$ for some $B \in \mathcal{B}_x$. Thus, we must show that there exists $i \in N$ such that $f^i(F) \subset B'$ for some $B' \in \mathcal{B}_x\,, B' \subset B\,, B' \neq B$. Observe that B cannot be the point x.

Let H be the boundary of B with respect to K. Then by the position of Sym_1 it is easy to see that for every compact set $A \subset B$, and for every $t \in [-r\,, r]$, the following result is valid:

$$\text{If } K_1(t) \cap (B\setminus A) \neq \emptyset\,, \text{ then } K_1(t) \cap H \subset H\setminus \mathrm{Sym}_1(A)\,. \tag{3.16}$$

Now since $F \neq B$, there exists an open cap V of K such that $V \subset B\setminus F$. Put $F' = B\setminus V \supset F$. If $K_1(t) \cap V \cap H \neq \emptyset$ for some $K_1(t)$, $t \in (-r\,, r]$, then (3.16) implies $\emptyset \neq H\setminus \mathrm{Sym}_1(F') \subset H\setminus \mathrm{Sym}_1(F)$. Hence we see that

$$H\setminus \mathrm{Sym}_1 \circ f(F) \neq \emptyset\,. \tag{3.17}$$

If $K_1(t) \cap V \cap H = \emptyset$ for every $K_1(t)\,, t \in [-r\,, r]$, then clearly $B\setminus \mathrm{Sym}_1(F')$ is an open cap of K, and further $B\setminus f(F')$ is an open cap V', whose centre is in $T_2 \cap K \subset K \cap L_1(0) = K_1(0)$. Thus, either $K_1(0) \cap V' \cap H \neq \emptyset$ or $H = \emptyset$. If $K_1(0) \cap V' \cap H \neq \emptyset$, then arguing as above we find that (3.17) is again valid.

If $H = \emptyset$, then $B = K$ and $B\setminus \mathrm{Sym}_1 \circ f(F')$ is an open cap V'' of K, and the centre of V'' is the point $K_1(0) \cap (T_1\setminus T_{1s}) = K \cap (T\setminus T_s)$, whence $\mathrm{Sym}_1 \circ f(F') = B\setminus V'' = B' \in \mathcal{B}$, which implies $f^2(F) \subset f^2(F') \subset B' \subset B\,, B' \neq B$, and there is nothing more to prove. So we can suppose that (3.17) is valid, and $H \neq \emptyset$.

To prove the rounding property of f it is sufficient to show that

$$H\setminus f^j(F) = H \text{ for some } j \in N\,. \tag{3.18}$$

Since $H \neq \emptyset$, it is a $(k-1)$-dimensional sphere so that for some $t_H \in (-r\,, r)$

$$H = [\{z + t_H a_0\} + E^k(a_1\,, a_2\,, \ldots\,, a_k)] \cap S^{n-1}(z\,, r)\,,$$

whence especially, $H = K_2(t_H)$. Now we write for every $A \subset H$

$$Z[A] = \bigcup_{K_1(t)\cap A \neq \emptyset} K_1(t) \cap H\,.$$

Then if U is an open cap of H with the centre y, $\{y\} = K_2(t_H) \cap (T_2 \setminus T_{2s})$, so

$$(3.19) \qquad m_{k-1}(Z[U]) \geqq 2m_{k-1}(U)\,, \quad \text{unless} \quad Z[U] = H\,.$$

On the other hand, if $H \setminus \mathrm{Sym}_1 \circ f^i(F) \neq \emptyset$ for some $i \in N$, then $H \setminus f^{i+1}(F) = U_i$ is an open cap of H with the centre y, since $H = K_2(t_H)$. Hence (3.16) implies

$$Z[U_i] \subset H \setminus \mathrm{Sym}_1 \circ f^{i+1}(F)\,,$$

whence by the definition of Sym_2 the inequality (3.19) yields

$$(3.20) \qquad m_{k-1}(U_{i+1}) \geqq 2m_{k-1}(U_i)\,, \quad \text{unless} \quad U_{i+1} = H\,.$$

Now $H \setminus \mathrm{Sym}_1 \circ f(F) \neq \emptyset$ by (3.17), whence we get from (3.20), $m_{k-1}(U_{i+1}) \geqq 2^i m_{k-1}(U_1)$ for every $i \in N$, unless $U_{i+1} = H$. Thus, (3.18) must be valid for some $j \in N$, which proves the lemma.

3.21. **Lemma.** *Let* Sym *be a k-dimensional cap symmetrization in* R^n, $2 \leqq k \leqq n-1$, *and let* Sym_1 *and* Sym_2 *be two $(k-1)$-dimensional cap symmetrizations associated with* Sym *as in Lemma* 3.12. *If* Sym_1 *and* Sym_2 *are closed and smoothing, then*

$$\mathrm{Sym}\,(F) = \lim_i\, (\mathrm{Sym}_2 \circ \mathrm{Sym}_1)^i(F)$$

for every $F \in \mathfrak{F}$.

Proof. Let J, J_1 and J_2 be the symmetrizing axes of Sym, Sym_1 and Sym_2, respectively, and let T_s be the half-plane which defines Sym. For every $x \in T_s$ let $K(x)$ and $\mathfrak{B}_x$ be as in Lemma 3.12 and put $f = \mathrm{Sym}_2 \circ \mathrm{Sym}_1$.

Choose $F \in \mathfrak{F}$. To prove the existence of $\lim_i f^i(F)$ we apply Lemma 3.9 for which we put $I = T_s$, $\{P_\alpha \mid \alpha \in I\} = \{K(x) \mid x \in T_s\}$, and $\mathfrak{B}_\alpha = \mathfrak{B}_x$ whenever $P_\alpha = K(x)$. Consider the validity of the assumptions of Lemma 3.9. First f is regular and by Lemma 2.10, continuous from the outside. It is easy to see that the system $(K(x), \mathfrak{B}_x)$, $x \in T_s$, is a continuous partition of R^n. The assumptions (i) and (ii) of 3.9 are valid by Lemma 3.12. Furthermore, by 3.12

$$(3.22) \qquad m_k(f(F) \cap K(x)) = m_k(F \cap K(x))\,,\; x \in T_s\,,\; F \in \mathfrak{F}\,.$$

Hence, if $F \in \mathfrak{F}$, then $\lim f^i(F) = \bigcup_{x \in T_s} B_x^*$ by Lemma 3.9, where

$$B_x^* = \begin{cases} \emptyset \ \text{ if } \ K(x) \cap F = \emptyset \,, \\ \bigcap_{i=1}^{\infty} \beta_x(f^i(F \cap K(x))) \in \mathcal{B}_x \ \text{ otherwise.} \end{cases}$$

Thus, $\operatorname{Sym}(F) = \lim_i f^i(F)$, if

$$\text{(3.23)} \qquad B_x^* = \operatorname{Sym}(F) \cap K(x) \,, \ \text{ for every } \ x \in T_s \,.$$

If $K(x) \cap F = \emptyset$, then $\operatorname{Sym}(F) \cap K(x) = \emptyset$, and (3.23) holds. Assume $K(x) \cap F \neq \emptyset$. Then $B_x^* \in \mathcal{B}_x$, and in addition, since f is smoothing and (3.22) is valid, we get by Lemma 3.5, $m_k(B_x^*) = m_k(F \cap K(x)) = m_k(\operatorname{Sym}(F) \cap K(x))$, whence, by the definition of Sym, the equation (3.23) is again valid, and this proves the lemma.

3.24. To prove a similar corollary for Steiner symmetrizations, we need the following lemma.

3.25. **Lemma.** *Let two straight lines L_1 and L_2 in R^2 intersect each others at a point x in an angle $\gamma > 0$ such that γ/π is irrational. If $r > 0$ and $I \subset S = S^1(x, r)$ is an open arc and $g_i : R^2 \to R^2$ a reflection in L_i, $i = 1, 2$, then for some $m' \in N$*

$$\bigcup_{m=1}^{m'} (g_2 \circ g_1)^m(I) = S \,.$$

Proof. We use complex notation with $i = \sqrt{-1}$ as the imaginary unit. We may assume that $S = \{e^{i\varphi} \mid \varphi \in [0, 2\pi)\}$, L_1 is the real axis and $L_2 = \{te^{i\gamma} \mid t \in R^1\}$. Hence for every $e^{i\varphi} \in S$, $g_1(e^{i\varphi}) = e^{-i\varphi}$ and $g_2(e^{i\varphi}) = e^{i(2\gamma - \varphi)}$, whence

$$\text{(3.26)} \qquad (g_2 \circ g_1)^m(e^{i\varphi}) = e^{i(2m\gamma + \varphi)} \,, \ m \in N \,.$$

Let the centre of the arc I be the point $e^{i\beta}$. To prove the lemma it suffices to show that the set $B = \{(g_2 \circ g_1)^m(e^{i\beta}) \mid m \in N\}$ is dense in S. Now $B = \{e^{i(2m\gamma + \beta)} \mid m \in N\}$ according to (3.26), and thus we need only show that the set $B' = \{e^{i2m\gamma} \mid m \in N\}$ is dense in S.

Now if $l, m \in N$ and $l > m$, then $e^{i2l\gamma} \neq e^{i2m\gamma}$ because otherwise $2l\gamma = 2m\gamma + k2\pi$, for some $k = 0, \pm 1, \pm 2, \ldots$, which implies $\gamma/\pi = \dfrac{k}{(l - m)}$, and this contradicts the irrationality of γ/π. Then B' is an infinite set and it has at least one accumalation point in S, which implies that every point of S is an accumalation point of B', since $B' \supset \{z^m \mid m \in N\}$ for every $z \in B'$. The lemma follows.

3.27. Let Sym be a k-dimensional Steiner symmetrization in R^n, $2 \leqq k \leqq n$, and T its symmetry plane. We associate with Sym two $(k-1)$-dimensional Steiner symmetrizations Sym_i, $i = 1, 2$, as follows. Choose two unit vectors b_1, b_2 such that they are perpendicular to T, angle $(b_1, b_2) = \gamma > 0$ and γ/π is irrational. Let Sym_i be the $(k-1)$-dimensional Steiner symmetrization defined by the plane $T + E^1(b_i)$, $i = 1, 2$. For the following lemma let $L(x)$, $x \in T$, denote the symmetrizing plane of Sym through x, and let $\mathcal{B}_x \subset \mathcal{F}$ denote the collection $\{L(x) \cap \bar{B}^n(x, r) \mid r > 0\} \cup \{x\}$.

3.28. **Lemma.** *If* Sym_i, $i = 1, 2$, *are closed, then* $f = \mathrm{Sym}_2 \circ \mathrm{Sym}_1$ *is a regular set transformation which is rounding with respect to* $\mathcal{B}_x$ *for every* $x \in T$, *and*

$$f(F) \cap L(x) = f(F \cap L(x)), \tag{3.29}$$

$$m_k(f(F) \cap L(x)) = m_k(F \cap L(x)), \quad \textit{for every} \quad F \in \mathcal{F}. \tag{3.30}$$

Proof. The set transformations Sym_1 and Sym_2 are regular because they are closed, and thus f is also regular.

To prove the other assertions let $a_1, \ldots, a_k$ be mutually orthogonal unit vectors such that every a_i is perpendicular to T and $a_i \in E^2(b_1, b_2)$ for i = 1,2, and let b_1' and b_2' be unit vectors in $E^2(b_1, b_2)$ such that b_i' is orthogonal to b_i for i = 1,2. If $x \in T$, then $L(x) = \{x\} + E^k(a_1, \ldots, a_k)$, and for every $t \in R^1$

$$L_1(t) = \{x + tb_1\} + E^{k-1}(b_1', a_3, \ldots, a_k),$$

$$L_2(t) = \{x + tb_2\} + E^{k-1}(b_2', a_3, \ldots, a_k)$$

are the symmetrizing planes of Sym_1 and Sym_2, respectively. Furthermore,

$$L(x) = \bigcup_{t \in R^1} L_i(t), \; i = 1, 2. \tag{3.31}$$

Now (3.31) clearly implies (3.29). By Fubini's theorem (3.30) follows from (3.31).

We must still prove that f is rounding with respect to $\mathcal{B}_x$. Clearly, by (3.31), $f(B) = B$ for every $B \in \mathcal{B}_x$. Let $F \in \mathcal{F}$ be such that $F \subset B$, $F \neq B$ for some $B \in \mathcal{B}_x$. Then we have to show that there exist $i \in N$ and $B' \in \mathcal{B}_x$ such that $f^i(F) \subset B' \subset B$, $B' \neq B$. Obviously B cannot be the point x.

Let H be the boundary of B with respect to $L(x)$. We need only prove that for some $i \in N$, $H \setminus f^i(F) = H$. Let $C = H \cap M$, where $M = \{x\} + E^2(b_1, b_2)$. By the position of the symmetrizations Sym_1

and Sym_2 we see that for every compact set $A \subset B$ and for every $L_i(t)$, $t \in R^1$, $i = 1, 2$, the following holds:

(3.32) if $L_i(t) \cap (B \setminus A) \neq \emptyset$, then $L_i(t) \cap H \subset H \setminus \mathrm{Sym}_i(A)$.

Let $g_i : M \to M$ be the reflection in the 2-plane M with respect to the straight line $\{x\} + E^1(b_i)$, $i = 1, 2$. If we define for every $A \subset C$,

$$Z_i(A) = \bigcup_{L_i(t) \cap A \neq \emptyset} L_i(t) \cap H, \; i = 1, 2,$$

then clearly $Z_i[A] = Z_i[A \cup g_i(A)]$, $i = 1, 2$.

Now since $F \neq B$, there exists an open ball $B^n(y, r)$, $y \in B$, $r > 0$, such that $B^n(y, r) \cap B \subset B \setminus F$. Hence there exists an open arc I of C such that if $L_1(t) \cap I \neq \emptyset$, then $L_1(t) \cap (B \setminus F) \neq \emptyset$. But this implies by (3.32) that $I \cup g_1(I) \subset Z_1[I] \subset H \setminus \mathrm{Sym}_1(F)$, whence also by (3.32)

$$H \setminus f(F) \supset Z_2[I \cup g_1(I)] = Z_2[I \cup g_1(I) \cup g_2(I \cup g_1(I))] \supset$$

$$Z_2[I \cup g_2 \circ g_1(I)].$$

Continuing in a similar way we get

(3.33) $$H \setminus f^i(F) \supset Z_2[I \cup \bigcup_{j=1}^{i} (g_2 \circ g_1)^j(I)], \; i = 1, 2, \ldots .$$

But now by Lemma 3.25 there exists $m \in N$ such that $C = \bigcup_{j=1}^{m} (g_2 \circ g_1)^j(I)$, whence by (3.33), $H \setminus f^m(F) \supset Z_2[C] = H$, which proves the lemma.

3.34. **Lemma.** *Let* Sym *be a k-dimensional Steiner symmetrization in* R^n, $2 \leq k \leq n$, *and let* Sym_1, Sym_2 *be the two* $(k-1)$*-dimensional Steiner symmetrizations associated with* Sym *as in Lemma* 3.28. *If* Sym_1 *and* Sym_2 *are closed and smoothing, then*

(3.35) $$\mathrm{Sym}(F) = \lim_i (\mathrm{Sym}_2 \circ \mathrm{Sym}_1)^i(F) \quad \textit{for every} \quad F \in \mathcal{F}.$$

Proof. Let T be the symmetry plane of Sym and let, for every $x \in T$, $L(x)$ and $\mathcal{B}_x$ be as in Lemma 3.28. Put $f = \mathrm{Sym}_2 \circ \mathrm{Sym}_1$ and choose $F \in \mathcal{F}$. To prove (3.35) we apply Lemma 3.9 to f, where $I = T$, $\{P_\alpha \mid \alpha \in I\} = \{L(x) \mid x \in T\}$, and $\mathcal{B}_x = \mathcal{B}_\alpha$, whenever $L(x) = P_\alpha$. Hence, arguing as in Lemma 3.21, we see that Lemmas 3.9 and 3.5 with the preceding Lemma 3.28 yield (3.35).

4. Further properties of cap and Steiner symmetrizations

In this chapter we prove that every symmetrization is open, closed, and smoothing. We first show that 1-dimensional cap symmetrizations have these properties. We then show that 1-dimensional Steiner symmetrizations have these properties by establishing a connection between 1-dimensional Steiner and cap symmetrizations. Finally, using the results 3.21 and 3.34 we prove by induction that all symmetrizations have these properties.

4.1. Every 1-dimensional cap symmetrization in R^n has, by definition, an $(n-1)$-dimensional symmetry plane and 1-dimensional symmetrizing spheres. To treat these symmetrizations in detail we consider some geometrical properties of an arbitrary 1-dimensional sphere $K \subset R^n$ with centre x_0 and radius $r > 0$.

For every m_1-measurable set $A \subset K$ we define the *angle measure* $\varphi(A)$ by the formula $\varphi(A) = \dfrac{1}{r} m_1(A)$. For every $0 < \theta \leqq \pi$ and $A \subset K$ we call the set

$$\operatorname{par}_\theta(A) = \bigcup_{z \in A} \{y \in K \mid \operatorname{angle}(y - x_0, z - x_0) \leqq \theta\} \tag{4.2}$$

the *parallel set* of A in K with the radius θ.

4.3. **Lemma.** (*1-dimensional Brunn-Minkowski inequality in spherical geometry*) *Let* $A \subset K$ *be a closed set and* I_A *a closed arc of* K *such that* $\varphi(A) = \varphi(I_A)$. *Then*

$$\varphi[\operatorname{par}_\theta(A)] \geqq \varphi[\operatorname{par}_\theta(I_A)],$$

for every θ, $0 < \theta \leqq \pi$.

Proof. If $A = K$, the lemma is trivial. If $A \neq K$, then $K \setminus A$ consists of countably many disjoint open arcs $I_1, I_2, \ldots$. If $\varphi(I_i) \leqq 2\theta$ for every I_i, then $\operatorname{par}_\theta(A) = K$ and the lemma holds. If $\varphi(I_i) > 2\theta$ for some I_i, then $\varphi(A) = \varphi(I_A) < 2\pi - 2\theta$, and $\varphi[\operatorname{par}_\theta(A)] \geqq \varphi(A) + 2\theta = \varphi[\operatorname{par}_\theta(I_A)]$.

4.4. **Theorem.** *Every* 1-*dimensional cap symmetrization is open, closed, and smoothing.*

Proof. By Lemma 2.14 we need only prove the smoothing property. Let Sym be a 1-dimensional cap symmetrization, F a closed set in R^n and $q > 0$. We must prove that

(4.5) $$\mathrm{Sym}\,(F + q\bar{B}^n) \supset \mathrm{Sym}\,(F) + q\bar{B}^n .$$

Let J be the symmetrizing axis of Sym. Choose any $x_0 \in \mathrm{Sym}\,(F)$ and let $K(z_0, r_0)$ denote the symmetrizing sphere of Sym through x_0 with $z_0 \in J$ and $r_0 \geqq 0$, and put

$$A = K(z_0, r_0) \cap F \quad \text{and} \quad A^* = K(z_0, r_0) \cap \mathrm{Sym}\,(F) .$$

Hence, to prove (4.5), we need only show that

(4.6) $$A^* + q\bar{B}^n \subset \mathrm{Sym}\,(A + q\bar{B}^n) .$$

Clearly we can suppose that $r_0 > 0$. Let $z \in J$, $r \geqq 0$, and let $K(z, r)$ be the corresponding symmetrizing sphere. We write

$$I = K(z, r) \cap \mathrm{Sym}\,(A + q\bar{B}^n) \quad \text{and} \quad I^* = K(z, r) \cap (A^* + q\bar{B}^n) .$$

Then to establish (4.6) it suffices to show that $I^* \subset I$, or equivalently,

(4.7) $$\varphi(I^*) \leqq \varphi(I) ,$$

since I and I^* are either simultaneously empty or two concentric, possibly degenerate, arcs in $K(z, r)$. We can suppose that $|z - z_0|^2 + |r - r_0|^2 \leqq q^2$, for otherwise $I = I^* = \emptyset$, and that $r > 0$, for the case $r = 0$ is trivial. Define a mapping $p : K(z_0, r_0) \to K(z, r)$ by setting

$$p(x) = z + \frac{r}{r_0}(x - z_0) ,\ x \in K(z_0, r_0) ,$$

and consider, in the spheres $K(z_0, r_0)$ and $K(z, r)$, the parallel sets defined by the formula (4.2) with the fixed radius $\Theta = \frac{1}{2}\varphi[K(z, r) \cap \bar{B}^n(x_0, q)]$. It is not difficult to verify the following results: for every m_1-measurable set $D \subset K(z_0, r_0)$

(4.8) $$K(z, r) \cap (D + q\bar{B}^n) = \mathrm{par}_\Theta\,[p(D)] ,$$

(4.9) $$\mathrm{par}_\Theta\,[p(D)] = p\,[\mathrm{par}_\Theta\,(D)] , \quad \text{and}$$

(4.10) $$\varphi\,[p(D)] = \varphi(D) .$$

Now applying first the definition of the 1-dimensional cap symmetrization and then the formulas (4.8—4.10) we get

(4.11) $$\varphi(I) = \varphi\,[K(z, r) \cap \mathrm{Sym}\,(A + q\bar{B}^n)] = \varphi[K(z, r) \cap (A + q\bar{B}^n)] = \varphi\,\{\mathrm{par}_\Theta\,[p(A)]\} = \varphi\,\{p\,[\mathrm{par}_\Theta\,(A)]\} = \varphi\,[\mathrm{par}_\Theta\,(A)] .$$

Similarly, we get

(4.12) $$\varphi(I^*) = \varphi\,[K(z, r) \cap (A^* + q\bar{B}^n)] = \varphi\,[\mathrm{par}_\Theta\,(A^*)] .$$

By the definition of Sym the set A^* is a closed arc and $\varphi(A) = \varphi(A^*)$. Hence, equations (4.11), (4.12), and Lemma 4.3 imply the inequality (4.7), and the proof is complete.

4.13. *The connection between* 1-*dimensional Steiner symmetrizations and* 1-*dimensional cap symmetrizations.* Let Sym be a 1-dimensional Steiner symmetrization defined by an $(n-1)$-dimensional plane $T \subset R^n$. We associate with Sym a 1-dimensional cap symmetrization S and a homeomorphism $g : G \to G'$, G and G' domains in R^n, such that

$$\text{(4.14)} \qquad \operatorname{Sym}(A) = g \circ S \circ g^{-1}(A)$$

for every $A \in \operatorname{Dom}(\operatorname{Sym}) \cap \{B \subset R^n \mid B \subset G'\}$.

Choose an $(n-2)$-dimensional subplane J of T and two mutually orthogonal unit vectors a, b such that a is perpendicular to T and b is perpendicular to J. Let S denote the 1-dimensional cap symmetrization defined by the half-plane $T_s = J + \{tb \mid t \geqq 0\} \subset T$. Hence, J is the symmetrizing axis of S and T is the common symmetry plane of S and Sym.

For every $x \in R^n \setminus J$ let $z(x) \in J$ and $r(x) > 0$ be such that x is in the symmetrizing sphere $K(z(x), r(x))$ of S. Put $\varphi(x) = \operatorname{sign}[(x - z(x)) \cdot a] \operatorname{angle}(x - z(x), b)$, and let $p(x)$ be the point $K(z(x), r(x)) \cap T_s$. Further, put $G = R^n \setminus \operatorname{cl}(T \setminus T_s)$ and $G' = \{x \in R^n \setminus J \mid |\varphi(x)| < \arctan \pi\}$, where $0 < \arctan \pi < \pi/2$, and define the mapping

$$\text{(4.15)} \qquad g : G \to G', \ g(x) = p(x) + \varphi(x) r(x) a, \ x \in G.$$

Clearly g is a homeomorphism and by the construction of g we see that (4.14) holds.

Using the above notation, we represent $\operatorname{Sym}(A)$, for every $A \in \operatorname{Dom}(\operatorname{Sym})$, in terms of S. Define, for every $i \in N$, the translation

$$\text{(4.16)} \qquad t_i : R^n \to R^n, \ t_i(x) = x + ib, \ x \in R^n.$$

Now for every $A \in \operatorname{Dom}(\operatorname{Sym})$ we get $\operatorname{Sym}(A) = t_i^{-1} \circ \operatorname{Sym} \circ t_i(A)$, $i = 1, 2, \ldots$, and, since A is bounded there exists $i_A \in N$ such that $t_i(A) \subset G'$ for every $i \geqq i_A$, whence by (4.14)

$$\text{(4.17)} \qquad \operatorname{Sym}(A) = t_i^{-1} \circ g \circ S \circ g^{-1} \circ t_i(A), \text{ for } i \geqq i_A.$$

Consider now the mappings $g : G \to G'$ and $g^{-1} : G' \to G$. By a straightforward calculation we easily see that for every $s > 0$ and $\varepsilon > 0$ there exists $i_0 \in N$ such that $g(x), g^{-1}(x) \in B^n(x, \varepsilon)$ for every $x \in t_i(B^n(z, s))$, $i \geqq i_0$ and $z \in J$. Using this result and the above notation, we easily get:

4.18. **Lemma.** *If* $0 < \varepsilon < s$, *then there exists* $i_0 \in N$ *such that if* $i \geqq i_0$, *then* $t_i(B^n(z, s)) \subset G'$ *for any* $z \in J$ *and for every compact set* $A \subset t_i(B^n(z, s))$

(i) $g(A) \subset A + \varepsilon \bar{B}^n$,

(ii) $g^{-1}(A) \subset A + \varepsilon \bar{B}^n$,

(iii) $A \subset g(A + \varepsilon \bar{B}^n)$ *and*

(iv) $A \subset g^{-1}(A + \varepsilon \bar{B}^n)$.

Using the above lemma we now prove:

4.19. **Lemma.** *Let* Sym *be a* 1-*dimensional Steiner symmetrization,* S *the* 1-*dimensional cap symmetrization associated with* Sym *as above and* $t_i : R^n \to R^n$, $i \in N$, *the translations in* (4.16). *Then, for any compact set* $F \subset R^n$ *and* $r > 0$, *there exists* $i_0 \in N$ *such that for all* $i \geqq i_0$

$$\mathrm{Sym}\,(F) \subset t_i^{-1} \circ S \circ t_i(F + r\bar{B}^n) \quad \textit{and} \tag{4.20}$$

$$t_i^{-1} \circ S \circ t_i(F) \subset \mathrm{Sym}\,(F + r\bar{B}^n)\,. \tag{4.21}$$

Proof. Let F be a compact set and $r > 0$. Let J be the symmetrizing axis of S, $z \in J$, and choose $\frac{s}{2} > r$ so that $F \subset B^n\left(z, \frac{s}{2}\right)$. Then, for s and $\varepsilon = \frac{r}{2}$, choose $i_0 \in N$ such that the conditions (i)—(iv) of Lemma 4.18 hold. Hence, $t_i(F) + \frac{r}{2}\bar{B}^n \subset t_i(B^n(z, s))$ for all $i \in N$, and by (ii) and (iv) of 4.18 we get for every $i \geqq i_0$

$$g^{-1} \circ t_i(F) \subset t_i(F) + \frac{r}{2}\bar{B}^n\,, \tag{4.22}$$

$$t_i(F) \subset g^{-1}\left(t_i(F) + \frac{r}{2}\bar{B}^n\right) \quad \text{and} \tag{4.23}$$

$$t_i\left(F + \frac{r}{2}\bar{B}^n\right) \subset g^{-1}\left(t_i\left(F + \frac{r}{2}\bar{B}^n\right) + \frac{r}{2}\bar{B}^n\right). \tag{4.24}$$

Since $z \in J$, we get $S(t_i(B^n(z, s))) = t_i(B^n(z, s))$. By (4.22) we get $g^{-1} \circ t_i(F) \subset t_i(B^n(z, s))$, and so $S \circ g^{-1} \circ t_i(F) \subset t_i(B^n(z, s))$ for all $i \geqq i_0$. Hence by (i) of 4.18

$$g(S \circ g^{-1} \circ t_i(F)) \subset S \circ g^{-1} \circ t_i(F) + \frac{r}{2}\bar{B}^n \quad \text{for } i \geqq i_0\,. \tag{4.25}$$

Similarly we find that $S \circ t_i(F) \subset t_i(B^n(z, s))$, $i \geqq i_0$, and then by (iii) of 4.18

$$S \circ t_i(F) \subset g\left(S \circ t_i(F) + \frac{r}{2}\bar{B}^n\right), \ i \geqq i_0 . \tag{4.26}$$

To prove (4.20) we fix $i \geqq i_0$ and observe that $t_i(F) \subset G'$. Then using (4.17), (4.25), the smoothing property of S proved in Theorem 4.4, and finally (4.22) we get

$$\mathrm{Sym}\,(F) = t_i^{-1} \circ g \circ S \circ g^{-1} \circ t_i(F) \subset t_i^{-1}\left(S \circ g^{-1} \circ t_i(F) + \frac{r}{2}\bar{B}^n\right) \subset$$

$$t_i^{-1} \circ S\left(g^{-1} \circ t_i(F) + \frac{r}{2}\bar{B}^n\right) \subset t_i^{-1} \circ S\left(t_i(F) + \frac{r}{2}\bar{B}^n + \frac{r}{2}\bar{B}^n\right) =$$

$$t_i^{-1} \circ S \circ t_i(F + r\bar{B}^n),$$

which proves (4.20). To prove (4.21) we again fix $i \geqq i_0$ and observe that $t_i(F + r\bar{B}^n) \subset t_i(B^n(z, s)) \subset G'$. Then using (4.24), the smoothing property of S and (4.26) we get

$$\mathrm{Sym}\,(F + r\bar{B}^n) = t_i^{-1} \circ g \circ S \circ g^{-1} \circ t_i(F + r\bar{B}^n) =$$

$$t_i^{-1} \circ g \circ S \circ g^{-1}\left(t_i\left(F + \frac{r}{2}\bar{B}^n\right) + \frac{r}{2}\bar{B}^n\right) \supset t_i^{-1} \circ g \circ S\left(t_i\left(F + \frac{r}{2}\bar{B}^n\right)\right) =$$

$$t_i^{-1} \circ g \circ S\left(t_i(F) + \frac{r}{2}\bar{B}^n\right) \supset t_i^{-1} \circ g\left(S \circ t_i(F) + \frac{r}{2}\bar{B}^n\right) \supset t_i^{-1} \circ S \circ t_i(F),$$

which proves (4.21).

4.27. **Theorem.** *Every* 1-*dimensional Steiner symmetrization is open, closed, and smoothing.*

Proof. We employ the same notation as above. If $A \in \mathrm{Dom}\,(\mathrm{Sym})$, then by (4.17) there exists $i \in N$ such that $t_i(A) \subset G'$ and $\mathrm{Sym}\,(A) = t_i^{-1} \circ g \circ S \circ g^{-1} \circ t_i(A)$. Since g and t_i are homeomorphisms and S preserves open and closed sets, $\mathrm{Sym}\,(A)$ is open or closed according as A is open or closed.

To prove the smoothing property of Sym we use Lemma 4.19. Let F be a compact set and $r > 0$. Then we have to show that $\mathrm{Sym}\,(F + r\bar{B}^n) \supset \mathrm{Sym}\,(F) + r\bar{B}^n$. For this, it suffices to show that

$$\mathrm{Sym}\,(F + r\bar{B}^n) \supset \mathrm{Sym}\,(F) + r'\bar{B}^n \tag{4.28}$$

for every r', $0 < r' < r$. To prove (4.28) put $r'' = \frac{1}{2}(r - r')$. By Lemma

4.19 we can choose $i \in N$ so that $\mathrm{Sym}\,[(F + (r' + r'')\bar{B}^n) + r''\bar{B}^n] \supset t_i^{-1} \circ S \circ t_i(F + (r' + r'')\bar{B}^n)$ and $t_i^{-1} \circ S \circ t_i(F + r''\bar{B}^n) \supset \mathrm{Sym}\,(F)$. These relations and the smoothing property of S yield

$$\mathrm{Sym}\,(F + r\bar{B}^n) = \mathrm{Sym}\,[(F + (r' + r'')\bar{B}^n) + r''\bar{B}^n] \supset$$

$$t_i^{-1} \circ S \circ t_i(F + (r' + r'')\bar{B}^n) = t_i^{-1} \circ S(t_i(F + r''\bar{B}^n) + r'\bar{B}^n) \supset$$

$$t_i^{-1} \circ S \circ t_i(F + r''\bar{B}^n) + r'\bar{B}^n \supset \mathrm{Sym}\,(F) + r'\bar{B}^n\,,$$

which proves (4.28).

We now are ready to establish the main properties of symmetrizations.

4.29. **Theorem.** *Every cap symmetrization in R^n is open, closed, and smoothing. Further, if* Sym *is a k-dimensional cap symmetrization, $2 \leq k \leq n - 1$, then there exist $(k - 1)$-dimensional cap symmetrizations* Sym_1 *and* Sym_2 *such that*

$$\mathrm{Sym}\,(F) = \lim_i\,(\mathrm{Sym}_2 \circ \mathrm{Sym}_1)^i(F) \tag{4.30}$$

for every compact non-empty set $F \subset R^n$.

Proof. We proceed by induction. By Theorem 4.4 every 1-dimensional cap symmetrization is open, closed, and smoothing. Let $2 \leq k \leq n - 1$, and assume that every $(k - 1)$-dimensional cap symmetrization is closed and smoothing. Let Sym be a k-dimensional cap symmetrization. Then by Lemma 3.21 we can choose two $(k - 1)$-dimensional cap symmetrizations Sym_i, $i = 1, 2$, such that (4.30) is valid. To prove that Sym is smoothing we observe that by (4.30) and Lemma 2.4 for every $F \in \mathcal{F}$

$$\mathrm{Sym}\,(F + r\bar{B}^n) \supset \mathrm{Sym}\,(F) + r\bar{B}^n\,, \tag{4.31}$$

since Sym_i, $i = 1, 2$, is smoothing. Let J be the symmetrizing axis of Sym. If A is any closed set in R^n, then $A = \bigcup_{i=1}^{\infty} F_i$, where $F_i = A \cap \bar{B}^n(z, i)$, $i \in N$, and $z \in J$. Hence, $\mathrm{Sym}\,(A) = \bigcup_{i=1}^{\infty} \mathrm{Sym}\,(F_i)$. If $r > 0$, we get by (4.31)

$$\mathrm{Sym}\,(A + r\bar{B}^n) = \mathrm{Sym}\,(\bigcup_{i=1}^{\infty}(F_i + r\bar{B}^n)) \supset \bigcup_{i=1}^{\infty}\mathrm{Sym}\,(F_i + r\bar{B}^n) \supset$$

$$\bigcup_{i=1}^{\infty}(\mathrm{Sym}\,(F_i) + r\bar{B}^n) = (\bigcup_{i=1}^{\infty}\mathrm{Sym}\,(F_i)) + r\bar{B}^n = \mathrm{Sym}\,(A) + r\bar{B}^n\,,$$

and thus Sym is smoothing. Hence, by Lemma 2.14, Sym is open and closed, and the proof of the theorem is complete.

4.32. **Theorem.** *Every Steiner symmetrization in* R^n *is open, closed, and smoothing. Further, if* Sym *is a* k*-dimensional Steiner symmetrization,* $2 \leqq k \leqq n$, *then there exist* $(k-1)$*-dimensional Steiner symmetrizations* Sym_1 *and* Sym_2 *such that*

$$\mathrm{Sym}\,(F) = \lim\,(\mathrm{Sym}_2 \circ \mathrm{Sym}_1)^i(F) \tag{4.33}$$

for every compact non-empty set $F \subset R^n$.

Proof. Proceeding by induction, similarly as in the previous theorem, we see that the theorem follows from Theorem 4.27 and Lemmas 3.34, 2.4, 2.10 and 2.3.

4.34. *Remark.* In the preceding theorems we used the 1-dimensional Brunn-Minkowski inequality to establish the smoothing property of 1-dimensional cap symmetrizations and, from this, we derived the smoothing property of all symmetrizations.

On the other hand, the general Brunn-Minkowski inequalities can be easily proved by the smoothing property of Steiner and cap symmetrizations. These inequalities are the following:

(4.35) (*k-dimensional Brunn-Minkowski inequality in spherical geometry,* see [10]) If K is a k-dimensional sphere in R^n, $1 \leqq k \leqq n-1$, A a closed set in K and $0 < \theta \leqq \pi$, then $m_k[\mathrm{par}_\theta\,(A)] \geqq m_k\,[\mathrm{par}_\theta\,(I_A)]$, where I_A is some closed cap of K with $m_k(I_A) = m_k(A)$ and $\mathrm{par}_\theta\,(B) = \bigcup_{x \in B} \{y \in K \mid \text{angle}\,(y - x_0\,, x - x_0) \leqq \theta\}$ for every $B \subset K$, where x_0 is the centre of K.

(4.36) (*Brunn-Minkowski inequality in euclidean geometry,* see [3, p. 174—175]) If A is a compact set in R^n and B a closed ball such that $m(A) = m(B)$ and $r > 0$, then $m(A + r\bar{B}^n) \geqq m(B + r\bar{B}^n)$.

In fact, to prove (4.35) let K, A, I_A and θ be as in (4.35), and let Sym be a k-dimensional cap symmetrization with K as a symmetrizing sphere and $I_A = \mathrm{Sym}\,(A)$. Clearly $r > 0$ can be chosen so that $\mathrm{par}_\theta\,(A) = (A + r\bar{B}^n) \cap K$ and $\mathrm{par}_\theta\,(I_A) = (I_A + r\bar{B}^n) \cap K$. Now $\mathrm{Sym}\,(A + r\bar{B}^n) \supset \mathrm{Sym}\,(A) + r\bar{B}^n = I_A + r\bar{B}^n$, since Sym is smoothing. Then

$$m_k\,[\mathrm{par}_\theta\,(A)] = m_k\,[(A + r\bar{B}^n) \cap K] = m_k\,[\mathrm{Sym}\,(A + r\bar{B}^n) \cap K] \geqq$$

$$m_k\,[(\mathrm{Sym}\,(A) + r\bar{B}^n) \cap K] = m_k\,[\mathrm{par}_\theta\,(I_A)]\,,$$

which proves (4.35). Similarly (4.36) can be proved using the smoothing property of n-dimensional Steiner symmetrizations.

The idea of proving (4.35) and (4.36) by symmetrizations is well-known and was used by Schmidt [10] and by Hadwiger [3]. Hadwiger, however, uses Steiner symmetrizations in a different way than we, and Schmidt's symmetrizations are not cap symmetrizations as we define them.

5. Condensers and p-capacity

In the first chapter we defined a condenser in R^n and its p-capacity for $p > 0$. In this chapter we give further results concerning these concepts and, in particular, an equivalent definition of the p-capacity.

5.1. **Lemma.** *If* Sym *is a cap or Steiner symmetrization and* (A, C) *a condenser such that* $A \in \mathrm{Dom}\,(\mathrm{Sym})$, *then* $(\mathrm{Sym}\,(A), \mathrm{Sym}\,(C))$, *the symmetrization of* (A, C) *under* Sym, *is also a condenser. If, in addition,* (A, C) *is ringlike, then* $(\mathrm{Sym}\,(A), \mathrm{Sym}\,(C))$ *is also ringlike.*

Proof. Since symmetrizations are open, monotone, and preserve compact sets, $(\mathrm{Sym}\,(A), \mathrm{Sym}\,(C))$ is a condenser. If (A, C) is ringlike, then by definition $A \setminus C$, $\bar{R}^n \setminus A$ and C are connected. Then A and $\bar{R}^n \setminus C$ are also connected, since $C \subset A$. Hence, by Lemma 2.15, $\mathrm{Sym}\,(A)$, $\mathrm{Sym}\,(C)$, $\bar{R}^n \setminus \mathrm{Sym}\,(A)$ and $\bar{R}^n \setminus \mathrm{Sym}\,(C)$ are connected. Thus the components of $\bar{R}^n \setminus (\mathrm{Sym}\,(A) \setminus \mathrm{Sym}\,(C))$ are $\mathrm{Sym}\,(C)$ and $\bar{R}^n \setminus \mathrm{Sym}\,(A)$. Since $\mathrm{Sym}\,(A)$ is open, $\mathrm{Sym}\,(C)$ is closed, $\mathrm{Sym}\,(C) \subset \mathrm{Sym}\,(A)$ and both $\mathrm{Sym}\,(A)$ and $\bar{R}^n \setminus \mathrm{Sym}\,(C)$ are connected, the Phragmen-Brouwer theorem [5, p. 359] implies that $\mathrm{Sym}\,(A) \setminus \mathrm{Sym}\,(C)$ is a domain. Hence $(\mathrm{Sym}\,(A), \mathrm{Sym}\,(C))$ is ringlike, and the proof is complete.

5.2. The condenser (A', C') is said to separate the condenser (A, C) if $A' \subset A$ and $C \subset C'$. If (A', C') separates (A, C), then for all $p > 0$

$$\mathrm{cap}_p\,(A, C) \leqq \mathrm{cap}_p\,(A', C')\,, \tag{5.3}$$

since then $W(A', C') \subset W(A, C)$, see (1.1).

We call $f : R^n \to R^n$ *orthogonal* if f is linear and $|f(x)| = |x|$ for all $x \in R^n$. If $f : R^n \to R^n$ is an orthogonal mapping or a translation, then for every condenser (A, C) and $p > 0$

$$\mathrm{cap}_p\,(A, C) = \mathrm{cap}_p\,(f(A), f(C))\,, \tag{5.4}$$

since $|\nabla(u \circ f)(x)| = |\nabla u(f(x))|$ whenever $u \in W(A, C)$ and $\nabla u(f(x))$ exists. In other words, the p-capacity is invariant under translations and orthogonal mappings.

We say that a condenser (A, C) is *bounded* if A is bounded.

5.5. **Lemma.** *If* (A, C) *is a condenser,* $p > 0$ *and* $\varepsilon > 0$, *then there exists an* $r > 0$ *and a bounded condenser* (A', C') *such that* $\bar{A}' + r\bar{B}^n \subset A$, $C + r\bar{B}^n \subset C'$ *and* $\operatorname{cap}_p(A, C) + \varepsilon \geqq \operatorname{cap}_p(A', C')$.

Proof. Choose $0 < a < 1$ such that $\left(\frac{1}{1-a}\right)^p (\operatorname{cap}_p(A, C) + a) \leqq \operatorname{cap}_p(A, C) + \varepsilon$, and choose $u \in W(A, C)$ such that $\int_{R^n} |\nabla u|^p \, dm \leqq \operatorname{cap}_p(A, C) + a$. Put $B = \{x \in R^n \mid 0 \leqq u(x) < 1\}$, whence $\bar{B}$ is a compact subset of A by the definition of $W(A, C)$. Let $r = \min\left\{\frac{r_1}{3}, r_2\right\}$, where $r_1 = d(\bar{B}, \complement A) > 0$ and $r_2 = d(C, \{x \in R^n \mid a \leqq u(x)\}) > 0$. If we now put $A' = B + rB^n$ and $C' = C + r\bar{B}^n$, then (A', C') is a bounded condenser such that $\bar{A}' + r\bar{B}^n \subset A$ and $C + r\bar{B}^n \subset C'$. Furthermore, if we define $v : R^n \to R^1$ by setting

$$v(x) = \begin{cases} 0, & \text{if } 0 \leqq u(x) \leqq a, \\ \dfrac{u(x) - a}{1 - a}, & \text{if } a < u(x), \end{cases}$$

then $v \in W(A', C')$, and thus

$$\operatorname{cap}_p(A', C') \leqq \int_{R^n} |\nabla v|^p \, dm \leqq \left(\frac{1}{1-a}\right)^p \left(\int_{R^n} |\nabla u|^p \, dm\right) \leqq$$

$$\left(\frac{1}{1-a}\right)^p (\operatorname{cap}_p(A, C) + a) \leqq \operatorname{cap}_p(A, C) + \varepsilon,$$

which proves the lemma.

For the sake of completeness, we have defined the p-capacity of a condenser for all $p > 0$. In fact, the following lemma shows that the only interesting case is $p \geqq 1$.

5.6. **Lemma.** *If* $0 < p < 1$, *then* $\operatorname{cap}_p(A, C) = 0$ *for every condenser* (A, C).

Proof. Choose a compact polyhedron P so that P is a finite union of closed cubes in R^n and $C \subset P \subset A$. Then $\operatorname{cap}_p(A, C) \leqq \operatorname{cap}_p(A, P)$ by the separation inequality (5.3). Let $0 < r < \min\{1, d(P, \complement A)\}$ and define $u : R^n \to R^1$ by $u(x) = \min\left\{1, \frac{1}{r} d(x, P)\right\}$ for $x \in R^n$. Then $u \in W(A, P)$ and

$$|\nabla u(x)| = \begin{cases} \dfrac{1}{r} \text{ for almost every } x \in (P + r\bar{B}^n) \setminus \operatorname{int} P \\ 0 \text{ for } x \in \complement((P + r\bar{B}^n) \setminus \operatorname{int} P) . \end{cases}$$

Then

$$\operatorname{cap}_p (A , C) \leqq \int_{R^n} |\nabla u|^p dm = \int_{(P+r\bar{B}^n)\setminus \operatorname{int} P} \frac{1}{r^p} dm \leqq \tag{5.7}$$
$$\sum_{i=1}^{k} \int_{Q_i + r\bar{B}^n} \frac{dm}{r^p} \leqq k(d + 2r)^{n-1} 2r \frac{1}{r^p} ,$$

where $Q_1 , \ldots , Q_k$ are the $(n - 1)$-dimensional cubes which form ∂P, and $d = \max \{d(Q_1) , \ldots , d(Q_k)\}$. Letting r tend to zero in (5.7), we obtain the lemma.

5.8. In the definition (1.1) of the p-capacity of a condenser (A , C) the set $W(A , C)$ can be replaced by several of its subsets without changing the value of $\operatorname{cap}_p (A , C)$. Next we define such a subset of a very special kind; it will be used, modifying an idea due to Anderson [1], in the proof of the capacity inequality for 1-dimensional cap symmetrizations.

By a *proper polyhedron* P in R^n we mean a compact set which is a finite union of n-dimensional simplices. A triangulation T of a proper polyhedron P is a finite collection of n-dimensional simplices such that $P = \bigcup_{\triangle \in T} \triangle$ and, for any $\triangle' , \triangle'' \in T$, the intersection $\triangle' \cap \triangle''$ is either empty or a $(k - 1)$-dimensional side or edge common to the simplices $\triangle'$ and $\triangle''$, where $1 \leqq k \leqq n$. A mapping $u : R^n \to R^m$ is said to be simplicial with respect to the triangulation T, if the restriction $u \mid \triangle$ is affine for every $\triangle \in T$.

5.9. **Definition.** *For a condenser (A , C) a function $u \in W_s(A , C)$ if and only if $u \in W(A , C)$ and there exist a proper polyhedron $P \subset A \setminus C$ and its triangulation T such that*

(i) *u is identically equal to 0 or 1 in every component of $\complement P$,*

(ii) *u is simplicial with respect to T, and*

(iii) *if $a_1 , a_2 , \ldots , a_k$ are the values assumed by u at the vertices of the simplices in T, then for every $a \in (0 , 1) \setminus \{a_1 , \ldots , a_k\}$ the preimage $u^{-1}\{a\}$ consists of finitely many $(n - 1)$-dimensional simplices whose normals are not perpendicular to the plane $E^2(e_{n-1} , e_n)$.*

5.10. **Lemma.** *For every condenser* $(A\,,C)$ *and* $p \geqq 1$

$$\operatorname{cap}_p(A\,,C) = \inf_{u\in W_s(A,C)} \int_{R^n} |\nabla u|^p dm\,.$$

Proof. Given $\varepsilon > 0$. Choose $v \in W(A\,,C)$ such that

$$\int_{R^n} |\nabla v|^p dm \leqq \operatorname{cap}_p(A\,,C) + \frac{\varepsilon}{2}\,. \tag{5.11}$$

Using Gehring's technique [2, p. 502, Remark 5 and p. 504, section 7] of truncating, smoothing by integral averages, and triangulation of functions, we can form a piecewise linear approximation w of v such that $w \in W(A\,,C)$,

$$\int_{R^n} |\nabla v|^p dm + \frac{\varepsilon}{2} \geqq \int_{R^n} |\nabla w|^p dm\,, \tag{5.12}$$

and for a proper polyhedron $P' \subset A \setminus C$ and for its triangulation T'

(5.13) w is identically equal to 0 or 1 in every component of $\complement P'$, and w is simplicial with respect to T'.

Now let $a_1, \ldots, a_k$ be the values assumed by w at the vertices of the simplices in T'. If $\triangle \in T'$ and $w \mid \triangle$ is not constant, then $w \mid \triangle$ is an affine mapping and its level surfaces are parallel $(n-1)$-dimensional simplices. Then if $a \in (0\,,1) \setminus \{a_1, \ldots, a_k\}$, the set $w^{-1}\{a\}$ consists of $(n-1)$-dimensional simplices whose normals belong to a finite fixed set of vectors, say $\{p_1, \ldots, p_m\}$, which is uniquely determined by T' and w. Combining finitely many suitably chosen rotations of R^n with respect to different 3-dimensional linear subspaces, we can construct an orthogonal mapping $f: R^n \to R^n$ with the properties:

(5.14) $f(p_i)$ is not perpendicular to $E^2(e_{n-1}\,, e_n)$ for $i = 1\,,2\,,\ldots,m$, and

$$|f(x) - x| < d(P'\,, \complement(A \setminus C)) \text{ for every } x \in P'\,. \tag{5.15}$$

Finally we put $u = w \circ f^{-1}$. By (5.15) and the orthogonality of f, then $u \in W(A, C)$ and $\int_{R^n} |\nabla u|^p dm = \int_{R^n} |\nabla w|^p dm$. Hence by (5.11) and (5.12),

$$\text{(5.16)} \qquad \operatorname{cap}_p (A , C) + \varepsilon \geqq \int_{R^n} |\nabla u|^p dm .$$

Furthermore, $u \in W_s(A , C)$. For if we put $P = fP'$ and $T = \{f\triangle \mid \triangle \in T'\}$, then clearly (i) and (ii) of 5.9 are valid for u . On the other hand, u assumes the values $a_1 , \ldots , a_k$ at the vertices of the simplices of T , and if $a \in (0 , 1)\setminus\{a_1 , \ldots , a_k\}$ the preimage $u^{-1}\{a\} = f[w^{-1}\{a\}]$ consists of finitely many $(n - 1)$-dimensional simplices whose normals belong to the set $\{f(p_1) , \ldots , f(p_m)\}$, and so none of them is perpendicular to $E^2(e_{n-1} , e_n)$. Hence (iii) of 5.9 is also valid. Thus $u \in W_s(A , C) \subset W(A , C)$, which, together with (5.16) proves the lemma.

6. The capacity inequality for 1-dimensional cap symmetrizations

6.1. *Cap symmetrization of functions.* Let $u : R^n \to R^1$ be a continuous function and let Sym be a cap symmetrization. For every $a \in R^1$ define $F_a = \{x \in R^n \mid u(x) \leqq a\}$ and $G_a = \{x \in R^n \mid u(x) < a\}$. Then for every $x \in R^n$

$$\text{(6.2)} \qquad \{a \in R^1 \mid x \in \operatorname{Sym} (F_a)\} \neq \emptyset .$$

For, if we write $K(x)$ for the symmetrizing sphere through $x \in R^n$ and $a(x) = \sup uK(x) < \infty$, then for every $x \in R^n$ we get $x \in K(x) = \operatorname{Sym} (K(x)) \subset \operatorname{Sym} (F_{a(x)})$, which implies (6.2). We define $u^* : R^n \to R^1$ by setting $u^*(x) = \inf \{a \in R^1 \mid x \in \operatorname{Sym} (F_a)\} , x \in R^n$. The function u^* is called the *symmetrization of* u *under* Sym *and we write,* $u^* = \operatorname{Sym} (u)$. The following holds for u^* :

6.3. **Lemma.** *For every* $a \in R^1$

$$\text{(6.4)} \qquad \{x \in R^n \mid u^*(x) \leqq a\} = \operatorname{Sym} (F_a) \quad \textit{and}$$

$$\text{(6.5)} \qquad \{x \in R^n \mid u^*(x) < a\} = \operatorname{Sym} (G_a) ,$$

whence, in particular, $\{x \in R^n \mid u^*(x) = a\} = \operatorname{Sym} (F_a)\setminus\operatorname{Sym} (G_a)$.

Proof. Fix $a \in R^1$ and put $F_a^* = \{x \in R^n \mid u^*(x) \leqq a\}$. By the definition of u^* , $\operatorname{Sym} (F_a) \subset F_a^*$. Assume $x \in F_a^*$. Then $u^*(x) \leqq a$, and so $x \in \operatorname{Sym} (F_b)$ for every $b > a$. Thus, for a sequence $\{b_i\} \subset R^1$ with $b_1 > b_2 > \ldots$ and $\lim_i b_i = a$, we get, by the continuity from the outside of Sym , see 2. 10,

$$x \in \bigcap_i \operatorname{Sym} (F_{b_i}) = \operatorname{Sym} (\bigcap_i F_{b_i}) = \operatorname{Sym} (F_a) .$$

Hence $F_a^* \subset \text{Sym}(F_a)$ and (6.4) holds.

For a sequence $\{b_i\} \subset R^1$ with $b_1 < b_2 < \ldots$ and $\lim_i b_i = a$, we get, by the continuity from the inside of Sym and by (6.4),

$$\text{Sym}(G_a) = \text{Sym}(\bigcup_i G_{b_i}) = \bigcup_i \text{Sym}(G_{b_i}) = \bigcup_i \text{Sym}(F_{b_i}) =$$

$$\bigcup_i \{x \in R^n \mid u^*(x) \leqq b_i\} = \{x \in R^n \mid u^*(x) < a\},$$

which proves (6.5).

6.6. In particular, the above lemma and the continuity of u implies that u^* is also continuous, and because $u^{*-1}\{a\} = \text{Sym}(F_a) \setminus \text{Sym}(G_a)$, we easily see that $uK = u^*K$ for every symmetrizing sphere K of Sym. Further, the function u^* is symmetric with respect to the symmetry plane of Sym.

Next we show that the symmetrization of a Lipschitz function is also Lipschitzian.

6.7. **Lemma.** *If $u : R^n \to R^1$ satisfies, for some $M > 0$ and for every $y_1, y_2 \in R^n$, the condition*

(6.8) $$|u(y_1) - u(y_2)| \leqq M |y_1 - y_2|,$$

then $u^* = \text{Sym}(u)$ satisfies the condition

(6.9) $$|u^*(x_1) - u^*(x_2)| \leqq M |x_1 - x_2|$$

for every $x_1, x_2 \in R^n$, where Sym *is a cap symmetrization.*

Proof. Let $x_1, x_2 \in R^n$ and assume $u^*(x_2) \geqq u^*(x_1) = a$. Hence we have to prove that

(6.10) $$u^*(x_2) \leqq a + Md, \quad \text{where } d = |x_1 - x_2| > 0.$$

Now $\{x \mid u(x) \leqq a + Md\} \supset \{x \mid u(x) \leqq a\} + d\bar{B}^n$ by (6.8). Then by (6.4) and the smoothing property of Sym

$$\{x \mid u^*(x) \leqq a + Md\} = \text{Sym}\{x \mid u(x) \leqq a + Md\} \supset$$

$$\text{Sym}[\{x \mid u(x) \leqq a\} + d\bar{B}^n] \supset \text{Sym}\{x \mid u(x) \leqq a\} + d\bar{B}^n =$$

$$[\{x \mid u^*(x) \leqq a\} + d\bar{B}^n] \ni x_2.$$

So $x_2 \in \{x \mid u^*(x) \leqq a + Md\}$, which implies (6.10), and the lemma is proved.

6.11. **Lemma.** *If* $u \in W_s(A, C)$ *for a condenser* (A, C), *then* $u^* = \mathrm{Sym}(u) \in W(\mathrm{Sym}(A), \mathrm{Sym}(C))$ *for every cap symmetrization* Sym.

Proof. Clearly $0 \leqq u^*(x) \leqq 1$ for every $x \in R^n$ because $0 \leqq u(x) \leqq 1$. Since $u \in W_s(A, C)$, it is Lipschitzian, and so by (6.7) u^* is also Lipschitzian and thus ACL. Since $C \subset \{x \mid u(x) \leqq 0\}$, $\mathrm{Sym}(C) \subset \mathrm{Sym}\{x \mid u(x) \leqq 0\} = \{x \mid u^*(x) \leqq 0\}$ by (6.4). And finally, $cl\,\{x \mid u^*(x) < 1\}$ is a compact subset of $\mathrm{Sym}(A)$, for $\{x \mid u^*(x) < 1\} = \mathrm{Sym}\{x \mid u(x) < 1\} \subset \mathrm{Sym}[cl\,\{x \mid u(x) < 1\}] \subset \mathrm{Sym}(A)$, where $\mathrm{Sym}[cl\,\{x \mid u(x) < 1\}]$ is compact. Hence $u^* \in W(\mathrm{Sym}(A), \mathrm{Sym}(C))$.

6.12. **Theorem.** *If* (A, C) *is a condenser in* R^n *and* Sym is *a* 1-*dimensional cap symmetrization, then*

$$\mathrm{cap}_p(A, C) \geqq \mathrm{cap}_p(\mathrm{Sym}(A), \mathrm{Sym}(C)), \quad \textit{for every } p \geqq 1. \tag{6.13}$$

Proof. By Lemmas 5.10 and 6.11 we need only show that for any $u \in W_s(A, C)$

$$\int_{R^n} |\nabla u|^p dm_n \geqq \int_{R^n} |\nabla u^*|^p dm_n, \tag{6.14}$$

where $u^* = \mathrm{Sym}(u)$. Because the orthogonal mappings and the translations of R^n do not change the p-capacity of a condenser, we may assume that Sym is defined by the half-plane $\{x \in R^n \mid x_{n-1} \geqq 0, x_n = 0\}$. Then we can identify the symmetrizing axis J of Sym with R^{n-2} and use in R^n the cylindrical coordinates (r, φ, z), where $z \in R^{n-2}$, $r = (x_{n-1}^2 + x_n^2)^{1/2}$ and $\varphi \in [0, 2\pi)$ is such that $x_{n-1} = r \cdot \cos\varphi$, $x_n = r \cdot \sin\varphi$. Hence every symmetrizing sphere $K(z, r)$ of Sym, $z \in J$, $r \geqq 0$, is the set $\{(r, \varphi, z) \mid \varphi \in [0, 2\pi)\}$.

Since $u \in W_s(A, C)$, there exists a polyhedron P and its triangulation T such that u is simplicial with respect to T, and $u \mid \complement P$ is identically equal to 0 or 1 in every component of $\complement P$. Let u assume the values $0 = a_0 < a_1 < \ldots < a_m = 1$ at the vertices of the simplices of T, and define $D_i = u^{-1}(a_{i-1}, a_i)$ and $D_i^* = u^{*-1}(a_{i-1}, a_i)$, $i = 1, 2, \ldots, m$. Then to establish (6.14) we need, by Fubini's theorem, only to show that for fixed $i \in N$, $0 < i \leqq m$, and for m_{n-1}-almost every $(r_0, z_0) \in \{r \in R^1 \mid r \geqq 0\} \times R^{n-2}$

$$\int_{K(z_0, r_0) \cap D_i} |\nabla u|^p dm_1 \geqq \int_{K(z_0, r_0) \cap D_i^*} |\nabla u^*|^p dm_1, \tag{6.15}$$

for it is not difficult to see that $\nabla u^* = 0$ almost everywhere in the sets $u^{*-1}\{a_i\}$, $i = 0, 1, \ldots, m$. Furthermore, we can assume in (6.15) that

$r_0 > 0$, the symmetrizing sphere $K = K(z_0, r_0)$ does not intersect any $(n-2)$-dimensional side of any simplex in T, and $K \cap D_i \neq \emptyset$.

Since $u \in W_s(A, C)$, the sphere K intersects every level surface $u^{-1}\{a\}$, $a \in (a_{i-1}, a_i)$, in at most finitely many points. Hence by the piecewise linearity of u, we can divide $D_i \cap K$ into arcs $H_1, \ldots, H_L$ such that at the interior points of every H_j, u is strictly monotone with respect to φ and $\dfrac{\partial u}{\partial \varphi}$ is continuous and not zero. Let $t_1 < t_2 < \ldots < t_{L'}$ be the values of u assumed at the end points of all the H_j. Hence, for a fixed k, $1 \leqq k < L'$,

(6.16) $u^{-1}(t_k, t_{k+1}) \cap K$ consists of an even number of open arcs, say $I_1, \ldots, I_{2M}$, such that u is strictly monotone with respect to φ in every I_j and $\dfrac{\partial u}{\partial \varphi} \neq 0$ at every point in I_j.

Consider the arcs I_j in (6.16). We may assume that they are chosen so that $I_j = \{(r_0, \varphi, z_0) \mid \alpha_j < \varphi < \beta_j\}$, $j = 1, 2, \ldots, 2M$, where $0 \leqq \alpha_1 < \beta_1 \leqq \alpha_2 < \beta_2 \leqq \ldots \leqq \alpha_{2M} < \beta_{2M} \leqq 2\pi$, and for instance,

$$(6.17) \qquad (-1)^{j+1} \frac{\partial u}{\partial \varphi}(r_0, \varphi, z_0) > 0 \quad \text{for } \alpha_j < \varphi < \beta_j, \ 1 \leqq j \leqq 2M.$$

On the other hand, the function u^* is symmetric with respect to $\{x \in R^n \mid x_n = 0\}$, and $u^*(r, \varphi, z)$ is increasing with respect to φ for $0 \leqq \varphi \leqq \pi$ and decreasing for $\pi \leqq \varphi < 2\pi$. Hence $u^{*-1}(t_k, t_{k+1}) \cap K$ consists of two symmetric arcs I^* and $I^{*\prime}$. Furthermore, u^* is strictly monotone with respects to φ in I^* and $I^{*\prime}$, since $m_1[K \cap u^{*-1}\{t\}] = m_1[K \cap u^{-1}\{t\}] = 0$ for every $t \in (t_k, t_{k+1})$. Let $I^* = \{(r_0, \varphi, z_0) \mid \alpha < \varphi < \beta\}$, where $0 \leqq \alpha < \beta \leqq \pi$.
Hence to prove (6.15), we need only show that

$$(6.18) \qquad \sum_{j=1}^{2M} \int_{\alpha j}^{\beta j} G(\varphi) d\varphi \geqq 2 \int_{\alpha}^{\beta} G^*(\varphi) d\varphi,$$

where $G(\varphi) = r_0 |(\nabla u)(r_0, \varphi, z_0)|^p = r_0 \left[\left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r_0^2}\left(\dfrac{\partial u}{\partial \varphi}\right)^2 + \sum_{i=1}^{n-2}\left(\dfrac{\partial u}{\partial z_i}\right)^2\right]^{p/2}$
and similarly for $G^*(\varphi)$ in terms of u^*.

Next we express the connection between u and u^* explicitly. Let $\tau_j : (\alpha_j, \beta_j) \to (t_k, t_{k+1})$ be the homeomorphism $\tau_j(\varphi) = u(r_0, \varphi, z_0)$, $\alpha_j < \varphi < \beta_j$, and put $\varphi_j = \tau_j^{-1} : (t_k, t_{k+1}) \to (\alpha_j, \beta_j)$, $j = 1, 2, \ldots, 2M$. Similarly, define $\tau^* : (\alpha, \beta) \to (t_k, t_{k+1})$, $\tau^*(\varphi) = u^*(r_0, \varphi, z_0)$, $\alpha < \varphi < \beta$, and put $\varphi^* = \tau^{*-1} : (t_k , t_{k+1}) \to (\alpha , \beta)$. By the definition of u^* and by the indexing in (6.17) we get

$$(6.19)\qquad \varphi^*(t) = \frac{1}{2r_0} m_1(u^{-1}[0 , t] \cap K) = \frac{1}{2}\left(2\pi - \sum_{j=1}^{2M} (-1)^j \varphi_j(t)\right),$$

and then by (6.17)

$$(6.20)\qquad \varphi_j'(t) = \left[\frac{\partial u}{\partial \varphi} (r_0 , \varphi_j(t) , z_0)\right]^{-1} = (-1)^{j+1} |\varphi_j'(t)| \neq 0 ,$$

$$(6.21)\qquad \varphi^{*\prime}(t) = \left[\frac{\partial u^*}{\partial \varphi} (r_0 , \varphi^*(t) , z_0)\right]^{-1} = \frac{1}{2} \sum_{j=1}^{2M} (-1)^{j+1} \varphi_j'(t) > 0 .$$

Using the functions $\varphi_j , 1 \leqq j \leqq 2M$, and φ^* to make a change of variables in the integrals of (6.18) we get

$$(6.22)\qquad \sum_{j=1}^{2M} \int_{\alpha_j}^{\beta_j} G(\varphi)d\varphi = \int_{t_k}^{t_{k+1}} \left(\sum_{j=1}^{2M} G(\varphi_j(t)) \mid \varphi_j'(t)|\right) dt ,$$

$$(6.23)\qquad \int_{\alpha}^{\beta} G^*(\varphi)d\varphi = \int_{t_k}^{t_{k+1}} G^*(\varphi^*(t)) \, |\varphi^{*\prime}(t)| \, dt .$$

Hence to prove (6.18) it is sufficient to show that for every $t \in (t_k , t_{k+1})$

$$(6.24)\qquad \Gamma(t) \geqq 2\Gamma^*(t) ,$$

where $\Gamma(t) = \sum_{j=1}^{2M} G(\varphi_j(t)) \, |\varphi_j'(t)|$ and $\Gamma^*(t) = G^*(\varphi^*(t)) \, |\varphi^{*\prime}(t)|$. We fix $t \in (t_k , t_{k+1})$. By an elementary geometric consideration, equation (6.19), for fixed t , can be extended smoothly into a small neighbourhood of (r_0 , z_0) . This means that there exists an $s > 0$ such that $C_j = u^{-1}\{t\} \cap B^n(p_j , s) , p_j = (r_0 , \varphi_j(t) , z_0) , j = 1 , 2 , \ldots , 2M$, is an $(n-1)$-dimensional open ball, and

$$(6.25)\qquad \Phi^*(r , z) = \frac{1}{2}\left(2\pi - \sum_{j=1}^{2M} (-1)^j \Phi_j (r , z)\right)$$

for $|(r , z) - (r_0 , z_0)| < s$, where $\Phi^*(r , z)$ is the φ-coordinate of the point $K(z , r) \cap u^{*-1}\{t\} \cap \{x \in R^n \mid x_n > 0\}$ and $\Phi_j(r , z)$ is the φ-coordinate of the point $K(z , r) \cap u^{-1}\{t\} \cap C_j$. Hence for $j = 1 , 2 , \ldots , 2M$,

$$(6.26)\qquad u(r , \Phi_j(r , z) , z) = u^*(r , \Phi^*(r , z) , z) = t$$

for $|(r, z) - (r_0, z_0)| < s$. Furthermore, $\Phi_j(r_0, z_0) = \varphi_j(t)$, $j = 1, 2, \ldots, 2M$, and $\Phi^*(r_0, z_0) = \varphi^*(t)$. Partial differentiation of (6.26) with respect to y_ν, $\nu = 0, 1, \ldots, n-2$, yields expressions for $\frac{\partial u}{\partial y_\nu}$ and $\frac{\partial u^*}{\partial y_\nu}$, $0 \leqq \nu \leqq n-2$, where $y_0 = r$ and $y_\nu = z_\nu$ for $1 \leqq \nu \leqq n-2$. Substituting these expressions into both sides of (6.24) we get by (6.20), (6.21) and (6.25)

$$\Gamma(t) = \sum_{j=1}^{2M} F\left\{\left[1 + \sum_{\nu=0}^{n-2} a_{j\nu}^2\right]^{1/2} (b_j)^{-1}\right\} b_j \quad \text{and} \tag{6.27}$$

$$\Gamma^*(t) = \frac{1}{2} F\left\{\left[1 + \sum_{\nu=0}^{n-2}\left(\frac{1}{2}\sum_{j=1}^{2M}(-1)^{j+1} a_{j\nu}\right)^2\right]^{1/2}\left(\frac{2}{\sum_{j=1}^{2M} b_j}\right)\right\}\sum_{j=1}^{2M} b_j \tag{6.28}$$

where $F : \{x \in R^1 \mid x \geqq 0\} \to R^1$ is defined by $F(t) = t^p$, $t \geqq 0$, and

$$a_{j\nu} = r_0 \left(\frac{\partial \Phi_j}{\partial y_\nu}\right)(r_0, \varphi_j(t), z_0), \; 1 \leqq j \leqq 2M, \; 0 \leqq \nu \leqq \mathrm{n} - 2,$$

$$b_j = r_0 \, |\varphi_j'(t)| = r_0(-1)^{j+1}\varphi_j'(t), \; 1 \leqq j \leqq 2M.$$

For $p \geqq 1$ the function F is convex, that is, $F(\sum_i \lambda_i t_i) \leqq \sum_i \lambda_i F(t_i)$ when $\sum_i \lambda_i = 1$, $\lambda_i \geqq 0$ and $t_i \geqq 0$.

Hence by the Minkowski inequality

$$\left[1 + \sum_{\nu=0}^{n-2}\left(\frac{1}{2}\sum_{j=1}^{2M}(-1)^{j+1}a_{j\nu}\right)^2\right]^{1/2} \leqq \frac{1}{2}\sum_{j=1}^{2M}\left(1 + \sum_{\nu=0}^{n-2} a_{j\nu}^2\right)^{1/2}.$$

Since F is convex and increasing, the former inequality yields

$$2\Gamma^*(t) \leqq F\left\{\left[\sum_{j=1}^{2M}\left(1 + \sum_{\nu=0}^{n-2} a_{j\nu}^2\right)^{1/2}\right]\left(\sum_{j=1}^{2M} b_j\right)^{-1}\right\}\sum_{j=1}^{2M} b_j \leqq$$
$$\sum_{j=1}^{2M} F\left\{\left(1 + \sum_\nu a_{j\nu}^2\right)^{1/2}(b_j)^{-1}\right\} b_j = \Gamma(t),$$

which proves (6.24), and thus the proof of the entire theorem is complete.

7. The main theorem

We say that a set transformation f *satisfies the capacity inequality*, if

$$\operatorname{cap}_p(A, C) \geqq \operatorname{cap}_p(f(A), f(C)), \text{ for all } p > 0, \tag{7.1}$$

for every condenser (A, C) such that $A, C \in \mathrm{Dom}(f)$ and $(f(A), f(C))$ is a condenser. In the previous chapter we proved that 1-dimensional cap symmetrizations satisfy the capacity inequality. In this chapter we derive from this result that all symmetrizations satisfy the capacity inequality. Because for $0 < p < 1$ the inequality (7.1) is trivially valid by Lemma 5.6, we need only consider the case $p \geqq 1$.

7.2. **Lemma.** *Suppose that*

(i) *f is an open and regular set transformation* (cf 3.1),

(ii) *$\{f_i\}$ is a sequence of regular,* (7.1) *satisfying set transformations such that* $\mathrm{Dom}(f_i)$ *includes open bounded sets,*

(iii) *for every $F \in \mathfrak{F}$ and $r > 0$ there exists $j_0 \in N$ such that*

$$f(F) \subset f_j(F + r\bar{B}^n) \text{ and } f(F + r\bar{B}^n) \supset f_j(F), \text{ for } j \geqq j_0. \tag{7.3}$$

Then f satisfies the capacity inequality. In particular, (iii) *is true, if* $\mathrm{Dom}(f)$ *includes open bounded sets, every f_i is smoothing and*

$$\lim_i f_i(F) = f(F) \text{ for all } F \in \mathfrak{F}. \tag{7.4}$$

Proof. Consider a condenser (A, C) such that $A \in \mathrm{Dom}(f)$, and $p \geqq 1$. Choose $\varepsilon > 0$. By Lemma 5.5 there exists a bounded condenser (A', C') and $r > 0$ such that $\bar{A}' + r\bar{B}^n \subset A$, $C + r\bar{B}^n \subset C'$ and $\mathrm{cap}_p(A, C) + \varepsilon \geqq \mathrm{cap}_p(A', C')$. Then by (iii) we can find f_j such that $f(C) \subset f_j(C + r\bar{B}^n) \subset f_j(C')$ and $f(A) \supset f(\bar{A}' + r\bar{B}^n) \supset f_j(\bar{A}') \supset f_j(A')$. Because f_j satisfies the capacity inequality we get by the separation ineaqulity (5.3)

$$\mathrm{cap}_p(A, C) + \varepsilon \geqq \mathrm{cap}_p(A', C') \geqq \mathrm{cap}_p(f_j(A'), f_j(C')) \geqq \mathrm{cap}_p(f(A), f(C)).$$

Letting ε tend to zero, we see that f satisfies the capacity inequality.

To prove the second part of the lemma, we choose $F \in \mathfrak{F}$ and $r > 0$. Then $d = d(f(F), \complement f(F + r\bar{B}^n)) > 0$ because f is open and regular. Let $s = \min\left\{\frac{d}{2}, r\right\}$. By (7.4) we can choose $j_0 \in N$ such that $f_j(F) \subset f(F) + s\bar{B}^n$ and $f(F) \subset f_j(F) + s\bar{B}^n$ for $j \geqq j_0$. Then $f_j(F) \subset f(F) + \frac{d}{2}\bar{B}^n \subset f(F + r\bar{B}^n)$, and the smoothing property of f_j yields $f(F) \subset f_j(F) + s\bar{B}^n \subset f_j(F + s\bar{B}^n) \subset f_j(F + r\bar{B}^n)$. So condition (7.3) is valid.

7.5. **The main theorem.** *Let* Sym *be any Steiner or cap symmetrization and (A, C) a condenser. If* Sym *is a Steiner symmetrization we also assume A is bounded. Then* $(\mathrm{Sym}(A), \mathrm{Sym}(C))$ *is also a condenser and*

$$\operatorname{cap}_p (A , C) \geqq \operatorname{cap}_p (\operatorname{Sym} (A) , \operatorname{Sym} (C)) \textit{ for every } p > 0 .$$

If, in addition, (A , C) *is ringlike, then* $(\operatorname{Sym} (A) , \operatorname{Sym} (C))$ *is also ringlike.*

Proof. By Lemma 5.1 symmetrizations preserve condensers and rings. Hence we need only prove that symmetrizations satisfy the capacity inequality. We first consider cap symmetrizations and proceed by induction. By Theorem 6.12 every 1-dimensional cap symmetrization satisfies the cap inequality. Now let $2 \leqq k \leqq n - 1$ and assume that every $(k - 1)$-dimensional cap symmetrization satisfies the capacity inequality. Let Sym be a k-dimensional cap symmetrization. Hence, by Theorem 4.29, there are two $(k - 1)$-dimensional cap symmetrizations Sym_i, $i = 1 , 2$, such that $\operatorname{Sym} (F) = \lim_i (\operatorname{Sym}_2 \circ \operatorname{Sym}_1)^i(F)$ for every non-empty compact $F \subset R^n$. Thus, by the induction assumption and Lemma 7.2, Sym satisfies the capacity inequality.

Next we consider Steiner symmetrizations and again proceed by induction. Let Sym be a 1-dimensional Steiner symmetrization. By Lemma 4.19 there exists a 1-dimensional cap symmetrization S and a sequence of translations $t_i : R^n \to R^n$, $i \in N$, such that the set transformations $f_i = t_i^{-1} \circ S \circ t_i$, $i \in N$, satisfy condition (7.3) of Lemma 7.2. Thus, by Lemma 7.2, Sym satisfies the capacity inequality, since every translation t_i preserves the p-capacity of a condenser and the symmetrization S satisfies the capacity inequality. Now let $2 \leqq k \leqq n$ and assume that every $(k - 1)$-dimensional Steiner symmetrization satisfies the capacity inequality. Let Sym be a k-dimensional Steiner symmetrization. Hence Theorem 4.32 and Lemma 7.2 imply that Sym satisfies the capacity inequality.

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Printed October 1972