Symmetry and quantization: Higher-order polarization and anomalies

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The concept of (full) polarization subalgebra in a Group Approach to Quantization on a Lie group $\widetilde{G}$ as a generalization of the analogous concept in geometric or standard quantization is discussed. The lack of full polarization subalgebras is considered as an anomaly of the corresponding system and related to its more conventional definition. A generalization of the subalgebra of (full) polarization is then provided, made out of higher-order differential operators in the enveloping algebra of $\widetilde{G}$. Higher-order polarizations can also be used to quantize nonanomalous theories in different "representations." Numerous examples are analyzed, including the finite-dimensional dynamics associated with the Schrödinger group, which presents an anomaly, and an infinite-dimensional anomalous system associated with the Virasoro group. In the last example, the operators in the higher-order polarization are in one-to-one correspondence with the null vectors in the Verma module approach.

I. INTRODUCTION

In a previous paper[1] the space of first-order polarized functions on the Virasoro Group was further reduced by enlarging the polarization subalgebra with higher-order differential operators lying in the left-enveloping algebra of the group. The new operators in this generalized or higher-order polarization were closely related to the null states appearing in the Verma module approach.[2] There the need for higher-order polarizations was associated with the anomalous character of the Virasoro group quantization itself, and that of the bosonic string quantization, where the Virasoro group also plays a crucial role.

In this paper we extend the concept of higher-order polarizations (HOP) to general cases, thus enriching the general setting of the group approach to quantization.[3,4] For some anomalous systems the group quantization can be achieved, nevertheless, without a HOP. They are characterized by having a vacuum, i.e., a highest-weight vector state (see Sec. III and Ref. 5). We shall study, in particular, the case of the Schrödinger group,[6] which possesses a (finite-dimensional) anomaly, and has two nonequivalent classes of representations: the conventional one without vacuum and a different class containing a highest-weight state. This is an interesting example of an anomalous group, because it has finitely many dimensions, but still suffers from the essential pathology of the (typically infinite-dimensional) anomalous physical systems.[7] The use of a HOP is not restricted to anomalous systems but can also be introduced in ordinary systems to quantize the theory in a "representation" different from the "natural" one(s). To be precise, a (first-order) polarization, whose role is that of reducing the representation by restricting the arguments of the wave functions, determines the arguments, η, on which the wave functions depend, and therefore leads to the quantum theory in the η representation. If there is no (first-order) polarization leading to a desired η representation then a HOP can probably be introduced to this end. For instance, the quantization of the free particle results in the p representation since the centrally extended Galilei algebra $\{\widehat{H},\widehat{p},\widehat{x},\widehat{I}\}$ only admits the polarization subalgebra $\{\widehat{H},\widehat{P}\sim\partial/\partial x\}$. In the same way the natural representation for the harmonic oscillator is the coherent state representation. In both cases there is a HOP leading to the x representation.

II. THE CONCEPT OF POLARIZATION

In quantum mechanics a polarization is an involutive set $\mathscr{P}$ of differential operators annihilating the wave functions. The condition $\overline{X}\Psi=0$, $\overline{X}\in\mathscr{P}$ is intended to reduce the quantum representation by restricting the arguments of the wave functions Ψ. For instance, in geometric quantization[8] the classical Poisson algebra is first repre-

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sented by extended Hamiltonian vector fields on the quantum manifold Q, which is a $U(1)$ principal bundle on the phase space manifold M, i.e., Q is locally diffeomorphic to the product $M\times S^1$. The carrier space is constituted by $U(1)$ functions on Q; in other words, complex functions on Q satisfying the condition of being homogeneous of degree 1 in the variable $\zeta\in S^1$. The representation thus obtained is not irreducible and corresponds to the Bohr approximation. At this stage the wave functions depend on x and p. The nonirreducibility of that representation is made evident by the existence of certain sets of nontrivial differential operators commuting with the basic operators $\hat{x}$ and $\hat{p}$. These sets, which are precisely the polarizations, are isotropic subspaces of dimension $n=\frac{1}{2}\dim M$ of the symplectic form ω ($\omega(X',X)=0$ if $\bar{X}'$, $\bar{X}\in\mathscr{P}$, where the overbar indicates lifting of vectors from M to Q with the connection form Θ, whose curvature $d\Theta$ is ω). In the finite-dimensional case the conditions $\bar{X}\Psi=0$, $\bar{X}\in\mathscr{P}$ restrict the arguments of Ψ to half of them, leading to the irreducible x representation, p representation, or any diagonal one.

In the group approach to quantization (GAQ), the quantum manifold Q is changed to a Lie group $\tilde{G}$ with a principal bundle structure with fibre a subgroup $U(1)$[3] [in Ref. 4 the structure group $U(1)$ is generalized to a bigger non-Abelian and even infinite-dimensional group]. The quotient $\tilde{G}/U(1)$ is no longer a symplectic manifold, yet it is endowed with a presymplectic form ω naturally defined as $d\Theta$, where Θ is a component, dual to the vertical $U(1)$ generator, of the left-invariant canonical one-form on $\tilde{G}$. The kernel of ω or, more precisely, $\mathrm{Ker}\, d\Theta\cap\mathrm{Ker}\,\Theta$ is a module (the characteristic module) generated by a left subalgebra of $\tilde{G}$, the so-called characteristic subalgebra $\mathscr{G}_\Theta$. If $\tilde{G}$ is a central extension of a Lie group G, as will mostly be the case, then $\mathscr{G}_\Theta$ is the kernel of the Lie algebra cocycle, $\Sigma:\mathscr{G}\times\mathscr{G}\to\mathbb{R}$, characterizing the extension. Now, instead of taking the quotient of $\tilde{G}$ by $\mathscr{G}_\Theta$ (the generalized equations of motion) to get rid of nonsymplectic variables it is far more advantageous to keep the whole set of variables (including the time variable, for instance) and generalize the notion of polarization, so as to include the subalgebra $\mathscr{G}_\Theta$. This results in additional polarizations conditions, which include the Schrödinger equation in the corresponding "representation." In this way no classical equations of motion have to be solved prior to quantization. *We define a full polarization in GAQ as a maximal left-subalgebra $\mathscr{P}$ of $\tilde{\mathscr{G}}$ containing $\mathscr{G}_\Theta$ and excluding the vertical U(1) generator.* The excluding the $U(1)$ generator ensures that no conjugate coordinate-momentum pairs are present in the polarization.

As mentioned above, in connection with geometric quantization, a (full) polarization is required (at least in the finite-dimensional case) for the quantum representation to be irreducible. The space of quantum wave functions is the space of complex $U(1)$-equivariant functions on $\tilde{G}$ satisfying the polarization conditions, i.e., $\Psi:\tilde{G}\to\mathbb{C}$, such that $\tilde{\Xi}\Psi=i\Psi$, Ξ being the $U(1)$ generator, and $\tilde{X}^L\Psi=0$, $\forall\ \tilde{X}^L\in\mathscr{P}$. The physical operators are the right-invariant generators of $\tilde{G}$, $\tilde{X}^R$, acting on the polarized (reduced) space of Ψ's.

To get familiar with the meaning of a (full) polarization, let us consider the simplest physical example of the free particle, whose basic symmetry is the centrally extended Galilei group.[3,4] For this group, translations and boosts play the role of canonically conjugate (or symplectic) variables, whereas the time translation and rotations are in the kernel of the cocycle $\eta\equiv e^{i\xi}:\mathbf{G}\times\mathbf{G}\to U(1)$ ($\Sigma:\mathscr{G}\times\mathscr{G}\to\mathbb{R}$). Only one (full) polarization is possible, and it is generated by the time translation, rotations, and space translations. The polarization conditions then imply that wave functions do not depend on x (the translation parameter), depend arbitrarily on p (the boost parameter), and the dependence on time is such that they obey the momentum-space Schrödinger equation $i\hbar(\partial\Psi/\partial t)=\mathbf{p}^2/2m\Psi$. To get the "$x$ representation" we would need the boost generators inside the full polarization but, unfortunately, they do not close a subalgebra with the time translation generator.

From the point of view of the physical operators, i.e., the right-invariant vector fields associated with each one of the group variables, the quantum observables corresponding to the characteristic subalgebra differ from the remainder in that the former must be expressed in terms of the latter once the full polarization conditions are imposed. This solvability of nonsymplectic operators as functions of the symplectic ones, say, $\hat{x}$ and $\hat{p}$, is nothing other than the quantum analog of the "classical" reduction of the generalized phase space to the true symplectic solution manifold, i.e., the solvability of Noether invariants (in GAQ they are written as the interior product of right-invariant vector fields with the form Θ, $i_{\tilde{X}^R}\Theta$) associated with the subalgebra $\mathscr{G}_\Theta$ in terms of the rest. Indeed, this mechanism is that which allows us to generalize geometric quantization by including nonsymplectic variables. These comments can immediately be verified for the case of the free particle. There the energy operator $i\hbar\tilde{X}_t^R\equiv i\hbar(\partial/\partial t)$ reduces to $\hat{p}^2/2m$ once the polarization has been taken. Likewise, the operator $\tilde{X}_\epsilon^R$ associated with rotations [$R^i_j\in SO(3)$ is parametrized by $\epsilon\in\mathbb{R}^3$ as $R^i_j(\epsilon)=(1-\epsilon^2/2)\delta^i_j+\sqrt{1-\epsilon^2/4}\,\eta^i_{\cdot jk}\epsilon^k+1/2\epsilon^i\epsilon_j$), which, prior to the polarization restriction, are written as $\tilde{X}^R_{\epsilon j}=[\sqrt{1-\epsilon^2/4}\delta^i_j-\eta^i_{\cdot jk}\,\epsilon^k/2]\partial/\partial\epsilon^i-\eta^i_{\cdot jk}x^k\partial/\partial x^i-\eta^i_{\cdot jk}v^k\partial/\partial v^i$, condense to the standard form $\hat{\mathbf{x}}\wedge\hat{\mathbf{p}}$.

III. ANOMALIES IN GROUP QUANTIZATION: HIGHER-ORDER POLARIZATIONS

Up to now we have been referring to a group $\tilde{G}$ for which a full polarization does exist. Nevertheless, this is not always the case, and we shall call anomalous a dynamical system without such a full polarization. It is easy to imagine what kind of structure renders a symmetry group anomalous. For instance, suppose a group $\widetilde{G}$ contains coordinate-momentum variables x^i, p_j and a subgroup of some sort of nondiagonal "rotations" mixing x's and p's. These "rotations" would be on the characteristic subalgebra (like ordinary rotations) and should therefore be in the full polarization subalgebra. However, no such subalgebra exists (see the example of the Schrödinger group in Sec. IV).

The nonexistence of full polarization parallels the absence of polarizations in a symplectic manifold[9] in standard geometric quantization. There, it constitutes an unavoidable obstruction to the quantization mechanism.

In GAQ we define a nonfull polarization as a maximal left subalgebra of $\widetilde{\mathscr{G}}$, excluding the vertical ($U(1)$) generator and containing a subalgebra of $\mathscr{G}_\Theta$. With this definition we can quantize as before, with the drawback that the space of polarized functions is not, in general, irreducible or, what is more significant, not all the physical operators can be solved in terms of the basic ones, even though all the classical Noether invariants are expressible in terms of the coordinates of the classical solution manifold (basic Noether invariants).

A way out of the problem of nonirreducibility and (or) the lack of solvability of some of the operators in terms of the basic ones was given in Refs. 5 and 10, in connection with the Virasoro group. The solution consisted in taking the orbit through the vacuum, i.e., a highest-weight vector among the polarized functions on the group, under the action of the right-enveloping algebra. This subspace is not only irreducible, but also provides the desired form of the physical operators as a function of the basic ones, although it is only possible for given values, *the quantum values*, of those constants characterizing the representation (central charge, vacuum expectation value of the energy, etc., see later on in this section).

A more general solution for the anomaly problem in a group $\widetilde{G}$ is to generalize the notion of full polarization subalgebra by allowing elements in the left-enveloping algebra of $\widetilde{G}$ to enter this subalgebra. We arrive at the definition of a *higher-order polarization as a maximal subalgebra of the left-enveloping algebra of $\widetilde{G}$ excluding the central generator and containing a (first-order or ordinary) polarization*. Once a higher-order polarization is given, the space of polarized functions on the group satisfying the $U(1)$-equivariance condition is irreducible. Like in the case of a non-full (first-order) polarization, when the group is anomalous the values of the constants characterizing the irreducible representation are not arbitrary. In fact, a higher-order polarization only exists for very special values of those constants, referred to as the quantum values.

Let us think in terms of coordinates and Lie algebra commutators to clarify this anomalous quantization and to relate it to the more conventional presentation of anomalies in terms of Poisson brackets, commutators, central charges, etc. Roughly speaking, the coordinates of an anomalous group $\widetilde{G}$ are of the form (x, p, ϵ^+, ϵ^-, ϵ^0, ζ), where x and p represent a family, perhaps infinite, of coordinates and momenta, satisfying Lie commutators of the type

$$[\widetilde{X}_x,\widetilde{X}_p]=a\widetilde{X}_{\epsilon^0}+b\widetilde{X}_\zeta, \quad a,b,\in\mathbb{R} \tag{1}$$

(analogous to the ordinary Poisson brackets $\{x,p\}=1$ for the Galilei group, or $\{x,p\}=p^0+1$ for the Poincaré group), ϵ^+, ϵ^-, and ϵ^0 parametrize a subgroup with a nondiagonal action on x and p, and ζ parametrizes the central $U(1)$ subgroup. The anomaly arises through the nondiagonality of the ϵ action, i.e., the fact that ϵ^+ and ϵ^- mix x and p. A typical communtator for the ϵ subgroup is

$$[\widetilde{X}_{\epsilon^+},\widetilde{X}_{\epsilon^-}]=\widetilde{X}_{\epsilon^0}. \tag{2}$$

Looking at formulas (1) and (2), we see that $\mathscr{G}_\Theta$, the kernel of the Lie algebra cocycle $\Sigma:\mathscr{G}\times\mathscr{G}\to\mathbb{R}$, is generated by $\langle X_{\epsilon^0},X_{\epsilon^+},X_{\epsilon^-}\rangle$. However, if $X_{\epsilon^\pm}$ mix X_x and X_p in a nondiagonalizable way, there is no full polarization containing $\mathscr{G}_\Theta$ and X_x or X_p, i.e., in our language there is an anomaly. We can only find, at most, (nonfull) polarizations subalgebras $\mathscr{P}$ containing just the subalgebra $\langle\widetilde{X}^L_{\epsilon^0}\rangle$ of $\mathscr{G}_\Theta$, actually,

$$\mathscr{P}^{\pm}=\langle\widetilde{X}^L_{x^\pm},\widetilde{X}^L_{\epsilon^0},\widetilde{X}^L_{\epsilon^\pm}\rangle, \tag{3}$$

where $x^\pm$ are certain combinations of x and p that are stable under the action of $\widetilde{X}_{\epsilon^\pm}$ (X_ϵ—transform x^+ into x^-; the same holds changing $+$ to $-$, otherwise a full polarization would exist).

Quantizing according to the (nonfull) polarization $\mathscr{P}^{\pm}$ causes the physical quantum operators $\widetilde{X}^R_{\epsilon^\pm}$ not to be expressible in terms of those operator that should be the basic ones, i.e., $\widetilde{X}^R_x$ and $\widetilde{X}^R_p$, in contrast to what occurs with the corresponding classical Noether invariants $i_{\widetilde{X}^R_{\epsilon^\pm}}\Theta$. It looks as if $\widetilde{X}^R_{\epsilon^\pm}$ really were basic operators, that is to say, as if the initial (classical) commutator (2) had been

$$[\widetilde{X}_{\epsilon^+},\widetilde{X}_{\epsilon^-}]=\widetilde{X}_{\epsilon^0}+c\widetilde{X}_\zeta. \tag{4}$$

Indeed, once we take an orbit of the vacuum according to the prescription mentioned above or a higher-order polarization $\mathscr{P}^{\mathrm{HO}\pm}$ containing $\mathscr{P}^{\pm}$, the reduced quantum operators $\widetilde{X}^R_\epsilon$ obey (4) for a concrete (quantum) value of c, $c(b)$, depending on the (classical) value of the central charge b in (1). Even more, no higher-order polarization completing $\mathscr{P}^{\pm}$ exists unless c takes that specific (quantum) value.

We must remark, nevertheless, that we could have started the quantization mechanism from a group $\widetilde{G}$ centrally extended by two independent constants b and c, i.e., from the commutators (1) and (4), and then we would have found natural to obtain the operators $\widetilde{X}^R_{\epsilon^\pm}$ as independent operators, because (4) says that they are associated with canonically conjugated variables ϵ^+, ϵ^-. Then, once the (for this case) full polarization $\langle \widetilde{X}^L_{x^+}, \widetilde{X}^L_{\epsilon^0}, \widetilde{X}^L_{\epsilon^+}\rangle$ (or with $+$ changed to $-$) would have been taken, we would have discovered that for special values of b and c, actually, b and $c=c(b)$ above, there would have been an automatic condensation mechanism in such a way that the operators $\widetilde{X}^R_{\epsilon^\pm}$ would be written as a function of $\widetilde{X}^R_{x^\pm}$

Let us summarize our view of anomalies as it emerges from the group quantization scheme. In the usual canonical quantization, anomalies "appear" as additional terms if one tries to translate the classical Poisson-bracket relations to quantum commutators. However, in the GAQ, where one is interested in (exact) unitary irreducible representations of certain dynamical groups $\widetilde{G}$, it does not make sense to introduce central extensions or, more generally, deformations (in addition to the ordinary central terms associated with the classical Poisson bracket structure) only after the quantization. They have to be present at the classical level. That is, to a given classical system we associate a Lie group $\widetilde{G}$ together with all its possible "extra" deformations labeled by $c_1,...,c_n$, i.e., by n (unspecified) arbitrary constants. Then we proceed with the quantization and, if no obstructions occur, we obtain unitary irreducible representations of the group $\widetilde{G}$ for any value of the parameters c_i. However, we may find that $\widetilde{G}$ cannot be quantized properly for all values of the c_i, for example, if there are c_i for which we cannot find a HOP. We call such a theory anomalous. Note that according to this philosophy an anomalous theory may still admit quantization, albeit only for certain values of the c_i (the so-called quantum values). It may even occur that $c_i\equiv 0$ are quantum values, in which case all deformations disappear from the original group law, and the theory is "quantizable" in the usual sense, without ever referring to possible anomalous terms.

However, in the general case this will not occur, and therefore it is essential for the understanding of the anomalies to consider all possible central extensions (or deformations) from the very beginning. In the GAQ, classical and quantum structures are intimately connected, which allows us to view anomalies not primarily as quantum objects, but also as an integral part of the underlying symmetry structure of the theory. Hence, an anomalous theory does not manifest itself by the appearance of central extensions, but rather by the fact that only for specific values of the corresponding extensions (deformations) the theory admits a proper quantization.

IV. EXAMPLES

In this section we shall discuss two types of examples. On the one hand, there are the genuinely anomalous theories (Schrödinger group, Virasoro group,[1] bosonic string), and on the other hand nonanomalous theories (free particle, harmonic oscillator, free particle on anti-de Sitter background) to which one can apply the HOP method to obtain different realizations of quantum representations. For the anomalous case we compare two possible ways of quantizing: one by considering orbits through vacuum vectors, the other by using HOP. Since it is easier to illustrate the method, we shall begin with the discussion of the nonanomalous examples.

A. The free particle

The starting point is the Galilei group with a standard central extension:

$$S''=S'S,\quad S\equiv\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix},\quad t\in\mathbb{R},$$
$$\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix}''=S\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix}'+\begin{pmatrix}\mathbf{x}\\ \mathbf{v}\end{pmatrix},\quad \mathbf{x},\mathbf{v}\in\mathbb{R}^3, \tag{5}$$

$$\zeta''=\zeta'\zeta e^{i(m/2)[\mathbf{x}'\cdot\mathbf{v}+t(\mathbf{v}'\cdot\mathbf{v}+(1/2)\mathbf{v}'^2)]}. \tag{6}$$

From this we derive the left-invariant vector fields,

$$\widetilde{X}^L_t=\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+m\,\frac{\mathbf{v}}{2\hbar}\,\Xi,\quad \widetilde{X}^L_{\mathbf{x}}=\frac{\partial}{\partial\mathbf{x}},$$
$$\widetilde{X}^L_{\mathbf{v}}=\frac{\partial}{\partial\mathbf{v}}+\frac{m\mathbf{x}}{\hbar}\,\Xi,\quad \widetilde{X}^L_\zeta=i\zeta\,\frac{\partial}{\partial\zeta}\equiv\Xi, \tag{7}$$

leading to the Lie algebra relations

$$[\widetilde{X}^L_t,\widetilde{X}^L_{\mathbf{x}}]=0,\quad [\widetilde{X}^L_t,\widetilde{X}^L_{\mathbf{v}}]=-\widetilde{X}^L_{\mathbf{x}},$$
$$[\widetilde{X}^L_{v^i},\widetilde{X}^L_{x^j}]=-\frac{m}{\hbar}\,\delta_{ij}\Xi. \tag{8}$$

[The rotation subgroup, which we have not written down explicitly, forbids the introduction of an additional central term in the first equation of (8).] This algebra is not anomalous because a full polarization exists, which is generated by $\widetilde{X}^L_t$ and $\widetilde{X}^L_{\mathbf{x}}$. The corresponding polarization conditions, together with the $U(1)$ equivariance condition $\Xi\Psi=i\Psi$, lead to the Schrödinger equation on wave functions in momentum space. In order to obtain the Schrödinger equation in configuration space, the operator $\widehat{X}^L_{\mathbf{v}}$ has to be in the polarization. However, there is no first-order polarization containing just $\widetilde{X}_t^L$ and $\widetilde{X}_{\mathbf{v}}^L$ and not $\widetilde{X}_{\mathbf{x}}^L$, so that, to get a nontrivial result, we have to resort to a higher-order polarization $P^{\mathrm{HO}}=\widetilde{X}_{\mathbf{v}}^L$, $\widetilde{X}_t^L-(i\hbar/2m)\widetilde{X}_{\mathbf{x}}^L\widetilde{X}_{\mathbf{x}}^L\rangle$. The higher-order polarization equations now read as

$$\left(\frac{\partial}{\partial\mathbf{v}}+\frac{im\mathbf{x}}{\hbar}\right)\Psi=0,$$
$$\left[\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial\mathbf{x}}+\frac{im\mathbf{v}^2}{2\hbar}\right)-\frac{i\hbar}{2m}\frac{\partial^2}{\partial\mathbf{x}^2}\right]\Psi=0. \tag{9}$$

The most general solution of (8) is $\Psi(\mathbf{x},\mathbf{v},t)=e^{-(im/\hbar)\mathbf{v}\cdot\mathbf{x}}\Phi(\mathbf{x},t)$, where Φ satisfies the Schrödinger equation in configuration space,

$$i\hbar\frac{\partial\Phi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial x^2}. \tag{10}$$

B. The harmonic oscillator

The corresponding dynamical group for the harmonic oscillator is given by the expression (1), where the matrix S is now

$$S=\begin{pmatrix}\cos\omega t & \omega^{-1}\sin\omega t\\ -\omega\sin\omega t & \cos\omega t\end{pmatrix}, \tag{11}$$

and the cocycle

$$\zeta''=\zeta'\zeta\exp i(m/2\hbar)[\mathbf{x}'\cdot\mathbf{v}\cos\omega t-\mathbf{v}'\cdot\mathbf{x}\cos\omega t+(\mathbf{v}\cdot\mathbf{v}'/\omega+\omega\mathbf{x}\cdot\mathbf{x}')\sin\omega t]. \tag{12}$$

The left-invariant vector fields are given by

$$\widetilde{X}_t^L=\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}-\omega^2\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{v}},$$
$$\widetilde{X}_{\mathbf{x}}^L=\frac{\partial}{\partial\mathbf{x}}-\frac{m}{2\hbar}\mathbf{v}\Xi, \tag{13}$$
$$\widetilde{X}_{\mathbf{v}}^L=\frac{\partial}{\partial\mathbf{v}}+\frac{m}{2\hbar}\mathbf{x}\Xi,$$

with the following commutation relations:

$$[\widetilde{X}_t^L,\widetilde{X}_{\mathbf{x}}^L]=\omega^2\widetilde{X}_{\mathbf{v}}^L,\quad [\widetilde{X}_t^L,\widetilde{X}_{\mathbf{v}}^L]=-\widetilde{X}_{\mathbf{x}}^L,$$
$$[\widetilde{X}_{v^i}^L,\widetilde{X}_{x^j}^L]=-\frac{m}{\hbar}\delta_{ij}\Xi. \tag{14}$$

As in the case of the free particle the group above is not anomalous, since a full polarization can be found. It is generated by $\langle\widetilde{X}_t^L,\widetilde{X}_{\mathbf{x}}^L+(i/m)\widetilde{X}_{\mathbf{v}}^L\rangle$, and leads to the coherent-state representation of the harmonic oscillator.[3] Nevertheless, there is no full polarization leading to the configuration space representation because the subalgebra $\widetilde{X}_{\mathbf{v}}^L$ cannot be enlarged further. The wave functions in the x representation can be obtained by imposing the polarization condition $\widetilde{X}_{\mathbf{v}}^L\Psi=0$ (that factors out the p dependence) and taking the orbit of the right-enveloping algebra on the vacuum $|0\rangle\equiv\Psi_0$, where $|0\rangle$ is the state annihilated by the operators $\widetilde{X}_t^R$ and $\widetilde{X}_{\mathbf{x}}^R+i/\omega\widetilde{X}_{\mathbf{v}}^R$ (in Ref. 11 the same method was applied to obtain the relativistic Hermite polynomials).

We now describe the higher-order polarization method to get the wave functions in configuration space. The higher-order polarization is $\langle\widetilde{X}_{\mathbf{v}}^L,\widetilde{X}_t^L-(i\hbar/2m)\widetilde{X}_{\mathbf{x}}^L\cdot\widetilde{X}_{\mathbf{x}}^L\rangle$ and the corresponding polarization equations are

$$\left(\frac{\partial}{\partial\mathbf{v}}+\frac{im\mathbf{x}}{2\hbar}\right)\Psi=0,$$
$$\left[\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}-\omega^2\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{v}}\right)-\frac{i\hbar}{2m}\left(\frac{\partial^2}{\partial\mathbf{x}^2}-\frac{im\mathbf{v}}{\hbar}\cdot\frac{\partial}{\partial\mathbf{x}}-\frac{m^2\mathbf{v}^2}{4\hbar^2}\right)\right]\Psi=0. \tag{15}$$

The most general solution of (15) is $\Psi(\mathbf{x},\mathbf{v},t)=\exp\{-im\mathbf{v}\cdot\mathbf{x}/2\hbar\}\Phi(\mathbf{x},t)$, where Φ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Phi}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\mathbf{x}^2}+\frac{1}{2}m\omega^2\mathbf{x}^2\right)\Phi. \tag{16}$$

C. Free particle in anti-de Sitter space

We now consider the quantization of an affine version of SO(2,1):

$$[\hat{E},\hat{X}]=-(i/m)\hat{P},\quad [\hat{E},\hat{P}]=im\omega^2\hat{X},$$
$$[\hat{X},\hat{P}]=i\left(1+\frac{1}{mc^2}\hat{E}\right). \tag{17}$$

This algebra was considered previously, in connection with two different physical interpretations: first, observe that the algebra (17), in the $c\to\infty$ limit goes to the harmonic oscillator algebra, and under $\omega\to 0$ to the free relativistic particle algebra (Poincaré in 1+1 dimensions). Hence, (17) can be considered as a relativistic harmonic oscillator (note, in passing, that the relativistic energy operator $\hat{E}$ in (17) has the rest energy substracted, in order to have the correct nonrelativistic limit for (17)). This interpretation of (17) as a relativistic oscillator has been completely worked out in Ref. 11.

On the other hand, (17) can be viewed as the 1+1-dimensional version of the SO(3,2) anti-de Sitter algebra (again with an energy operator with the rest energy mc^2

substracted). Taking this into account, the quantization of (17) should lead to the quantum mechanics of a free particle in $1+1$ anti-de Sitter space. This view was considered in Ref. 12.

Here we want to begin by observing that (17) admits a full polarization given, e.g., by the left version of the two operators,

$$\hat{E}, \quad \sqrt{\frac{m\omega}{2}}\,\hat{X}+\frac{i}{\sqrt{2m\omega}}\,\hat{P}. \tag{18}$$

Quantization of (17) was carried out in Ref. 12 by imposing this polarization; the point now is that this polarization does not lead to a solution in configuration space. Such a kind of solution requires imposing the polarization condition $\hat{X}\Psi=0$, which by itself is a nonfull polarization. Thus, there is a need for further reduction of the representation, which can be achieved, as we mentioned before, by determining the vacuum and taking an orbit through it; this was done in Ref. 11.

Another way of getting a solution in configuration space, which we want to use out now, is to look for a HOP that includes the condition $\hat{X}\Psi=0$. In particular, wave functions in configuration space should satisfy a general-relativistic wave equation to be determined by this HOP.

To apply the GAQ to the system we are considering, we need a group law having (17) as its Lie algebra. In Ref. 12 a group law was given that describes the free particle in anti-de Sitter space. There, the metric in $1+1$-dimensional space was the one induced by the metric in three-dimensional space,

$$c^2\,dx^{0^2}-dx^2-R^2\,dz^2=c^2\,d\tau^2, \tag{19a}$$

under restriction to the hypersurface

$$c^2x^{0^2}-x^2-R^2z^2=R^2, \tag{19b}$$

where R is the *anti-de Sitter radius*, related to the frequency in (17) by $R\omega=c$.

Here, we introduce a reparametrization in time, such that the new time is given by

$$t=\frac{1}{\omega}\sin^{-1}\frac{\omega x^0}{\sqrt{c^2+\omega^2x^2}}. \tag{20}$$

With this time variable, the metric in (19), which is nondiagonal in $1+1$ space (see Refs. 12 and 13), obtains the diagonal form

$$(c^2+\omega^2x^2)dt^2-\frac{1}{1+\omega^2x^2/c^2}\,dx^2=c^2\,d\tau^2. \tag{21}$$

Applying the change in (20) to the group law in Ref. 12, Eq. (4.2a), we get the following group law in coordinates t, x, p:

$$x''=\frac{p'\alpha}{m\omega}\sin\omega t+\alpha x'\cos\omega t+\frac{xP'^0}{mc},$$

$$p''=\frac{\omega xp}{c^2\alpha}\left(\frac{p'}{m}\sin\omega t-\omega x'\cos\omega t\right)+\frac{P^0}{c\alpha}\left(\frac{p'}{m}\cos\omega t-\omega x'\sin\omega t\right)+\frac{pP'^0}{mc},$$

$$\sin\omega t''=\frac{\omega}{\alpha''}\left(\frac{\alpha}{mc^2\alpha'}p'x'\sin\omega t'\sin\omega t+\frac{\alpha P'^0}{m\omega c\alpha}\cos\omega t'\sin\omega t+\frac{\omega}{\alpha' mc^3}xx'P'^0\sin\omega t'+\frac{\alpha\alpha'}{\omega}\cos\omega t\sin\omega t'+\frac{p'x}{mc^2\alpha'}\cos\omega t'\right), \tag{22a}$$

where

$$\alpha\equiv\sqrt{1+\frac{\omega^2x^2}{c^2}}, \quad P^0\equiv\sqrt{m^2c^2+p^2+m^2\omega^2x^2}. \tag{23}$$

Here we shall use the (pseudo)extension of the group law (22a) given by

$$\begin{aligned}&\zeta''=\zeta'\zeta e^{i/\hbar(\sigma''-\sigma'-\sigma)}, \quad \sigma\equiv-mc^2t-f,\\ &f=-\frac{2mc^2}{\omega}\arctan\left[\frac{mc^2}{\omega px}(x-1)\left(\frac{P^0}{mc}-x\right)\right].\end{aligned} \tag{22b}$$

From the group law (22) one derives the following left generators:

$$\begin{aligned}\tilde{X}^L_{(x)}&=\frac{P^0}{mc}\frac{\partial}{\partial x}+\frac{P}{mc^2\alpha^2}\frac{\partial}{\partial t}-\frac{Pmc}{P^0+mc}\frac{1}{\hbar}\,\Xi,\\ \tilde{X}^L_{(p)}&=\frac{P^0}{mc}\frac{\partial}{\partial p}+\frac{mcx}{P^0+mc}\frac{1}{\hbar}\,\Xi,\\ \tilde{X}^L_{(t)}&=\frac{p}{m}\frac{\partial}{\partial x}-m\omega^2x\frac{\partial}{\partial p}+\frac{P^0}{mc\alpha^2},\end{aligned} \tag{24}$$

which obey, under the identification

$$\hat{E}\to i\hbar\tilde{X}^L_{(t)};\quad \hat{X}\to i\hbar\tilde{X}^L_{(p)};\quad \hat{P}\to i\hbar\tilde{X}^L_{(x)}, \tag{25}$$

the commutation relations (17).

In order to obtain wave functions in configuration space, we must, as was said before, impose $\hat{X}\Psi\equiv\tilde{X}^L_{(p)}\Psi=0$ as a first polarization condition. This is a nonfull polarization, so we have to look for another higher-order condition, including the operator $\hat{E}\sim\tilde{X}^L_{(t)}$ (see the previous examples).

From the commutation relations in (17) it can be seen that the combination

$$-2mc^2\hat{E}+\hat{E}^2-m^2\omega^2c^2\hat{X}^2-c^2\hat{P}^2$$

$$=(\hat{E}-mc^2)^2-m^2\omega^2c^2\hat{X}^2-c^2\hat{P}-m^2c^4, \quad (26)$$

commutes with both $\hat{X}$ and $\hat{P}$ [from the right-hand side in (26) it is evident that this is just the Casimir operator of the Lie algebra], and thus completes the HOP, which, in terms of the left generators, is given by

$$\left\langle \tilde{X}^L_{(p)}, \frac{-2imc^2}{\hbar}\tilde{X}^L_{(t)}-(\tilde{X}^L_{(t)})^2+c^2(\tilde{X}^L_{(x)})^2 -m^2\omega^2c^2(\tilde{X}^L_{(p)})^2\right\rangle. \quad (27)$$

The first condition,

$$\tilde{X}^L_{(p)}\Psi(x,p,t)=0\to\Psi=e^{if/\hbar}\varphi(x,t), \quad (28)$$

results in a factorization of the p dependence, which is the same for all wave functions [f is given in (22b)]. The remaining condition,

$$\left[\frac{2imc^2}{\hbar}\tilde{X}^L_{(t)}-(\tilde{X}^L_{(t)})^2+c^2(\tilde{X}^L_{(x)})^2\right]e^{if/\hbar}\varphi(x,t)=0, \quad (29)$$

can be seen to lead (after a long, but straightforward calculation) to the differential equation

$$\frac{1}{c^2\alpha^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{i}{\hbar}\frac{2m}{\alpha^2}\frac{\partial\varphi}{\partial t}-\frac{2\omega^2x}{c^2}\frac{\partial\varphi}{\partial x}-\alpha^2\frac{\partial^2\varphi}{\partial x^2} +\frac{m^2\omega^2x^2}{\hbar^2\alpha^2}\varphi=0. \quad (30)$$

In order to put this into a more familiar form, recall that the "energy" operator used in (17) and in the polarization had the rest energy mc^2 substracted [see the rhs of (26)]. We can restore the rest energy in the wave functions by means of the transformation

$$\Phi=e^{-im^2t/\hbar}\varphi, \quad (31)$$

in such a way that Φ becomes the true relativistic wave functions. Under (31), the differential equation (30) transforms into

$$\frac{1}{c^2\alpha^2}\frac{\partial^2\Phi}{\partial t^2}-\frac{2\omega^2x}{c^2}\frac{\partial\Phi}{\partial x}-\alpha^2\frac{\partial^2\Phi}{\partial x^2}=-\frac{m^2c^2}{\hbar^2}\Phi. \quad (32)$$

This is our final result. The left-hand side of (32) turns out to be precisely the D'Alembertian defined by the metric in (21). Thus we see how, in this case, the usage of a HOP leads directly to the wave equation for the wave functions of a free particle in curved space-time (see, for example, Ref. 14).

D. The Schrödinger group

This group is the maximal kinematical invariance group of the Schrödinger equation for a particle moving in an arbitrary potential of the form Ax^2+Bx+C (see Ref. 6 and references therein). Mathematically, it can be obtained from the Galilei group by replacing the time parameter with the three-parameter group $SL(2,\mathbb{R})$. The Schrödinger group is the first known example of a finite-dimensional group that does not possess a full polarization.

We discuss the group in two different coordinations. The first one uses the coordinates of Ref. 6, which, strictly speaking, correspond to taking the universal covering group of the $SL(2,\mathbb{R})$ subgroup. This is necessary if we want to interpret t as a noncompact time parameter. For the quantization we have to resort to a HOP, leading to a representation of the Schrödinger group on the carrier space of wave functions for the Galilean particle.

In the second one, keeping the compact $U(1)$ parameter in the $SL(2,\mathbb{R})$ subgroup, we can find an additional, unequivalent nonfull polarization containing this compact parameter.

Since there is a distinguished vacuum vector in the space of wave functions we use the orbit method in the quantization procedure, leading to highest-weight representations of the Schrödinger group on the carrier space of wave functions for the harmonic oscillator. We could equally well have performed the quantization by using a HOP, which, however, would have led to equivalent quantum representations.

We begin with the Schrödinger group as given in Ref. 6, which is of the form (5), with S being now an element of $SL(2,\mathbb{R})$,

$$S=\begin{pmatrix} a^{-1}(1+ct) & at \\ a^{-1} & a\end{pmatrix}$$

$$=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix},\quad a,c,t\in\mathbb{R}. \quad (33)$$

For the cocycle we use the expression

$$\zeta''=\zeta'\zeta\exp-im\left\{\frac{1}{2}a^2t\mathbf{v}'^2+\frac{c}{2a^2}(1+ct)\mathbf{x}'^2+\frac{c}{a}\mathbf{x}'\cdot\mathbf{x} +ct\mathbf{x}'\cdot\mathbf{v}'+a\mathbf{x}\cdot\mathbf{v}'\right\}\exp i\alpha\ln\frac{a'}{a}(a^2+ct'), \quad (34)$$

where the exponential containing the real parameter α corresponds to a trivial extension. The left-invariant vector fields are

$$\tilde{X}^L_t = \frac{\partial}{\partial \tau} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{1}{2} m\mathbf{v}\Xi,$$

$$\tilde{X}^L_c = ta\frac{\partial}{\partial a} + (1+2ct)\frac{\partial}{\partial c} - t^2\frac{\partial}{\partial t} + \mathbf{x}\cdot\frac{\partial}{\partial \mathbf{v}} + \left(-\frac{1}{2}m\mathbf{x}^2 + \alpha t\right)\Xi, \tag{35}$$

$$\tilde{X}^L_a = a\frac{\partial}{\partial a} + 2c\frac{\partial}{\partial c} - \mathbf{x}\cdot\frac{\partial}{\partial \mathbf{x}} - 2t\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{v}},$$

$$\tilde{X}^L_{\mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} - m\mathbf{v}\Xi, \quad \tilde{X}^L_{\mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}, \quad \tilde{X}^L_\zeta = i\zeta\frac{\partial}{\partial \zeta} \equiv \Xi,$$

satisfying the algebra relations

$$\begin{aligned}
&[\tilde{X}^L_t, \tilde{X}^L_a] = -2\tilde{X}^L_t, \qquad [\tilde{X}^L_t, \tilde{X}^L_{\mathbf{x}}] = 0,\\
&[\tilde{X}^L_t, \tilde{X}^L_c] = \tilde{X}^L_a + \alpha\Xi, \qquad [\tilde{X}^L_t, \tilde{X}^L_{\mathbf{v}}] = -\tilde{X}^L_{\mathbf{x}},\\
&[\tilde{X}^L_a, \tilde{X}^L_c] = 2\tilde{X}^L_c, \qquad [\tilde{X}^L_a, \tilde{X}^L_{\mathbf{x}}] = \tilde{X}^L_{\mathbf{x}},\\
&[\tilde{X}^L_a, \tilde{X}^L_{\mathbf{v}}] = -\tilde{X}^L_{\mathbf{v}}, \qquad [\tilde{X}^L_c, \tilde{X}^L_{\mathbf{v}}] = 0,\\
&[\tilde{X}^L_c, \tilde{X}^L_{\mathbf{x}}] = -\tilde{X}^L_{\mathbf{v}}, \qquad [\tilde{X}^L_{\mathbf{x}}, \tilde{X}^L_{\mathbf{v}}] = \mathbb{I}m\Xi.
\end{aligned} \tag{36}$$

By duallity on (35) we are able to work out the left-invariant canonical one-form, and, in particular, the component dual to the $\tilde{X}^L_\zeta \equiv \Xi$ generator, Θ:

$$\begin{aligned}
\Theta = {}& m\mathbf{v}\cdot d\mathbf{x} - \tfrac{1}{2} m\mathbf{v}^2\, dt + (\tfrac{1}{2} m\mathbf{v}^2 t^2 - m\mathbf{x}\cdot\mathbf{v}t\\
&+ \tfrac{1}{2} m\mathbf{x}^2 - \alpha t)dc + [m(1+2ct)\mathbf{x}\cdot\mathbf{v}\\
&- m(1+ct)\mathbf{v}^2 t - cm\mathbf{x}^2 + 2\alpha ct]da + \frac{d\zeta}{i\zeta}.
\end{aligned} \tag{37}$$

This group does not have a full polarization and the only polarization we can find is a nonfull one,

$$\mathscr{P} = \langle \tilde{X}^L_{\mathbf{x}}, \tilde{X}^L_t, \tilde{X}^L_a \rangle. \tag{38}$$

We need to include the operator $\tilde{X}^L_c$ in the polarization, but this is only possible if we modify $\tilde{X}^L_c$ by a term from the left-enveloping algebra. The operator $\tilde{X}^L_c + (i/2m)\tilde{X}^L_{\mathbf{v}}\cdot\tilde{X}^L_{\mathbf{v}}$, together with (38), form a closing, higher-order polarization algebra if α assumes the value $\alpha = \frac{3}{2}i$.

The three polarization conditions $\tilde{X}^L_{\mathbf{x},t,a}\Psi = 0$ have as general solutions wave functions of the form

$$\Psi = e^{im[\mathbf{x}\cdot\boldsymbol{\eta}a - (1/2)\eta^2 a^2 t]}\varphi(\boldsymbol{\eta},\xi), \tag{39}$$

where $\boldsymbol{\eta} = \mathbf{v}/a$, $\xi = c/a^2$, and φ satisfies the Schrödinger equation,

$$\left[\frac{\partial}{\partial \xi} + \frac{i}{2m}\frac{\partial^2}{\partial \eta^2}\right]\varphi = 0. \tag{40}$$

It is straightforward to show that the action of the right vector fields on the wave functions φ reduces to

$$\begin{aligned}
&\tilde{X}^R_t \varphi(\boldsymbol{\eta},\xi) = \left(-\frac{im}{2}\eta^2 - \frac{3}{2}\xi - \xi^2\frac{\partial}{\partial \xi} - \xi\boldsymbol{\eta}\cdot\frac{\partial}{\partial \boldsymbol{\eta}}\right)\varphi\\
&\qquad \equiv \frac{i}{2m}\tilde{X}^R_{\mathbf{x}}\cdot\tilde{X}^R_{\mathbf{x}}\varphi,\\
&\tilde{X}^R_{\mathbf{x}}\varphi = \left(im\boldsymbol{\eta} + \xi\frac{\partial}{\partial \boldsymbol{\eta}}\right)\varphi,\\
&\tilde{X}^R_{\mathbf{v}}\varphi = \frac{\partial}{\partial \boldsymbol{\eta}}\varphi, \quad \tilde{X}^R_c\varphi = \frac{\partial}{\partial \xi}\varphi = -\frac{i}{2m}\tilde{X}^R_{\mathbf{v}}\cdot\tilde{X}^R_{\mathbf{v}}\varphi,\\
&\tilde{X}^R_a\varphi = \left(-\boldsymbol{\eta}\cdot\frac{\partial}{\partial \boldsymbol{\eta}}\right)\varphi = \frac{i}{m}\tilde{X}^R_{\mathbf{x}}\cdot\tilde{X}^R_{\mathbf{v}}\varphi.
\end{aligned} \tag{41}$$

The action is well defined, since all the $\tilde{X}^R$ operators map solutions of (39) into themselves.

From the right-invariant vector fields,

$$\begin{aligned}
&\tilde{X}^R_t = \frac{1}{a^2}\frac{\partial}{\partial t} + \frac{1}{a^2}c^2\frac{\partial}{\partial c} + \frac{c}{a}a\frac{\partial}{\partial a} + \frac{c}{a^2}\alpha\Xi,\\
&\tilde{X}^R_{\mathbf{x}} = \frac{1}{a}(1+ct)\frac{\partial}{\partial \mathbf{x}} + \frac{c}{a}\frac{\partial}{\partial \mathbf{v}} - m\frac{c}{a}\mathbf{x}\Xi,\\
&\tilde{X}^R_{\mathbf{v}} = a\frac{\partial}{\partial \mathbf{v}} + at\frac{\partial}{\partial \mathbf{x}} - ma\mathbf{x}\Xi,\\
&\tilde{X}^R_a = a\frac{\partial}{\partial a}, \quad \tilde{X}^R_c = a^2\frac{\partial}{\partial c},
\end{aligned} \tag{42}$$

we can derive the classical Noether invariants,

$$\mathbf{p}_{\rm cl}\sim\Theta(\tilde{X}^R_{\mathbf{x}})=\frac{1}{a}m\mathbf{v}+\frac{c}{a}m(\mathbf{v}t-\mathbf{x}),$$

$$\mathbf{x}_{\rm cl}\sim\Theta(\tilde{X}^R_{\mathbf{v}})=am(\mathbf{v}t-\mathbf{x}),$$

$$E_{\rm cl}\sim\Theta(\tilde{X}^R_t)=-\frac{1}{2a^2}m\mathbf{v}^2(ct+1)^2+\frac{c}{a^2}m\mathbf{x}\cdot\mathbf{v}(ct+1)$$
$$-\frac{mc^2}{2a^2}\mathbf{x}^2+\frac{\alpha c}{a^2}(ct+1), \tag{43}$$

$$\Theta(\tilde{X}^R_a)=-m\mathbf{v}^2t(ct+1)+m\mathbf{x}\cdot\mathbf{v}(2ct+1)-cm\mathbf{x}^2$$
$$+2\alpha ct,$$

$$\Theta(\tilde{X}^R_c)=\tfrac{1}{2}a^2m\mathbf{v}^2t^2-a^2m\mathbf{x}\cdot\mathbf{v}t+\tfrac{1}{2}a^2m\mathbf{x}^2-\alpha a^2t.$$

We can perform the classical reduction for any value of α, but only for $\alpha=0$ we obtain the maximal reduction, which enables us to express all quantities in terms of just the basic variables x and p, in agreement with our interpretation of the classical theory.

Hence, similar to the quantum theory, one value of α is singled out, but it is different from the value singled out in the quantum case.

Alternatively, we can write the Schrödinger group law in terms of the variables C, $C^\dagger$, η, z, z^*, ζ, with C and $C^\dagger$ corresponding to harmonic oscillator creation and annihilation operators, and η, z, z^* parametrizing the $SL(2,\mathbb{R})$ subgroup. The connection with the old variables is given by

$$a=\sqrt{\frac{1+\kappa}{2}}\left\{\frac{\eta+\eta^*}{2}-\frac{1}{1+\kappa}\frac{z\eta+z^*\eta^*}{2}\right\},$$

$$\mathbf{x}=\frac{1}{\sqrt{2m\omega}}(\mathbf{C}+\mathbf{C}^\dagger);$$

$$a^{-1}c=\omega\sqrt{\frac{1+\kappa}{2}}\left\{\frac{\eta-\eta^*}{2i}-\frac{1}{1+\kappa}\frac{z\eta-z^*\eta^*}{2i}\right\}, \tag{44}$$

$$\mathbf{v}=\frac{-i\omega}{\sqrt{2m\omega}}(\mathbf{C}-\mathbf{C}^\dagger);$$

$$at=\frac{1}{\omega}\sqrt{\frac{1+\kappa}{2}}\left\{\frac{\eta-\eta^*}{2i}-\frac{1}{1+\kappa}\frac{z\eta-z^*\eta^*}{2i}\right\};$$

$$\kappa\equiv\sqrt{1+zz^*}.$$

These coordinates are well suited if one uses a polarization containing the compact, and not the affine, subgroup inside $SL(2,\mathbb{R})$. The group law is

$$\mathbf{C}''=\mathbf{C}+\sqrt{\frac{1+\kappa}{2}}\eta\mathbf{C}'+\sqrt{\frac{2}{1+\kappa}}\frac{z^*}{2}\eta^*\mathbf{C}'^\dagger,$$

$$\mathbf{C}''^\dagger=\mathbf{C}^\dagger+\sqrt{\frac{1+\kappa}{2}}\eta^*\mathbf{C}'^\dagger+\sqrt{\frac{2}{1+\kappa}}\frac{z}{2}\eta\mathbf{C}',$$

$$z''=z'\eta^{-2}+\kappa'z+\frac{z}{2(1+\kappa)}[z^*z'\eta^{-2}+z'^*z\eta^2],$$

$$z''^*=z'^*\eta^2+\kappa'z^*+\frac{z^*}{2(1+\kappa)}[zz'^*\eta^2+z'z^*\eta^{-2}], \tag{45}$$

$$\eta''=\sqrt{\frac{2}{1+\kappa''}}\left\{\sqrt{\frac{1+\kappa}{2}}\sqrt{\frac{1+\kappa'}{2}}\eta\eta'+\sqrt{\frac{2}{1+\kappa}}\sqrt{\frac{2}{1+\kappa'}}\frac{z^*z'}{4}\eta^*\eta'\right\},$$

$$\zeta''=\zeta'\zeta\exp i\xi(g',g)\cdot(\eta''\eta'^{-1}\eta^{-1})^\gamma,$$

$$\xi(g',g)\equiv\frac{i}{2}\left[\sqrt{\frac{1+\kappa}{2}}\eta\mathbf{C}'\cdot\mathbf{C}^\dagger+\sqrt{\frac{2}{1+\kappa}}\frac{z^*}{2}\mathbf{C}'^\dagger\cdot\mathbf{C}^\dagger-\sqrt{\frac{2}{1+\kappa}}\frac{z}{2}\mathbf{C}\cdot\mathbf{C}'-\sqrt{\frac{1+\kappa}{2}}\eta^{-1}\mathbf{C}\cdot\mathbf{C}'^\dagger\right].$$

The left- and right-invariant vector fields read as

$$\tilde{X}^L_\eta=i\eta\frac{\partial}{\partial\eta}-2iz\frac{\partial}{\partial z}+2iz^*\frac{\partial}{\partial z^*}+i\mathbf{C}\cdot\frac{\partial}{\partial\mathbf{C}}-i\mathbf{C}^\dagger\cdot\frac{\partial}{\partial\mathbf{C}^\dagger},$$

$$\tilde{X}^L_{\mathbf{C}}=\frac{\partial}{\partial\mathbf{C}}-\frac{i}{2}\mathbf{C}^\dagger\Xi,\quad \tilde{X}^L_{\mathbf{C}^\dagger}=\frac{\partial}{\partial\mathbf{C}^\dagger}+\frac{i}{2}\mathbf{C}\Xi,$$

$$\tilde{X}^L_z=\kappa\frac{\partial}{\partial z}+\frac{i}{2(1+\kappa)}\frac{z^*}{2}i\eta\frac{\partial}{\partial\eta}+\frac{1}{2}\mathbf{C}\cdot\frac{\partial}{\partial\mathbf{C}^\dagger}+\frac{i\gamma z^*}{4(1+\kappa)}\Xi, \tag{46a}$$

$$\tilde{X}^L_{z^*}=\kappa\frac{\partial}{\partial z^*}-\frac{i}{2(1+\kappa)}\frac{z}{2}i\eta\frac{\partial}{\partial\eta}+\frac{1}{2}\mathbf{C}^\dagger\cdot\frac{\partial}{\partial\mathbf{C}}-\frac{i\gamma z}{4(1+\kappa)}\Xi,$$

$$\tilde{X}^L_\zeta=i\zeta\frac{\partial}{\partial\zeta}\equiv\Xi,$$

$$\tilde{X}^R_\eta = i\eta \frac{\partial}{\partial\eta},$$

$$\tilde{X}^R_z = \frac{\eta^{-2}}{2(1+\kappa)}\left\{(1+\kappa)^2\frac{\partial}{\partial z}+z^{*2}\frac{\partial}{\partial z^*}-i\frac{z^*}{2}i\eta\frac{\partial}{\partial\eta} -i\gamma\frac{z^*}{2}\Xi\right\},$$

$$\tilde{X}^R_{z^*} = \frac{\eta^{2}}{2(1+\kappa)}\left\{(1+\kappa)^2\frac{\partial}{\partial z^*}+z^{2}\frac{\partial}{\partial z}+i\frac{z}{2}i\eta\frac{\partial}{\partial\eta} +i\gamma\frac{z}{2}\Xi\right\},$$

$$\tilde{X}^R_{\mathbf{C}} = \sqrt{\frac{1+\kappa}{2}}\,\eta\frac{\partial}{\partial\mathbf{C}}+\sqrt{\frac{2}{1+\kappa}}\frac{z}{2}\eta\frac{\partial}{\partial\mathbf{C}^\dagger} -\frac{i}{2}\left(\sqrt{\frac{2}{1+\kappa}}\,\eta\frac{z\mathbf{C}}{2}-\sqrt{\frac{1+\kappa}{2}}\,\eta\mathbf{C}^\dagger\right)\Xi, \tag{46b}$$

$$\tilde{X}^R_{\mathbf{C}^\dagger} = \sqrt{\frac{1+\kappa}{2}}\,\eta\frac{\partial}{\partial\mathbf{C}^\dagger}+\sqrt{\frac{2}{1+\kappa}}\frac{z^*}{2}\eta^*\frac{\partial}{\partial\mathbf{C}} +\frac{i}{2}\left(\sqrt{\frac{2}{1+\kappa}}\,\eta^*\frac{z^*\mathbf{C}^\dagger}{2}-\sqrt{\frac{1+\kappa}{2}}\,\eta^*\mathbf{C}\right)\Xi,$$

$$\tilde{X}^R_\zeta = i\zeta\frac{\partial}{\partial\zeta}\equiv\Xi.$$

We choose to work with the (nonfull) polarization

$$\mathscr{P}=\langle\tilde{X}^L_\eta,\tilde{X}^L_{\mathbf{C}},\tilde{X}^L_{z^*}\rangle. \tag{47}$$

We could again use a HOP for quantizing, but this time we will construct an irreducible representation by taking the orbit through the vacuum.

The general form of a wave function projected out is

$$\Psi=e^{-\mathbf{C}^\dagger\mathbf{C}/2}e^{\mathbf{C}^{\dagger 2}(\kappa-1)/2z}(1+\kappa)^{-\gamma/2}\sum_n \eta^n(1+\kappa)^{-n/2} \times \sum_{\substack{n_1,n_2\\ n_1+n_2=n}} A_{n_1n_2}z^{-n_1/2}, \tag{48}$$

and we find a distinguished vacuum vector,

$$|0\rangle=e^{-\mathbf{C}^\dagger\mathbf{C}/2}e^{\mathbf{C}^{\dagger 2}(\kappa-1)/2z}(1+\kappa)^{-\gamma/2}, \tag{49}$$

which is annihilated by the operators $\tilde{X}^R_\eta$, $\tilde{X}^R_{\mathbf{C}^\dagger}$, and $\tilde{X}^R_z$. The orbit through the vacuum is obtained by applying all right-invariant vector fields to (49). The quantum reduction (i.e., expressing all operators in terms of the basic ones $\tilde{X}^R_{\mathbf{C}}$ and $\tilde{X}^R_{\mathbf{C}^\dagger}$) is obtained if we set $\gamma=-\frac{1}{2}$, in which case $\tilde{X}^R_{z^*}-\frac{1}{4}(\tilde{X}^R_{\mathbf{C}})^2=0$ is fulfilled on all states lying in the orbit. Similarly, we find an analogous expression for the operator $\tilde{X}^R_z$. The fact that the value of γ is a half-integer means that the $SL(2,\mathbb{R})$ subgroup of the Schrödinger group appears covered twice.

Note that if we had used a Verma module approach, as in the infinite-dimensional anomalous case, we also would have obtained null vectors generated from the vacuum by the operator analogous to our $\tilde{X}^R_{z^*}-\frac{1}{4}(\tilde{X}^R_{\mathbf{C}})^2$. The advantage of the method presented here is that these vectors vanish identically, so we avoid having to quotient out by the space of null vectors.

E. The bosonic string in Minkowski space.

This typical infinite-dimensional anomalous system has already been analyzed in the context of HOP in Ref. 1.

The underlying symmetry for this dynamical system is characterized by the centrally extended semidirect product diff $S^1\otimes$ (loops on $\mathbb{R}^{1,d-1}$), whose Lie algebra is given by

$$\begin{aligned}
&[\tilde{X}^L_{\alpha^\mu_n},\tilde{X}^L_{\alpha^\nu_m}]=n\,\delta_{n,-m}\Xi,\quad \Xi\equiv\text{central generator},\\
&[\tilde{X}^L_{l_n},\tilde{X}^L_{\alpha^\mu_m}]=-mi\tilde{X}^L_{\alpha^\mu_{n+m}},\\
&[\tilde{X}^L_{l_n},\tilde{X}^L_{l_m}]=-i(n-m)\tilde{X}^L_{l_{n+m}}-\tfrac{1}{12}(cn^3-c'n)\delta_{n,-m}\Xi,
\end{aligned} \tag{50}$$

where c' is related with the vacuum expectation value of the L_0 operator, $h=(c-c')/24$. For the classical values $c=0=c'$ the basic functions α^μ_n cannot be quantized without violating the above symmetry. The reason for this obstruction is that the group diff S^1 does not preserve the polarization $\{\tilde{X}^L_{\alpha^\mu_{n<0}}\}$ of the loop space, just in the same way as the $SL(2,\mathbb{R})$ subgroup of the Schrödinger group does not preserve the x or p polarization of the corresponding phase space. In other words, the generators $\langle\tilde{X}^L_{l_m},\tilde{X}^L_{\alpha^\mu_{n<0}},m\in Z\rangle$ do not close a first-order polarization subalgebra. We have to consider only half of the Virasoro generators (half of the characteristic subalgebra) to close a first-order polarization subalgebra $\mathscr{P}=\langle\tilde{X}^L_{l_{m<0}},\tilde{X}^L_{\alpha^\mu_{n<0}},m\in Z\rangle$, even though all Virasoro generators have Noether invariants that can be solved in terms of α^μ_n's (the basic ones). Quantizing with the nonfull polarization $\mathscr{P}$ for $c=0=c'$ results in a representation that, although irreducible, prevents the solvability of the Virasoro operators, $\tilde{X}^R_{l_m}$, in terms of the basic ones $\tilde{X}^R_{\alpha^\mu_n}$.

A higher-order polarization $\mathscr{P}^{\rm HO}$ can be defined that "contains" all the Virasoro generators as the leading terms in higher-order operators of the left-enveloping algebra. However, such a higher-order polarization $\mathscr{P}^{\mathrm{HO}}$ exists only for $c=c'=d$ (=dimension of Minkowski space), thus modifying the classical values. $\mathscr{P}^{\mathrm{HO}}$ is a left ideal of the enveloping algebra of (50), generated by the first-order polarization $\mathscr{P}=\langle \tilde{X}^L_{l_{m<0}}, \tilde{X}^L_{\alpha^\mu_{n<0}}\rangle$ and the higher-order operators $\tilde{Z}^L_{N>0}$,

$$\tilde{Z}^L_N=\tilde{X}^L_{l_N}-\frac{1}{2}\sum_{\substack{n+m=N\\ 1<n,m}}\eta^{\mu\nu}\tilde{X}^L_{\alpha^\mu_m}\tilde{X}^L_{\alpha^\nu_n}. \tag{51}$$

These expressions are closely related to the Sugawara construction of the Virasoro algebra (see Ref. 1 for technical details). As in the finite-dimensional examples studied above, the space of higher-order polarized functions carries the irreducible representations of the group diff $S^1\otimes$ (loops on $\mathbf{R}^{1,d-1}$) and coincides with the orbit of the right-enveloping algebra through the vacuum.

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