Symmetry-adapted classification of aberrations

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Optical systems produce canonical transformations on phase space that are nonlinear. When a power expansion of the coordinates is performed around a chosen optical axis, the linear part is the paraxial approximation, and the nonlinear part is the ideal of aberrations. When the optical system has axial symmetry, its linear part is the symplectic group $Sp(2, R)$ represented by 2×2 matrices. It is used to provide a classification of aberrations into multiplets of spin that are irreducible under the group, in complete analogy with the quantum harmonic-oscillator states. The "magnetic" axis of the latter may be chosen to adapt to magnifying systems or to optical fiberlike media. There seems to be a significant computational advantage in using the symplectic classification of aberrations.

1. INTRODUCTION

In the past few years the methods of Lie algebras and groups were applied to optics, from radar detection to magnetic optics and to coherent states. Much of the literature is presented or contained in Ref. 1. The purpose of this paper is to show how the symmetry methods of Lie optics apply for the classification of aberrations based on the paraxial properties of the system. The language used here resembles that used to describe harmonic-oscillator models in physics.

Optical phase space, we should recall,[2] is the manifold of rays crossing a reference screen normal to a chosen optical axis z. We mark Cartesian coordinates (x, y) on the screen. The position coordinates of a ray $\mathbf{q} = (q_x, q_y)^T$ determine the intersection with the screen. The momentum coordinates $\mathbf{p} = (p_x, p_y)^T$ conjugate to the former are the projection of a three-vector along the ray direction of length $n(\mathbf{q})$ (the refractive index of the medium at $\mathbf{q}$) on the plane of the screen. Optical phase space is the manifold of four-vectors[3,4]:

$$\mathbf{w} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}. \tag{1.1}$$

The optical Hamiltonian is the component of the above three-vector that is normal to the reference screen. Snell's law for differential variations of the refractive index leads to the Hamilton equations of motion,[5] which are familiar from studies of mechanics.[6] Optical phase space differs from mechanical phase space only in its global properties: momentum is bounded by $|\mathbf{p}| \leq n(\mathbf{q})$ in the former but has no such restriction in the latter.

We follow the common practice of regarding a coordinate patch in a neighborhood of the optical axis in the momentum subspace and perform analytic continuation in any expression to all value of $\mathbf{p}$ in the plane. This permits the usual separation into paraxial and aberration optics through power-series expansion of both $\mathbf{q}$ and $\mathbf{p}$ coordinates, with the direct analogy of the former regime with mechanics: the free particle with free propagation in homogeneous optical media and the harmonic oscillator with quadratic- or elliptic-profile fibers.[7] Still in the paraxial regime, harmonic oscillator "kicks" correspond to quasi-flat refracting surfaces.

The standard[2] elements of the Lie-optics formalism that view optical systems as transformations of phase space realized through exponential operators factorized into aberration orders, as done by Dragt[8] and by Dragt and Finn,[9] are described in Section 2.

Optical systems that are symmetric under rotations around the optical axis (and reflections across planes containing the latter) are succinctly described by functions over a sphere. The spherical-harmonic expansion of the functions over a sphere leads to the classification of the aberrations of the optical systems under study.[10] In Section 3 this construction is detailed for the monomial basis,[2] and in Section 4 the symplectic (or spherical-harmonic) basis is introduced. This is possible because of the complex homomorphism between the two-dimensional symplectic group $Sp(2, R)$ of paraxial transformations and the group $SO(3)$ of three-dimensional rotations. It is pointed out that the traditional Seidel classification of aberrations in imaging systems distinguishes one north pole of the sphere. Systems whose paraxial part is that of a fiber distinguish another direction, related to the former by a rotation of the sphere corresponding to the (complex) Bargmann transform of phase space. In this context, in Section 5 aberrations adapted to fiberlike systems are classified.

The behavior of the third-order aberration coefficients of a quartic-profile fiber along the optical axis in closed form is calculated in Section 6.

In Lie optics, optical elements are concatenated through multiplication of the corresponding group elements. This is done by computer algorithms for aberration orders higher than 3. In Section 7 we discuss the economy of the symplectic over the monomial aberration classification and the economy of adapting the symplectic classification to the paraxial part of the system. The conceptual economy of group-theoretical classification schemes consists of reducing the rather formidable theory of aberrations to the mathematics of the quantum harmonic oscillator, perhaps at the reasonable price of some abstraction.

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2. LIE TRANSFORMATIONS OF OPTICAL PHASE SPACE

The elements of Lie theory in optics are the following. Consider smooth functions f and g of phase space [Eq. (1.1)]. Their Poisson bracket[6] is

$$\{f, g\}(\mathbf{p}, \mathbf{q}) = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{q}} = (:f:g)(\mathbf{p}, \mathbf{q}). \qquad (2.1)$$

For arbitrary g, the last equality defines the Lie operator[8] $:f:$ associated with the function f. The exponential of this operator is the Lie transformation generated by f and is given by the series

$$[\exp(z:f:)g](\mathbf{p}, \mathbf{q}) = \left[\sum_{m=0}^{\infty} \frac{1}{m!} (z:f:)^m g\right](\mathbf{p}, \mathbf{q})$$

$$= \sum_{m=0}^{\infty} \frac{z^m}{m!} \{f, \{f, \{\cdots\{f, g\}\cdots\}\}\}(\mathbf{p}, \mathbf{q})$$

$$= g(\mathbf{p}, \mathbf{q}; z). \qquad (2.2)$$

The three-variable function $g(\mathbf{p}, \mathbf{q}; z)$ then satisfies the following differential equation and boundary condition:

$$\frac{\mathrm{d}}{\mathrm{d}z} g(\mathbf{p}, \mathbf{q}; z) = :f:g(\mathbf{p}, \mathbf{q}; z), \qquad g(\mathbf{p}, \mathbf{q}; 0) = g(\mathbf{p}, \mathbf{q}). \quad (2.3)$$

Conversely, if Eqs. (2.3) hold and f is independent of z, then the formal solution [Eq. (2.2)] follows.

In particular, if $f(\mathbf{p}, \mathbf{q}) = -H(\mathbf{p}, \mathbf{q})$ is (minus) the optical Hamiltonian[2,5] of a medium of refractive index $n(\mathbf{q})$ that is homogeneous under translations along the z axis [i.e., a fiber of profile $n(\mathbf{q})$],

$$H(\mathbf{p}, \mathbf{q}) = -[n(\mathbf{q})^2 - p^2]^{1/2}, \qquad (2.4)$$

then Eqs. (2.3), for $g(\mathbf{p}, \mathbf{q}) = \mathbf{p}, \mathbf{q}$, are the Hamilton equation of motion for rays in the fiber. (This includes free propagation when n is constant.)

Lie transformations of immediate interest are the following: Linear functions of phase space $f = \mathbf{x} \cdot \mathbf{p} + \mathbf{y} \cdot \mathbf{q}$ generate translations of phase space:

$$[\exp(:\mathbf{x} \cdot \mathbf{p} + \mathbf{y} \cdot \mathbf{q}:)g](\mathbf{p}, \mathbf{q}) = g(\mathbf{p} + \mathbf{y}, \mathbf{q} - \mathbf{x}). \qquad (2.5)$$

Quadratic functions generate Lie transformations of phase space that are linear and that may be written in matrix form as[11]

$$\exp(:\alpha p^2 + \beta \mathbf{p} \cdot \mathbf{q} + \gamma q^2:) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}$$

$$= \begin{bmatrix} \cos u + \beta \operatorname{sinc} u & 2\gamma \operatorname{sinc} u \\ -2\alpha \operatorname{sinc} u & \cos u - \beta \operatorname{sinc} u \end{bmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}, \qquad (2.6a)$$

$$u = \pm(4\alpha\gamma - \beta^2)^{1/2}, \qquad \operatorname{sinc} x = x^{-1} \sin x. \qquad (2.6b)$$

A few other functions f lead to Lie transformations that can be written in closed form. We are generally content to know that functions f that are of order higher than second in $\mathbf{p}$ and $\mathbf{q}$ will yield through Eq. (2.2) an expansion in powers of $\mathbf{p}$ and $\mathbf{q}$ when acting on the coordinates of phase space.

Let us now give some further basic results[12] from the theory of Lie transformations: First, Lie transformations are canonical (i.e., Poisson brackets are conserved); second, refracting surfaces also produce canonical transformations of phase space.[13] The composition of canonical transformations is a canonical transformation. Third, a result that may be regarded as a converse to the first is the Dragt–Finn factorization theorem,[9] the statement that canonical transformations $\mathfrak{m}$ leaving the origin of phase space invariant (i.e., referred to the system's optical axis) may be written, locally, in the factorized form

$$\mathfrak{m} = \cdots \exp:f_5: \exp:f_4: \exp:f_3: \exp:f_2: \qquad (2.7)$$

generated by polynomials $f_m(\mathbf{p}, \mathbf{q})$ that are homogeneous of degree $m = 2, 3, \ldots$ in the components of $\mathbf{p}$ and $\mathbf{q}$. When we restrict the transformation $\mathfrak{m}$ to be axis symmetric (representing optical systems invariant under rotations around a common optical axis and under reflections across planes containing this axis[2,4]), then the f_m present in Eq. (2.7) may be functions of only the quadratic combinations[10]

$$p^2 = \sqrt{2}\xi_+ = -(\xi_1 + i\xi_2), \qquad (2.8a)$$

$$\mathbf{p} \cdot \mathbf{q} = \xi_0 = \xi_3, \qquad (2.8b)$$

$$q^2 = \sqrt{2}\xi_- = \xi_1 - i\xi_2, \qquad (2.8c)$$

so that $f_m = 0$ for all odd m. We thus define the vector $\boldsymbol{\xi}$.

We should stress the fact that the general linear axis-symmetric Lie transformation is generated by f_2 and given by Eqs. (2.6). Such transformations are privileged since they form a group of three parameters; these we may choose as (α, β, γ) or the 2×2 unimodular matrix of Eqs. (2.6). This group is called the two-dimensional real symplectic group, denoted by $Sp(2, R)$. In dimension 2, the accident occurs that this is the same as the group $SL(2, R)$ of real 2×2 unimodular matrices. Further, as transformations of the three functions ξ_i in Eqs. (2.8), this group[14] is recognized to be 2:1 homeomorphic to the group of pseudo-orthogonal matrices $SO(2, 1)$ with the metric $(1, 1, -1)$. Finally, $SO(2, 1)$ has the same complex extension as $SO(3)$, the ordinary rotation group in three dimensions.

Aberrations, as is stated above, constitute the nonlinear part of an optical transformation. For axis-symmetric systems, all aberrations have the generic form

$$\mathfrak{m}_f = \cdots \exp:f_8: \exp:f_6: \exp:f_4: \qquad (2.9)$$

and are generated by the polynomials f_{2k} of degree k in $\boldsymbol{\xi}$, as defined in Eqs. (2.8). Aberrations compose to aberrations, the neutral element exists, and optical elements are associative; the last axiom, the inverse of Eq. (2.9), exists (but one must be careful with limits, nevertheless). The set of elements [Eq. (2.9)] forms an infinite-parameter group parameterized by the coefficients of the polynomials f_{2k}, $k = 2, 3, 4, \ldots$. Moreover, from the Baker–Campbell–Hausdorff formulas[12] and the property $\{f_{2n}, f_{2m}\} = g_{2(n+m-1)}$ of the Poisson bracket, we have

$$\exp:f_{2n}: \exp:f_{2m}: = \cdots \exp:g_{2(n+m-1)}: \exp:f_{2m}: \exp:f_{2n}:. \qquad (2.10)$$

It follows that the group of aberrations has a nested structure that permits us to form the factor group of Nth-order aberrations, modulo aberrations of order higher than N.

Aberrating optical systems have a paraxial part described by the 2×2 matrix $\mathbf{M}$ and an aberration part [Eq. (2.9)]. Its restriction to aberration order $N = 2k - 1$ permits us to parameterize the elements through $\mathbf{f} = \{f_{2\kappa}\}_{\kappa=2}^{k}$, where $\mathbf{f}$ is a vector of polynomials or, alternatively, a vector with their coefficients in some basis,

$$G(\mathbf{f};\mathbf{M}) = G(f_{2k}, f_{2k-2}, \cdots, f_4; \mathbf{M}) = G(\mathbf{f}; 1)\exp{:}f_2{:}. \tag{2.11}$$

Composing optical elements means multiplying group elements:

$$G(\mathbf{f};\mathbf{M})G(\mathbf{g};\mathbf{N}) = G(\mathbf{f}\#[\exp{:}f_2{:}\mathbf{g}]; \mathbf{MN}). \tag{2.12}$$

The two paraxial parts are composed by ordinary matrix multiplication, and # denotes aberration composition.[4,12] The structure of Eq. (2.12) indicates that, within the group of axis-symmetric Nth-order optical transformations, the subgroup of aberrations forms an ideal, i.e., is acted on by linear (paraxial) transformations but does not act on them. This permits us to use $Sp(2, \mathrm{R})$ to classify aberrations into irreducible vector subspaces, as we shall show below.

3. MONOMIAL (SEIDEL) CLASSIFICATION OF ABERRATIONS

The action of linear transformations on the aberration polynomials is partially reduced: the former do not map the latter out of their aberration order, so it will suffice to work with a single, generic aberration polynomial of degree k in ξ, for aberration order $N = 2k - 1$:

$$f_{2k}(\mathbf{p},\mathbf{q}) = \sum_{k_++k_0+k_-=k} v^{(k)}_{k_+,k_0,k_-} M_{k_+,k_0,k_-}(\xi), \tag{3.1a}$$

$$M_{k_+,k_0,k_-}(\xi) = (p^2)^{k_+}(\mathbf{p}\cdot\mathbf{q})^{k_0}(q^2)^{k_-}. \tag{3.1b}$$

The v's are the monomial aberration coefficients. As shown for order 3 by Dragt,[2] they correspond directly to the traditional Seidel aberrations. Following Eq. (2.2) to the second term in the series for any single aberration $v^{(k)}_{k+k0k-}$ acting on object position $g = \mathbf{q}$, we find

$$\exp{:}f_{2k}{:}\mathbf{q} = \mathbf{q} - v^{(k)}_{k_+k_0k_-}[2M_{k_+-1,k_0,k_-}\mathbf{p} + M_{k_+,k_0-1,k_-}\mathbf{q}] + \ldots. \tag{3.2}$$

This we may compare with previously obtained results[15] to arrive at the following identifications for general aberration order:

$v_{k,0,0}$,	spherical aberration (S),
$v_{k-1,1,0}$,	circular coma (C),
$v_{k-2,2,0}$, $v_{k-1,0,1}$,	oblique spherical aberration,
$v_{k-3,3,0}$, $v_{k-2,1,1}$	(nameless),
...	
v_{0k0}, $v_{1,k-2,1}$, ...,	$v_{k/2,0,k/2}$ or $v_{(k-1)/2,1,(k-1)/2}$,
...	
$v_{0,3,k-3}$, $v_{1,1,k-2}$,	elliptical coma,
$v_{0,2,k-2}$, $v_{1,0,k-1}$,	curvature of field (F), astigmatism (A),
$v_{0,1,k-1}$,	distortion (D),
$v_{0,0,k}$	(nameless) (P).

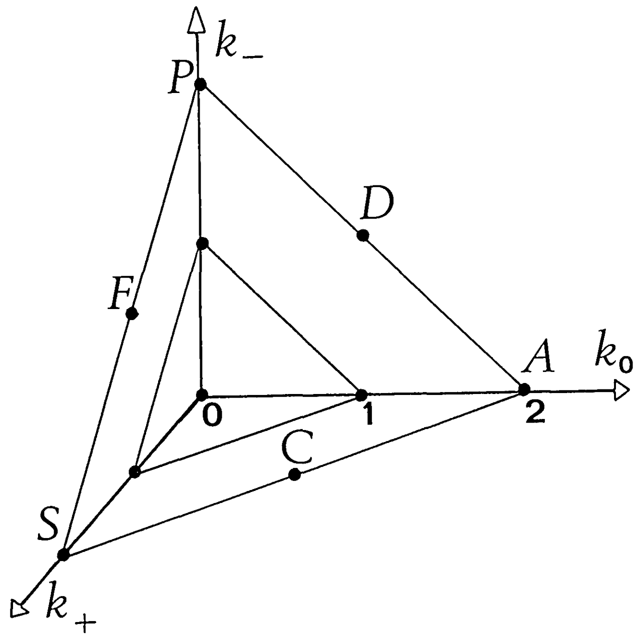

Fig. 1. The sextet of third-order aberrations in the monomial (Seidel) classification: S, spherical aberration; C, coma; A, astigmatism; F, curvature of field; D, distortion; P, pocus. Fourier conjugation reflects across the A–F line.

The letters in parentheses (S, C, F, A, D, P) identify the classic third-order aberrations in Fig. 1, in which $k = 0, 1, 2$ is plotted for the unit, paraxial, and aberration generators. This diagram is nothing more than one of the standard diagrams to display the quantum three-dimensional harmonic-oscillator states by number of quanta along the three axes. The (nameless) P aberration does not affect the position coordinate in Eq. (3.2), since the term in brackets is zero, but it does affect the direction of arrival, as we may see if we write the $\mathbf{p}$ analog of that equation. Fourier transformation ($\mathbf{q} \mapsto \mathbf{p}$, $\mathbf{p} \mapsto -\mathbf{q}$) is a reflection across the F–A line, and P is the Fourier conjugate of S, the spherical aberration.[16] In Ref. 17 it was playfully called pocus, since it $\mathbf{p}$-unfocuses a position-perfect image and produces a diminishing depth of field at an increasing distance from the optical center. We should note that, except for S, C, D, and P, the traditional names refer to parametric families of monomial-classified aberrations.

Lastly, in Lie optics the full exponential series is a canonical transformation, where position and momentum are on equal footing. The above identification holds for only the first term of the series beyond the identity. Lie and traditional aberrations differ beyond, by terms of at least the order of v^2.

4. SYMPLECTIC CLASSIFICATION OF ABERRATIONS

For aberration orders 3, 5, 7, ..., $2k - 1$, there are 6, 10, 15, ..., $\frac{1}{2}k(k + 1)$ independent aberrations, as we may verify by summing triangular numbers. These are the dimensions of the submatrices in Eq. (2.12) when we write $\exp{:}f_2{:}$ in matrix form and acting on the aberration coefficients $v^{(k)}_{k+k0k-}$ as entries in the vector $\mathbf{g}$ of dimension $6 + 10 + \cdots + \frac{1}{2}k(k + 1)$. These matrices can be reduced further by a better choice of basis. This amounts to classifying the harmonic-oscillator states by angular momentum. Note that the squared radius of a sphere in ξ space is

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = \xi_0^2 - 2\xi_+\xi_- = (\mathbf{p}\cdot\mathbf{q})^2 - p^2q^2 = -(\mathbf{p}\times\mathbf{q})^2, \tag{4.1}$$

namely, minus the Petzval invariant. Paraxial transformations only rotate the sphere [this is a complex rotation, owing to Eqs. (2.8)]. The plan is thus to use solid spherical harmonics and powers of Eq. (4.1) as a basis alternative to the monomial basis (3.1) that will reduce the action of the symplectic transformation $\exp{:}f_2{:}$ to blocks classified by the symplectic spin j, which is entirely analogous to the angular momenta $j = k, k-2, \ldots, 1$ or 0, contained in a k-quantum oscillator shell in nuclear models: third aberration order corresponds to the six-state $2s$-$1d$ shell, whose familiar diagram is shown in Fig. 2 with the aberration labels of Fig. 1.

An explicit choice of magnetic axis is that of pure magnifying paraxial systems:

$$\exp(\alpha{:}\mathbf{p}\cdot\mathbf{q}{:})\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{bmatrix} e^{\alpha} & 0\\ 0 & e^{-\alpha}\end{bmatrix}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}, \tag{4.2a}$$

i.e.,

$$\exp(\alpha{:}\xi_0{:})g(\xi_+, \xi_0, \xi_-) = g(e^{2\alpha}\xi_+, \xi_0, e^{-2\alpha}\xi_-). \tag{4.2b}$$

For this reason we choose the 0 component of ξ to be $\mathbf{p}\cdot\mathbf{q}$ in Eqs. (2.8); this is the "north pole" of image-forming optical systems.

Solid spherical harmonics[18] $\mathcal{Y}^j_m(\xi)$ have square-root factors that are inconvenient for symbolic and numeric computation. The polynomials

$$\begin{aligned} {}^{j}\mathcal{X}^{j}_{m}(\xi) &= \left[\frac{4\pi(2j+1)(j+m)!(j-m)!}{(2j-1)!!}\right]^{1/2}\mathcal{Y}^j_m(\xi)\\ &= \frac{(j+m)!(j-m)!}{2^{m/2}(2j-1)!!}\\ &\quad\times\sum_{\nu}\frac{1}{2^{\nu}}\,\frac{\xi_+^{m+\nu}}{(m+\nu)!}\,\frac{\xi_0^{j-m-2\nu}}{(j-m-2\nu)!}\,\frac{\xi_-^{\nu}}{\nu!}, \end{aligned}\tag{4.3a}$$

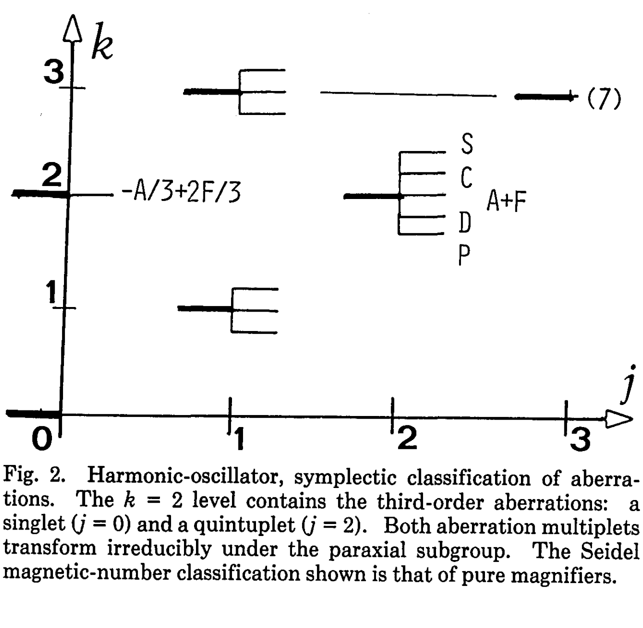

Fig. 2. Harmonic-oscillator, symplectic classification of aberrations. The $k = 2$ level contains the third-order aberrations: a singlet ($j = 0$) and a quintuplet ($j = 2$). Both aberration multiplets transform irreducibly under the paraxial subgroup. The Seidel magnetic-number classification shown is that of pure magnifiers.

$${}^{k}\mathcal{X}^{j}_{m}(\xi) = (-\xi^2)^{(k-j)/2}\,{}^{j}\mathcal{X}^{j}_{m}(\xi) \tag{4.3b}$$

were defined previously.[4,10] These form a basis for the space of polynomials of degree k in ξ:

$$f_{2k}(\mathbf{p}, \mathbf{q}) = \sum_{j=k,(-2)}^{1 \text{ or } 0}\ \sum_{m=+j}^{-j} {}^{k}x^{j}_{m}\ {}^{k}\mathcal{X}^{j}_{m}(\xi). \tag{4.4}$$

Aberrations are consequently classified by their order $2k-1$; their symplectic spin $j = k, k-2, \ldots, 1$ or 0; and their Seidel weight $m = j, j-1, j-2, \ldots, -j$. See Fig. 2 for aberration order 3, in which S, C, D, and P are ${}^2\mathcal{X}^2_2$, ${}^2\mathcal{X}^2_1$, ${}^2\mathcal{X}^2_{-1}$, and ${}^2\mathcal{X}^2_{-2}$ and the familiar curvature of field-astigmatism degeneracy is resolved into the coefficients of

$${}^2\mathcal{X}^0_0(\xi) = 2\xi_+\xi_- - \xi_0^2 = p^2q^2 - (\mathbf{p}\cdot\mathbf{q})^2 = (\mathbf{p}\times\mathbf{q})^2, \tag{4.5a}$$

$$\begin{aligned} {}^2\mathcal{X}^2_0(\xi) &= -\tfrac{1}{3}\xi^2 + \xi_0^2 = \tfrac{2}{3}(\xi_+\xi_- + \xi_0^2)\\ &= \tfrac{1}{3}p^2q^2 + \tfrac{2}{3}(\mathbf{p}\cdot\mathbf{q})^2. \end{aligned}\tag{4.5b}$$

Under the paraxial part of the system, ${}^2\mathcal{X}^0_0$ is invariant (this is well known), and ${}^2\mathcal{X}^2_0$ transforms as an element of a $j = 2$ quintuplet of quadrupole states. In Ref. 17 these terms are called astigmature and curvatism, respectively. For higher-order aberrations, this scheme is never degenerate: in aberration order 5 we have a septuplet and a triplet; in aberration order 7 we have a nonuplet, a quintuplet, and a singlet, etc. The number of active matrix coefficients in the group-composition algorithm due to the action of the paraxial part of the first system on the aberrations of the second is thus reduced, for orders 3, 5, and 7, from 36, 100, and 225 to 26, 58, and 107.

5. COHERENT-STATE ABERRATION BASIS FOR FIBERS

When the paraxial part of the system is that of an optical fiber, a harmonic oscillator in mechanics, its generator is the Hamiltonian of the form

$$H^{\mathrm{osc}} = \tfrac{1}{2}(p^2 + q^2) = \sqrt{2}(\xi_+ + \xi_-) = -i\xi_2, \tag{5.1}$$

or it may be brought to this form by a simple change of scale with Eqs. (4.2). Evolution along the optical axis (complex) rotates the sphere around the ξ_2 axis and mixes aberrations within each (k, j) multiplet. It is clear, however, that the main oscillation[19] in magnifier-classified aberration coefficients [Eq. (4.4)] is due to the rotation of the north pole around the new axis. In phase space, the q_x–p_x and q_y–p_y planes are rotating, and we should pass to the "coherent-state" coordinates $\mathbf{q} \pm i\mathbf{p}$. This is to bring the former ξ_0 axis onto the new ξ_2 axis by means of a rotation by $\pi/2$ around the ξ_1 axis, namely, the complex Bargmann transformation,[20,21] which we write here as

$$\mathcal{B} = \exp(-\tfrac{1}{8}i\pi{:}p^2 - q^2{:}). \tag{5.2}$$

The Bargmann transformation acts on phase space as

$$\mathcal{B}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i\\ i & 1\end{bmatrix}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}(\mathbf{p} + i\mathbf{q})/\sqrt{2}\\ (\mathbf{q} + i\mathbf{p})/\sqrt{2}\end{pmatrix} = \begin{pmatrix}\eta\\ \zeta\end{pmatrix} \tag{5.3a}$$

and on the ξ sphere through

$$\mathcal{B}\begin{pmatrix}\xi_+\\ \xi_0\\ \xi_-\end{pmatrix} = \begin{bmatrix} \tfrac{1}{2} & i/\sqrt{2} & -\tfrac{1}{2}\\ i/\sqrt{2} & 0 & i/\sqrt{2}\\ -\tfrac{1}{2} & i/\sqrt{2} & \tfrac{1}{2}\end{bmatrix}\begin{pmatrix}\xi_+\\ \xi_0\\ \xi_-\end{pmatrix} = \begin{pmatrix}\sigma_+\\ \sigma_0\\ \sigma_1\end{pmatrix}, \quad (5.3b)$$

defining new coordinates σ in place of Eqs. (2.8):

$$\sigma_1 = (\sigma_- - \sigma_+)/\sqrt{2} = \xi_1 = -\tfrac{1}{2}(p^2 - q^2), \quad (5.4a)$$

$$\sigma_2 = i(\sigma_- + \sigma_+)/\sqrt{2} = -\xi_3 = -\mathbf{p}\cdot\mathbf{q}, \quad (5.4b)$$

$$\sigma_3 = \sigma_0 = \xi_2 = i(\tfrac{1}{2})(p^2 + q^2). \quad (5.4c)$$

On the symplectic-basis polynomials [Eqs. (4.3)], the Bargmann transform effects

$$\mathcal{B}\,{}^k\mathcal{X}^j_m(\xi) = \sum_{m'=-j}^{j} B^j_{mm'}\,{}^k\mathcal{X}^j_{m'}(\xi) = {}^k\mathcal{X}^j_m(\sigma), \quad (5.5a)$$

$$B^j_{mm'} = \frac{(j+m)!(j-m)!}{(-2)^j}\, i^{m+m'} \times \sum_\nu \frac{(-1)^\nu}{(\nu - m - m')(j + m' - \nu)!(j + m - \nu)!\nu!}. \quad (5.5b)$$

Equations (5.3a) and (5.3b) are the $j = 1/2$ and $j = 1$ cases.

We may now expand the aberration polynomials f_{2k} as

$$f_{2k}(\mathbf{p}, \mathbf{q}) = \sum_{j=k,(-2)}^{1 \text{ or } 0} \sum_{m=-j}^{j} {}^ks^j_m\,{}^k\mathcal{X}^j_m(\sigma), \quad (5.6)$$

and we refer to the coefficients ${}^ks^j_m$ as the coherent-state-basis aberration coefficients. They relate to the magnifier symplectic aberration coefficients in Eq. (4.4) through

$${}^ks^j_m = \sum_{m'} {}^kx^j_{m'}\, B^j_{m'm}, \quad (5.7a)$$

$${}^kx^j_m = \sum_{m'} {}^ks^j_{m'}\, B^{j*}_{m'm}. \quad (5.7b)$$

6. THIRD-ORDER ABERRATIONS IN QUARTIC-PROFILE FIBERS

In this section we provide an example of the description of an optical system in terms of its coherent-state aberration coefficients (cf. Ref. 19, in which the same example is given with symplectic aberration coefficients). We consider a fiberlike medium with a refractive index expressed as

$$n(\mathbf{q}) = [n_0^2 - q^2 - \beta(q^2)^2 + \ldots]^{1/2}. \quad (6.1)$$

To work in the third aberration order, we build the ruling Hamiltonian from Eq. (2.4) and (6.1), keeping up to fourth-order terms in the phase-space components:

$$\begin{aligned} H^f &= -n_0 + \frac{1}{2n_0}(p^2 + q^2) + \frac{1}{8n_0^3}(p^2+q^2)^2 + \frac{\beta}{2n_0}(q^2)^2 \\ &= -n_0 - \frac{i}{n_0}\,{}^1\mathcal{X}^1_0(\sigma) + \frac{\beta}{8n_0}[{}^2\mathcal{X}^2_2(\sigma) + {}^2\mathcal{X}^2_{-2}(\sigma)] \\ &\quad + \frac{i\beta}{2n_0}[{}^2\mathcal{X}^2_1(\sigma) - {}^2\mathcal{X}^2_{-1}(\sigma)] - \left(\frac{1}{2n_0^3} + \frac{3\beta}{4n_0}\right){}^2\mathcal{X}^2_0(\sigma) \\ &\quad + \frac{1}{6n_0^3}\,{}^2\mathcal{X}^0_0(\sigma). \end{aligned} \quad (6.2)$$

Although the first expression in terms of $\mathbf{p}$ and $\mathbf{q}$ appears shorter than the second in σ, the latter's transformation properties under $H^{\rm osc} = -i\,{}^1\mathcal{X}^1_0(\sigma)$ are simpler. To the third order it holds that[12,19]

$$\exp(:\omega\mathcal{X}^1_0 + \mathbf{a}\cdot\mathcal{X}^2 + b\mathcal{X}^0_0:) \simeq \exp(:\mathbf{a}'\cdot\mathcal{X}^2 + b\mathcal{X}^0_0:) \times \exp(:\omega\mathcal{X}^1_0:), \quad (6.3a)$$

$$a'_m = a_m\frac{e^{2m\omega} - 1}{2m\omega}, \quad m = 2, 1, 0, -1, -2 \quad (a'_0 = a_0), \quad (6.3b)$$

where we have dropped the anterior index k of the ${}^k\mathcal{X}$ terms for brevity and $\mathbf{a}$ and $\mathbf{a}'$ are five-vectors (quintuplets) of third-order aberration; the $\simeq$ indicates that this is an equality for aberration order 3. In the group notation [Eq. (2.12)] no such proviso needs statement; Lie-operator notation is used here, since it retains phase-space variables and may be clearer to the reader as such.

The evolution of phase space along the fiber is given by the operator $\exp(:-zH^f:)$. We may use the last expression in Eq. (6.2), with $\omega = iz/n_0$ and a values read off correspondingly; Eq. (6.3b) tells us that their evolution is governed by the exponential oscillating factor $\exp(2imz_0/n) - 1$ and that the coherent-state-classified a_0, the members of the quintuplet, and the singlet b aberration coefficients increase linearly in time; that is,

$$s^2_2 = -\frac{1}{32}i\beta[1 - \exp(4iz/n_0)] = s^{2*}_{-2}, \quad (6.4a)$$

$$s^2_1 = \tfrac{1}{4}\beta[1 - \exp(2iz/n_0)] = s^{2*}_{-1}, \quad (6.4b)$$

$$s^2_0 = \left(\frac{1}{2n_0^3} + \frac{3\beta}{4n_0}\right)z, \qquad s^0_0 = \frac{1}{6n_0^3}z. \quad (6.4c)$$

Having entered the complex field, we can satisfy ourselves that the conjugation properties above are the guarantees of real transformations in phase space. We may apply the evolution operator $\exp(:-zH^f:)$ to the coherent-state variables [Eqs. (5.3)] through the third order, using also Eqs. (2.6) for the paraxial part, to find

$$\begin{aligned} \exp(:-zH^f:)\zeta \simeq \exp(-iz/n_0)[\zeta &+ (-4s^2_2\eta^2 - 2s^2_1\eta\cdot\zeta \\ &- \{\tfrac{2}{3}s^2_0 + \tfrac{1}{2}s^0_0\}\zeta^2)\eta \\ &+ (-s^2_1\eta^2 - \{\tfrac{4}{3}s^2_0 - \tfrac{1}{2}s^0_0\}\eta\cdot\zeta - s^2_{-1}\zeta^2)\zeta]. \end{aligned} \quad (6.5)$$

The complex-conjugate equation yields the transformation for $\zeta^* = -i\eta$, so we need compute only once with complex numbers to find the transformation of the phase-space variables $\mathbf{p}$ and $\mathbf{q}$.

7. CONCLUDING REMARKS

The understanding of the symmetry behind the apparently complicated subject of optical aberrations is part of the purpose of this paper. To show that it is computationally advantageous to use the group-theoretic classification, we have so far centered the arguments on the transformations of the pure aberration subgroup under the paraxial part of the system: matrices reduce to block-diagonal form according to symplectic spin-aberration multiplets. The # operation in Eq. (2.12), denoting aberration composition, deserves additional discussion.

The composition of Lie transformations [see Eq. (2.10)] proceeds through Poisson brackets of the generating polynomials. When the monomial basis [Eqs. (3.1)] is used, we can easily compute

$$\{M_{abc}, M_{a'b'c'}\} = 4(ca' - ac')M_{a+a'-1,b+b'+1,c+c'-1} + 2(ba' - ab' + cb' - bc') \times M_{a+a',b+b'-1,c+c'}. \qquad (7.1)$$

When the symplectic basis [Eq. (4.4)] of spherical harmonics is used, the corresponding basic Poisson bracket is of the structure

$$\{{}^{k}\mathcal{X}^{j}_{m}, {}^{k'}\mathcal{X}^{j'}_{m'}\} = \sum_{\substack{j''=|m_1+m_2| \\ k+k'+j''\text{odd}}}^{j+j'-1} S^{jj'j''}_{mm'}\, {}^{k+k'-1}X^{j''}_{m+m'}. \qquad (7.2)$$

Note that the Seidel weight index m sums and that the number of terms in the right-hand side of Eq. (7.2) is the number of symplectic spin values consistent with that sum. For the four extreme members of any k multiplet, this number is 1. [The coefficients $S^{jj'j''}_{mm'}$ were shown in Ref. 4 to be a reduced $SO(3)$ Wigner coefficient times a sum of three such coefficients, but these need to be calculated once and for all.] In which basis do we have a shorter composition rule?

At the Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas/Cuernavaca, a set of muSIMP functions was developed to handle symbolically the various mathematical objects described in this paper, in terms of phase-space variables, monomials, symplectic (spherical) harmonics, and optical group elements parametrically defined, to seventh aberration order; and optical elements such as free-space propagation, refracting surfaces (eight order, axis symmetric), and fibers (with refractive index as fourth-degree polynomials in q^2). Including paraxial parts, we can compose, invert, and bring to paraxial focus optical systems defined through the set of their (paraxial and) aberration parameters. The composition function # in Eq. (2.12) that yields the third-order aberrations is a single sum in each aberration vector component, and the monomial and symplectic classification schemes have equal complexity (six sums). The composition to fifth order is a sum of fifth-order coefficients with bilinear products of the third-order coefficients of the factors. For the symplectic basis there is a total of 42 sums (27 for the septuplet, 15 for the triplet); the monomial basis has 44 sums in the composition function. For seventh-order composition functions we have a sum of the aberration coefficients of the factors, plus bilinear products of the fifth and third orders, plus trilinear products of third-order coefficients of the factors. The symplectic basis yields a total of 303 sums in that order (169 in the nonuplet, 133 in the quintuplet, and 1 in the singlet); the monomial basis yields 407 sums. A seventh-order composition of pure aberrations involves thus 351 sums in the symplectic basis versus 457 sums in the monomial basis. Operation with the paraxial part on the aberration ideal, as described in Section 2, is distinctly faster in the symplectic basis of aberrations. These two arguments compounded would seem to give a significant computational edge to the use of the symplectic basis presented in Section 4.

Some virtue has been found in using muSIMP, which will run on a 256-kilobyte computer (IBM PC), since it is a working tool even in the dismal local conditions. Early work was done in REDUCE-2 (e.g., Ref. 22 and work reported in Ref. 4) to the ninth aberration order, but the machine (FOONLY F2) is no longer in operation. Results for the composition of optical elements at aberration order 7 with too many free, symbolic parameters can easily exceed the available memory of the microcomputer.

The monomial basis of aberration coefficients, however, has definite computational and theoretical advantages. Of the latter, the selection rules that were found for refracting-surface aberration coefficients[4,10] are clearly stated as zeros of the monomial coefficients, while for the symplectic ones part of these only bind different-j coefficients. Of the former, the monomial basis easily generalizes to non-axis-symmetric systems such as magnetic-optics systems, where the program MARYLIE, developed in MACSYMA by Dragt,[23] has had significant influence in the optics of accelerators. Asymmetric aberrations were classified in Ref. 24 with attention to $Sp(4, R) \supset Sp(2, R)$ symmetry, but no systematic computational work has been carried out by the author in that direction. In any perspective, there seems to be a good case for the application of the symmetry arguments of Lie algebras and groups to optics in the realm of aberration phenomena, as compared with the traditional rendering of the subject.

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