December, 2011

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E. Kirr
Panos Kevrekidis
D. E. Pelinovsky

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E. Kirr[1], P.G. Kevrekidis[2], and D.E. Pelinovsky[3]

[1] Department of Mathematics, University of Illinois, Urbana–Champaign, Urbana, IL 61801

[2] Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003

[3] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

December 20, 2010

Abstract

We consider the focusing (attractive) nonlinear Schrödinger (NLS) equation with an external, symmetric potential which vanishes at infinity and supports a linear bound state. We prove that the symmetric, nonlinear ground states must undergo a symmetry breaking bifurcation if the potential has a non-degenerate local maxima at zero. Under a generic assumption we show that the bifurcation is either subcritical or supercritical pitchfork. In the particular case of double-well potentials with large separation, the power of nonlinearity determines the subcritical or supercritical character of the bifurcation. The results are obtained from a careful analysis of the spectral properties of the ground states at both small and large values for the corresponding eigenvalue parameter.

We employ a novel technique combining concentration–compactness and spectral properties of linearized Schrödinger type operators to show that the symmetric ground states can either be uniquely continued for the entire interval of the eigenvalue parameter or they undergo a symmetry–breaking pitchfork bifurcation due to the second eigenvalue of the linearized operator crossing zero. In addition we prove the appropriate scaling for the L^q, $2 \leq q \leq \infty$ and H^1 norms of any stationary states in the limit of large values of the eigenvalue parameter. The scaling and our novel technique imply that all ground states at large eigenvalues must be localized near a critical point of the potential and bifurcate from the soliton of the focusing NLS equation without potential localized at the same point.

The theoretical results are illustrated numerically for a double-well potential obtained after the splitting of a single-well potential. We compare the cases before and after the splitting, and numerically investigate bifurcation and stability properties of the ground states which are beyond the reach of our theoretical tools.

1 Introduction

Over the past few years, there has been a remarkable growth of interest in the study of nonlinear Schrödinger (NLS) equations with external potentials. This has been fueled, to a considerable extent, by the theoretical and experimental investigation of Bose-Einstein condensates (BECs) [26, 28]. Localized waveforms emerge within the atom-trapping potentials in such ultracold systems

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[4]. Another major area of applications for NLS equations is nonlinear optics, in particular photonic crystals and optical waveguides [21, 14].

One generic type of the external potential for the NLS equation that has drawn considerable attention is the symmetric double-well potential. This is due to its relative simplicity which often makes it amenable to analytical considerations, but also due to the wealth of phenomenology that even such a relatively simple system can offer. Such potentials in the atomic physics setting of BECs have already been experimental realized [1] through the combination of routinely available parabolic and periodic (optical lattice) potentials. Among the interesting phenomena studied therein were Josephson oscillations and tunneling for a small number of atoms, or macroscopic quantum self-trapping and an asymmetric partition of the atoms between the wells for sufficiently large numbers of atoms. Double well potentials were also examined in the context of nonlinear optics, e.g. in twin-core self-guided laser beams in Kerr media [3], optically induced dual-core waveguiding structures in a photorefractive crystal [15], and trapped light beams in a structured annular core of an optical fiber [25].

In the present work, we address the NLS equation with a symmetric potential as the prototypical mathematical model associated with the above experimental settings. For simplicity we will focus on the case of one space dimension. We write the equation in the normalized form

$$iu_t = -u_{xx} + V(x)u + \sigma|u|^{2p}u, \qquad (1.1)$$

where $u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is the wave function, $p > 0$ is the nonlinearity power, $\sigma \in \mathbb{R}$ determines the defocusing (repulsive), respectively focusing (attractive), character of the nonlinearity when $\sigma > 0$, respectively $\sigma < 0$, and $V(x) : \mathbb{R} \mapsto \mathbb{R}$ is an external real-valued, *symmetric* (even in x) potential satisfying:

(H1) $V(x) \in L^\infty(\mathbb{R})$,

(H2) $\lim_{|x|\to\infty} V(x) = 0$,

(H3) $V(-x) = V(x)$ for all $x \in \mathbb{R}$.

Hypothesis (H1) implies that $-\partial_x^2 + V(x)$ is a self-adjoint operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We will make the following spectral assumption:

(H4) $L_0 = -\partial_x^2 + V(x)$ has the lowest eigenvalue $-E_0 < 0$.

It is well known from the Sturm-Liouville theory that all eigenvalues of L_0 are simple, the corresponding eigenfunctions can be chosen to be real valued and the one corresponding to the $k+1$-th eigenvalue has exactly k zeroes, and, because of the symmetry (H3), is symmetric (even in x) if k is even and anti-symmetric if k is odd. We can choose a normalized eigenfunction, ψ_0, corresponding to the eigenvalue $-E_0$, which will satisfy:

$$-\psi_0''(x) + V(x)\psi_0(x) + E_0\psi_0(x) = 0, \quad \psi_0(x) > 0, \ \psi_0(-x) = \psi_0(x), \ x \in \mathbb{R}, \ \|\psi_0\|_{L^2} = 1. \quad (1.2)$$

We are interested in understanding properties of stationary, symmetric and asymmetric states of (1.1), i.e. solutions of the form $u(t,x) = e^{iEt}\phi(x)$, where ϕ satisfies the stationary NLS equation

$$-\phi''(x) + V(x)\phi(x) + \sigma|\phi(x)|^{2p}\phi(x) + E\phi(x) = 0, \quad x \in \mathbb{R}, \qquad (1.3)$$

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and $E \in \mathbb{R}$ is an arbitrary parameter. We recall the following basic facts about solutions of the stationary NLS equation in one dimension.

(i) Via standard regularity theory, if $V(x) \in L^\infty(\mathbb{R})$, then any weak solution $\phi(x) \in H^1(\mathbb{R})$ of the stationary equation (1.3) belongs to $H^2(\mathbb{R})$.

(ii) All solutions of the stationary equation (1.3) in $H^2(\mathbb{R}) \hookrightarrow \mathcal{C}^1(\mathbb{R})$ are real-valued up to multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$.

(iii) If $E > 0$, all solutions of (1.3) in $H^2(\mathbb{R})$ decay exponentially fast to zero as $|x| \to \infty$.

Numerically we will focus on a one parameter double well potential $V \equiv V_s$ constructed from splitting the single-well potential $V_0(x) = -\operatorname{sech}^2(x)$:

$$V_s(x) = V_0(x+s) + V_0(-x+s), \quad s \geq 0. \tag{1.4}$$

The general theory of bifurcations from a simple eigenvalue of the linearized operator [24] implies that solutions with small H^2 norm of the stationary equation (1.3) exist for E near E_0. The symmetry hypothesis (H3) implies that these solutions are symmetric (even in x). Variants of the local bifurcation analysis near $E = E_0$, including the fact that $E > E_0$ if $\sigma < 0$, and $E < E_0$ if $\sigma > 0$, have already appeared in [31, 27], as well as in many recent publications. We review this analysis in Section 2 to give readers a complete picture.

Orbital stability [32] of the stationary state $e^{iEt}\phi(x)$, $\phi(x) \in \mathbb{R}$, is closely related to the linearization of the time-dependent NLS equation (1.1) at the stationary state, which, in the direction $e^{iEt}[u_1(x,t) + iu_2(x,t)]$, is given by:

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & L_-(\phi, E) \\ -L_+(\phi, E) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $L_\pm$ are self adjoint linear Schrödinger operators with domains $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$:

$$\begin{cases} L_+(\phi, E) = -\partial_x^2 + E + V(x) + \sigma(2p+1)|\phi|^{2p}(x), \\ L_-(\phi, E) = -\partial_x^2 + E + V(x) + \sigma|\phi|^{2p}(x). \end{cases} \tag{1.5}$$

Sufficient conditions for orbital stability and orbital instability, which we will use throughout this paper, were obtained in [32, 9, 8].

Definition 1. *If (ϕ, E) solves (1.3) and zero is the lowest eigenvalue of $L_-(\phi, E)$ we call ϕ a ground state of (1.3).*

Remark 1. *Note that, for any solution (ϕ, E) of (1.3), zero is an eigenvalue of $L_-(\phi, E)$ with eigenfunction ϕ. Via standard theory of second order elliptic operators the above definition is equivalent to the one requiring a ground state to be strictly positive or strictly negative.*

In particular it is known that solutions (ϕ, E) of the stationary NLS equation (1.3) with $|E - E_0|$ and $\|\phi\|_{H^1}$ small are orbitally stable ground states, see Section 2. We remark as a side note that, for critical and supercritical nonlinearities, $p \geq 2$, and more restrictive hypotheses on the potential

$V(x)$, one can show asymptotic stability of these solutions in the space of one dimension [6, 23]. For subcritical nonlinearities asymptotic stability is proven only in dimensions higher than one, see [17, 18, 19].

Kirr *et al.* [16] showed that the symmetric ground states undergo a symmetry–breaking bifurcation at $E = E_* > E_0$, in the focusing case $\sigma < 0$, with $p = 1$, or other cubic like nonlinearities, provided the first two eigenvalues of $L_0 = -\partial_x^2 + V(x)$ are sufficiently close to each other. In particular, the result is applicable to double-well potentials such as (1.4) for sufficiently large separation parameter s between the two wells. Furthermore the authors show that the symmetric states become unstable for $E > E_*$, and, a new pair of orbitally stable, asymmetric ground states exist for $E > E_*$. The proofs rely on a Lyapunov-Schmidt type projection onto the two eigenvectors corresponding to the lowest eigenvalues of L_0, which exists for small $\|\phi\|_{H^1}$, combined with a normal form analysis of the reduced system valid to all orders. Marzuola & Weinstein [22] used a time dependent normal form valid for finite time to extract interesting properties of the dynamics of solutions of the NLS equation (1.1) near the bifurcation point $E = E_*$. We also mention that [30] uses a similar finite time, normal form technique to study the solutions and predict bifurcations of the first excited (anti-symmetric) state for defocusing NLS equation ($\sigma > 0$) with a symmetric double well potential which is essentially brought in the large separation regime by passing to the semi-classical limit and assuming $\sigma \searrow 0$ at a specific rate.

To our knowledge there are very few results for bifurcations of NLS stationary states in non-perturbative regimes. Rose & Weinstein [29] use variational methods to show that the stationary NLS equation (1.3) with $\sigma < 0$, and potential satisfying (H1), (H2) and (H4), has at least one solution for any $E > E_0$. Jeanjean & Stuart [11] prove that for $\sigma < 0$, the symmetric states bifurcating from the lowest eigenvalue of L_0 can be uniquely continued for all $E > E_0$, hence there are no bifurcations along this branch, provided $V(x)$ is monotonically increasing for $x > 0$, and C^1 in addition to satisfying (H1)-(H4). In particular, the result applies to the potential (1.4) if $s \leq s_*$, where

$$s_* = \frac{1}{2}\mathrm{arccosh}(2) = \mathrm{arccosh}(\sqrt{3}/\sqrt{2}) \approx 0.6585, \tag{1.6}$$

because for $x > 0$ and $0 \leq s \leq s_*$:

$$V_s'(x) = V_0'(x-s) + V_0'(x+s) = \frac{\sinh(2x)}{\cosh^3(x-s)\cosh^3(x+s)}[2 + \cosh(2x)\cosh(2s) - \cosh^2(2s)] > 0.$$

Results on continuation of branches of stationary states in the defocusing case $\sigma > 0$, but without reference to existence or non-existence of bifurcation points can be found in [12, 13]. In [2] the authors rely on variational techniques to deduce symmetry–breaking of the ground states in Hartree equations. Their method can be adapted to our problem and implies the emergence of asymmetric, ground state branches in the focusing case $\sigma < 0$ provided the nonlinearity is subcritical, $p < 2$, and $V(x)$ is continuous, bounded, and has at least two separated minima. In particular, assuming $p < 2$, asymmetric ground state branches will appear for the potential (1.4) as soon as it becomes a double well, i.e., for $s > s_*$, but the method cannot tell whether the asymmetric branches are connected to the symmetric branch of ground states bifurcating from the lowest eigenvalue of L_0. Jackson & Weinstein [10] use a topological shooting method for the case $p = 1$ and Dirac type double-well potential, i.e. $V_0(x) = \delta(x)$ in (1.4), to show that the asymmetric branches emerge from the symmetric ones via a pitchfork bifurcation and they all coexists past a certain value of E.

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Our main result extends the ones in [16] to non-perturbative regimes and the ones in [2] to critical and supercritical nonlinearities, $p \geq 2$, while proving that the asymmetric ground states emerge from the symmetric ones via a pitchfork bifurcation. The main theorem is formulated as follows.

Theorem 1. *Consider the stationary NLS equation* (1.3) *in the focusing case* $\sigma < 0$, *with* $V(x)$ *satisfying* (H1)-(H4). *Then the* C^1 *curve* $E \mapsto \phi = \psi_E \in H^2$, $E > E_0$, $\psi_E(x) > 0$ *of symmetric, real valued solutions bifurcating from the zero solution at* $E = E_0$, *undergoes another bifurcation at a finite* $E = E_* > E_0$, $\psi_{E_*} \in H^2$, *provided* $V(x)$ *has a non-degenerate maxima at* $x = 0$, *and*

(H5) $xV'(x) \in L^\infty(\mathbb{R})$.

The bifurcation is due to the second eigenvalue $\lambda(E)$ *of* $L_+(\psi_E, E)$ *crossing zero at* $E = E_*$. *Moreover, if* $p \geq 1/2$, *and the following non-degeneracy condition holds:*

$$\frac{d\lambda}{dE}(E_*) \neq 0,$$

then the bifurcation is of pitchfork type: the set of real valued solutions $(\phi, E) \in H^2 \times \mathbb{R}$ *in a neighborhood of* (ψ_{E_*}, E_*) *consists of exactly two orthogonal* C^1 *curves: the symmetric branch* $E \mapsto \phi = \psi_E$ *which continues past* $E = E_*$, *but becomes orbitally unstable, and an asymmetric branch* $(\phi(a), E(a))$:

$$E(a) = E_* + \frac{Q}{2}a^2 + o(a^2), \quad \phi(a) = \psi_{E_*} + a\phi_* + O(a^2), \quad a \in \mathbb{R}, \; |a| \text{ small}$$

where ϕ_* *is the eigenfunction corresponding to the second eigenvalue of* $L_+(\psi_{E_*}, E_*)$, *and* Q *can be calculated from* ψ_{E_*} *and* ϕ_*, *see* (3.35). *The asymmetric solutions are orbitally stable if* $Q > 0$ *and* $\|\phi(a)\|_{L^2}$ *is increasing as* $E(a)$ *increases, but they are orbitally unstable if* $\|\phi(a)\|_{L^2}$ *is decreasing with* $E(a)$ *or if* $Q < 0$.

In particular, the result applies to the potential (1.4) for $s > s_* = \operatorname{arccosh}(\sqrt{3}/\sqrt{2})$ because

$$V_s'(0) = V_0'(s) - V_0'(s) = 0, \; V_s''(0) = 2V_0''(s) = 12\operatorname{sech}^4(s) - 8\operatorname{sech}^2(s) < 0, \quad s > s_*,$$

and implies that a pitchfork bifurcation occurs along the branch of symmetric states. Recall that for $s \leq s_*$ this branch can be uniquely continued for all $E > E_0$ due to the result in [11], see (1.6). Moreover, in the large separation limit $s \to \infty$, the branch of asymmetric states is orbitally stable near the pitchfork bifurcation if $p < p_*$, where

$$p_* = \frac{1}{2}\left(3 + \sqrt{13}\right) \approx 3.3028, \tag{1.7}$$

and orbitally unstable for $p > p_*$, see Corollary 2. The threshold power p_* of the nonlinearity was predicted in [30] but we justify this result with rigorous analysis.

We emphasize that hypotheses (H1) and (H5) can be relaxed to $V(x), xV'(x) \in L^q(\mathbb{R})$ for some $q \geq 1$, at the expense of slightly complicating the proofs in this paper. Moreover, our results extend to more than one dimension $x \in \mathbb{R}^n$, $n \geq 2$, and other symmetries in $\mathbb{R}^n$, provided that the symmetries still prevent the solutions to concentrate at $|x| = \infty$, see Remark 7. Note that

for classifying the bifurcation, we will have to assume that the second eigenvalue of $L_+(\psi_E, E)$ is simple. To completely remove any symmetry assumptions, or the spectral assumption (H4), or the simplicity of the second eigenvalue of L_+, is a much more difficult task, see [20] for partial results.

The proof of the main result relies on Theorems 2, 3, 5, and 6 which, viewed individually, are important themselves. Properly generalized they could completely describe the set of all solutions of the stationary NLS equation (1.3) for any dimension, arbitrary potentials and more general nonlinearities in terms of the critical points of the potential. In Section 2 we prove the following dichotomy: the branch of stationary solutions (ψ_E, E) bifurcating from the lowest eigenvalue $-E_0$ of $-\partial_x^2 + V(x)$ can either (a) be uniquely continued for all $E > E_0$ or (b) there exists a finite $E_* > E_0$ such that zero is an accumulation point for the discrete spectrum of the linearized operator $L_+(\psi_E, E)$, as $E \nearrow E_*$. The result essentially eliminates the possibility that $\|\psi_E\|_{H^2}$ diverges to infinity as E approaches E_* with $\|L_+^{-1}(\psi_E, E)\|_{L^2 \mapsto L^2}$ remaining uniformly bounded, and relies on the differential estimates for the mass (charge) and energy of the NLS equation (1.1).

In Section 3 we use a novel technique combining concentration–compactness, see for example [5], and the spectral properties of the linearized operator L_+, to show that, in case (b), the states ψ_E must converge in H^2 to a nonzero state ψ_{E_*} for $E = E_*$. By continuity we deduce that the linearized operator $L_+(\psi_{E_*}, E_*)$ has zero as a simple eigenvalue, then we use a Lyapunov-Schmidt decomposition and the Morse Lemma, see for example [24], to show that a pitchfork bifurcation occurs at (ψ_{E_*}, E_*). The symmetry hypothesis (H3) implies that $\psi_E(x)$ are even in x which is essential in showing that the limit $\lim_{E \nearrow E_*} \psi_E$ exists. It turns out that without assuming (H3) ψ_E may drift to infinity as E approaches E_*, i.e. there exists $y_E \in \mathbb{R}$, $\lim_{E \nearrow E_*} y_E = \pm\infty$, such that $\lim_{E \nearrow E_*} \psi_E(\cdot - y_E) = \psi_{E_*}$ where ψ_{E_*} is now a solution of the stationary NLS equation (1.3) with $V(x) \equiv 0$ and $E = E_*$, see [20] for a more detailed discussion of this phenomenon.

In Section 4, we obtain new, rigorous results on the behavior of all stationary states in the focusing case $\sigma < 0$, for large E. In Theorem 5 we combine Pohozaev type identities with differential estimates for mass, energy and the L^{2p+2} norm of the stationary solutions to prove how the relevant norms of the solutions scale with E as $E \to \infty$. In particular, we obtain the behavior of the L^2 norm of the symmetric states, which was numerically and heuristically predicted in [29]. Moreover, by combining these estimates with our novel concentration–compactness/spectral technique we show that, modulo a re-scaling, the symmetric branches of solutions, along which L_+ has only one negative eigenvalue, converge to a non-trivial solution of the constant-coefficient stationary NLS equation,

$$- \phi''(x) + \sigma|\phi(x)|^{2p}\phi(x) + \phi(x) = 0, \quad \phi \in H^2(\mathbb{R}). \tag{1.8}$$

Since the set of solutions of the latter in dimension one is well known, we adapt and extend the bifurcation analysis of Floer & Weinstein [7] to our problem and obtain detailed information on *all* stationary solutions, which, modulo a re-scaling, bifurcate from a non-trivial solution of equation (1.8). We show that such solutions can only be localized near a critical point of the potential $V(x)$, and they are always orbitally unstable for supercritical nonlinearities, $p > 2$. If the critical point is a non-degenerate minimum, respectively maximum, then there is exactly one branch of solutions localized at that point, and these solutions are orbitally stable if and only if we are in the critical and subcritical regimes, $p \leq 2$. We note that compared to the semi-classical analysis in [7], completed with orbital stability analysis in [9, Example C], we are forced to make a precise analysis up to order four, instead of two, in the relevant small parameter. In addition, we prove non-existence

of solutions localized near regular points of the potential $V(x)$, uniqueness of solutions localized near non-degenerate minima and maxima, and we recover the stability for the critical nonlinearity $p = 2$. All these results can now be extended without modifications to the problem studied in [7].

In Section 5, we illustrate the main theoretical results numerically for the potential (1.4), subcritical nonlinearity $p = 1$ and supercritical nonlinearities $p = 3$, $p = 5$ in the focusing case $\sigma < 0$. We note that $p = 3 < p_*$ and $p = 5 > p_*$, where p_* is defined by (1.7). We will show that both subcritical and supercritical pitchfork bifurcations occur depending on the value of p. For this potential we will also show numerically that, except for the bifurcation predicted by our main result, there are no other bifurcations along any of the ground state branches, a result beyond the grasp of our current theoretical techniques.

In what follows, we shall use notations $\mathcal{O}(\varepsilon)$ and $o(\varepsilon)$ as $\varepsilon \to 0$ in the sense

$$\Lambda = \mathcal{O}(\varepsilon) \quad \Leftrightarrow \quad \lim_{\varepsilon \to 0} \varepsilon^{-1} \Lambda(\varepsilon) = \Lambda_\infty \in \mathbb{R} \quad \text{and} \quad \Lambda = o(\varepsilon) \quad \Leftrightarrow \quad \lim_{\varepsilon \to 0} \varepsilon^{-1} \Lambda(\varepsilon) = 0.$$

We will denote ε-independent constants by C, which may change from one line to another line. We will also use the standard Hilbert space $L^2(\mathbb{R})$ of the real valued square integrable functions on a real line and the Sobolev space $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ of the real valued functions on $\mathbb{R}$ which are square integrable together with their first and second order weak derivatives.

2   Local bifurcations of symmetric ground states

In this section we trace the manifold of symmetric ground states of the stationary problem (1.3) from its local bifurcation from the linear eigenmode of $L_0 = -\partial_x^2 + V(x)$ near $E = E_0$ up to its next bifurcation. We will show that the symmetric state exists in an interval to the right of E_0 if $\sigma < 0$ and in an interval to the left of E_0 if $\sigma > 0$. In the case of $\sigma < 0$, we will further find necessary and sufficient conditions for the symmetric state to be extended for all values of $E > E_0$ or suffer a symmetry–breaking bifurcation.

Let us rewrite the stationary equation (1.3) for real-valued solutions $\phi(x)$ as the root-finding equation for the functional $F(\phi, E) : H^2(\mathbb{R}) \times \mathbb{R} \mapsto L^2(\mathbb{R})$ given by

$$F(\phi, E) = (-\partial_x^2 + V(x) + E)\phi + \sigma|\phi|^{2p}\phi. \tag{2.1}$$

We recall the following result describing the existence of symmetric ground states near E_0.

Proposition 1. *Let $-E_0 < 0$ be the smallest eigenvalue of $L_0 = -\partial_x^2 + V(x)$. There exist $\epsilon > 0$ and $\delta > 0$ such that for each E on the interval $\mathcal{I}_\epsilon$, $\mathcal{I}_\epsilon = (E_0 - \epsilon, E_0)$ for $\sigma > 0$, $\mathcal{I}_\epsilon = (E_0, E_0 + \epsilon)$ for $\sigma < 0$, the stationary equation (1.3) has exactly two nonzero, real valued solutions $\pm\psi_E(x) \in H^2(\mathbb{R})$, satisfying $\|\psi_E\|_{H^2} < \delta$. Moreover*

$$\|\psi_E\|_{H^2} \leq C|E - E_0|^{\frac{1}{2p}}.$$

for some $C > 0$, the map $E \mapsto \psi_E$ is C^1 from $\mathcal{I}_\epsilon$ to H^2 and $\psi_E(x) = \psi_E(-x)$ for each $x \in \mathbb{R}$ and $E \in \mathcal{I}_\epsilon$.

Proof. We sketch the main steps. For any $p > 0$, the functional $F : H^2(\mathbb{R}) \times \mathbb{R} \mapsto L^2(\mathbb{R})$ is C^1 , i.e. it is continuous with continuous Frechet derivative

$$D_\phi F(\phi, E) = -\partial_x^2 + V(x) + E + (2p+1)\sigma|\phi|^{2p}.$$

We note that $D_\phi F(0, E) = L_0 + E$. Let ψ_0 be the L^2-normalized eigenfunction of $D_\phi F(0, E) = L_0 + E_0$ corresponding to its zero eigenvalue. Then $D_\phi F(0, E_0)$ is a Fredholm operator of index zero with $\mathrm{Ker}(L_0 + E_0) = \mathrm{span}\{\psi_0\}$ and $\mathrm{Ran}(L_0 + E_0) = [\mathrm{Ker}(L_0 + E_0)]^\perp$. Let

$$P_\| \phi = \langle \psi_0, \phi \rangle_{L^2} \psi_0, \quad P_\perp \phi = \phi - \langle \psi_0, \phi \rangle_{L^2} \psi_0$$

be the two orthogonal projections associated to this Lyapunov-Schmidt decomposition. Then $F(\phi, E) = 0$ with $\phi = a\psi_0 + \varphi$, where $a = \langle \psi_0, \phi \rangle$ and $\varphi = P_\perp \phi$, is equivalent to two equations

$$P_\perp (L_0 + E) P_\perp \varphi + \sigma P_\perp |a\psi_0 + \varphi|^{2p} (a\psi_0 + \varphi) = 0 \tag{2.2}$$
$$(E - E_0) a + \sigma \langle \psi_0, |a\psi_0 + \varphi|^{2p} (a\psi_0 + \varphi) \rangle = 0 \tag{2.3}$$

Since $P_\perp (L_0 + E) P_\perp$ has a bounded inverse for any E near E_0, the Implicit Function Theorem states that there exists a unique C^1 map $\mathbb{R}^2 \ni (a, E) \mapsto \varphi = \varphi_*(a, E) \in H^2$ for sufficiently small $|a|$ and $|E - E_0|$ such that φ solves equation (2.2) and

$$\exists C > 0 : \quad \|\varphi_*(a, E)\|_{H^2} \le C|a|^{2p+1}.$$

Hence (2.3) becomes a scalar equation in variables $(a, E) \in \mathbb{R}^2$ given by

$$a(E - E_0) + \sigma \langle \psi_0, |a\psi_0 + \varphi|^{2p} (a\psi_0 + \varphi) \rangle_{L^2} = 0. \tag{2.4}$$

Dividing (2.4) by a and invoking again the Implicit Function Theorem for functions, we obtain the existence of a unique continuous map $\mathbb{R} \ni a \mapsto E = E_*(a) \in \mathbb{R}$ for sufficiently small $|a|$ such that E solves (2.4) and
$$\exists C > 0 : \quad |E_*(a) - E_0 + \sigma \|\psi_0\|_{L^{2p+2}}^{2p+2} |a|^{2p}| \le C|a|^{4p}.$$

Therefore, $E > E_0$ if $\sigma < 0$ and $E < E_0$ if $\sigma > 0$. Moreover, the map $a \mapsto E_*(a)$ is C^1 and invertible for $a > 0$, rendering the map $E \mapsto \psi_E := a\psi_0 + \varphi_*(a, E_*(a))$ to be C^1. The negative branch $-\psi_E$ is obtained from the negative values of a. Moreover, since for any solution $\psi_E(x)$ of (1.3) we have that $\psi_E(-x)$ is also a solution, uniqueness and continuity in a imply that $\psi_E(-x) = \psi_E(x)$. $\square$

Remark 2. *A result similar to Proposition 1 can be obtained for the anti-symmetric state of the stationary equation (1.3), which bifurcates from the second eigenvalue $-E_1$ of $L_0 = -\partial_x^2 + V(x)$, if the second eigenvalue of L_0 exists. Moreover, $(0, E) \in H^2(\mathbb{R}) \times \mathbb{R}$ is the only solution of the stationary NLS equation (1.3) in a small neighborhood of $(0, E_*) \in H^2(\mathbb{R}) \times \mathbb{R}$ for any $E_* > 0$, if $-E_*$ is not an eigenvalue of L_0.*

Let us introduce operators L_+ and L_- along the branch of symmetric states (ψ_E, E) according to definition (1.5) with $\phi = \psi_E$. Since they depend C^1 on $E \in \mathcal{I}_\epsilon$ and continuously at E_0, their isolated eigenvalues depend C^1 on $E \in \mathcal{I}_\epsilon$ and continuously at E_0.

Since $L_- \psi_E = 0$ and $L_- = L_0 + E_0$ for $E = E_0$, 0 is the lowest eigenvalue of L_- for all $E \in \mathcal{I}_\epsilon$. On the other hand, we have

$$L_+ - L_- = \sigma 2 p \psi_E^{2p}.$$

Hence, $L_+ < L_-$ if $\sigma < 0$, while $L_+ > L_-$ if $\sigma > 0$, and, via eigenvalue comparison principle, the lowest eigenvalue of L_+ is strictly negative if $\sigma < 0$ and strictly positive if $\sigma > 0$. Consequently, 0 is not in the discrete spectrum of L_+ nor in the essential spectrum for $E \in \mathcal{I}_\epsilon$. The latter follows from $V(x) \in L^\infty(\mathbb{R})$, $\lim_{|x| \to \infty} V(x) = 0$, and $|\psi_E|^{2p} \in L^2(\mathbb{R})$, since $\psi_E \in H^2(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $2 \leq q \leq \infty$. Together they imply that $L_+ - E$ is a relatively compact perturbation of $-\partial_x^2$, hence, via Weyl's theorem, the essential spectrum of L_+ is the $[E, \infty)$ interval.

The following result shows that we can continue the branch of symmetric states (ψ_E, E) as long as 0 is not in the spectrum of L_+.

Lemma 1. *Let $\psi_{E_1}(x) \in H^2(\mathbb{R})$ be a real valued solution of the stationary equation (1.3) for $E = E_1$ and assume 0 is not in the spectrum of $L_+(\psi_{E_1}, E_1)$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that for each $E \in (E_1 - \epsilon, E_1 + \epsilon)$ the stationary equation (1.3) has a unique, real valued, nonzero solution $\psi_E(x) \in H^2(\mathbb{R})$ satisfying*

$$\|\psi_E - \psi_{E_1}\|_{H^2} \leq \delta.$$

Moreover the map $E \mapsto \psi_E$ is C^1 from $(E_1 - \epsilon, E_1 + \epsilon)$ to H^2.

Proof. The result follows from the Implicit Function Theorem for $F(\phi, E) = 0$ at (ψ_{E_1}, E_1). $\qquad\square$

Combining the Proposition and Lemma, we get the following maximal result for the branch of symmetric modes bifurcating from the point E_0. For definiteness, we state and prove this result only for the focusing case $\sigma < 0$.

Theorem 2. *If $\sigma < 0$, then the branch of solutions (ψ_E, E) of (1.3) which bifurcates from the lowest eigenvalue $-E_0$ of L_0 can be uniquely continued to a maximal interval (E_0, E_*) such that either:*

(a) $E_* = \infty$;

   or

(b) $E_* < \infty$ and there exists a sequence $\{E_n\}_{n \in \mathbb{N}} \subset (E_0, E_*)$ such that $\lim_{n \to \infty} E_n = E_*$ and $L_+(\psi_{E_n}, E_n)$ has an eigenvalue λ_n satisfying $\lim_{n \to \infty} \lambda_n = 0$.

Proof. Define:

$$E_* = \sup\{\tilde{E}: \quad \tilde{E} > E_0, \ E \mapsto \psi_E \text{ is a } C^1 \text{ extension on } (E_0, \tilde{E}) \text{ of the map in Proposition 1,}$$
$$\text{for which 0 is not in the spectrum of } L_+\}$$

Proposition 1 and the discussion following it guarantees that the set above is not empty. Assume neither (a) nor (b) hold for $E_* > E_0$ defined above. Then we can fix any E_1, $E_0 < E_1 < E_*$ and find that, for $E \in [E_1, E_*)$, the spectrum of $L_+(\psi_E, E)$ (which is real valued since L_+ is self-adjoint) has no points in the interval $[-d, d]$ for some $0 < d < E_1$. Indeed, as discussed after Proposition 1 the essential spectrum of L_+ at E is $[E, \infty)$, and if no $d > 0$ exists, there must be a sequence of eigenvalues λ_n for L_+ at $E_n \in [E_1, E_*)$ such that $\lim_{n \to \infty} \lambda_n = 0$. But $[E_1, E_*]$ is compact since (a) does not hold, hence there exists a subsequence E_{n_k} of E_n converging to $E_2 \in [E_1, E_*]$. However, $E_2 \neq E_*$ because we assumed that (b) does not hold, so $E_2 \in [E_1, E_*)$ and by continuous dependence of the eigenvalues of L_+ on $E \in [E_1, E_*)$ we get that 0 is an eigenvalue of L_+ at $E_2 < E_*$ which contradicts the choice of E_*.

Consequently $L_+^{-1} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is bounded with uniform bound $K = 1/d$ on $[E_1, E_*)$. Moreover, by differentiating (1.3) with respect to E we have:

$$L_+ \partial_E \psi_E = -\psi_E \quad \Rightarrow \quad \partial_E \psi_E = -L_+^{-1} \psi_E, \quad E \in (E_0, E_*), \tag{2.5}$$

hence

$$\|\partial_E \psi_E\|_{L^2} \leq K \|\psi_E\|_{L^2}, \quad E \in [E_1, E_*). \tag{2.6}$$

and, by Cauchy-Schwarz inequality:

$$\frac{d}{dE} \|\psi_E\|_{L^2}^2 = 2\langle \partial_E \psi_E, \psi_E \rangle \leq 2K \|\psi_E\|_{L^2}^2, \quad E \in [E_1, E_*).$$

The latter implies

$$\|\psi_E\|_{L^2}^2 \leq \|\psi_{E_1}\|_{L^2}^2 e^{2K(E - E_1)}, \quad E \in [E_1, E_*), \tag{2.7}$$

which combined with $E_* < \infty$ and bound (2.6) gives that both $\partial_E \psi_E$ and ψ_E have uniformly bounded L^2 norms on $[E_1, E_*)$. By the Mean Value Theorem there exists $\psi_{E_*}(x) \in L^2(\mathbb{R})$ such that

$$\lim_{E \nearrow E_*} \|\psi_E - \psi_{E_*}\|_{L^2} = 0. \tag{2.8}$$

We claim that $\psi_{E_*}(\mathbb{R}) \in H^1(\mathbb{R})$ is a weak solution of the stationary equation (1.3) with $E = E_*$, hence it is in $H^2(\mathbb{R})$ since $V \in L^\infty(\mathbb{R})$. Indeed, consider the energy functional

$$\mathcal{E}(E) = \int_{\mathbb{R}} |\nabla \psi_E(x)|^2 dx + \int_{\mathbb{R}} V(x)|\psi_E(x)|^2 dx + \frac{\sigma}{p+1} \int_{\mathbb{R}} |\psi_E(x)|^{2p+2} dx. \tag{2.9}$$

Note that because, ψ_E is a weak solution of the stationary equation (1.3) for any $E \in (E_0, E_*)$ we have

$$\frac{d\mathcal{E}}{dE} = -2E \langle \psi_E, \partial_E \psi_E \rangle_{L^2} \tag{2.10}$$

and, via Cauchy-Schwarz inequality:

$$\left| \frac{d\mathcal{E}}{dE} \right| \leq E \|\psi_E\|_{L^2} \|\partial_E \psi_E\|_{L^2}.$$

Hence, from the uniform bounds (2.6) and (2.7), the derivative of $\mathcal{E}(E)$, and hence $\mathcal{E}(E)$, is uniformly bounded on $[E_1, E_*)$. On the other hand, from the weak formulation of solutions of (1.3), we get

$$\|\nabla \psi_E\|_{L^2}^2 + \int_{\mathbb{R}} V(x)|\psi_E(x)|^2 dx + \sigma \|\psi_E\|_{L^{2p+2}}^{2p+2} + E \|\psi_E\|_{L^2}^2 = 0 \tag{2.11}$$

Subtracting the latter from $(p+1)\mathcal{E}$ we get that there exists an $M > 0$ such that

$$\left| p\|\nabla\psi_E\|_{L^2}^2 + p\int_{\mathbb{R}} V(x)|\psi_E(x)|^2 dx \right| \leq M, \qquad \text{for all } E \in [E_1, E_*). \qquad (2.12)$$

From Hölder inequality we obtain:

$$\left| \int_{\mathbb{R}} V(x)|\psi_E(x)|^2 dx \right| \leq \|V\|_{L^\infty}\|\psi_E\|_{L^2}.$$

Using the inequality in (2.12) we deduce that $\|\nabla\psi_E\|_{L^2}$ has to be uniformly bounded. Consequently, there exists $M > 0$ such that

$$\|\psi_E\|_{H^1} \leq M, \qquad \text{for all } E \in [E_1, E_*). \qquad (2.13)$$

Because of the embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and the interpolation

$$\|f\|_{L^q} \leq \|f\|_{L^2}^{2/q}\|f\|_{L^\infty}^{1-2/q}, \quad q \geq 2,$$

bound (2.13) together with convergence (2.8) imply that as $E \nearrow E_*$ we have:

$$\left.\begin{array}{ll} \psi_E \to \psi_{E_*}, & \text{in } L^2(\mathbb{R}) \\ \psi_E \rightharpoonup \psi_{E_*}, & \text{in } H^1(\mathbb{R}) \end{array}\right\} \quad \Rightarrow \quad \psi_E \to \psi_{E_*}, \quad \text{in } L^q(\mathbb{R}),\ q \geq 2.$$

Now, by passing to the limit in the weak formulation of the stationary equation (1.3), we conclude that $\psi_{E_*}(x) \in H^1(\mathbb{R})$ is a weak solution. Moreover, the linearized operator L_+ depends continuously on E on the interval $[E_1, E_*]$. By the standard perturbation theory, the discrete spectrum of L_+ depends continuously on E. Since 0 is not in the spectrum of L_+ for $E \in [E_1, E_*)$ and we assumed that (b) does not hold we deduce that 0 is not an eigenvalue of L_+ at E_*. Moreover, since the essential spectrum of L_+ is $[E, \infty)$, 0 is not in the spectrum of L_+ at $E = E_*$. Applying now Lemma 1 we can continue the C^1 branch (ψ_E, E) past $E = E_*$ which contradicts the choice of E_*.

The theorem is now completely proven. □

Remark 3. *Let $\sigma < 0$ and L_+ be computed at the branch points (ψ_E, E) for $E \in (E_0, E_*)$, where E_* is given by Theorem 2. Then, L_+ has exactly one (strictly) negative eigenvalue. This follows from the fact that the eigenvalues of L_+ depend C^1 on $E \in (E_0, E_*)$, 0 is not in the spectrum of L_+ and for E near E_0, L_+ has exactly one strictly negative eigenvalue, see the discussion after Proposition 1.*

3 Symmetry-breaking transitions to asymmetric states

In this section we show that, in the focusing case $\sigma < 0$, the second alternative in Theorem 2 occurs if and only if the second eigenvalue of L_+, $\lambda(E)$ crosses the zero value at $E = E_*$. Moreover, at $E = E_*$, under the generic assumption of $\lambda'(E_*) \neq 0$, the branch of symmetric states suffers a symmetry–breaking bifurcation of pitchfork type with a new branch of asymmetric states emerging. The new branch consists of solutions of (1.3) that are neither even nor odd in x. Depending on the

sign of a quantities Q and R, see Theorem 4, which can be numerically computed, the asymmetric solutions are either orbitally stable or orbitally unstable with respect to the full dynamics of the NLS equation (1.1). For double well potentials with large separation, e.g. (1.4) with $s \to \infty$, the orbital stability of the asymmetric branch is determined by the power of the nonlinearity, see Corollary 2. Regarding the branch of symmetric states, it continues past the bifurcation point but it is always orbitally unstable.

Note that sufficient conditions for the symmetry-breaking bifurcation at E_* were presented by Kirr *et al.* [16] under the assumption that E_* was sufficiently close to E_0 and $\|\psi_E\|_{H^1}$ was small for all $E \in (E_0, E_*)$. Moreover, only the case of cubic nonlinearity ($p = 1$) was considered. We now present a generalization of this result which allows large values of $|E_* - E_0|$, large norms $\|\psi_E\|_{H^1}$, and any power nonlinearities $p \geq 1/2$.

Remark 3 showed that, in the focusing case $\sigma < 0$, the operator L_+ on the branch of symmetric states has exactly one negative eigenvalue. We first show that this eigenvalue cannot approach zero as E approaches E_*.

Lemma 2. *Let $\sigma < 0$ and consider the branch of solutions (ψ_E, E) of (1.3) which bifurcates from the lowest eigenvalue $-E_0$ of L_0. Let (E_0, E_*) be the maximal interval on which this branch can be uniquely continued. If $\lambda_0(E)$ is the lowest eigenvalue of L_+ along this branch then there exist $\delta, \ d > 0$ such that:*

$$\lambda_0(E) \leq -d < 0, \quad \text{for all } E \in [E_* - \delta, E_*) \text{ if } E_* < \infty, \text{ or for all } E > E_0, \text{ if } E_* = \infty.$$

Proof. Assume the contrary, that there exists a sequence $E_n \nearrow E_*$ such that $\lambda_0(E_n) \nearrow 0$. From the min-max principle we have for all integers n :

$$
\begin{aligned}
\lambda_0(E_n) &= \inf_{\xi \in H^2, \ \|\xi\|_{L^2}=1} \langle \xi, L_+\xi \rangle \leq \frac{1}{\|\psi_{E_n}\|_{L^2}^2} \langle \psi_{E_n}, L_+\psi_{E_n} \rangle \\
&= \frac{1}{\|\psi_{E_n}\|_{L^2}^2} \langle \psi_{E_n}, L_-\psi_{E_n} + 2p\,\sigma|\psi_{E_n}|^{2p}\psi_{E_n} \rangle = 2p\,\sigma \frac{\|\psi_{E_n}\|_{L^{2p+2}}^{2p+2}}{\|\psi_{E_n}\|_{L^2}^2} < 0,
\end{aligned}
$$

where we used the definitions (1.5) and $L_-\psi_E = 0$ for all $E \in (E_0, E_*)$. From the above inequalities and $\lambda_0(E_n) \nearrow 0$ we conclude that:

$$\lim_{n \to \infty} \frac{\|\psi_{E_n}\|_{L^{2p+2}}^{2p+2}}{\|\psi_{E_n}\|_{L^2}^2} = 0.$$

But (ψ_{E_n}, E_n) solves the stationary NLS equation (1.3). Plugging in and taking the L^2 scalar product with $\frac{\psi_{E_n}}{\|\psi_{E_n}\|_{L^2}^2}$ we get:

$$-E_n - \sigma \frac{\|\psi_{E_n}\|_{L^{2p+2}}^{2p+2}}{\|\psi_{E_n}\|_{L^2}^2} = \left\langle \frac{\psi_{E_n}}{\|\psi_{E_n}\|_{L^2}}, \ (-\partial_x^2 + V(x))\frac{\psi_{E_n}}{\|\psi_{E_n}\|_{L^2}} \right\rangle \geq -E_0,$$

where the last inequality follows from $-E_0$ being the lowest eigenvalue of $L_0 = -\partial_x^2 + V(x)$. Passing to the limit when $n \to \infty$ we get the contradiction $E_0 \geq E_*$. $\qquad \square$

Lemma 2 combined with concentration compactness method and the spectral theory of Schrödinger type operators enables us to deduce the following important result regarding the behavior of ψ_E for E near E_*.

Theorem 3. *Let $\sigma < 0$ and consider the branch of solutions (ψ_E, E) of (1.3) which bifurcates from the lowest eigenvalue $-E_0$ of L_0. Let (E_0, E_*) be the maximal interval on which this branch can be uniquely continued. Denote*

$$N(E) = \|\psi_E\|_{L^2}^2, \quad E \in (E_0, E_*).$$

If $E_* < \infty$, then:

  (i) *$N(E)$ is bounded on $[E_0, E_*)$, $N_* := \lim_{E \to E_*} N(E)$ exists, and $0 < N_* < \infty$;*

  (ii) *there exists $\psi_{E_*} \in H^2(\mathbb{R})$ such that (ψ_{E_*}, E_*) solves the stationary NLS equation (1.3) and:*

$$\lim_{E \to E_*} \|\psi_E - \psi_{E_*}\|_{H^2} = 0.$$

Proof. For part (i) denote $\xi_E^0 \in H^2(\mathbb{R})$, $\|\xi_E^0\|_{L^2} = 1$ the normalized eigenfunction of L_+ corresponding to its lowest eigenvalue, $\lambda_0(E)$. We use the orthogonal decomposition:

$$\psi_E = \langle \xi_E^0, \psi_E \rangle\, \xi_E^0 + \psi_E^\perp, \quad \text{where } \langle \xi_E^0, \psi_E^\perp \rangle = 0, \ E \in (E_0, E_*).$$

Then

$$
\begin{aligned}
\frac{1}{2}\frac{dN}{dE} = \langle \psi_E, \partial_E \psi_E \rangle &= -\langle \psi_E, L_+^{-1}\psi_E \rangle = -\frac{|\langle \xi_E^0, \psi_E \rangle|^2}{\lambda_0(E)} - \langle \psi_E^\perp, L_+^{-1}\psi_E^\perp \rangle \\
&\leq \frac{|\langle \xi_E^0, \psi_E \rangle|^2}{-\lambda_0(E)} \leq \frac{N(E)}{-\lambda_0(E)},
\end{aligned}
\tag{3.1}
$$

where we used (2.5), $\lambda_0(E) < 0$, and the fact that $L_+ > 0$ on the orthogonal complement of ξ_E^0, see Remark 3. The above inequality implies that

$$N(E) \leq N(E_1) e^{\int_{E_1}^E \frac{dE}{-\lambda_0(E)}}, \quad \text{for any } E_0 < E_1 < E < E_*.$$

Using now the bound $\frac{1}{-\lambda_0} \leq \frac{1}{d}$, $E \in [E_* - \delta, E_*)$ given by Lemma 2, together with the fact that $N(E)$ is continuous on $[E_0, E_* - \delta]$, see Proposition 1 and Theorem 2, we deduce that there exists $N > 0$ such that:

$$0 \leq N(E) \leq N, \quad \text{for all } E \in [E_0, E_*).$$

To show that $N(E)$ converges actually to a finite value as $E \nearrow E_*$ we go back to the bound (3.1) and integrate it from $E_1 = E_* - \delta$ to any E, $E_1 < E < E_*$. We get

$$
\begin{aligned}
\int_{E_1}^E \langle \psi_E^\perp, L_+^{-1}\psi_E^\perp \rangle dE &= \int_{E_1}^E \frac{|\langle \xi_E^0, \psi_E \rangle|^2}{-\lambda_0(E)} dE + \frac{1}{2}N(E_1) - \frac{1}{2}N(E) \\
&\leq \frac{N}{d}\delta + \frac{1}{2}N(E_1)
\end{aligned}
\tag{3.2}
$$

But the integrand on the left hand side is non-negative, hence the uniform bound implies that the integral on the left hand side converges as $E \nearrow E_*$. Since the same holds for the integral on the right hand side, we deduce from (3.2) that $N(E)$ must converge to a finite limit as $E \nearrow E_*$.

Moreover, since both integrals in (3.2) are now convergent on $[E_1, E_*)$ we deduce that the derivative of $N(E)$ is absolutely convergent on the same interval, see (3.1), consequently the derivative of the energy functional, see (2.9) and (2.10), is absolutely convergent and the energy functional remains uniformly bounded on $[E_1, E_*)$. By repeating the argument in the proof of Theorem 2 we get uniform bounds in H^1 norm, see (2.13), i.e. there exists $M > 0$ such that:

$$\|\psi_E\|_{H^1} \leq M, \qquad \text{for all } E \in [E_1, E_*). \tag{3.3}$$

Now, if $\lim_{E \to E_*} N(E) = 0$, then, equivalently, $\lim_{E \to E_*} \|\psi_E\|_{L^2} = 0$. Because of the bound (3.3), Sobolev imbedding and interpolation in L^q spaces, we get $\lim_{E \to E_*} \|\psi_E\|_{L^q} = 0$, $q \geq 2$. Hence L_+ depends continuously on $E \in [E_1, E_*]$, and L_+ at E_* becomes $L_0 + E_*$. Then the eigenvalues of L_+ should converge to the eigenvalues of $L_0 + E_*$ as $E \nearrow E_*$, so 0 is an eigenvalue of $L_0 + E_*$, because we are in case (b) of Theorem 2. Equivalently $-E_* < -E_0$ is an eigenvalue of L_0, which leads to a contradiction.

Part (i) is now completely proven. For part (ii) we will use the following lemma.

Lemma 3. *Under the assumptions of Theorem 3, if there exist a sequence on the branch (ψ_{E_n}, E_n) and a function $\psi_{E_*} \in H^1(\mathbb{R})$ such that*

$$E_n \quad \nearrow \quad E_* \tag{3.4}$$

$$\psi_{E_n} \quad \overset{H^1}{\rightharpoonup} \quad \psi_{E_*} \tag{3.5}$$

$$\psi_{E_n} \quad \overset{L^q}{\to} \quad \psi_{E_*}, \quad \text{for all } 2 < q \leq \infty, \tag{3.6}$$

then $\psi_{E_n} \overset{H^1}{\to} \psi_{E_*}$. Moreover, ψ_{E_*} is a solution of (1.3) and:

$$\lim_{E \to E_*} \|\psi_E - \psi_{E_*}\|_{H^2} = 0.$$

Proof. The equation (1.3) satisfied by the elements on the branch can be rewritten:

$$\psi_{E_n} = (-\partial_x^2 + E_n)^{-1}[-V(x)\psi_{E_n}(x) - \sigma|\psi_{E_n}(x)|^{2p}\psi_{E_n}], \tag{3.7}$$

and, from (3.6), we have

$$|\psi_{E_n}(x)|^{2p}\psi_{E_n} \overset{L^2}{\to} |\psi_{E_*}(x)|^{2p}\psi_{E_*}.$$

To show that $V(x)\psi_{E_n}(x)$ also converges in $L^2(\mathbb{R})$ we use the following compactness argument. Fix $\epsilon > 0$ and choose $R > 0$ sufficiently large such that

$$\|V(x)\|_{L^\infty(\{|x| \geq R\})} < \frac{\epsilon}{4M},$$

where $M > 0$ is a bound for the sequence $\|\psi_{E_n}\|_{H^1}$, and such a bound exists because the sequence ψ_{E_n} is weakly convergent in $H^1(\mathbb{R})$. Then, by Rellich's compactness theorem,

$$\lim_{n \to \infty} \|\psi_{E_n} - \psi_{E_*}\|_{L^2(\{|x| < R\})} = 0,$$

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hence there exists n_0 such that

$$\|\psi_{E_n} - \psi_{E_*}\|_{L^2(\{|x|<R\})} < \frac{\epsilon}{2\|V\|_{L^\infty(\mathbb{R})}}, \qquad \text{for } n \geq n_0,$$

and, consequently,

$$
\begin{aligned}
\|V\psi_{E_n} - V\psi_{E_*}\|_{L^2(\mathbb{R})} &\leq \|V\|_{L^\infty(\mathbb{R})}\|\psi_{E_n} - \psi_{E_*}\|_{L^2(\{|x|<R\})} + \|V\|_{L^\infty(\{|x|\geq R\})}\|\psi_{E_n} - \psi_{E_*}\|_{L^2(\mathbb{R})} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4M}2M \leq \epsilon, \qquad \text{for } n \geq n_0
\end{aligned}
$$

which proves $V\psi_{E_n} \xrightarrow{L^2} V\psi_{E_*}$. Now both terms in the bracket of the right hand side of (3.7) converge in $L^2(\mathbb{R})$, and $(-\partial_x^2 + E_n)^{-1} \to (-\partial_x^2 + E_*)^{-1}$ in the space of bounded linear operators from $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$, we obtain ψ_{E_n} convergent in $H^2(\mathbb{R})$. But $H^2(\mathbb{R})$ is continuously imbedded in $L^q(\mathbb{R})$, for $2 \leq q \leq \infty$, hence the limit of ψ_{E_n} in H^2 should be the same as its limit in L^q, $2 < q \leq \infty$. Therefore, (3.6) implies that ψ_{E_n} converges to ψ_{E_*} in $H^2(\mathbb{R})$. Moreover, by passing to the limit in (3.7), we get

$$\psi_{E_*} = (-\partial_x^2 + E_*)^{-1}[-V(x)\psi_{E_*}(x) - \sigma|\psi_{E_*}(x)|^{2p}\psi_{E_*}],$$

or equivalently, (ψ_{E_*}, E_*) is a solution of the stationary NLS equation (1.3).

In addition, the eigenvalues of L_+ calculated at (ψ_{E_n}, E_n) will converge to the eigenvalues of L_+ calculated at (ψ_{E_*}, E_*). In particular, at (ψ_{E_*}, E_*), L_+ will have the lowest eigenvalue strictly negative, followed by 0 as the second eigenvalue, see Lemma 2 and case (b) in Theorem 2. By Sturm-Liouville theory 0 is a simple eigenvalue with corresponding eigenfunction being odd. Hence L_+ at (ψ_{E_*}, E_*) is invertible with continuous inverse as an operator restricted to even functions. The implicit function theorem applied to (1.3) gives a unique C^1 in $H^2(\mathbb{R})$ branch of symmetric (even in x) solutions in a neighborhood of (ψ_{E_*}, E_*). By uniqueness, this branch must contain (ψ_E, E), $E \in (E_1, E_*)$. In particular:

$$\lim_{E \to E_*} \|\psi_E - \psi_{E_*}\|_{H^2} = 0,$$

and Lemma 3 is completely proven. $\qquad \square$

We now return to the proof of Theorem 3. Based on (3.3) and on the weak relative compactness of bounded sets in the Hilbert space $H^1(\mathbb{R})$ we can construct a sequence $E_n \nearrow E_*$ and $\psi_{E_*} \in H^1(\mathbb{R})$ such that (3.4) and (3.5) are satisfied. It suffices now to show that at least for a subsequence (3.6) is satisfied. Define

$$\Psi_n = \frac{\psi_{E_n}}{\|\psi_{E_n}\|_{L^2}}, \tag{3.8}$$

and the concentration function:

$$\rho(\phi, t) = \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} |\phi(x)|^2 dx, \quad \phi \in L^2(\mathbb{R}),$$

see [5, Section 1.7] for its main properties. Let

$$\mu = \lim_{t\to\infty} \liminf_{n\to\infty} \rho(\Psi_n, t).$$

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Then, from (3.3) and part (i) of Theorem 3, the sequence Ψ_n is bounded in $H^1(\mathbb{R})$ and normalized in $L^2(\mathbb{R})$. According to the concentration compactness theory, see for example [5, Section 1.7], we have the following three possible cases.

1. - *Vanishing:* if $\mu = 0$, then there is a subsequence Ψ_{n_k} convergent to zero in $L^q(\mathbb{R})$, $2 < q \leq \infty$, and, using (3.8) and part(i):

$$\psi_{E_{n_k}} \overset{L^q}{\to} 0, \quad \text{for all } 2 \leq q \leq \infty.$$

Via Lemma 3 we now get

$$\psi_{E_{n_k}} \overset{H^2}{\to} 0$$

which contradicts the fact that $\lim_{E \to E_*} \|\psi_{E_n}\|_{L^2}^2 = N_* \neq 0$, see part (i) of the theorem.

2. - *Splitting:* if $0 < \mu < 1$, then there is a subsequence Ψ_{n_k}, the sequences v_k, w_k, $z_k \in H^2(\mathbb{R})$ all of them bounded in H^1, the sequence $y_k \in \mathbb{R}$ and the function $v_* \in H^1(\mathbb{R})$ such that:

$$\psi_{E_{n_k}} = \|\psi_{E_{n_k}}\|_{L^2} \Psi_{n_k} = v_k + w_k + z_k, \tag{3.9}$$

$$\text{supp } v_k \cap \text{ supp } w_k = \emptyset, \quad \lim_{k \to \infty} \text{dist}(y_k, \text{supp } w_k) = \infty \tag{3.10}$$

$$v_k(\cdot - y_k) \overset{H^1}{\rightharpoonup} v_*, \quad v_k(\cdot - y_k) \overset{L^q}{\to} v_*, \ 2 \leq q \leq \infty, \tag{3.11}$$

$$\|v_*\|_{L^2}^2 = \mu N_* > 0, \quad \lim_{k \to \infty} \|w_k\|_{L^2}^2 = (1 - \mu)N_* > 0 \tag{3.12}$$

$$z_k \overset{H^1}{\rightharpoonup} 0, \quad z_k \overset{L^q}{\to} 0, \ 2 \leq q \leq \infty. \tag{3.13}$$

If, in addition, the sequence $y_k \in \mathbb{R}$ is unbounded, then, by possible passing to a subsequence, we have $\lim_{k \to \infty} y_k = \pm\infty$. Both cases can be treated exactly the same so let us assume $\lim_{k \to \infty} y_k = \infty$. Fix a compactly supported smooth function $\phi \in C_0^\infty(\mathbb{R})$. Then, from the weak formulation(1.3) we have:

$$\begin{aligned} 0 &= \langle \phi'(\cdot), (v_k + w_k + z_k)'(\cdot - y_k) \rangle + \langle \phi(\cdot), (V(\cdot - y_k) + E_{n_k})(v_k + w_k + z_k)(\cdot - y_k) \rangle \\ &\quad + \sigma \langle \phi(\cdot), |v_k + w_k + z_k|^{2p}(v_k + w_k + z_k)(\cdot - y_k) \rangle. \end{aligned}$$

By passing to the limit when $k \to \infty$, the terms containing v_k will converge to v_*, see (3.11), the terms containing w_k respectively z_k will converge to zero due to (3.10) respectively (3.13), and the term containing $V(\cdot - y_k)$ converges to zero due to $\lim_{|x| \to \infty} V(x) = 0$. Hence

$$\langle \phi', v_*' \rangle + \langle \phi, E_* v_* \rangle + \sigma \langle \phi, |v_*|^{2p} v_* \rangle = 0,$$

and $v_* \in H^1(\mathbb{R})$ is a weak solution of

$$-v_*'' + E_* v_* + \sigma |v_*|^{2p} v_* = 0, \tag{3.14}$$

therefore a strong, exponentially decaying solution, with v_*' also exponentially decaying. Let

$$\xi_k(x) = \frac{v_*(x + y_k) - v_*(-x + y_k)}{\sqrt{2\mu N_*}}.$$

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$\xi_k(x)$ is obviously odd in x, and the exponential decay of v_* and its derivative implies:

$$\lim_{k\to\infty} \langle v_*(x + y_k), v_*(-x + y_k)\rangle = 0, \quad \lim_{k\to\infty} \langle v_*(x + y_k), L_+v_*(-x + y_k)\rangle = 0,$$

where L_+ is calculated at E_{n_k}. Consequently:

$$\lim_{k\to\infty} \|\xi_k\|_{L^2}^2 = 1,$$

$$\lim_{k\to\infty} \langle \xi_k, L_+\xi_k\rangle = \frac{1}{\mu N_*} \lim_{k\to\infty} \langle v_*(x + y_k), L_+v_*(x + y_k)\rangle,$$

where we also used (3.12) for the first identity, and the fact that L_+ is invariant under the transformation $x \mapsto -x$ because both $\psi_{E_{n_k}}(x)$ and $V(x)$ are even in x, see also (1.5). Employing again the splitting (3.9), the convergence (3.11), (3.13), and the separation (3.10) we have:

$$\lim_{k\to\infty} \langle v_*(x+y_k), L_+v_*(x+y_k)\rangle = \langle v_*, -v_*'' + E_*v_* + (2p+1)\sigma|v_*|^{2p}v_*\rangle = 2p\,\sigma \int_{\mathbb{R}} |v_*|^{2p+2}(x)dx < 0$$

where we also used (3.14). All in all we have

$$\lim_{k\to\infty} \langle \xi_k, L_+\xi_k\rangle = \frac{2p\,\sigma}{\mu N_*} \int_{\mathbb{R}} |v_*|^{2p+2}(x)dx < 0$$

which together with ξ_k odd implies that for k sufficiently large the second eigenvalue of L_+ at E_{n_k} must become strictly negative, in contradiction with Remark 3.

If, on the other hand, the sequence $y_k \in \mathbb{R}$ is bounded, then, by possible passing to a subsequence, we have $\lim_{k\to\infty} y_k = y_* \in \mathbb{R}$. Let $\tilde{v} = v_*(\cdot + y_*)$. We will now rename $\tilde{v}$ to be v_* and we get from (3.11)

$$v_k \xrightarrow{H^1} v_*, \quad v_k \xrightarrow{L^q} v_*, \ 2 \le q \le \infty, \tag{3.15}$$

and (3.12) remains valid. As before the convergence implies that v_* satisfies weakly, then strongly:

$$-v_*'' + V(x)v_* + E_*v_* + \sigma|v_*|^{2p}v_* = 0, \tag{3.16}$$

hence, both $v_*(x)$ and $v_*'(x)$ decay exponentially in x. Consider

$$\tilde{\mu} = \lim_{t\to\infty} \liminf_{k\to\infty} \rho(w_k, t), \quad \text{where } 0 \le \tilde{\mu} \le (1 - \mu)N_*.$$

If $\tilde{\mu} > 0$ then w_k splits:

$$w_k = \tilde{v}_k + \tilde{w}_k + \tilde{z}_k$$

where the sequences $\tilde{v}_k, \tilde{w}_k, \tilde{z}_k \in H^2(\mathbb{R})$ have properties (3.10)–(3.13). In particular there exist $\tilde{v}_* \in H^1(\mathbb{R})$, $\|\tilde{v}_*\|_{L^2} = \tilde{\mu}$ and the sequence $\tilde{y}_k \in \mathbb{R}$ such that

$$\tilde{v}_k(\cdot - \tilde{y}_k) \xrightarrow{H^1} \tilde{v}_*, \quad \tilde{v}_k(\cdot - \tilde{y}_k) \xrightarrow{L^q} \tilde{v}_*, \ 2 \le q \le \infty.$$

But now, the sequence $\tilde{y}_k$ is definitely unbounded since $\tilde{y}_k \in \text{supp } w_k$ and y_k is bounded, see (3.10). Using

$$\xi_k(x) = \frac{\tilde{v}_*(x + \tilde{y}_k) - \tilde{v}_*(-x + \tilde{y}_k)}{\sqrt{2\tilde{\mu}}},$$

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we get as before

$$\lim_{k\to\infty} \langle \xi_k, L_+\xi_k \rangle = \frac{2p\,\sigma}{\tilde{\mu}} \int_{\mathbb{R}} |\tilde{v}_*|^{2p+2}(x)dx < 0$$

which contradicts the fact that the second eigenvalue of L_+ should remain non-negative, see Remark 3.

If, on the other hand, $\tilde{\mu} = 0$ then, by possibly passing to a subsequence, we have

$$w_k \overset{H^1}{\rightharpoonup} 0, \quad w_k \overset{L^q}{\to} 0,\ 2 < q \le \infty.$$

Moreover, from (3.9), (3.13) and (3.15) we get

$$\psi_{E_{n_k}} \overset{H^1}{\rightharpoonup} v_*, \quad \psi_{E_{n_k}} \overset{L^q}{\to} v_*,\ 2 < q \le \infty,$$

which, via Lemma 3, implies $\lim_{k\to\infty} \|\psi_{E_{n_k}} - v_*\|_{H^2} = 0$. The latter is in contradiction with

$$\|v_*\|_{L^2}^2 = \mu N_* < N_* = \lim_{k\to\infty} \|\psi_{E_{n_k}}\|_{L^2}.$$

3. - *Compactness* If $\mu = 1$ then there is a subsequence $\psi_{E_{n_k}}$, a sequence $y_k \in \mathbb{R}$ and a function $\psi_* \in H^1(\mathbb{R})$, $\|\psi_*\|_{L^2} = N_*$ such that:

$$\psi_{E_{n_k}}(\cdot - y_k) \overset{H^1}{\rightharpoonup} \psi_*, \quad \psi_{E_{n_k}}(\cdot - y_k) \overset{L^q}{\to} \psi_*,\ 2 \le q \le \infty.$$

But the sequence y_k must be bounded, otherwise, by possibly passing to a subsequence, $\lim_{k\to\infty} y_k = \pm\infty$. Both cases are treated in the same way, so let us assume $\lim_{k\to\infty} y_k = \infty$. Then, by the symmetry of $\psi_{E_{n_k}}$ we have:

$$\psi_{E_{n_k}}(x) = \frac{\psi_{E_{n_k}}(x) + \psi_{E_{n_k}}(-x)}{2} \overset{L^2}{\to} \frac{\psi_*(x + y_k) + \psi_*(-x + y_k)}{2} = \xi_k(x),$$

and we get the contradiction:

$$N_* = \lim_{k\to\infty} \|\psi_{E_{n_k}}(x)\|_{L^2}^2 \ne \frac{N_*}{2} = \lim_{k\to\infty} \|\xi_k\|_{L^2}^2,$$

where the last identity is a consequence of $\lim_{k\to\infty} \langle \psi_*(x + y_k), \psi_*(-x + y_k) \rangle = 0$.

So, the sequence $y_k \in \mathbb{R}$ must be bounded, and, by possible passing to a subsequence, we have $\lim_{k\to\infty} y_k = y_*$, and

$$\psi_{E_{n_k}} \overset{H^1}{\rightharpoonup} \psi_{E_*}, \quad \psi_{E_{n_k}} \overset{L^q}{\to} \psi_{E_*},\ 2 \le q \le \infty,$$

where $\psi_{E_*}(x) = \psi_*(x + y_*)$. Therefore the hypotheses of Lemma 3 are verified and part (ii) is proven.

Theorem 3 is now completely proven. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Remark 4. *The proof of Theorem 3 can be greatly simplified if one knows apriori that, with the exception of the two lowest eigenvalues, the spectrum of L_+ is bounded away from zero.*

Indeed, in this case only the second eigenvalue of L_+ can approach zero, see Lemma 2. Hence L_+ restricted to even functions is invertible with uniformly bounded inverse. In particular

$$\partial_E \psi_E = -L_+^{-1}\psi_E, \quad \|\partial_E \psi_E\|_{L^2} \leq K\|\psi_E\|_{L^2}$$

The argument in Theorem 2 can now be repeated to show directly that ψ_E has a limit in L^q, $q \geq 2$, and the proof is finished by applying Lemma 3.

Corollary 1. *Under the assumptions of Theorem 3 we have*

(i) *the second eigenvalue of $L_+(\psi_E, E)$, $E < E_*$, denote it by $\lambda(E)$, and only the second eigenvalue approaches 0 as $E \nearrow E_*$;*

(ii) *if in addition $p > 1/2$ and the derivative of the second eigenvalue of L_+ satisfies:*

$$\lambda'(E_*) = \lim_{E \to E_*} \frac{d\lambda}{dE}(E) \neq 0$$

then the set of real valued solutions of (1.3) in a $H^2 \times \mathbb{R}$ neighborhood of (ψ_{E_*}, E_*) consists of exactly two curves of class at least $C^{[2p]-1}$ intersecting only at (ψ_{E_*}, E_*).

Proof. For part (i), Theorem 3 part (ii) implies that L_+ depends continuously on $E \in [E_0, E_*]$, hence its isolated eigenvalues will depend continuously on E. In particular, at E_*, L_+ will have the lowest eigenvalue strictly negative, followed by 0 as the second eigenvalue, see Lemma 2 and case (b) in Theorem 2. If any other continuous branch of eigenvalues of L_+ approaches zero as $E \nearrow E_*$, then 0 becomes a multiple eigenvalue for L_+ at E_* in contradiction with Sturm-Liouville theory.

Part (ii) is a standard bifurcation result which uses Lyapunov-Schmidt decomposition and Morse Lemma, see for example [24]. More precisely, we continue working with the functional $F(\phi, E)$: $H^2(\mathbb{R}) \times \mathbb{R} \mapsto L^2(\mathbb{R})$ given by (2.1) which has the Frechet derivative:

$$D_\phi F(\phi, E) = -\partial_x^2 + V(x) + E + (2p+1)\sigma|\phi|^{2p} = L_+(\phi, E).$$

For $p > 0$ the functional is C^1 while for $p > \frac{1}{2}$ it is C^2. By part (i) $D_\phi F(\psi_{E_*}, E_*)$ has zero as a simple eigenvalue. Let ϕ_* be the L^2-normalized eigenfunction of $L_* = D_\phi F(\psi_{E_*}, E_*)$ corresponding to its zero eigenvalue. Then L_* is a Fredholm operator of index zero with $\mathrm{Ker}L_* = \mathrm{span}\{\phi_*\}$ and $\mathrm{Ran}L_* = [\mathrm{Ker}L_*]^\perp$. Let

$$P_\parallel \phi = \langle \phi_*, \phi \rangle_{L^2} \phi_*, \quad P_\perp \phi = \phi - \langle \phi_*, \phi \rangle_{L^2} \phi_*$$

be the two orthogonal projections associated to the decomposition $L^2 = \mathrm{Ker}L_* \oplus \mathrm{Ran}L_*$. By the standard Lyapunov-Schmidt procedure, see for example [24], we get the following result.

Lemma 4. *(Lyapunov-Schmidt decomposition) There exists a neighborhood $W \subset H^2 \times \mathbb{R}$ of (ψ_{E_*}, E_*), a neighborhood $U \subset \mathbb{R}^2$ of $(0, E_*)$, and an unique C^1 map $h : U \mapsto L^2 \cap \{\phi_*\}^\perp$ such that for any solution $(\phi, E) \in W$ of $F(\phi, E) = 0$ there exists a unique $a \in \mathbb{R}$ satisfying:*

$$(a, E) \in U, \qquad \phi = \psi_{E_*} + a\phi_* + h(a, E)$$

and

$$F_{\parallel}(a, E) = \langle \phi_*, F(\psi_{E_*} + a\phi_* + h(a, E), E)\rangle = 0 \tag{3.17}$$

In addition, for all $(a, E) \in U$ we have:

$$h(0, E_*) = 0, \tag{3.18}$$

$$\frac{\partial h}{\partial a}(a, E) = -(P_\perp L_+)^{-1} P_\perp L_+ \phi_*, \tag{3.19}$$

$$\frac{\partial h}{\partial E}(a, E) = -(P_\perp L_+)^{-1} P_\perp [\psi_{E_*} + h], \tag{3.20}$$

where $L_+ = L_+(\psi_{E_*} + a\phi_* + h(a, E), E)$.

Moreover, if $p > \frac{1}{2}$ then h is C^2 and for all $(a, E) \in U$ we have:

$$\frac{\partial^2 h}{\partial a^2}(a, E) = -(P_\perp L_+)^{-1} P_\perp (\partial_a L_+) \left[\phi_* + \frac{\partial h}{\partial a}\right], \tag{3.21}$$

$$\frac{\partial^2 h}{\partial a \partial E}(a, E) = -(P_\perp L_+)^{-1} P_\perp \left[\frac{\partial h}{\partial a} + (\partial_a L_+)\frac{\partial h}{\partial E}\right], \tag{3.22}$$

$$\frac{\partial^2 h}{\partial E^2}(a, E) = -(P_\perp L_+)^{-1} P_\perp \left[\frac{\partial h}{\partial E} + (\partial_E L_+)\frac{\partial h}{\partial E}\right], \tag{3.23}$$

where

$$\partial_a L_+ = (2p+1)2p\sigma|\psi_{E_*} + a\phi_* + h|^{2p-1}\mathrm{sign}(\psi_{E_*} + a\phi_* + h)\left(\phi_* + \frac{\partial h}{\partial a}\right),$$

$$(\partial_E L_+)\frac{\partial h}{\partial E} = \frac{\partial h}{\partial E} + (2p+1)2p\sigma|\psi_{E_*} + a\phi_* + h|^{2p-1}\mathrm{sign}(\psi_{E_*} + a\phi_* + h)\left(\frac{\partial h}{\partial E}\right)^2.$$

So, the solutions of (1.3) in W are given by the solutions of (3.17). From (3.18) and $F(\psi_{E_*}, E_*) = 0$ we have $F_{\parallel}(0, E_*) = 0$. Differentiating (3.17) we get:

$$\frac{\partial F_{\parallel}}{\partial a}(a, E) = \langle \phi_*, \ L_+[\phi_* + \frac{\partial h}{\partial a}]\rangle, \tag{3.24}$$

$$\frac{\partial F_{\parallel}}{\partial E}(a, E) = \langle \phi_*, \ \psi_{E_*} + a\phi_* + h + L_+\frac{\partial h}{\partial E}\rangle = a + \langle \phi_*, \ L_+\frac{\partial h}{\partial E}\rangle. \tag{3.25}$$

In particular, from (3.19)-(3.20), $L_*\phi_* = 0$, and L_* self adjoint, we have

$$\frac{\partial F_{\parallel}}{\partial a}(0, E_*) = 0, \tag{3.26}$$

$$\frac{\partial F_{\parallel}}{\partial E}(0, E_*) = 0. \tag{3.27}$$

Since the gradient of $F_{\parallel}$ at $(0, E_*)$ is zero, the number of solutions of (3.17) in a small neighbor-

hood of $(0, E_*)$ is determined by the Hessian at $(0, E_*)$. For $p > \frac{1}{2}$ we can calculate:

$$\frac{\partial^2 F_\|}{\partial a^2}(a, E) = \langle \phi_*, \ (2p+1)2p\sigma|\psi_{E_*} + a\phi_* + h|^{2p-1}\text{sign}(\psi_{E_*} + a\phi_* + h)(\phi_* + \frac{\partial h}{\partial a})^2 + L_+\frac{\partial^2 h}{\partial a^2}\rangle,$$

$$\frac{\partial^2 F_\|}{\partial a\partial E}(a, E) = 1 + \langle \phi_*, \ (2p+1)2p\sigma|\psi_{E_*} + a\phi_* + h|^{2p-1}\text{sign}(\psi_{E_*} + a\phi_* + h)(\phi_* + \frac{\partial h}{\partial a})\frac{\partial h}{\partial E}\rangle$$

$$+\langle \phi_*, \ L_+\frac{\partial^2 h}{\partial a\partial E}\rangle$$

$$\frac{\partial^2 F_\|}{\partial E^2}(a, E) = \langle \phi_*, \ \frac{\partial h}{\partial E} + (2p+1)2p\sigma|\psi_{E_*} + a\phi_* + h|^{2p-1}\text{sign}(\psi_{E_*} + a\phi_* + h)(\frac{\partial h}{\partial E})^2 + L_+\frac{\partial^2 h}{\partial E^2}\rangle,$$

In particular

$$\frac{\partial^2 F_\|}{\partial a^2}(0, E_*) = \langle \phi_*, \ (2p+1)2p\sigma\psi_{E_*}^{2p-1}\phi_*^2\rangle + \langle \phi_*, \ L_*\frac{\partial^2 h}{\partial a^2}\rangle = 0, \tag{3.28}$$

$$\frac{\partial^2 F_\|}{\partial a\partial E}(0, E_*) = 1 + \langle \phi_*, \ (2p+1)2p\sigma\psi_{E_*}^{2p-1}\phi_*\frac{\partial h}{\partial E}\rangle + \langle \phi_*, \ L_*\frac{\partial^2 h}{\partial a\partial E}\rangle$$

$$= 1 + (2p+1)2p\sigma\int_{\mathbb{R}} \psi_{E_*}^{2p-1}\phi_*^2\frac{dh}{dE}(0, E_*)dx \tag{3.29}$$

$$\frac{\partial^2 F_\|}{\partial E^2}(0, E_*) = \langle \phi_*, \ \frac{\partial h}{\partial E}\rangle + \langle \phi_*, \ (2p+1)2p\sigma\psi_{E_*}^{2p-1}(\frac{\partial h}{\partial E})^2\rangle + \langle \phi_*, \ L_*\frac{\partial^2 h}{\partial E^2}\rangle = 0, \tag{3.30}$$

where we used (3.18)-(3.20), the fact that ψ_{E_*} is even while ϕ_* is odd, the fact that h and its partial derivatives are in $\{\phi_*\}^\perp$, and the fact that L_* is self adjoint with $L_*\phi_* = 0$. We next show:

$$\lambda'(E_*) = 1 + (2p+1)2p\sigma\int_{\mathbb{R}} \psi_{E_*}^{2p-1}\phi_*^2\frac{dh}{dE}(0, E_*)dx \equiv \frac{\partial^2 F_\|}{\partial a\partial E}(0, E_*). \tag{3.31}$$

The second eigenvalue of $L_+(\psi_E, E)$ along the C^1 symmetric branch (ψ_E, E) satisfies the equation

$$L_+(\psi_E, E)\phi_E = \lambda(E)\phi_E, \quad \|\phi_E\|_{L^2} \equiv 1.$$

Differentiating with respect to E, we get:

$$\phi_E + (2p+1)2p\sigma\psi_E^{2p-1}\frac{d\psi_E}{dE}\phi_E + L_+\frac{d\phi_E}{dE} = \lambda'(E)\phi_E + \lambda(E)\frac{d\phi_E}{dE},$$

and, by taking the scalar product with ϕ_E, we obtain

$$1 + (2p+1)2p\sigma\int_{\mathbb{R}} \psi_E^{2p-1}\phi_E^2\frac{d\psi_E}{dE}dx = \lambda'(E). \tag{3.32}$$

Using the continuous dependence of the spectral decomposition of $L_+(\psi_E, E)$ with respect to $E \in (E_0, E_*]$, we have $\lim_{E \nearrow E_*} \|\phi_E - \phi_*\|_{H^2} = 0$. Moreover, from (2.5) and (3.20) we have

$$\lim_{E \nearrow E_*} \|\frac{d\psi_E}{dE} - \frac{dh}{dE}(0, E_*)\|_{H^2} = \lim_{E \nearrow E_*} \|L_+^{-1}\psi_E - L_*^{-1}\psi_{E_*}\|_{H^2} = 0.$$

Passing now to the limit $E \nearrow E_*$ in the identity above we get (3.31).

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From (3.28)-(3.31) we have

$$\nabla^2 F_{\parallel}(0, E_*) = \begin{bmatrix} 0 & \lambda'(E_*) \\ \lambda'(E_*) & 0 \end{bmatrix}$$

Since by hypothesis $\lambda'(E_*) \neq 0$, the Hessian of $F_{\parallel}$ is nonsingular and negative definite at $(0, E_*)$, hence by Morse Lemma, see [24, Theorem 3.1.1 and Corollary 3.1.2], the set of solutions of $F_{\parallel}(a, E) = 0$ in a neighborhood of $(0, E_*)$ consists of exactly two curves of class $C^{[2p]-1}$ intersecting only at $(0, E_*)$. This finishes the proof of the corollary. $\qquad\square$

Unfortunately, for $1/2 \leq p < 3/2$, the corollary does not guarantee that the two curves of solutions are C^2 which turns out to be necessary for determining their orbital stability with respect to the full dynamical system (1.1). However a more careful analysis of equation (3.17) recovers the full regularity and stability of these solutions:

Theorem 4. *Let $\sigma < 0$, $p \geq 1/2$, and consider the symmetric (even in x) branch of solutions (ψ_E, E) of (1.3) which bifurcates from the lowest eigenvalue $-E_0$ of L_0. Let (E_0, E_*) be the maximal interval on which this branch can be uniquely continued. Assume $E_* < \infty$, and*

$$\lambda'(E_*) = \lim_{E \nearrow E_*} \frac{d\lambda}{dE}(E) \neq 0, \tag{3.33}$$

where $\lambda(E)$ is the second eigenvalue of $L_+(\psi_E, E)$. Then, the set of real valued solutions $(\phi, E) \in H^2 \times \mathbb{R}$ of the stationary NLS equation (1.3) in a small neighborhood of $(\psi_{E_*}, E_*) \in H^2 \times \mathbb{R}$ consists of exactly two C^2 curves intersecting only at (ψ_{E_*}, E_*) :

(i) the first curve can be parameterized by $E \mapsto \phi = \psi_E$, $E \in (E_0, E_* + \epsilon)$ for some small $\epsilon > 0$, it is a C^2 continuation past the bifurcation point $E = E_*$ of the symmetric branch, it has ψ_E even for all E and orbitally unstable for $E > E_*$;

(ii) the second curve is of the form $(\phi(a), E(a))$, $a \in \mathbb{R}$ small, where the parameter can be chosen to be the projection of $\phi - \psi_{E_*}$ onto $\ker L_* = \operatorname{span}\{\phi_*\}$ i.e. $\exists \epsilon > 0$ such that for $|a| < \epsilon$:

$$E = E(a) = E_* + \frac{Q}{2}a^2 + o(a^2), \quad \phi(a) = \psi_{E_*} + a\phi_* + \tilde{h}(a), \tag{3.34}$$

where $\tilde{h}(a) = O(a^2) \in \{\phi_*\}^{\perp}$, and

$$Q = -\frac{2p(2p+1)\sigma^2}{\lambda'(E_*)} \left(\frac{2p-1}{3\sigma} \langle \phi_*^2, \psi_{E_*}^{2p-2}\phi_*^2 \rangle_{L^2} - 2p(2p+1)\langle \psi_{E_*}^{2p-1}\phi_*^2, L_*^{-1}\psi_{E_*}^{2p-1}\phi_*^2 \rangle_{L^2} \right) \tag{3.35}$$

with $L_* = L_+(\psi_{E_*}, E_*)$, along this curve ϕ is neither even nor odd with respect to x, and is orbitally stable if

$$Q > 0, \quad \text{and} \quad R > 0$$

and orbitally unstable if

$$Q < 0, \quad \text{or} \quad Q > 0 \text{ and } R < 0,$$

where

$$R = \lim_{E \to E_*} \frac{d\|\phi\|_{L^2}^2}{dE} = 2\frac{\lambda'(E_*)}{Q} + N'(E_*), \quad N(E) = \|\psi_E\|_{L^2}^2. \tag{3.36}$$

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Proof. We continue to rely on the Lyapunov-Schmidt decomposition described Lemma 4, but we remark that $\psi_{E_*} = \lim_{E \nearrow E_*} \psi_E$ given by Theorem 3 is even, since ψ_E, $E < E_*$, were even. Hence the Frechet derivative of $F(\psi, E)$ with respect to ψ at (ψ_{E_*}, E_*) :

$$D_\psi F(\psi_{E_*}, E_*)[\phi] = L_*[\phi] = (-\partial_x^2 + V + E_*)\phi + (2p+1)|\psi_{E_*}|^{2p}\phi,$$

transforms even functions into even functions. By Corollary 1 part (i), 0 is the second eigenvalue of L_* hence its eigenfunction ϕ_* is odd via Sturm-Liouville theory. Consequently

$$L_*|_{even} : H^2_{even} \mapsto L^2_{even}$$

is an isomorphism. Implicit function theorem for $F(\psi, E) = 0$ at (ψ_{E_*}, E_*) implies that the set of even, real valued solutions of (1.3) in a neighborhood of (ψ_{E_*}, E_*) consists of a unique C^1 curve, $E \mapsto \psi_E$. Moreover $\psi_E > 0$, because ψ_E is also the eigenvector corresponding to the lowest eigenvalue of

$$L_- = -\partial_x^2 + V + E_* + |\psi_{E_*}|^{2p},$$

and, consequently, $F(E) = F(\psi_E, E)$ becomes C^2 even for $p = 1/2$. Hence $L_+ = D_\psi F$ is C^1 in E along this curve. Differentiating once $F(\psi_E, E) \equiv 0$ we get:

$$\frac{d\psi_E}{dE} = -(L_+)^{-1}\psi_E$$

and the curve $E \mapsto \psi_E$ is C^2 because the right hand side is C^1. Moreover, the second eigenvalue of L_+ along the curve, $\lambda(E)$, is C^1 in E, and, because:

$$\lambda(E_*) = 0, \qquad \frac{d\lambda}{dE}(E_*) < 0,$$

see Corollary 1 part (i) and (3.33), we deduce that $\lambda(E) < 0$, $E > E_*$.

For $E > E_*$ we now have that L_+ has two strictly negative eigenvalues while L_- has none along the symmetric (even in x) branch of solutions of (1.3). These imply that $\exp(iEt)\psi_E$ is an orbitally unstable solution of (1.1), see [8], and finishes part (i) of the theorem.

For part (ii) we will rely on the curve of solutions discovered in part (i) to do a sharper analysis of equation (3.17) compared to the one provided by Morse Lemma. The solutions (ψ_E, E) of (1.3) discovered in part (i) satisfy $a = \langle \phi_*, \psi_E \rangle = 0$ because ϕ_* is odd while ψ_E is even. Consequently one set of solutions of (3.17) is given by $a \equiv 0$, in particular:

$$\psi_E = \psi_{E_*} + h(0, E) > 0 \tag{3.37}$$
$$F_\|(0, E) = \langle \phi_*, F(\psi_{E_*} + h(0, E), E) \rangle \equiv 0 \tag{3.38}$$

Hence, "a" can be factored out in the left hand side of equation (3.17), and, solutions $a \neq 0$ of this equation satisfy:

$$g(a, E) = 0 \tag{3.39}$$

where

$$g(a, E) = \begin{cases} \frac{F_\|(a,E) - F_\|(0,E)}{a}, & \text{if } a \neq 0 \\ \frac{\partial F_\|}{\partial a}(0, E), & \text{if } a = 0 \end{cases} \tag{3.40}$$

We will show that:

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(a) $g(0, E_*) = 0$,

(b) g is C^1 in a $\mathbb{R} \times \mathbb{R}$ neighborhood of $(a = 0, E = E_*)$,

(c) $\frac{\partial g}{\partial E}(0, E_*) = \lambda'(E_*) \neq 0$, $\quad \frac{\partial g}{\partial a}(0, E_*) = 0$

Implicit function theorem will then imply that the set of solutions of (3.39) in a neighborhood of $(0, E_*)$ consist of a unique C^1 curve: $a \mapsto E(a)$, $|a| < \epsilon$ for some $\epsilon > 0$, with $E(0) = E_*$ and

$$\frac{dE}{da}(0) = -\frac{\frac{\partial g}{\partial a}(0, E_*)}{\frac{\partial g}{\partial E}(0, E_*)} = 0.$$

Then we will show that the curve $a \mapsto E(a)$ is C^2 with

$$\frac{d^2 E}{da^2}(0) = Q \tag{3.41}$$

where Q is given by (3.35). Hence the nonsymmetric solutions of (1.3), are given by: $a \mapsto (\phi(a), E(a))$ with:

$$E = E(a) = E_* + \frac{Q}{2}a^2 + o(a^2), \quad \phi(a) = \psi_{E_*} + a\phi_* + h(a, E(a)) > 0 \tag{3.42}$$

and $a \mapsto \phi(a)$ is C^2 (from $\mathbb{R}$ with values in H^2) because from (3.42) and (3.18):

$$\frac{d\psi_E}{da}(a) = \phi_* + \frac{dh}{da} + E'(a)\frac{dh}{dE} = \phi_* - (P_\perp L_+)^{-1} P_\perp [L_+ \phi_* + E'(a)(\phi_* + h)]$$

and the right hand side is C^1 in a because E, h are C^1, and L_+ is also C^1 when calculated along $(\phi(a) > 0, E(a))$. The orbital stability of this curve of solutions follows from the theory developed in [8], [9] and:

$$\lambda_1(a) = -\lambda'(E_*)Qa^2 + o(a^2) \tag{3.43}$$

$$\|\phi(a)\|_{L^2}^2 = N(E_*) + \frac{1}{2}\left(2\lambda'(E_*) + QN'(E_*)\right)a^2 + o(a^2) \tag{3.44}$$

where $\lambda_1(a)$ is the second eigenvalue of $L_+(\phi(a), E(a))$. Indeed, for $Q < 0$, (3.43) shows that L_+ has two strictly negative eigenvalues while we know that L_- has none, hence the result in [8] implies $\phi(a)$ is orbitally unstable. For $Q > 0$, L_+ has exactly one strictly negative eigenvalue, L_- has none, and the result in [9] implies that $\phi(a)$ is orbitally stable if $\frac{d\|\phi\|_{L^2}}{dE} > 0$ and unstable if $\frac{d\|\phi\|_{L^2}}{dE} < 0$ which via (3.44) and (3.42) is equivalent to $R > 0$, respectively $R < 0$, where R is defined by (3.36).

Now (3.44) follows from (3.42), (3.31) and properties (3.18)-(3.23) of $h(a, E)$, while (3.43) follows from the same relations by differentiating twice the equation for the second eigenvalue:

$$L_+(\psi_E(a), E(a))[\phi_1(a)] = \lambda_1(a)\phi_1(a), \qquad \|\phi_1(a)\|_{L^2} \equiv 1.$$

It remains to prove (a)-(c) and (3.41). (a) follows from (3.40) and (3.26). For $a \neq 0$ (b) follows from F being C^1. For $a = 0$ it suffices to prove:

$$\lim_{\substack{a \neq 0 \\ a \to 0, E \to E_0}} \frac{\partial g}{\partial a}(a, E) \quad \text{exists}, \quad \text{and} \quad \lim_{\substack{a \neq 0 \\ a \to 0, E \to E_0}} \frac{\partial g}{\partial E}(a, E) = \frac{\partial^2 F_\parallel}{\partial E \partial a}(0, E_0).$$

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Note that $\frac{\partial^2 F_\parallel}{\partial E \partial a}(0, E_0)$ exists and it is continuous in E_0 because the partial derivative with respect to E of the right hand side of (3.24) is the derivative along the symmetric branch $\psi_E = \psi_{E_*} + h(0, E) > 0$ which we already know is C^2. We will prove that the first limit exists, the argument for the second is similar. In what follows we use the shortened notation:

$$L_+(a, E) = L_+(\psi_{E_*} + a\phi_* + h(a, E), E) = D_\psi F(\psi_{E_*} + a\phi_* + h(a, E), E).$$

For $a \neq 0$ we have

$$\frac{\partial g}{\partial a}(a, E) = -\frac{1}{a^2}\langle\phi_*, \ F(\psi_{E_*}+a\phi_*+h(a, E), E)-F(\psi_{E_*}+h(0, E), E)\rangle+\frac{1}{a}\langle\phi_*, \ L_+(a, E)[\phi_*+\frac{\partial h}{\partial a}(a, E)]\rangle.$$

We add and subtract $\frac{1}{a}\langle\phi_*, \ L_+(0, E)[\phi_* + \frac{\partial h}{\partial a}(0, E)]\rangle$ to get

$$\lim_{\substack{a\neq 0 \\ a\to 0, E\to E_0}} \frac{\partial g}{\partial a}(a, E)$$

$$= \lim_{\substack{a\neq 0 \\ a\to 0, E\to E_0}} -\frac{F_\parallel(a, E) - F_\parallel(0, E) + a\langle\phi_*, \ L_+(0, E)[\phi_* + \frac{\partial h}{\partial a}(0, E)]\rangle}{a^2}$$

$$+ \lim_{\substack{a\neq 0 \\ a\to 0, E\to E_0}} \frac{\langle\phi_*, \ L_+(a, E)[\phi_* + \frac{\partial h}{\partial a}(a, E)] - L_+(0, E)[\phi_* + \frac{\partial h}{\partial a}(0, E)]\rangle}{a} = I_1 + I_2$$

The differential form of the intermediate value theorem gives:

$$I_1 = -\frac{1}{2} \lim_{\substack{a\neq 0 \\ a\to 0, E\to E_0}} \frac{\langle\phi_*, \ L_+(a', E)[\phi_* + \frac{\partial h}{\partial a}(a', E)] - L_+(0, E)[\phi_* + \frac{\partial h}{\partial a}(0, E)]\rangle}{a'}$$

where $a' = a'(a, E)$ is between 0 and a. So $I_1 = -\frac{1}{2}I_2$ provided I_2 exists.

For the limit I_2, when $p > 1/2$ we can use F is C^2 hence L_+ is C^1 and h is C^2 to get:

$$I_2 = \langle\phi_*, \ \partial_a L_+(0, E_0)[\phi_* + \frac{\partial h}{\partial a}(0, E_0)] + L_+(0, E_0)\frac{\partial^2 h}{\partial a^2}(0, E_0)\rangle = 0,$$

see (3.21) for an explicit expression for both $\partial_a L_+$ and $\frac{\partial^2 h}{\partial a^2}$, and notice that they are both even at $(0, E_0)$ while ϕ_* is odd. When $p = 1/2$ we rewrite I_2 as two limits:

$$I_2 = \lim_{\substack{a\neq 0 \\ a\to 0, E\to E_0}} \frac{\langle\phi_*, \ (L_+(a, E) - L_+(0, E))[\phi_* + \frac{\partial h}{\partial a}(a, E)]\rangle}{a}$$

$$+ \lim_{\substack{a\neq 0 \\ a\to 0, E\to E_0}} \frac{\langle\phi_*, \ L_+(0, E)[\frac{\partial h}{\partial a}(a, E) - \frac{\partial h}{\partial a}(0, E)]\rangle}{a} = I_3 + I_4$$

But

$$(L_+(a, E) - L_+(0, E))[v] = 2\sigma(|\psi_{E_*} + a\phi_* + h(a, E)| - |\psi_{E_*} + h(0, E)|)v$$

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and by $-|a - b| \le |a| - |b| \le |a - b|$ and for each $x \in \mathbb{R}$:

$$|h(a, E) - h(0, E)|(x) \le \left|\frac{\partial h}{\partial a}(a', E)(x)\right| |a|, \text{ for some } |a'| < |a|,$$

we get that the integrand in the expression for the limit I_3 is bounded by:

$$\left|\frac{\phi_*(x)(L_+(a, E) - L_+(0, E))[\phi_* + \frac{\partial h}{\partial a}(a, E)](x)}{a}\right| \le |\phi_*|^2(x) + |\phi_*|(x)\left|\frac{\partial h}{\partial a}(a', E)\right|(x)\left|\phi_* + \frac{\partial h}{\partial a}(a, E)\right|$$

where the right hand side can further be bounded by an integrable function, since $h(a, E) \in H^2$ is C^1 in a neighborhood of $(0, E_*)$.

In addition, since $(a, E) \mapsto h(a, E) \in H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ is continuous and $\psi_{E_*} + h(0, E_0)) > 0$, see (3.37), we get that for each $x \in \mathbb{R}$ there exists a $\delta(x) > 0$ such that $\psi_{E_*} + a\phi_* + h(a, E)) > 0$, for $|a|, |E - E_0| < \delta$. So, we have the pointwise convergence:

$$\lim_{\substack{a \ne 0 \\ a \to 0, E \to E_0}} \frac{|\psi_{E_*} + a\phi_* + h(a, E)|(x) - |\psi_{E_*} + h(0, E)|(x)}{a}$$

$$= \lim_{\substack{a \ne 0 \\ a \to 0, E \to E_0}} \frac{a\phi_*(x) + (h(a, E) - h(0, E))(x)}{a} = \phi_*(x) + \frac{\partial h}{\partial a}(0, E_0)(x) \; \forall x \in \mathbb{R},$$

which combined with the Lebesque Dominated Convergence Theorem implies

$$I_3 = \langle \phi_*, \; 2\sigma(\phi_* + \frac{\partial h}{\partial a}(0, E_0))^2 \rangle = 0,$$

since ϕ_* is odd and $(\phi_* + \frac{\partial h}{\partial a}(0, E_0))^2$ is even.

Similarly, from (3.19) we get the pointwise convergence:

$$\lim_{\substack{a \ne 0 \\ a \to 0, E \to E_0}} \frac{\frac{\partial h}{\partial a}(a, E)|(x) - \frac{\partial h}{\partial a}(0, E)(x)}{a} = -(P_\perp L_+)^{-1} P_\perp(\partial_a L)[\phi_* + \frac{\partial h}{\partial a}(0, E_0)](x) \forall x \in \mathbb{R},$$

and, again by Lebesque Dominated Convergence Theorem:

$$I_4 = -2\sigma\langle \phi_*, \; L_+(0, E_0)(P_\perp L_+)^{-1} P_\perp[\phi_* + \frac{\partial h}{\partial a}(0, E_0)]^2 \rangle = 0,$$

since ϕ_* is odd while the other factor is even.

In conclusion, for $p \ge 1/2$ we have

$$\lim_{\substack{a \ne 0 \\ a \to 0, E \to E_0}} \frac{\partial g}{\partial a}(a, E) = \frac{1}{2}\langle \phi_*, \; \partial_a L_+(0, E_0)[\phi_* + \frac{\partial h}{\partial a}(0, E_0)] + L_+(0, E_0)\frac{\partial^2 h}{\partial a^2}(0, E_0) \rangle = 0.$$

A similar argument shows

$$\lim_{\substack{a \ne 0 \\ a \to 0, E \to E_0}} \frac{\partial g}{\partial E}(a, E) = \frac{\partial^2 F_\parallel}{\partial E \partial a}(0, E_0)$$

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where

$$\frac{\partial^2 F_\|}{\partial E \partial a}(0, E_0) = 1 + (2p+1)2p\sigma \langle \phi_*, (\psi_{E_*} + h(0, E_0))^{2p-1}(\phi_* + \frac{\partial h}{\partial a}(0, E_0))\frac{\partial h}{\partial E}(0, E_0) + L_+ \frac{\partial^2 h}{\partial E \partial a}(0, E_0) \rangle.$$

So, $g(a, E)$ is C^1 with

$$g(0, E_*) = 0, \quad \text{and} \quad \frac{\partial g}{\partial E}(0, E_*) = 1 + (2p+1)2p\sigma \int_\mathbb{R} \psi_{E_*}^{2p-1} \phi_*^2 \frac{\partial h}{\partial E}(0, E_*)dx = \lambda'(E_*) \neq 0$$

see (3.31) and (3.33). Hence the set of solutions of $g(a, E) = 0$ in a neighborhood of $(0, E_*)$ consists of a unique C^1 curve $a \mapsto E(a)$ with $E(0) = E_*$ and

$$\frac{dE}{da}(0) = -\frac{\frac{\partial g}{\partial a}}{\frac{\partial g}{\partial E}}(0, E_*) = 0.$$

The curve is in fact C^2, because for $a \neq 0$, $p > 1/2$ we have that $g(a, E)$ is C^2 while for $a \neq 0$, $p = 1/2$ a similar argument as the one above involving pointwise convergence to $\psi_E(a) = \psi_{E_*} + a\phi_* + h(a, E(a)) > 0$ can be employed. For $a = 0$ we have from definition of the derivative:

$$E''(0) = \lim_{\substack{a \neq 0 \\ a \to 0}} -\frac{\frac{\partial g}{\partial a}(a, E(a))}{a\frac{\partial g}{\partial E}(a, E(a))} = \frac{1}{-\lambda'(E_*)} \lim_{\substack{a \neq 0 \\ a \to 0}} \frac{1}{a}\frac{\partial g}{\partial a}(a, E(a)),$$

where

$$\lim_{\substack{a \neq 0 \\ a \to 0}} \frac{1}{a}\frac{\partial g}{\partial a}(a, E(a)) = \frac{1}{3}(2p+1)2p(2p-1)\sigma\langle\phi_*^2, \psi_{E_*}^{2p-2}\phi_*^2\rangle - (2p+1)^2(2p)^2\sigma^2\langle\psi_{E_*}^{2p-1}\phi_*^2, L_*^{-1}\psi_{E_*}^{2p-1}\phi_*^2\rangle.$$

The last identity follows from the same argument as for $\lim_{a \to 0, E \to E_0} \frac{\partial g}{\partial a}(a, E)$ with the only difference that for $1/2 < p < 1$ Lebesque Dominated Convergence Theorem requires:

$$|\psi_{E_*} + a\phi_* + h(a, E(a))|^{2p-2}|\phi_*| \in L^2 \bigcap L^\infty.$$

But since both ϕ_* and $\psi_E(a) = \psi_{E_*} + a\phi_* + h(a, E(a)) > 0$ are solutions of the uniform elliptic equations:

$$L_*\phi_* = 0, \qquad L_-\psi_E(a) = 0,$$

we have via upper and lower bounds for uniform elliptic equations:

$$|\phi_*(x)| \leq C(\delta)\exp(-\sqrt{E_* - \delta}|x|), \text{ for } \delta > 0, \text{ and } \psi_E(a) \geq C(\epsilon)\exp(-\sqrt{E_* + \epsilon}|x|), \text{ for } \epsilon > E(a) - E_*.$$

Hence, for $1/2 < p < 1$ we get:

$$|\psi_E(a)|^{2p-2} \leq C\exp(-\tilde{\epsilon}|x|)$$

for any $\tilde{\epsilon} > 0$ such that:

$$E_* + \tilde{\epsilon} < \frac{E_*}{2 - 2p}.$$

This finishes (a)-(c) and (3.41), consequently Theorem 4. $\qquad\square$

We note that Theorem 4 shows that the branch of symmetric states goes through a pitchfork bifurcation at (ψ_{E_*}, E_*), which is classified, based on stability analysis, as supercritical when $Q > 0$ and $R > 0$, and subcritical when either $Q < 0$ or $Q > 0$ and $R < 0$. We do not have a general result determining which one occurs *except for the double-well potentials with large separation,* e.g. (1.4) with s large:

Corollary 2. *Consider the equation* (1.3) *with $\sigma < 0$ and potential of the form:*

$$V \equiv V_s(x) = V_0(x + s) + V_0(-x + s), \qquad x \in \mathbb{R}, \ s > 0.$$

Assume V_0 satisfies (H1), (H2) *and* (H4). *Then there exists $s_* > 0$ such that for all $s \geq s_*$ the branch of real valued, even in x solutions (ψ_E, E) bifurcating from the lowest eigenvalue of $-\Delta + V_s$ undergoes a pitchfork bifurcation at $E = E_*$, where $\lim_{s \to \infty} E_* = E_0$. Moreover, the asymmetric branch emerging at the bifurcation point is orbitally stable if $p < p_* = \frac{3+\sqrt{13}}{2}$ and orbitally unstable if $p > p_*$, while the symmetric branch (ψ_E, E) continues past the bifurcation point but becomes orbitally unstable.*

Remark 5. *We remark that the case $p = 1$ has already been obtained in* [16]. *In what follows we present a more direct argument to obtain the same result for all $p \geq 1/2$. The argument can be easily adapted to higher space dimensions, see* [20], *and to the case $\sigma > 0$ in which we can rigorously show that a pitchfork bifurcation occurs along the first excited state (the branch bifurcating from the second lowest eigenvalue of $-\Delta + V_s$). This result has been predicted in* [30].

Proof. Under hypotheses (H1), (H2) and (H4) for V_0, it is known that the spectrum Σ of $-\partial_x^2 + V_s$ has the lowest eigenvalues $-E_{0,s} < -E_{1,s}$ satisfying

$$\lim_{s \to \infty} |E_{k,s} - E_0| = 0, \ k = 0, 1, \quad \text{and} \quad \exists \ s_*, d > 0 : \ \text{dist}(E_{0,s}, \Sigma \setminus \{E_{1,s}\}) \geq d \ \forall s \geq s_*. \quad (3.45)$$

Moreover, the normalized eigenfunctions $\psi_{0,s}, \psi_{1,s}$ corresponding to these eigenvalues satisfy:

$$\lim_{s \to \infty} \left\| \psi_{0,s}(x) - \frac{\psi_0(x + s) + \psi_0(-x + s)}{\sqrt{2}} \right\|_{H^2} = 0, \quad (3.46)$$

$$\lim_{s \to \infty} \left\| \psi_{1,s}(x) - \frac{\psi_0(x + s) - \psi_0(-x + s)}{\sqrt{2}} \right\|_{H^2} = 0, \quad (3.47)$$

where ψ_0 is the eigenfunction of $-\partial_x^2 + V_0$ corresponding to its lowest eigenvalue $-E_0$, see [16, Appendix] and references therein. Proposition 1 shows that for each $s > 0$, a unique curve (ψ_E, E) of real valued, nontrivial solutions of

$$-\partial_x^2 \phi + V_s \phi + E \phi + \sigma |\phi|^{2p} \phi = 0,$$

bifurcates from $(0, E_0)$ and can be parametrized by $a = \langle \psi_{0,s}, \psi_E \rangle$, i.e. there exists $\varepsilon > 0$, such that for $|a| < \varepsilon$ we have:

$$E = E_0 - \sigma \|\psi_{0,s}\|_{L^{2p+2}}^{2p+2} |a|^{2p} + O(|a|^{4p}), \text{ i.e. } |E - E_0 - \sigma \|\psi_{0,s}\|_{L^{2p+2}}^{2p+2} |a|^{2p}| \leq C_1 |a|^{4p} \quad (3.48)$$

$$\psi_E = a\psi_{0,s} + O(|a|^{2p+1}), \text{ i.e. } \|\psi_E - a\psi_{0,s}\|_{H^2} \leq C_2 |a|^{2p+1}. \quad (3.49)$$

Moreover, $\psi_E(x)$ is even in x. We will rely on the fact that the distance between the two lowest eigenvalues of $\partial_x^2 + V_s$ converges to zero as $s \to \infty$, see (3.45), to show that there exists $s_* > 0$ such that for each $s \geq s_*$ the second eigenvalue $\lambda(a,s)$ of $L_+(\psi_E, E)$ must cross zero at $a = a_* = a_*(s)$, where $0 < a_* < \varepsilon$, and

$$\lim_{s\to\infty} a_*(s) = 0. \tag{3.50}$$

Moreover, for each $s \geq s_*$ we have

$$\frac{d\lambda}{dE}(a_*, s) \leq -p < 0. \tag{3.51}$$

Hence the hypotheses of Theorem 4 are satisfied and a pitchfork bifurcation occurs at $E_* = E(a_*)$ for each $s \geq s_*$. We will also calculate $Q = Q(s)$, $R = R(s)$ at these bifurcation points and show they are continuous on the interval $s \in [s_*, \infty)$ with

$$\lim_{s\to\infty} a_*^{2-2p}(s)Q(s) = -\sigma\frac{2^{2-p}}{3}(2p+1)(p+1)\|\psi_0\|_{L^{2p+2}}^{2p+2} \tag{3.52}$$

$$\lim_{s\to\infty} a_*^{2p-2}(s)R(s) = \frac{2^p(-p^2 + 3p + 1)}{-\sigma(2p+1)(p+1)p\|\psi_0\|_{L^{2p+2}}^{2p+2}}. \tag{3.53}$$

Therefore, by choosing a larger s_* if necessary, we have $Q(s) > 0$, for all $s \geq s_*$ and, if $p < p_* = \frac{3+\sqrt{13}}{2}$ then $R(s) > 0$, for all $s \geq s_*$ while if $p > p_*$ then $R(s) < 0$, for all $s \geq s_*$. The proof of the corollary is now finished.

It remains to prove (3.50)-(3.53). They follow from rather tedious calculations involving the spectral properties (3.45)-(3.47), and bifurcation estimates (3.48)-(3.49). We include them for completeness. First we note that there exists $s_* > 0$ such that the estimates (3.48)–(3.49) are uniform in $s \geq s_*$, i.e. the constants C_1, C_2 can be chosen independent of $s \geq s_*$. The reason is that the estimates rely on contraction principle applied to the operator $K(\phi, E, s) : H^2 \mapsto H^2$ given by:

$$K(\phi, E, s) = \mathbb{I} - (-\partial_x^2 + V_s + E_{0,s})^{-1} P_{\perp,s}(-\partial_x^2 + V_s + E + (2p+1)\sigma|\phi|^{2p}),$$

where $P_{\perp,s}$ denotes the orthogonal (in L^2) projection onto $\psi_{0,s}^\perp$. Since the spectrum of $(-\partial_x^2 + V_s + E_{0,s})P_{\perp,s}$ *restricted to even functions* remains bounded away from zero for s sufficiently large and $V_s : H^2 \mapsto L^2$ is uniformly bounded, we can choose the Lipschitz constant for K, hence the constants C_1, C_2 above, independent of s.

Now, if $\lambda(a,s)$ denotes the second eigenvalue of $L_+(\psi_E, E)$, then we have:

$$\lambda(a,s) = E_{0,s} - E_{1,s} + \frac{d\lambda}{dE}(0,s)(E - E_{0,s}) + O(|E - E_{0,s}|^2), \tag{3.54}$$

where, as before, the constant hidden in the $O(|E - E_{0,s}|^2)$ term can be chosen independent of s for large s. Using now (3.32) we get:

$$\frac{d\lambda}{dE}(0,s) = \lim_{a\to 0}\left[1 + (2p+1)\sigma\int_R \phi_E^2 \frac{d}{dE}|\psi_E|^{2p}dx\right] = 1 - \frac{2p+1}{\|\psi_{0,s}\|_{L^{2p+2}}^{2p+2}}\int_R \psi_{1,s}^2(x)\psi_{0,s}^{2p}(x)dx$$

where we relied on the expansions (3.48), (3.49), and on the continuous dependence with respect to a of the spectral decomposition of L_+ which implies that the eigenfunction ϕ_E corresponding

to the second eigenvalue of L_+ converges to $\psi_{1,s}$. Moreover, using the fact that the expansions of ψ_E, ϕ_E are uniform in s for large s, we get from (3.46)–(3.47):

$$\lim_{\substack{s\to\infty \\ a\to 0}} a^{-q} \int_{\mathbb{R}} \psi_E^q(x)\phi_E^{2k}(x)dx = 2^{1-k-q/2}\|\psi_0\|_{L^{q+2k}}^{q+2k}, \quad \text{for all } q \geq 0, \text{ and } k = 1, 2, \ldots, \tag{3.55}$$

which in particular implies

$$\lim_{\substack{s\to\infty \\ a\to 0}} \frac{d\lambda}{dE}(a, s) = -2p < 0. \tag{3.56}$$

From (3.45), (3.54) and (3.56) we now have

$$\lim_{s\to\infty} \lambda(a, s) = 0 + 2p\sigma 2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2}|a|^{2p} + O(|a|^{4p}).$$

Hence, because $\sigma < 0$, there exists $s_* > 0$ and $\epsilon > 0$ such that:

$$\lambda(\epsilon, s) < 0, \quad \text{and} \quad \frac{d\lambda}{dE}(a, s) \leq -p < 0, \quad \text{for all } s \geq s_* \text{ and } |a| \leq \epsilon, \ a \neq 0.$$

Using now $\lambda(0, s) = E_{0,s} - E_{1,s} > 0$, $s > 0$ we infer that for each $s \geq s_*$, $\lambda(a, s)$ changes sign exactly once in the interval $a \in [0, \epsilon]$ at

$$a = a_*(s) \approx \sqrt[2p]{\frac{E_{0,s} - E_{1,s}}{-\lambda'(0, s)}}.$$

Hence (3.50) and (3.51) hold.

We now compute Q and R in the limit $s \to \infty$, relying on (3.50). In formula (3.35) which defines Q we have already showed:

$$\lim_{s\to\infty} \lambda'(E_*) = \lim_{\substack{s\to\infty \\ a\to 0}} \frac{d\lambda}{dE}(a, s) = -2p, \tag{3.57}$$

and

$$\lim_{s\to\infty} \langle \phi_*^2, \psi_{E_*}^{2p-2}\phi_*^2 \rangle = \lim_{\substack{s\to\infty \\ a\to 0}} a^{2-2p} \int_{\mathbb{R}} \psi_E^{2p-2}(x)\phi_E^4(x)dx = 2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2},$$

see (3.56) and (3.55). For the remaining scalar product we use

$$L_*\psi_{E_*} = L_+(\psi_{E_*}, E_*)\psi_{E_*} = \sigma 2p\psi_{E_*}^{2p+1}$$

which for even functions is equivalent to:

$$L_*^{-1}\psi_{E_*}^{2p+1} = \frac{1}{2p\sigma}\psi_{E_*}.$$

We note that from (3.55) we have

$$\psi_{E_*}^{2p-1}\phi_*^2 = \psi_{E_*}^{2p-1}\left[\frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2} + \left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)\right] = \frac{\psi_{E_*}^{2p+1}}{\|\psi_{E_*}\|_{L^2}^2} + o(\frac{1}{s}),$$

hence

$$\langle \psi_{E_*}^{2p-1}\phi_*^2, L_*^{-1}\psi_{E_*}^{2p-1}\phi_*^2\rangle = \frac{1}{\|\psi_{E_*}\|_{L^2}^4}\langle \psi_{E_*}^{2p+1}, L_*^{-1}\psi_{E_*}^{2p+1}\rangle + \frac{2}{\|\psi_{E_*}\|_{L^2}^2}\langle L_*^{-1}\psi_{E_*}^{2p+1}, \psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)\rangle$$

$$+ \langle \psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right), L_*^{-1}\psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)\rangle$$

$$= \frac{1}{2p\sigma\|\phi_{E_*}\|_{L^2}^4}\int_R \phi_{E_*}^{2p+2}(x)dx + \frac{1}{p\sigma\|\phi_{E_*}\|_{L^2}^2}\int_R \psi_E^{2p}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)dx$$

$$+ \langle \psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right), L_*^{-1}\psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)\rangle.$$

Passing to the limit when $s \to \infty$ and using (3.55) we have:

$$\lim_{s\to\infty} a_*^{2-2p}\langle \psi_{E_*}^{2p-1}\phi_*^2, L_*^{-1}\psi_{E_*}^{2p-1}\phi_*^2\rangle = \frac{2^{-p}}{2p\sigma}\|\psi_0\|_{L^{2p+2}}^{2p+2}$$

$$+ \lim_{s\to\infty} a_*^{2-2p}\langle \psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right), L_*^{-1}\psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)\rangle.$$

Because the first two and only the first two eigenvalues of L_* approach zero as $s \to \infty$ we need to expand the quadratic form in the last limit in terms of the associated spectral projections. Since the quadratic form involves only even functions and the eigenfunction corresponding to the second eigenvalue of L_* is odd we only need to worry about the projection onto the eigenfunction ξ_* corresponding to the first eigenvalue $\lambda_0(E_*)$ of L_*. We have:

$$\lim_{s\to\infty} a_*^{2-2p}\langle \psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right), L_*^{-1}\psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right)\rangle = \lim_{s\to\infty} a_*^{2-2p}\frac{|\langle \psi_{E_*}^{2p-1}\left(\phi_*^2 - \frac{\psi_{E_*}^2}{\|\psi_{E_*}\|_{L^2}^2}\right), \xi_*\rangle|^2}{\lambda_0(E_*)}$$

The latter can be calculated via L'Hospital where, as in (3.56),

$$\lim_{s\to\infty} \lambda_0(E_*) = \lim_{\substack{s\to\infty \\ a\to 0}} \frac{d\lambda_0}{dE}(a,s) = -2p < 0$$

and the derivative of the denominator converges to zero. All in all we get (3.52).

Finally, to compute $\lim_{s\to\infty} R$ we use the definition (3.36). We have

$$N'(E_*) = 2\langle \frac{d\psi_E}{da}\left(\frac{dE}{da}\right)^{-1}, \psi_{E_*}\rangle = \frac{2a_* + O(|a_*|^{2p+1})}{-\sigma 2p\|\psi_{0,s}\|_{L^{2p+2}}^{2p+2}a_*^{2p-1} + O(|a_*|^{4p-1})},$$

where we used (3.48)-(3.49). Consequently

$$\lim_{s\to\infty} a_*^{2p-2}N'(E_*) = \frac{2^p}{-\sigma p\|\psi_0\|_{L^{2p+2}}^{2p+2}}$$

which combined with (3.36), (3.57) and (3.52) gives (3.53). $\qquad\square$

4 Behavior of the symmetric and asymmetric states for large E

In this section we show that if the branch of symmetric states (ψ_E, E) bifurcating from $(0, E_0)$ can be uniquely continued on the interval $E \in (E_0, \infty)$, i.e. case (a) in Theorem 2 holds, then, modulo re-scaling, this branch must bifurcate from a nontrivial, even solution of the constant–coefficient NLS equation:

$$- u_\infty''(x) + \sigma |u_\infty|^{2p} u_\infty(x) + u_\infty(x) = 0, \quad u_\infty \in H^1. \tag{4.1}$$

Since the above equation has exactly one such solution:

$$u_\infty(x) = \left(\frac{1+p}{-\sigma} \right)^{\frac{1}{2p}} \mathrm{sech}^{\frac{1}{p}}(px), \tag{4.2}$$

we infer essential properties of the branch (ψ_E, E) via bifurcation theory. In particular we show that when $V(x)$ has a non-degenerate local maximum at $x = 0$ then L_+ computed at (ψ_E, E) has two negative eigenvalues for E large, contradicting Remark 3. This finishes the proof of our main theorem.

However the arguments developed in this section tell much more about all solutions (ψ_E, E) of the stationary NLS equation (1.3) for large E. Certain scaling of the $\|\psi_E\|_{L^2}$, $\|\psi_E\|_{L^{2p+2}}$ and $\|\nabla \psi_E\|_{L^2}$ norms for large E emerges from Theorem 5. Combined with the concentration compactness arguments, these norms imply that the stationary solutions either bifurcate from solutions (4.2) translated to be centered at a critical point of $V(x)$, or centered at infinity, see Remarks 7 and 8. The stationary solutions bifurcating from a finite translation of (4.2) are localized near a critical point of $V(x)$, and if the latter is non-degenerate the orbital stability of these solutions can be determined, see Theorem 6 and Remark 6.

Theorem 5. *Let $\sigma < 0$, and consider a C^1 branch of stationary solutions (ψ_E, E) for $E \in (E_1, \infty)$. If $V(x)$ satisfies*

$$V(x) \in L^\infty(\mathbb{R}), \quad \lim_{|x| \to \infty} V(x) = 0, \quad x V'(x) \in L^\infty(\mathbb{R}), \tag{4.3}$$

then

(i) there exists $0 < b < \infty$ such that

$$\lim_{E \to \infty} \frac{\|\psi_E\|_{L^{2p+2}}^{2p+2}}{E^{\frac{1}{2} + \frac{1}{p}}} = b \tag{4.4}$$

$$\lim_{E \to \infty} \frac{\|\psi_E\|_{L^2}^2}{E^{\frac{1}{p} - \frac{1}{2}}} = \frac{-\sigma}{2} \frac{p+2}{p+1} b \tag{4.5}$$

$$\lim_{E \to \infty} \frac{\|\nabla \psi_E\|_{L^2}^2}{E^{\frac{1}{2} + \frac{1}{p}}} = \frac{-\sigma}{2} \frac{p}{p+1} b, \tag{4.6}$$

and, after the change of variables:

$$u_E(x) = R^{1/p} \psi_E(Rx), \quad R = \frac{1}{\sqrt{E + V(0)}}, \tag{4.7}$$

u_E satisfies:

$$\lim_{E \to \infty} \|u_E\|_{H^1} = -\sigma b > 0 \tag{4.8}$$

$$-u_E''(x) + R^2(V(Rx) - V(0))u_E(x) + u_E(x) + \sigma|u_E|^{2p}u_E(x) = 0. \tag{4.9}$$

(ii) if in addition $\psi_E(x)$ is even in x and L_+ computed at (ψ_E, E) has exactly one negative eigenvalue for all $E \in (E_1, \infty)$ then u_E defined above converges:

$$\lim_{E \to \infty} \|u_E - u_\infty\|_{H^2} = 0 \tag{4.10}$$

where $u_\infty \neq 0$ satisfies (4.1) and is given by (4.2).

Before we prove the theorem let us note that it implies that, for the case (a) of Theorem 2, the branch of symmetric states, under the re-scaling (4.7): $(\psi_E, E) \mapsto (u_E, R)$, bifurcates from the solution $(u_E = u_\infty, R = 0)$ of equation (4.9), where u_∞ is given by (4.2). This bifurcation can be analyzed in detail:

Theorem 6. *Consider $x_0 \in \mathbb{R}$, the equation:*

$$G(u, R) = -u'' + R^2(V(Rx + x_0) - V(x_0))u + u + \sigma|u|^{2p}u = 0, \tag{4.11}$$

and the solution $(u = u_\infty, R = 0)$, with u_∞ given by (4.2).

(i) If V is differentiable at x_0 but $V'(x_0) \neq 0$ then the set of solutions of $G(u, R) = 0$ in a neighborhood of $(u_\infty, 0) \in H^1 \times \mathbb{R}$ is given by $R = 0$ and translations of u_∞ :

$$\{(u_\infty(\cdot - s), 0) \; : \; s \in \mathbb{R}\}.$$

(ii) If V is twice differentiable at x_0 and x_0 is a non-degenerate critical point of V, i.e. $V'(x_0) = 0$, $V''(x_0) \neq 0$, then the set of solutions of $G(u, R) = 0$ in a neighborhood of $(u_\infty, 0) \in H^1 \times \mathbb{R}$ consists of two orthogonal C^1 curves:

$$\{(u_\infty(\cdot - s), 0) \; : \; s \in \mathbb{R}\} \quad and \quad \{(u_R, R) \; : \; R \in \mathbb{R}, \; |R| \; small\},$$

where

$$\|u_R\|_{L^2}^2 = \|u_\infty\|_{L^2}^2 + \left(\frac{1}{2p} - \frac{3}{4}\right) V''(x_0)\|xu_\infty(x)\|_{L^2}^2 R^4 + o(R^4).$$

Moreover, if x_0 is a local maximum (respectively local minimum) for V then L_+ computed at (u_R, R) has exactly two (respectively exactly one) negative eigenvalues.

We now outline a few remarks.

Remark 6. *The last two theorems combined with* Remark 3 *finish the proof of our main theorem. Moreover, for the generic potential (1.4) with $s > s_*$, which has exactly two non-degenerate local minima at $x = \pm x_0$ and one non-degenerate local maximum at $x = 0$, the above theorem gives two more branches of asymmetric states:*

$$E = R^{-2} - V(\pm x_0), \qquad \psi_E(x) = R^{-\frac{1}{p}} u_R\left(\frac{x \mp x_0}{R}\right) \tag{4.12}$$

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localized near the two minima. Both branches are orbitally stable for $p \leq 2$ and orbitally unstable for $p > 2$. This is because the operator $L_+(\psi_E, E)$ has exactly one negative eigenvalue, see part (ii) of Theorem 6, *and, according to the general theory in [9], the sign of $\partial_E \|\psi_E\|_{L^2}^2$ determines the orbital stability. Using (4.12) we have*

$$
\begin{aligned}
\partial_E \|\psi_E\|_{L^2}^2 &= \left(\frac{1}{p} - \frac{1}{2}\right)(E + V(\pm x_0))^{\frac{1}{p} - \frac{3}{2}}\|u_R\|_{L^2}^2 - \frac{1}{2}(E + V(\pm x_0))^{\frac{1}{p} - 2}\partial_R \|u_R\|_{L^2}^2 \\
&= E^{\frac{1}{p} - \frac{3}{2}}\left[\left(\frac{1}{p} - \frac{1}{2}\right)\|u_\infty\|_{L^2}^2 + O(E^{-\frac{1}{2}})\right]
\end{aligned}
$$

Hence $\partial_E \|\psi_E\|_{L^2}^2 > 0$, for $p < 2$, implying stability, $\partial_E \|\psi_E\|_{L^2}^2 < 0$, for $p > 2$, implying instability, while for $p = 2$ we have

$$
\|\psi_E\|_{L^2}^2 = \|u_R\|_{L^2}^2 = \|u_\infty\|_{L^2}^2 - \frac{1}{2}V''(\pm x_0)\|xu_\infty(x)\|_{L^2}^2 R^4 + o(R^4),
$$

i.e. $\|\psi_E\|_{L^2}^2$ is increasing with $E \sim R^{-2}$, implying stability.

Remark 7. *The proof of* Theorem 5 part (ii) *shows that in the absence of hypothesis ψ_E even, the re-scaled u_E may concentrate at $\pm\infty$, i.e.*

$$
\exists y_E \in \mathbb{R} \text{ such that } \lim_{E \to \infty} y_E = \pm\infty \text{ and } \lim_{E \to \infty} \|u_E(\cdot - y_E) - u_\infty\|_{H^2} = 0.
$$

This possibility prevents us to claim that for the double well potential (1.4) with $s > s_*$ the two branches of asymmetric states concentrated near its two minima, which exist for large E via Theorem 6, *are in fact the continuation of the two branches of asymmetric states emerging from the pitchfork bifurcation along the symmetric branch at $E = E_* < \infty$, via* Theorem 4. *Numerical simulations strongly support this claim but a proof of* Theorem 6 *which includes the case $x_0 = \pm\infty$ is needed for a rigorous resolution of the problem. Note that, for usual potentials V with $\lim_{|x| \to \infty} V(x) = 0$, $x_0 = \pm\infty$ is actually a degenerate critical point, i.e.*

$$
\lim_{x \to \pm\infty} \frac{d^k V}{dx^k}(x) = 0, \qquad k = 1, 2, 3, \ldots.
$$

Remark 8. *Moreover, in the absence of the spectral hypothesis that $L_+(\psi_E, E)$ has exactly one negative eigenvalue, the proof of* Theorem 5 part (ii) *shows that u_E may split in two or more functions, depending on the number of negative eigenvalues of L_+, which concentrate at points further and further apart, i.e there exist $u_E^1, u_E^2, \ldots, u_E^N \in H^2$, and $y_E^1, y_E^2, \ldots, y_E^N \in \mathbb{R}$ such that:*

$$
\lim_{E \to \infty} |y_E^j - y_E^k| = \infty \text{ for } j \neq k, \quad \lim_{E \to \infty} \|u_E^k(\cdot - y_E^k) - u_\infty\|_{H^2} = 0, \text{ for } k = 1, 2 \ldots N,
$$

and

$$
u_E = \sum_{k=1}^{n} u_E^k + v_E, \qquad \text{where } \lim_{E \to \infty} \|v_E\|_{L^q} = 0, \text{ for all } 2 \leq q \leq \infty.
$$

Such nonlocal bifurcations are excluded for ground states but are relevant for excited states of the stationary NLS equation (1.3), see [20] for partial results in this direction.

We now proceed with the proofs of the above theorems.

Proof of Theorem 5.   Recall from (2.9):

$$\mathcal{E}(E) = \int_{\mathbb{R}} |\nabla \psi_E(x)|^2 dx + \int_{\mathbb{R}} V(x)|\psi_E(x)|^2 dx + \frac{\sigma}{p+1} \int_{\mathbb{R}} |\psi_E(x)|^{2p+2} dx,$$

and from (2.10)

$$\frac{d\mathcal{E}}{dE} = -E \frac{dN}{dE},$$

where $N(E) = \|\psi_E\|_{L^2}^2$. Also, from (2.11):

$$\|\nabla \psi_E\|_{L^2}^2 + \int_{\mathbb{R}} V(x)|\psi_E(x)|^2 dx + \sigma \|\psi_E\|_{L^{2p+2}}^{2p+2} + E\|\psi_E\|_{L^2}^2 = 0$$

which can be rewritten:

$$\mathcal{E}(E) + \frac{\sigma p}{p+1} \|\psi_E\|_{L^{2p+2}}^{2p+2} = -EN(E).$$

Differentiating the latter with respect to E we get:

$$\frac{d}{dE} \|\psi_E\|_{L^{2p+2}}^{2p+2} = \frac{p+1}{-\sigma p} N. \tag{4.13}$$

Now, from ψ_E a weak solution of (1.3) using $x\psi_E'(x)$ as a test function and integrating by parts we get the Pohozaev type identity:

$$-\|\nabla \psi_E\|_{L^2}^2 + \int_{\mathbb{R}} (V(x) + xV'(x))|\psi_E(x)|^2 dx + \frac{\sigma}{p+1} \|\psi_E\|_{L^{2p+2}}^{2p+2} = -E\|\psi_E\|_{L^2}^2 \tag{4.14}$$

which added to (2.11) implies:

$$2EN + \int_{\mathbb{R}} (2V(x) + xV'(x))|\psi_E(x)|^2 dx = -\sigma \frac{p+2}{p+1} \|\psi_E\|_{L^{2p+2}}^{2p+2}.$$

The integral term in the identity above can be bounded via Hölder inequality:

$$-[2\|V\|_{L^\infty} + \|x \cdot \nabla V\|_{L^\infty}] N \leq \int_{\mathbb{R}} (2V(x) + xV'(x))|\psi_E(x)|^2 dx \leq [2\|V\|_{L^\infty} + \|xV'\|_{L^\infty}] N,$$

hence, for $C = 2\|V\|_{L}^\infty + \|xV'\|_{L^\infty}$, we have:

$$2(E - C)N(E) \leq -\sigma \frac{p+2}{p+1} \|\psi_E\|_{L^{2p+2}}^{2p+2} \leq 2(E + C)N(E). \tag{4.15}$$

Plugging (4.15) in (4.13) and using the notation $Q(E) = \|\psi_E\|_{L^{2p+2}}^{2p+2}$ we get:

$$\frac{p+2}{2p} \frac{Q(E)}{E + C} \leq \frac{dQ}{dE}(E) \leq \frac{p+2}{2p} \frac{Q(E)}{E - C}.$$

Fix $E_2 > \max\{C, E_1\}$ and integrate on $[E_2, E]$:

$$\frac{Q(E_2)}{(E_2 + C)^{1/2+1/p}} (E + C)^{1/2+1/p} \leq Q(E) \leq \frac{Q(E_2)}{(E_2 - C)^{1/2+1/p}} (E - C)^{1/2+1/p},$$

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hence:

$$\frac{Q(E_2)}{(E_2 + C)^{1/2+1/p}} \leq \frac{\|\psi_E\|_{L^{2p+2}}^{2p+2}}{E^{1/2+1/p}} \leq \frac{Q(E_2)}{(E_2 - C)^{1/2+1/p}},$$

which implies (4.4) since

$$\lim_{E_2 \to \infty} \frac{\frac{Q(E_2)}{(E_2-C)^{1/2+1/p}}}{\frac{Q(E_2)}{(E_2+C)^{1/2+1/p}}} = 1.$$

Now, (4.5) follows from dividing (4.15) by $E^{1/2+1/p}$ and passing to the limit $E \to \infty$, while (4.6) follows from dividing (2.11) by $E^{1/2+1/p}$ and passing to the limit $E \to \infty$.

Note that (4.7) and the fact that ψ_E solves (1.3) already implies (4.9). Moreover (4.7) combined with (4.4)-(4.6) shows that:

$$\|u_E\|_{L^{2p+2}}^{2p+2} = R^{1+2/p}\|\psi_E\|_{L^{2p+2}}^{2p+2} \overset{E\to\infty}{\to} b$$

$$\|u_E\|_{L^2}^2 = R^{2/p-1}\|\psi_E\|_{L^2}^2 \overset{E\to\infty}{\to} -\frac{\sigma}{2}\frac{p+2}{p+1}b$$

$$\|\nabla u_E\|_{L^2}^2 = R^{2/p+1}\|\nabla\psi_E\|_{L^2}^2 \overset{E\to\infty}{\to} -\frac{\sigma}{2}\frac{p}{p+1}b.$$

Adding the last two we get (4.8). Part (i) is now completely proven.

For part (ii) we use concentration compactness:

1. -*Vanishing*, i.e. $\lim_{E\to\infty}\|u_E\|_{L^q} = 0$, $2 < q \leq \infty$, cannot happen. Indeed, assuming the contrary, from (4.9) we get:

$$u_E = (-\Delta + 1)^{-1}[-R^2(V(Rx) - V(0))u_E - \sigma|u_E|^{2p}u_E]. \tag{4.16}$$

Hence, using that $(-\Delta + 1)^{-1} : L^2 \mapsto H^2$ is unitary, we have:

$$\|u_E\|_{H^2} \leq R^2\|V(Rx) - V(0)\|_{L^\infty}\|u_E\|_{L^2} + |\sigma|\|u_E\|_{L^{4p+2}}^{2p+1}.$$

Since the right hand side converges to zero we get $\|u_E\|_{H^1} \leq \|u_E\|_{H^2} \overset{E\to\infty}{\to} 0$ which contradicts (4.8).

2. -*Splitting* cannot happen. The argument is a slight adaptation of the one we used to exclude splitting in the proof of Theorem 3 part (ii) and relies on the hypothesis that $L_+(\psi_E, E)$ has exactly one negative eigenvalue for all $E \in (E_1, \infty)$.

3. -*Compactness* is the only case possible and implies that for any sequence $E_n \to \infty$ there exists a subsequence E_{n_k} and $y_k \in \mathbb{R}$, $\tilde{u}_\infty \in H^1$ such that:

$$\lim_{k\to\infty}\|u_{E_k}(\cdot - y_k) - \tilde{u}_\infty\|_{H^1} = 0.$$

As in the compactness part of the proof of Theorem 3 the symmetry of u_{E_k} implies that $y_k \in \mathbb{R}$ must be a bounded sequence. By possibly choosing a subsequence we have $\lim_{k\to\infty} y_k = y_\infty \in \mathbb{R}$, hence

$$\lim_{k\to\infty}\|u_{E_k} - \tilde{u}_\infty(\cdot + y_\infty)\|_{H^1} = 0.$$

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By plugging u_{E_k} in (4.16) and passing to the limit $k \to \infty$ we infer that $u_\infty(\cdot + y_\infty)$ is an even H^1 solution of (4.1), hence $u_\infty(\cdot + y_\infty) = u_\infty$ given by (4.2).

We have just showed that for each sequence $E_n \to \infty$ there exists a subsequence E_{n_k} such that

$$\lim_{k \to \infty} \|u_{E_k} - u_\infty\|_{H^1} = 0.$$

Since the limit is unique we have:

$$\lim_{E \to \infty} \|u_E - u_\infty\|_{H^1} = 0$$

and (4.16) now shows that the convergence is in H^2.

Theorem 5 is now completely proven. $\qquad\square$

Proof of Theorem 6: Since the Frechet derivative with respect to u of $G(u, R)$:

$$D_u G(u, R)[v] = L_+(u, R)[v] = -v'' + R^2(V(Rx + x_0) - V(x_0))v + v + (2p + 1)\sigma |u|^{2p} v$$

at $u = u_\infty$, $R = 0$ has kernel spanned by u'_∞ we could use the standard Lyapunov-Schmidt decomposition with respect to $L_+(u_\infty, 0)$ to reduce (4.11) to finding the zeroes of a map from $\mathbb{R}^2$ to $\mathbb{R}$. However the latter will have zero gradient and Hessian at $(0,0)$ because $G(0, R) = O(R^3)$ in case (i) and $G(0, R) = O(R^4)$ in case (ii). A generalization of the Morse Lemma and calculations of derivatives up to order four will be required to fully analyze the reduced problem. In particular $V(x)$ will need to be C^4 or in $W^{4,\infty}(\mathbb{R})$ to be able to carry on the analysis. We avoid this unnecessary complication by using a decomposition similar to the one in [7]:

Lemma 5. *There exists $\epsilon, \delta > 0$ such that for any $u \in L^2$ with $\|u - u_\infty\|_{L^2} < \epsilon$ there exists a unique $s \in \mathbb{R}$, $|s| < \delta$, with the property that:*

$$u = u_\infty(\cdot - s) + v, \qquad v \perp u'_\infty(\cdot - s).$$

Moreover the map $u \mapsto s$ is C^2 from L^2 to $\mathbb{R}$ and there exists $C > 0$ such that

$$|s| \leq C \|u - u_\infty\|_{L^2}.$$

The Lemma follows directly from applying the implicit function theorem to the problem of finding the zeroes of the C^2 map $F : L^2 \times \mathbb{R} \mapsto \mathbb{R}$ given by

$$F(u, s) = \langle u'_\infty(\cdot - s), u - u_\infty(\cdot - s) \rangle$$

in a neighborhood of $(u_\infty, 0)$, because:

$$\frac{\partial F}{\partial s}(u_\infty, 0) = -\langle u''_\infty, u_\infty - u_\infty \rangle + \langle u'_\infty, u'_\infty \rangle = \|u'_\infty\|_{L^2}^2 \neq 0.$$

Returning now to the equation (4.11) we note that $(u_\infty(\cdot - s), 0)$, $s \in \mathbb{R}$ is the set of all solutions when $R = 0$. To find other solutions in a $H^1 \times \mathbb{R}$ neighborhood of $(u_\infty, 0)$ we decompose them according to the above lemma:

$$u(x) = u_\infty(x - s) + v(x), \qquad v \perp u'_\infty(\cdot - s) \tag{4.17}$$

and rewrite (4.11) in the equivalent form:

$$\begin{aligned}
G_\perp(v, R, s) &= -v'' + P_\perp(s)R^2[V(Rx + x_0) - V(x_0)][u_\infty(x - s) + v] + v \\
&\quad + (2p+1)\sigma P_\perp(s)|u_\infty(x - s)|^{2p}v + P_\perp(s)N(s, v) = 0 \quad\quad (4.18)\\
G_\parallel(v, R, s) &= \langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)][u_\infty(x - s) + v] + N(s, v)\rangle = 0, \quad (4.19)
\end{aligned}$$

where $P_\perp(s)$ is the projection onto the orthogonal complement of $u'_\infty(\cdot - s)$ in L^2, and

$$N(s, v) = \sigma|u_\infty(\cdot - s) + v|^{2p}(|u_\infty(\cdot - s) + v) - \sigma|u_\infty(\cdot - s)|^{2p}u_\infty(\cdot - s) - (2p+1)\sigma|u_\infty(\cdot - s)|^{2p}v. \quad (4.20)$$

Note that, for all $x \in \mathbb{R}$ and some $0 \leq t \leq 1$, we have:

$$\begin{aligned}
|N(s, v_1)(x) - N(s, v_2)(x)| &\leq (2p+1)2p|\sigma|\left(|u_\infty(x-s)| + \max\{|v_1(x)|, |v_2(x)|\}\right)^{2p-1} \\
&\quad \times |tv_1(x) + (1-t)v_2(x)|\,|v_1(x) - v_2(x)|,
\end{aligned}$$

hence

$$\begin{aligned}
\|N(s, v_1) - N(s, v_2)\|_{L^2} &\leq (2p+1)2p|\sigma|\left(\|u_\infty\|_{L^\infty} + \max\{\|v_1\|_{L^\infty}, \|v_2\|_{L^\infty}\}\right)^{2p-1} \\
&\quad \times \max\{\|v_1\|_{L^\infty}, \|v_2\|_{L^\infty}\}\|v_1 - v_2\|_{L^2}
\end{aligned}$$

in particular, if we assume $\|v_{1,2}\|_{H^2} \leq 1$ then we can find a constant $C_N > 0$ such that:

$$\|N(s, v_1) - N(s, v_2)\|_{L^2} \leq C_N \max\{\|v_1\|_{H^2}, \|v_2\|_{H^2}\}\|v_1 - v_2\|_{H^2} \quad\quad (4.21)$$

and

$$\|N(s, v_1)\|_{L^2} \leq C_N\|v_1\|_{H^2}^2. \quad\quad (4.22)$$

Using now the notation $L_\infty(s)[v] = L_+(u_\infty(\cdot - s), 0)[v] = -v'' + v + (2p+1)\sigma|u_\infty(\cdot - s)|^{2p}v$ we can rewrite (4.18) in the fixed point form:

$$\begin{aligned}
v &= -L_\infty^{-1}(s)P_\perp(s)R^2[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s) \\
&\quad - L_\infty^{-1}(s)P_\perp(s)\left[R^2(V(Rx + x_0) - V(x_0))v + N(s, v)\right], \quad\quad (4.23)\\
&= v_0(s, R) + K_{s,R}(v)
\end{aligned}$$

where:

- $L_\infty^{-1}(s) : L^2 \cap \{u'_\infty(\cdot - s)\}^\perp \mapsto H^2 \cap \{u'_\infty(\cdot - s)\}^\perp$ is linear, bounded, with bound independent on $s \in \mathbb{R}$;

- $P_\perp(s)R^2[V(Rx + x_0) - V(x_0)] : H^2 \mapsto L^2 \cap \{u'_\infty(\cdot - s)\}^\perp$ is linear and bounded by $2R^2\|V\|_{L^\infty}$ uniformly for $s \in \mathbb{R}$;

- $P_\perp(s)N(s, \cdot) : H^2 \mapsto L^2 \cap \{u'_\infty(\cdot - s)\}^\perp$ is locally Lipschitz with Lipschitz constant independent on $s \in \mathbb{R}$, $Lip(r) \leq C_N r$ in the ball of radius $r \leq 1$ centered at origin in H^2, see (4.21).

Contraction principle can now be applied to (4.23) in the ball: $B(0, r) = \{v \in H^2 : \|v\|_{H^2} \leq r\}$ provided $\|v_0(s, R)\|_{H^2} \leq r/2$ and $K_{s,R}$ is a contraction on $B(0, r)$ with Lipschitz constant $Lip \leq 1/2$. Based on the above estimates it suffices to require $r \leq 1$ and:

$$2R^2\|L_\infty^{-1}\|_{L^2 \mapsto H^2}\|V\|_{L^\infty}\|u_\infty\|_{H^2} \leq \frac{r}{2}$$

$$\|L_\infty^{-1}\|_{L^2 \mapsto H^2}(2R^2\|V\|_{L^\infty} + C_N r) \leq \frac{1}{2}$$

which can be accomplished by choosing:

$$r = \min\{1, (2\|L_\infty^{-1}\|_{L^2 \mapsto H^2}C_N + 1/\|u_\infty\|_{H^2})^{-1}\}, \qquad R \leq \sqrt{r(4\|L_\infty^{-1}\|_{L^2 \mapsto H^2}\|V\|_{L^\infty}\|u_\infty\|_{H^2})^{-1}} = R_0.$$

Hence (4.23), and consequently (4.18), has a unique solution $v = v(s, R)$ in $B(0, r)$ for each $s \in \mathbb{R}$ and $0 \leq R \leq R_0$. This solution depends C^1 on s and R since (4.23) is C^1 in these parameters. $v(s, R)$ can be obtained by successive approximations:

$$v_0(s, R), \; v_1(s, R) = v_0(s, R) + K_{s,R}(v_0(s, R)), \; v_2(s, R) = v_0(s, R) + K_{s,R}(v_1(s, R)), \ldots \xrightarrow{H^2} v(s, R)$$

Moreover, from the contraction principle we have:

$$\|v(s, R) - v_1(s, R)\|_{H^2} \leq \frac{Lip}{1 - Lip}\|v_1 - v_0\|_{H^2} \leq \|L_\infty^{-1}\|_{L^2 \mapsto H^2}(2R^2\|V\|_{L^\infty} + C_N\|v_0\|_{H^2})\|v_0\|_{H^2}.$$

In what follows we will show that for $|s| \leq 1$ we have:

$$\|v_0(s, R)\|_{H^2} = \begin{cases} O(R^3) & \text{if } V'(x_0) \neq 0 \\ O(R^4) & \text{if } V'(x_0) = 0 \end{cases} \tag{4.24}$$

where in the second case we assume V is twice differentiable at x_0. Consequently:

$$v(s, R) = v_0(s, R) + \begin{cases} O(R^5) & \text{if } V'(x_0) \neq 0 \\ O(R^6) & \text{if } V'(x_0) = 0 \end{cases} \tag{4.25}$$

For (4.24) we use the definition (4.23) of v_0 :

$$\|v_0\|_{H^2} \leq R^2\|L_\infty^{-1}\|_{L^2 \mapsto H^2}\|[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s)\|_{L^2},$$

and

$$\|[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s)\|_{L^2}^2 = \int_\mathbb{R}[V(Rx + x_0) - V(x_0)]^2 u_\infty^2(x - s)dx$$

$$= \int_\mathbb{R} u_\infty^2(x)[V(Rx + Rs + x_0) - V(x_0)]^2 dx.$$

Since $V(x)$ is differentiable at $x = x_0$ and bounded for almost all x, there exists $C_1 > 0$ such that almost everywhere: $|V(x) - V(x_0)| \leq C_1|x - x_0|$. Moreover if $V(x)$ is twice differentiable at $x = x_0$ with $V'(x_0) = 0$ then there exists $C_2 > 0$ such that almost everywhere: $|V(x) - V(x_0)| \leq C_2|x - x_0|^2$. Plugging in above and using $\int_\mathbb{R}|x|^n u_\infty^2(x)dx < \infty$, $n = 0, 1, 2\ldots$ due to the well known exponential decay of u_∞, we get (4.24).

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Since (4.11) is equivalent with the system (4.18)-(4.19), and (4.18) has the unique solution $v(s, R)$ in a neighborhood of zero for all $s \in \mathbb{R}$, $0 \le R \le R_0$, all solutions of (4.11) in a small neighborhood of $(u_\infty, R = 0)$ will be given by the solutions of (4.19) with $v = v(s, R)$ and (s, R) in a small neighborhood of $(0, 0)$:

$$
\begin{aligned}
G_\|(s, R) &= G_\|(v(s, R), R, s) = \langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s)\rangle \quad (4.26) \\
&+ \langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]v(s, R) + N(s, v(s, R))\rangle = 0.
\end{aligned}
$$

Note that by the estimate (4.24)-(4.25) we have, for $V'(x_0) \ne 0$ and $|s| \le 1$:

$$
\langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]v(s, R) + N(s, v(s, R))\rangle = O(R^6)
$$

while

$$
\begin{aligned}
\langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s)\rangle &= R^2 \int_\mathbb{R} u'_\infty(x)u_\infty(x)[V(Rx + Rs + x_0) - V(x_0)]dx \\
&= -\frac{R^3}{2} \int_\mathbb{R} u^2_\infty(x)V'(Rx + Rs + x_0)dx \\
&= -\frac{V'(x_0)\|u_\infty\|^2_{L^2}}{2}R^3 + o(R^3)
\end{aligned}
$$

where in the last step we used $\lim_{R \to 0} V'(Rx + Rs + x_0) = V'(x_0)$ and Lebesque dominated convergence theorem. Hence, for $V'(x_0) \ne 0$ and $|s| \le 1$:

$$
0 = G_\|(s, R) = -\frac{V'(x_0)\|u_\infty\|^2_{L^2}}{2}R^3 + o(R^3)
$$

which has only the $R = 0$, $s \in \mathbb{R}$ in a small neighborhood of $(s, R) = (0, 0)$. However, for $V'(x_0) = 0$ and $|s| \le 1$, we have

$$
\begin{aligned}
\langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s)\rangle &= -\frac{R^3}{2} \int_\mathbb{R} u^2_\infty(x)V'(Rx + Rs + x_0)dx \\
&= -\frac{V''(x_0)}{2}R^4 \int_\mathbb{R} u^2_\infty(x)(x + s)dx + o(R^4) \\
&= -R^4\frac{V''(x_0)}{2}\|u_\infty\|^2_{L^2}s + o(R^4),
\end{aligned}
$$

hence

$$
\tilde{G}_\|(s, R) = R^{-4}G_\|(s, R) = -\frac{V''(x_0)}{2}\|u_\infty\|^2_{L^2}s + o(1) = 0
$$

for which the implicit function theorem can be applied at $(s, R) = (0, 0)$ where $\partial_s\tilde{G}_\|(0, 0) = -\frac{V''(x_0)}{2}\|u_\infty\|^2_{L^2} \ne 0$. Note that the $o(1)$ term is differentiable with respect to s because $v(s, R)$ is and it remains $o(1)$ in R.

The estimate

$$
\|u_R\|^2_{L^2} = \|u_\infty\|^2_{L^2} + \left(\frac{1}{2p} - \frac{3}{4}\right)V''(x_0)R^4\|xu_\infty(x)\|^2_{L^2} + o(R^4).
$$

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can be obtained from $u_R = u_\infty(\cdot - s(R)) + v(s(R), R)$, see (4.17), which implies:

$$\|u_R\|_{L^2}^2 = \|u_\infty\|_{L^2}^2 + 2\langle u_\infty(\cdot - s), v\rangle + \|v\|_{L^2}^2 = \|u_\infty\|_{L^2}^2 + 2\langle u_\infty(\cdot - s), v_0\rangle + O(R^6)$$

From (4.23) we have:

$$
\begin{aligned}
\langle u_\infty(\cdot - s), v_0\rangle &= -\langle L_\infty^{-1} u_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)] u_\infty(\cdot - s)\rangle \\
&= -\frac{1}{2} R^4 V''(x_0) \int_{\mathbb{R}} (x + s)^2 u_\infty(x) L_\infty^{-1}[u_\infty] dx + o(R^4),
\end{aligned}
$$

and the integral term can be computed using $-L_\infty^{-1}[u_\infty] = \frac{1}{2p} u_\infty + \frac{1}{2} x u_\infty'$.

The main part of the theorem is now finished. It remains to prove the spectral properties of the linear operator:

$$L_+(R)[\phi] = D_u G(u(R), R)[\phi] = -\phi'' + R^2(V(Rx + x_0) - V(x_0))\phi + \phi + (2p + 1)\sigma|u(R)|^{2p}\phi,$$

where $u(R) = u_\infty(\cdot - s(R)) + v(s, R)$. It is well known that:

$$L_+(0)[\phi] = L_\infty(0)[\phi] = -\phi'' + \phi + (2p + 1)\sigma|u_\infty|^{2p}\phi,$$

has exactly one, strictly negative eigenvalue which is simple and zero is the next eigenvalue which is also simple with corresponding eigenvector u_∞'. Since $u(R)$ depends continuously on R, $L_+(R)$ is continuous with respect to R in the resolvent sense, hence the isolated eigenvalues and eigenvectors depend continuously on R. To establish the sign of the second eigenvalue we use the following expansion for its eigenvector:

$$\phi(R) = (1 + a(R))u_\infty'(\cdot - s(R)) + \Phi(R), \qquad \Phi(R) \perp u_\infty'(\cdot - s(R)),$$

and the eigenvalue equation:

$$L_+(R)[\phi(R)] = \lambda(R)\phi(R)$$

which is equivalent to:

$$
\begin{aligned}
\lambda(R)\phi(R) = {}& L_\infty(s)\phi(R) \\
& + \underbrace{\left[R^2(V(Rx + x_0) - V(x_0)) + (2p + 1)2p\sigma|u_\infty(\cdot - s)|^{2p-1} v(s, R)\right]}_{V(s,R)=O(R^4)} \phi(R) + DN(R)\phi(R)
\end{aligned}
$$

where

$$
\begin{aligned}
DN(R) &= (2p + 1)\sigma \left(|u_\infty(\cdot - s) + v(s, R)|^{2p} - |u_\infty(\cdot - s)|^{2p} - 2p|u_\infty(\cdot - s)|^{2p-1} v(s, R)\right) \\
&= \begin{cases} v(s, R) & \text{if } p = 1/2 \\ O(R^{8p}) & \text{if } 1/2 < p < 1 \\ O(R^8) & \text{if } p \geq 1. \end{cases}
\end{aligned}
$$

We project the eigenvalue equation onto $u_\infty'(\cdot - s(R))$ and its orthogonal complement in L^2 to obtain the equivalent system of two equations:

$$
\begin{aligned}
\left[L_\infty(s) - \lambda(R) + P_\perp(s)V(s, R) + P_\perp(s)DN(R)\right]\Phi(R) &= -P_\perp(s)\left[V(s, R) + DN(R)\right](1 + a(R))u_\infty(\cdot - s) \\
\lambda(R)(1 + a(R))\|u_\infty'\|_{L^2}^2 &= \langle u_\infty'(\cdot - s(R)), V(s, R)\phi(R)\rangle + o(R^4)
\end{aligned}
$$

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As before, from the first equation we deduce $\Phi(R) = O(R^4)$, while replacing now in the second one $\phi(R) = (1 + o(1))u'_\infty(\cdot - s(R)) + O(R^4)$, we get:

$$\lambda(R) = \|u'_\infty\|_{L^2}^{-2}\langle u'_\infty(\cdot - s), V(s, R)u'_\infty(\cdot - s)\rangle + o(R^4).$$

To calculate

$$
\begin{aligned}
\langle u'_\infty(\cdot - s), V(s, R)u'_\infty(\cdot - s)\rangle &= \langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]u'_\infty(\cdot - s)\rangle \qquad (4.27)\\
&+ \langle u'_\infty(\cdot - s), (2p + 1)2p\sigma|u_\infty(\cdot - s)|^{2p-1}v(s, R)u'_\infty(\cdot - s)\rangle
\end{aligned}
$$

we use $v(s, R) = v_0(s, R) + O(R^6)$ and the equation satisfied by v_0, see (4.23):

$$L_\infty(s)v_0 = -P_\perp(s)R^2[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s).$$

Taking its space derivative:

$$
\begin{aligned}
L_\infty(s)v'_0 &+ (2p + 1)2p\sigma|u_\infty(\cdot - s)|^{2p-1}u'_\infty(\cdot - s)v_0\\
&= -R^3V'(Rx + x_0)u_\infty(\cdot - s) - R^2[V(Rx + x_0) - V(x_0)]u'_\infty(\cdot - s)\\
&+ \langle u'_\infty(\cdot - s), R^2[V(Rx + x_0) - V(x_0)]u_\infty(\cdot - s)\rangle u''_\infty(\cdot - s)
\end{aligned}
$$

then its scalar product with $u'_\infty(\cdot - s)$ we get

$$
\begin{aligned}
\langle u'_\infty(\cdot - s), (2p + 1)2p\sigma|u_\infty(\cdot - s)|^{2p-1}&v_0(s, R)u'_\infty(\cdot - s)\rangle\\
= -R^3\langle u'_\infty(\cdot - s), V'(Rx + x_0)u_\infty(\cdot - s)\rangle &- R^2\langle u'_\infty(\cdot - s), V(Rx + x_0) - V(x_0)]u'_\infty(\cdot - s)\rangle
\end{aligned}
$$

which plugged into (4.27) leads to:

$$
\begin{aligned}
\langle u'_\infty(\cdot - s), V(s, R)u'_\infty(\cdot - s)\rangle &= -R^3\langle u'_\infty(\cdot - s), V'(Rx + x_0)u_\infty(\cdot - s)\rangle + O(R^6)\\
&= \frac{R^4}{2}V''(x_0)\|u_\infty\|_{L^2}^2 + o(R^4)
\end{aligned}
$$

All in all we have:

$$\lambda(R) = \frac{1}{2}V''(x_0)R^4 + o(R^4),$$

which shows that the second eigenvalue of $L_+(R)$ becomes negative (respectively positive) if x_0 is a local maxima (respectively local minima) for the potential $V(x)$. The theorem is now completely proven. $\qquad\square$

5   Numerical results

To illustrate the results and further investigate the behavior of the ground state branches we have performed a series of numerical computations on the equation

$$-\frac{1}{2}\psi''(x) + \frac{1}{2}V(x)\psi(x) - |\psi(x)|^{2p}\psi(x) + E\psi(x) = 0,$$

which is equivalent with (1.3) with $\sigma = -2$ if one regards the eigenvalue parameter E in this section as being half the parameter used in the previous sections. The potential $V \equiv V_s$ is given by (1.4).

When $s < s_* \approx 0.6585$, the potential is a single well but it becomes a double well for $s > s_*$. We also recall that $p = p_* \approx 3.3028$ is the theoretical threshold power for the nonlinearity that separates different pitchfork bifurcations in the regime $s \to \infty$, see Corollary 2. The principal conclusions of our investigations for $p = 1$, $p = 3$, and $p = 5$ can be summarized as follows:

1. When the potential is a single well (that is $s < s_*$), the symmetric ground state exists for all $E > E_0$ and the operator L_+ at this states has a single negative eigenvalue for all $E > E_0$. There are no bifurcations along this branch which is consistent with the result in [11].

2. When the potential is a double well (that is $s > s_*$), the symmetric ground state exists for all $E > E_0$ and the second negative eigenvalue of L_+ along the branch of symmetric states emerges for $E > E_*$, where E_* depends on s and $E_* > E_0$. The asymmetric states bifurcate at $E = E_*$ and exist for all $E > E_*$. The second eigenvalue of L_+ along the branch of asymmetric states is positive for all $E > E_*$. The numerical results are in agreement with Theorem 1. Furthermore they show that there are no other bifurcations along these branches past E_*, that, as $E \to \infty$, one branch of asymmetric states localizes in the left well while the other localizes in the right well, and, modulo re-scaling, they both converge to the NLS soliton localized in the left, respectively right, minima of the potential, see Remark 6.

3. When $p < p_*$, the pitchfork bifurcation is supercritical and the branch of asymmetric states has bigger L^2 norm than the one for the symmetric state at $E = E_*$. When $p > p_*$, the pitchfork bifurcation is subcritical and the branch of asymmetric states has smaller L^2 norm for $E \gtrsim E_*$ than the one for the symmetric state at $E = E_*$. The numerical results are consistent with Corollary 2 but also suggest that the separation between wells does not have to be large, i.e., as soon as the potential has two wells, the supercritical/subcritical character of the bifurcation is controlled by the nonlinearity.

The conclusions are showcased in the following five figures. Figure 1 illustrates the cases of $s = 0.6$ and $s = 0.7$, for $p = 1$, that straddle the critical point $s_* \approx 0.6585$. The top panel presents the dependence of the squared L^2 norm of the symmetric state on E, while the bottom panel presents the second eigenvalue of the operator L_+ as a function of E. It is clear that for $s < s_*$, the second eigenvalue of L_+ tends asymptotically to 0, without ever crossing over to negative values (solid line), while for $s > s_*$, such a crossing exists (dashed line), occurring for $E_* \approx 10.68$. On the other hand, to examine whether a secondary crossing may exist for larger values of E, we have continued the $s = 0.7$ branch to considerably higher values of E in the bottom right panel of the figure, observing the eventual convergence of the eigenvalue to $\lambda = 0$, without any trace of a secondary crossing to positive values.

Figure 2 further clarifies the bifurcation structure of the asymmetric states for $s = 0.7 > s_*$, $p = 1$. As a relevant diagnostic, we monitor the location of the center of mass of the solution

$$x_{CM} = \frac{\int_{\mathbb{R}} x|\psi_E|^2 dx}{\int_{\mathbb{R}} |\psi_E|^2 dx}.$$

We can clearly see from the top left panel that beyond the critical threshold of $E_* \approx 10.68$, two asymmetric states with $x_{CM} \neq 0$, corresponding to $a < 0$ and $a > 0$ in Theorem 1, bifurcate out of the symmetric state with $x_{CM} = 0$, with the latter becoming unstable as per the crossing of
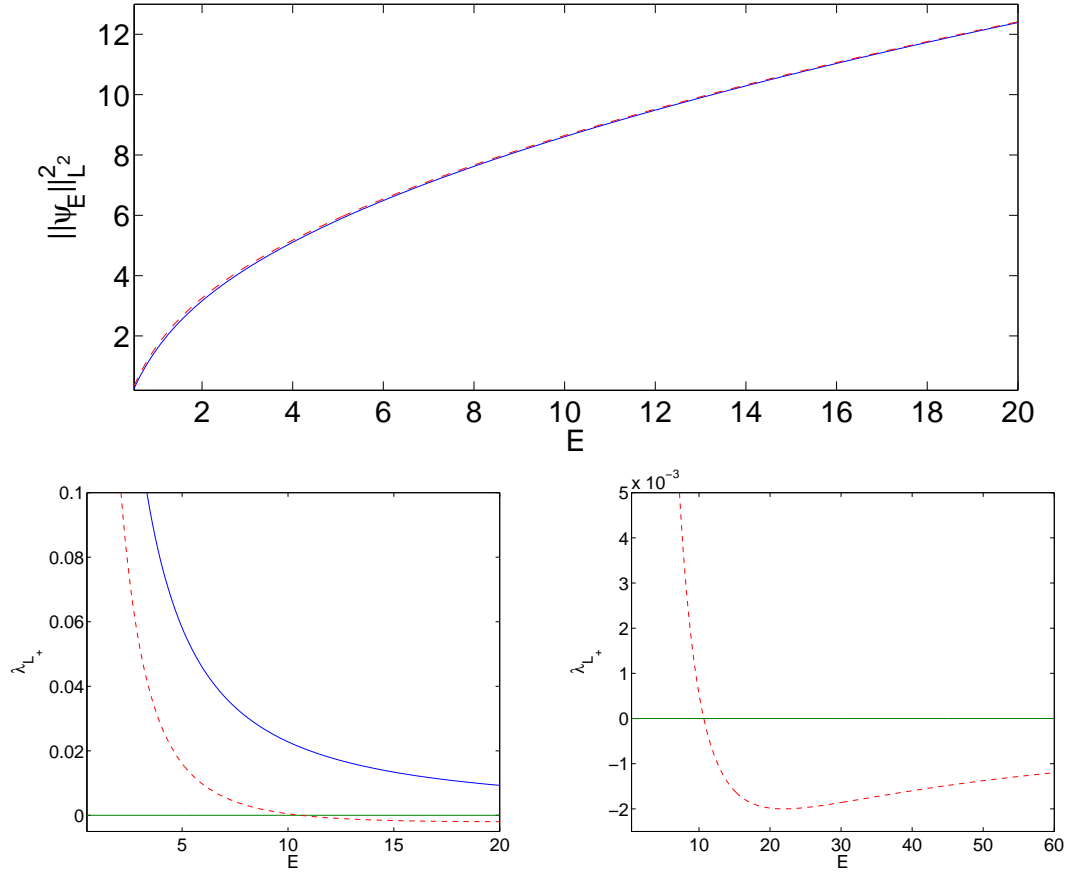
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Figure 1: The top panel shows the dependence of the squared L^2 norm of the symmetric state on the parameter E for $p = 1$. The bottom left panel shows the trajectory of the second eigenvalue of L_+ for the cases of $s = 0.6 < s_*$ (blue solid line) and $s = 0.7 > s_*$ (red dashed line). The bottom right panel shows an expanded plot of the second case for considerably larger values of E.
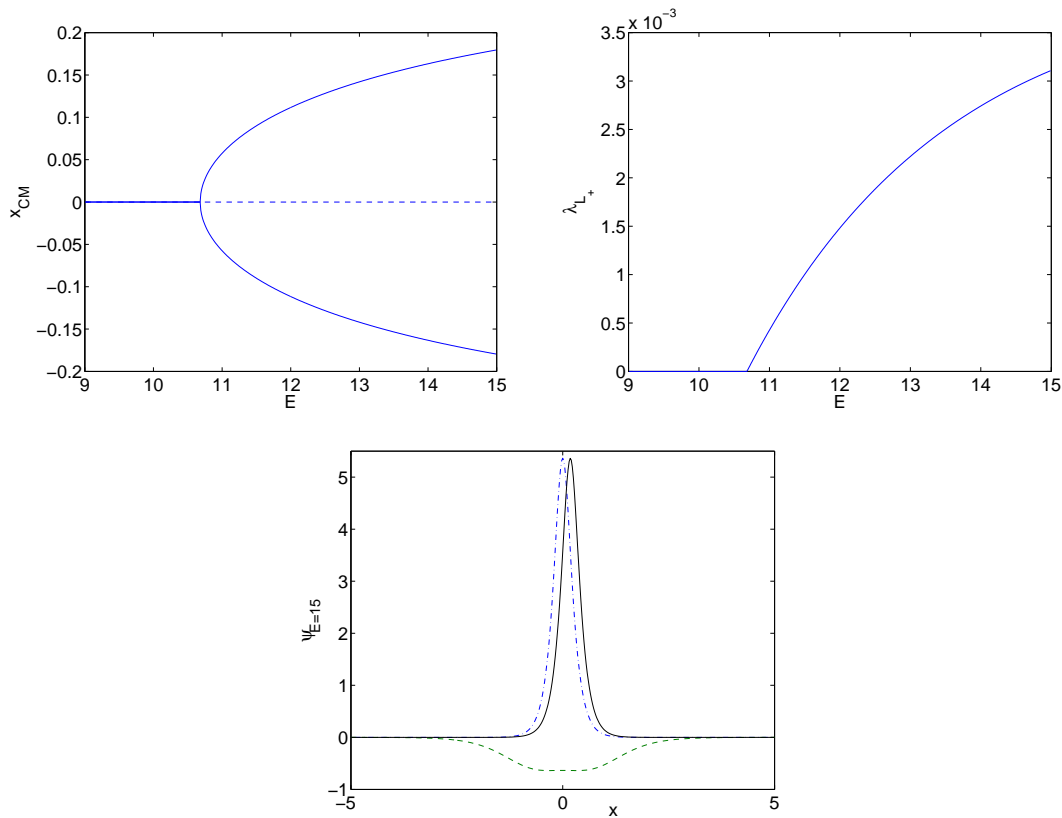
Figure 2: The top left panel shows the pitchfork bifurcation of asymmetric states for $s = 0.7 > s_*$ and $p = 1$. Two asymmetric ($x_{CM} \neq 0$) states emerge for $E > E_* \approx 10.68$. The second eigenvalue of L_+ for such asymmetric states is positive as shown in the top right panel. The bottom panel shows symmetric and asymmetric states versus x for $E = 15$.

the second eigenvalue of L_+ to the negative values. For the asymmetric branches with $x_{CM} \neq 0$ emerging past the bifurcation point, the second eigenvalue of L_+ is shown in the top right panel of the figure, with its positivity indicating the stability of asymmetric states. These panels corroborate the supercritical pitchfork bifurcation scenario $Q > 0$, $R > 0$ in Theorem 4. The bottom panel shows both symmetric and asymmetric states for $E = 15$.

The relevant computations are repeated for higher values of p. The corresponding numerical results for $p = 3$ are shown in Figure 3. Again illustrating the cases $s = 0.6 < s_*$ and $s = 0.7 > s_*$, we observe that a crossing of the relevant eigenvalue occurs in the latter but not in the former. Notice that in the latter case of $s = 0.7$, as shown in the bottom left panel of Fig. 3, the second eigenvalue crossing occurs for $E_* \approx 7$, i.e., for a smaller value of E than in the $p = 1$ case. Generally, we have found that the higher the p, the earlier the relevant crossing occurs and also the more computationally demanding the relevant numerical problem becomes, as the solution narrows and it becomes challenging to appropriately resolve it even with a fairly fine spatial grid for large values of E. This is clearly illustrated in the bottom right panel of Fig. 3, where it can be seen that

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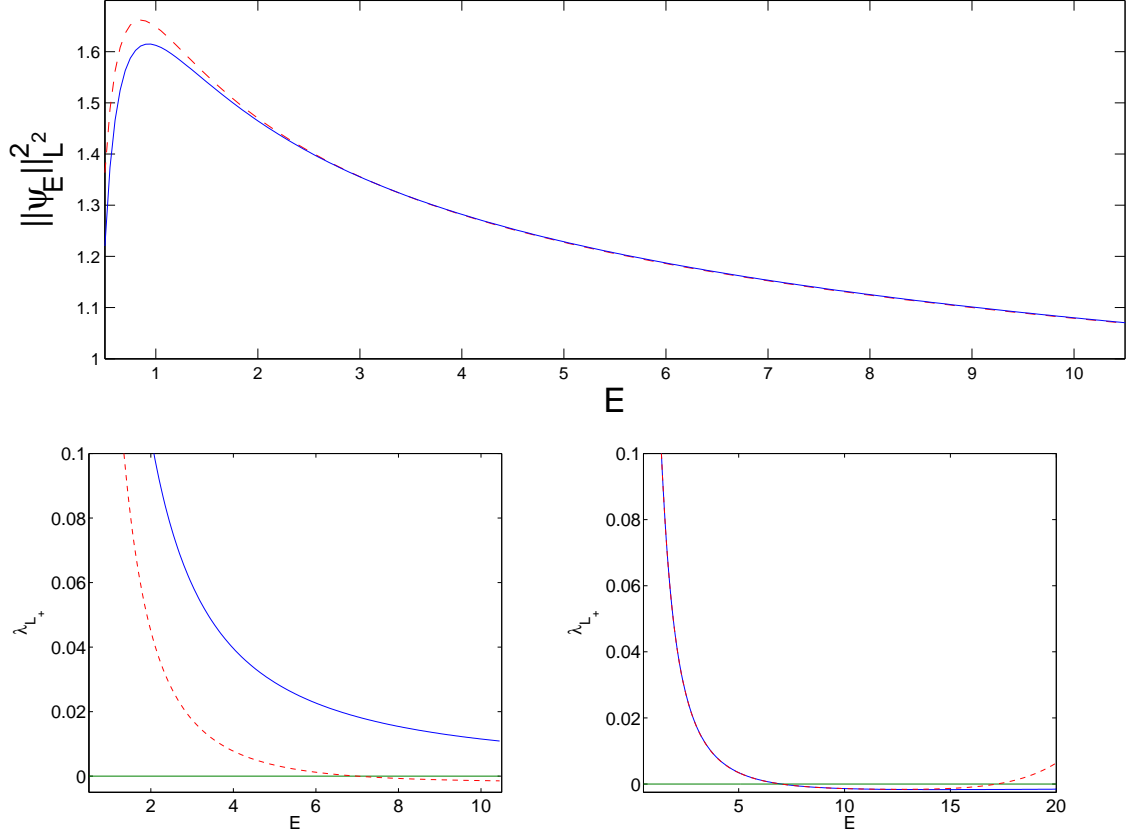Figure 3: The top and bottom left panels are similar to Fig. 1, but now for $p = 3$. The blue solid line corresponds to the case of $s = 0.6 < s_*$, while the red dashed to $s = 0.7 > s_*$. The bottom right panel shows the importance of sufficiently fine discretization in resolving this second eigenvalue for large E. Here the red dashed line corresponds to a spatial grid spacing of $\Delta x = 0.025$, while the solid blue line is obtained for $\Delta x = 0.0125$.

in the absence of sufficient resolution (dashed line for larger spacing of the spatial discretization), a *spurious* secondary crossing is observed for the second eigenvalue of L_+. This secondary crossing is eliminated by finer discretizations (solid line).

It should be pointed out that the symmetric states become unstable in the supercritical case $p = 3$ for $E \approx 0.95$, $s = 0.6$ and $E \approx 0.85$, $s = 0.7$, due to the change of the slope of $N(E) = \|\psi_E\|_{L^2}^2$ (see the top panel of Fig. 3). Therefore, the asymmetric states bifurcating from the symmetric ones at $E_* \approx 7$ will also be orbitally unstable, because $Q > 0, \lambda'(E_*) < 0$, and $N'(E_*) < 0$ will imply $R < 0$ in Theorem 4. However, as s becomes larger, the value of E_* becomes smaller and $E_* \to E_0$ as $s \to \infty$, see Corollary 2. Figure 4 shows the dependence of the squared L^2 norm for the symmetric, asymmetric, and anti-symmetric stationary states for p = 3 and s = 10. The pitchfork bifurcation occurs while the slope of $\|\psi_E\|_{L^2}^2$ is still positive and leading to orbitally stable asymmetric states and a change of stability along the symmetric states from stable for $E < E_*$ to unstable for $E > E_*$
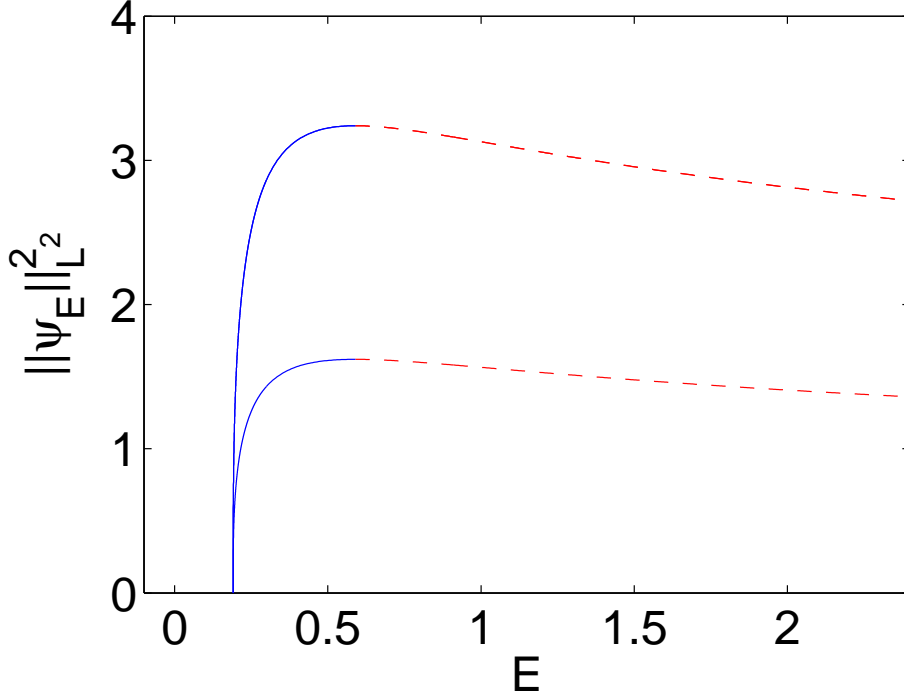
46

Figure 4: The graph shows the dependence of the squared L^2 norm of the stationary states on the parameter E for $p = 3$ and $s = 10$. The left most (blue solid) line are the anti-symmetric (exited) states emerging from zero at the second lowest eigenvalue $E_1 = 0.191042$. Almost on top of it is the (blue solid) line of symmetric ground states emerging from zero at $E_0 = 0.191046$. The latter bifurcates at $E_* \gtrsim E_0$ into the dashed red line (symmetric states) and the right most blue solid line (asymmetric states). At $E \approx 0.5908$ the slope of the L^2 norm becomes negative for all branches. The solid lines denote linearly stable branches, while dashed ones denote unstable branches.

as stipulated in Corollary 2. Note that the numerical simulations suggest that the asymmetric branches can be continued for all $E > E_*$, hence the slope of their L^2 norm square will change for large E and they will become unstable, see Fig. 4 and Remark 6.

The numerical results for $p = 5 > p_*$ and $s = 4 > s_*$ are shown in Figure 5. In this case, the subcritical pitchfork bifurcation occurs at $E_* \approx 0.196$ at the positive slope of $\|\psi_E\|_{L^2}^2$ with respect to E. The top panels show the behavior of squared L^2 norms for symmetric, anti-symmetric, and asymmetric stationary states. The blowup on the top right panel shows that the asymmetric states have decreasing L^2 norm for $E_* < E < E^*$, corresponding to the case $Q > 0$, $R < 0$ in Theorem 4, and in agreement with Corollary 2. However, their L^2 norm becomes increasing for $E > E^*$, where $E^* \approx 0.202$, hence the asymmetric states are orbitally stable in this regime. The bottom panels show the squared eigenvalues of the stability problem associated with the symmetric (left) and asymmetric (right) states. The symmetric state is unstable for any $E > E_*$ (because of the second negative eigenvalue of the operator L_+). It becomes even more unstable for $E > \tilde{E}_*$, where
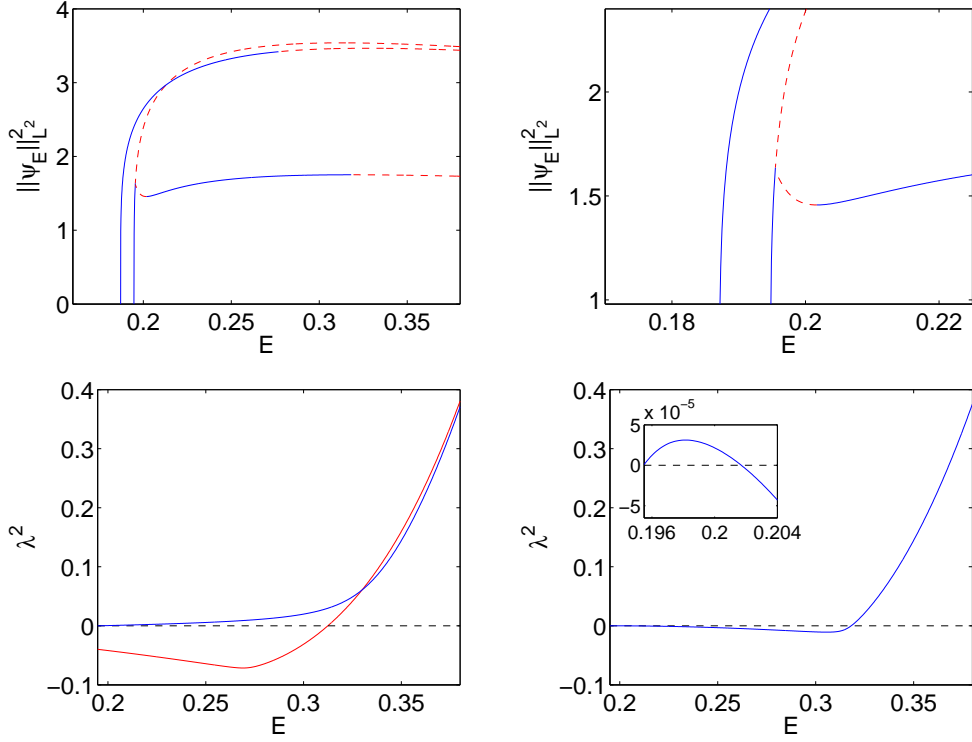
Figure 5: The top panels show dependence of the squared L^2 norm of the stationary states on the parameter E for $p = 5$, $s = 4$, where the right panel is a blowup of the left panel. The leftmost branch is the anti-symmetric (excited) states and the other branch corresponds to the symmetric ground states which bifurcate into asymmetric states. The bottom panels show the squared eigenvalue of the linearization spectrum associated with the symmetric (left) and asymmetric (right) branches. The insert on the bottom right panel gives a blowup of the figure to illustrate instability of asymmetric states near the subcritical pitchfork bifurcation. The solid lines on the top panels denote linearly stable branches, while dashed ones denote unstable branches.

$\tilde{E}_* \approx 0.32$, when another unstable eigenvalue appears (because of the negative slope of the L^2 norm). The asymmetric state is unstable for $E_* < E < E^*$ (because of the negative slope of the L^2 norm) and stable for $E > E^*$. We note that the asymmetric state becomes unstable past $E \approx 0.32$ because of the negative slope of the L^2 norm, similarly to the branch of symmetric states and consistent with Remark 6.

While the results presented herein formulate a relatively comprehensive picture of the one-dimensional phenomenology in the context of double wells, some questions still remain open for future investigations. One of them, raised in Remark 7, is associated with states of the form of u_∞ spatially concentrated at $\pm\infty$. Our numerics for the potential (1.4) has not revealed such states presently, but it would be relevant to lend this subject separate consideration. Additionally, we have not seen the case $Q < 0$ in Theorem 4, i.e. the eigenvalues E are decreasing along the asymmetric branch. Another important question concerns the generalization of the results presented herein

to higher dimensional settings. There, the bifurcation picture is expected to be more complicated when the eigenvalues crossing zero are not simple.

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