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Editorial

Symmetry and Approximation Methods

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The overwhelming majority of mathematical problems, describing realistic systems and processes, contain two parts: first, the problem needs to be characterized by an effective mathematical model and, second, the appropriate solutions are to be found. Both these stages usually require resorting to the aid of symmetry notions. From one side, symmetry properties help in modeling the considered problem. Additionally, from the other side, the knowledge of solution symmetries makes it easier to find these solutions.

At the same time, mathematical models are usually so much complicated that they allow only for the treatment by some approximation methods providing approximate solutions that, however, are assumed to respect the required symmetries. Thus, the notion of symmetry often neighbours the notion of approximation. The interplay between symmetries and approximations is what the Editors are keeping in mind presenting the collection of papers in the Special Issue. The Special Issue contains 9 original papers and 2 reviews.

In the paper "Features of Fermion Dynamics Revealed by SU(2) Symmetry" [1] by A. Plastino, G.L. Ferri, and A.R. Plastino, the authors apply the notion of statistical order to a system of interacting fermions endowed with an SU(2) $\times$ SU(2) symmetry. The discussion takes place in a thermal quantum statistical scenario. Two distinct fermion–fermion interactions are at play. One of them is a well-known spin–flip interaction. The other is the pairing interaction responsible for nuclear superconductivity. Novel statistical quantifiers are suggested that yield insights regarding changes in the statistical order produced when the values of the pertinent coupling constants vary. In particular, judicious manipulation of the energy cost associated with statistical order variations with the fermion number is shown to be the key to understanding important details of the associated dynamics.

The article by U. Amato and B.D. Vecchia "New Progressive Iterative Approximation Techniques for Shepard-Type Curves" [2] studies the use of this technique, abbreviated as PIA, for data fitting. In the modeling, if the given data points are taken as initial control points, the process generates a series of shaping curves by adjusting the control points iteratively, while the limit curve interpolates the data points. Such a format was used successfully for Shepard-type curves. Simple variants of the method for Shepard-type curves are constructed producing novel curves modeling data points, so that the designer can choose among several pencils of shapes outlining original control polygon. Matrix formulations, convergence results, error estimates, algorithmic formulations, critical comparisons, and numerical tests are shown. An application to a progressive modeling format by truncated wavelet transform is also presented, improving in some sense analogous process by truncated Fourier transform. By playing on two shapes handles—the number of base wavelet transform functions and the iteration level of PIA algorithm—several new contours, modeling the given control points, are constructed.

In the paper "Methods of Retrieving Large-Variable Exponents" [3] by V.I. Yukalov and S. Gluzman, the methods are analysed allowing for the determination, based on small-variable asymptotic expansions, of the characteristic exponents for variables tending to infinity. The following methods are considered: diff-log Padé summation, self-similar factor approximation, self-similar diff-log summation, self-similar Borel summation, and self-similar Borel–Leroy summation. Several typical problems are treated. The comparison

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Citation: Andrianov, I.V.; Gluzman, S.; Yukalov, V.I. Symmetry and Approximation Methods. *Symmetry* **2023**, *15*, 106. https://doi.org/10.3390/sym15010106

Received: 22 December 2022
Accepted: 26 December 2022
Published: 30 December 2022

of the results shows that all these methods provide close estimates for the large-variable exponents. The reliable estimates are obtained when different methods of summation are compatible with each other.

The paper "Online Streaming Features Selection via Markov Blanket" [4] by W. Khan, L. Kong, B. Brekhna, L. Wang, and H. Yan, studies the methods of feature selection. Streaming feature selection has always been an excellent method for selecting the relevant subset of features from high-dimensional data and overcoming learning complexity. However, little attention has been paid to online feature selection through the Markov Blanket (MB). Several studies based on traditional MB learning presented low prediction accuracy and used fewer datasets as the number of conditional independence tests is high and consumes more time. A novel algorithm, called Online Feature Selection Via Markov Blanket (OFSVMB), based on a statistical conditional independence test, offers high accuracy and less computation time. It reduces the number of conditional independence tests and incorporates the online relevance and redundant analysis to check the relevancy between the upcoming feature and target variable T, discard the redundant features from Parents–Child (PC) and Spouses (SP) online, and find PC and SP simultaneously. The performance OFSVMB is compared with traditional MB learning algorithms including IAMB, STMB, HITON-MB, BAMB, and EEMB, and streaming feature selection algorithms including OSFS, Alpha-investing, and SAOLA on 9 benchmark Bayesian Network (BN) datasets and 14 real-world datasets. For the performance evaluation, precision and recall measures are used with a significant level of 0.01 and 0.05 on benchmark BN and real-world datasets, including 12 classifiers keeping a significant level of 0.01. On benchmark BN datasets with 500 and 5000 sample sizes, OFSVMB achieved significant accuracy, as compared to IAMB, STMB, HITON-MB, BAMB, and EEMB in terms of F1, precision, recall, and running faster. It finds more accurate MB regardless of the size of the features set. In contrast, OFSVMB offers substantial improvements based on mean prediction accuracy regarding 12 classifiers with small and large sample sizes on real-world datasets, as compared to OSFS, Alpha-investing, and SAOLA, but slower than OSFS, Alpha-investing, and SAOLA because these algorithms only find the PC set but not SP. Furthermore, the sensitivity analysis shows that OFSVMB is more accurate in selecting the optimal features.

The article "The Altes Family of Log-Periodic Chirplets and the Hyperbolic Chirplet Transform" [5] by D. Daly and D. Sornette revisits a class of biomimetically inspired waveforms introduced by R.A. Altes in the 1970s for use in sonar detection. Similar to the chirps used for echolocation by bats and dolphins, these waveforms are log-periodic oscillations, windowed by a smooth decaying envelope. Log-periodicity is associated with the deep symmetry of discrete scale invariance in physical systems. Furthermore, there is a close connection between such chirping techniques, and other useful applications, such as wavelet decomposition for multi-resolution analysis. Motivated to uncover additional properties, the authors propose an alternative, simpler parameterization of the original Altes waveforms. From this, it becomes apparent that we have a flexible family of hyperbolic chirps suitable for the detection of accelerating time-series oscillations. The proposed formalism reveals the original chirps to be a set of admissible wavelets with desirable properties of regularity, infinite vanishing moments, and time-frequency localisation. As they are self-similar, these "Altes chirplets" allow for efficient implementation of the scale-invariant hyperbolic chirplet transform (HCT), whose basis functions form hyperbolic curves in the time-frequency plane. Compared with the rectangular time-frequency tilings of both the conventional wavelet transform and the short-time Fourier transform, the HCT can better facilitate the detection of chirping signals, which are often the signature of critical failure in complex systems. A synthetic example is presented to illustrate this useful application of the HCT.

The paper "A New Constructive Method for Solving the Schrödinger Equation" [6] by K. Rajchel proposes a new method for the exact solution of the stationary, one-dimensional Schrödinger equation. Application of the method leads to a three-parametric family of exact solutions, previously known only in the limiting cases. The method is based on solutions of the Ricatti equation in the form of a quadratic function with three parameters. The logarithmic derivative of the wave function transforms the Schrödinger equation to the Ricatti equation with arbitrary potential. The Ricatti equation is solved by exploiting the particular symmetry, where a family of discrete transformations preserves the original form of the equation. The method is applied to a one-dimensional Schrödinger equation with a bound states spectrum. By extending the results of the Ricatti equation to the Schrödinger equation the three-parametric solutions for wave functions and energy spectrum are obtained. This three-parametric family of exact solutions is defined on compact support, as well as on the whole real axis in the limiting case, and corresponds to a uniquely defined form of potential. Celebrated exactly solvable cases of special potentials, such as harmonic oscillator potential, Coulomb potential, infinite square well potential with corresponding energy spectrum and wave functions follow from the general form by appropriate selection of parameters values. The first two of these potentials with corresponding solutions, which are defined on the whole axis and half axis, respectively, are achieved by taking the limit of general three-parametric solutions, where one of the parameters approaches a certain, well-defined value.

In the paper "Continued Roots, Power Transform and Critical Properties" [7] by S. Gluzman the problem is considered of calculating the critical amplitudes at infinity by means of the self-similar continued root approximants. Region of applicability of the continued root approximants is extended from the determinate (convergent) problem with well-defined conditions studied before by Gluzman and Yukalov (Phys. Lett. A 377, 124, 2012), to the indeterminate (divergent) problem by means of power transformation. Most challenging indeterminate for the continued roots problems of calculating critical amplitudes, can be successfully attacked by performing proper power transformation to be found from the optimization imposed on the parameters of power transform. The self-similar continued roots were derived by systematically applying the algebraic self-similar renormalization to each and every level of interactions with their strength increasing, while the algebraic renormalization follows from the fundamental symmetry principle of functional self-similarity, realized constructively in the space of approximations. The approach to the solution of an indeterminate problem consists in replacing it with the determinate problem, but with some unknown control parameter b in place of the known critical index β. From optimization conditions b is found in the way making the problem determinate and convergent. The index β is hidden under the carpet and replaced by b. The idea is applied to various, mostly quantum-mechanical problems. In particular, the method allows us to solve the problem of Bose–Einstein condensation temperature with good accuracy.

The article "Revisiting the Predictability of the Haicheng and Tangshan Earthquakes" [8] by D. Sornette, E. Mearns, and S. Wheatley analyzes a set of precursory data measured before but compiled in retrospect of the MS7.5 Haicheng earthquake in February 1975 and the MS7.6–7.8 Tangshan earthquake in July 1976. A robust and simple coarse-graining method is proposed that aggregates and counts how all the anomalies together (levelling, geomagnetism, soil resistivity, earth currents, gravity, earth stress, well water radon, well water level) develop as a function of time. A strong evidence is demonstrated for the existence of an acceleration of the number of anomalies leading up to the major Haicheng and Tangshan earthquakes. In particular for the Tangshan earthquake, the frequency of occurrence of anomalies is found to be well described by the log-periodic power law singularity (LPPLS) model, previously proposed for the prediction of engineering failures and later adapted to the prediction of financial crashes. Using a mock real-time prediction experiment and simulation study, based on this methodology of monitoring accelerated rates of physical anomalies measured at the surface, the potential for an early warning system with a lead time of a few days is shown.

The paper "Optimal Random Packing of Spheres and Extremal Effective Conductivity" [9] by V. Mityushev and Z. Zhunussova establishes close relation between the optimal packing of spheres in R_d and minimal energy E (effective conductivity) of composites with ideally conducting spherical inclusions is established. The location of inclusions of the optimal-design problem yields the optimal packing of inclusions. The geometrical-packing and physical-conductivity problems are stated in a periodic toroidal d-dimensional space with an arbitrarily fixed number n of non-overlapping spheres per periodicity cell. Energy E depends on Voronoi tessellation (Delaunay graph) associated with the centres of spheres a_k (k = 1, 2, . . ., n). All Delaunay graphs are divided into classes of isomorphic periodic graphs. For any fixed n, the number of such classes is finite. Energy E is estimated in the framework of structural approximations and reduced to the study of a function of n variables. The minimum of E over locations of spheres is attained at the optimal packing within a fixed class of graphs. The optimal-packing location is unique within a fixed class up to translations and can be found from linear algebraic equations. Such an approach is useful for random optimal packing where an initial location of balls is randomly chosen; hence, a class of graphs is fixed and can dynamically change following prescribed packing rules. A finite algorithm for any fixed n is constructed to determine the optimal random packing of spheres in R_d.

The review article "On Markov Moment Problem, Polynomial Approximation on Unbounded Subsets, and Mazur-Orlicz Theorem" [10] by O. Olteanu surveys the earlier and recent results on the Markov moment problem and related polynomial approximation on unbounded subsets. These results allow to prove the existence and uniqueness of the solutions for some Markov moment problems. This is the first aim of the paper. The considered solutions have a codomain space a commutative algebra of (linear) symmetric operators acting from the entire real or complex Hilbert space H to H; this algebra of operators is also an order complete Banach lattice. In particular, Hahn–Banach type theorems for the extension of linear operators having a codomain such a space can be applied. The truncated moment problem is briefly discussed by means of reference citations. This is the second purpose of the paper. In the end, a general extension theorem for linear operators with two constraints is recalled and applied to concrete spaces. Here, polynomial approximation plays no role. This is the third aim of this work.

A thorough review "Padé Approximants, Their Properties, and Applications to Hydrodynamic Problems" [11] is given by I. Andrianov and A. Shatrov. The paper is devoted to an overview of the basic properties of the Padé transformation and its generalizations. The merits and limitations of the described approaches are discussed. Particular attention is paid to the application of Padé approximants in the mechanics of liquids and gases. One of the disadvantages of asymptotic methods is that the standard ansatz in the form of a power series in some parameter usually does not reflect the symmetry of the original problem. The search for asymptotic ansatzes that adequately take into account this symmetry has become one of the most important problems of asymptotic analysis. The most developed technique from this point of view is the Padé approximation.

We hope that the reader can find useful information in the papers of this Special Issue.

Conflicts of Interest: The authors declare no conflict of interest.

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