SYMMETRY-BREAKING BIFURCATION OF
ANALYTIC SOLUTIONS TO FREE BOUNDARY PROBLEMS:
AN APPLICATION TO A MODEL OF TUMOR GROWTH

AVNER FRIEDMAN AND FERNANDO REITICH

ABSTRACT. In this paper we develop a general technique for establishing analyticity of solutions of partial differential equations which depend on a parameter ε. The technique is worked out primarily for a free boundary problem describing a model of a stationary tumor. We prove the existence of infinitely many branches of symmetry-breaking solutions which bifurcate from any given radially symmetric steady state; these asymmetric solutions are analytic jointly in the spatial variables and in ε.

1. THE MODEL AND MAIN RESULT

In this paper we present a general technique for establishing analyticity of solutions of systems of partial differential equations which depend analytically on a parameter ε. The method works not only for boundary value problems but also for free boundary problems. In this latter context it can be used to establish long time existence of transient solutions, and also to study the existence of spatially asymmetric steady solutions. Since free boundary problems are typically more challenging than their boundary-value counterparts, we shall concentrate here on a free boundary problem from developmental biology, namely, a model of tumor growth. To further exemplify the generality of our approach an instance of a boundary value problem (in a fixed domain) is presented in the last section of the paper. A variety of other problems are amenable to the same analysis, including, in particular, the Hele-Shaw model of fluid flow [11].

Within the last several decades a number of mathematical models have been developed that aimed at describing the evolution of carcinomas (see. e.g., [1, 5, 6, 8, 12, 13] and the references cited there). The main objective of these models has been to qualitatively describe, under various simplifying assumptions, the growth and stability of tumor tissue. Analysis and simulations of such models are helping to assess the relative importance of various mechanisms affecting tumor growth as well as the efficacy of certain cancer treatments. On the other hand, the description of the stationary (dormant) configurations that arise from the models has only been addressed in the case of spherical tumors, but otherwise it remains largely unexplored. In this paper we develop a method for establishing analyticity of

solutions of PDEs which depend on a parameter ε which we use, in this context, to establish the existence of non-spherical families of dormant states for a model of non-necrotic vascularized tumors. For the sake of clarity we shall perform the analysis on two-dimensional geometries, where we demonstrate the existence of infinitely many branches of non-radially symmetric states that bifurcate from any given radial equilibrium; the solutions are analytic jointly in the spatial variables and in ε.

The model we shall study is a natural extension of that proposed in [7] and further analyzed in [10]. We consider the tumor occupying a region Ω, and denote a nutrient concentration by σ and the internal pressure that causes the motion of cellular material by p. The cell proliferation rate is assumed to have the form $\mu(\sigma - \widetilde{\sigma})$ where μ and $\widetilde{\sigma}$ are positive constants. Then, after non-dimensionalization, σ and p are assumed to satisfy the diffusion equations

$$(1.1) \qquad\qquad \Delta\sigma - \sigma = 0 \ \text{ in } \ \Omega,$$

$$(1.2) \qquad\qquad \Delta p = -\mu(\sigma - \widetilde{\sigma}) \ \text{ in } \ \Omega.$$

These equations are supplemented with the boundary conditions

$$(1.3) \qquad\qquad \sigma = \bar{\sigma} \ \text{ on } \ \partial\Omega,$$

$$(1.4) \qquad\qquad \frac{\partial p}{\partial n} = 0 \ \text{ on } \ \partial\Omega,$$

$$(1.5) \qquad\qquad p = \gamma\kappa \ \text{ on } \ \partial\Omega$$

where $\partial/\partial n$ denotes the normal derivative, κ is the curvature at $\partial\Omega$, and $\bar{\sigma}, \gamma$ are positive constants.

The simplest solutions to the (free-boundary) problem (1.1)–(1.5) are the radially symmetric ones, whereby

$$(1.6) \qquad\qquad \sigma = \sigma_0(r), \ p = p_0(r), \ \Omega = \{r < R_0\}.$$

As we show in §3, if $0 < \widetilde{\sigma} < \frac{1}{2}\bar{\sigma}$, then there exists a unique radial solution, with R_0 depending only on $\widetilde{\sigma}/\overline{\sigma}$. A natural question with obvious implications in the study of transient solutions, is whether the model admits non-radial steady states. In our search for such configurations we shall further specialize to the question of whether there exists a bifurcation branch $(\sigma_\varepsilon, \rho_\varepsilon, \Omega_\varepsilon, \gamma_\varepsilon)$ of solutions with

$$(1.7) \qquad \begin{aligned} r &= R_0 + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \cdots \equiv R_0 + f(\theta, \varepsilon), \\ \sigma_\varepsilon(r, \theta) &= \sigma_0(r) + \varepsilon\sigma_1(r, \theta) + \varepsilon^2\sigma_2(r, \theta) + \cdots, \\ p_\varepsilon(r, \sigma) &= p_0(r) + \varepsilon p_1(r, \theta) + \varepsilon^2 p_2(r, \theta) + \cdots, \\ \gamma_\varepsilon &= \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \cdots \end{aligned}$$

which bifurcates from a radial solution (1.6) for some value $\gamma = \gamma_0$.

We shall prove that indeed such a branch exists and that the above series are convergent and form analytic solutions in (r, θ, ε), or more precisely in (x, y, ε), for (x, y) in a neighborhood of $\{r \leq R_0\}$ and $|\varepsilon|$ small. For this we may fix $f_1(\theta) = \cos(\ell\theta)$, for some $\ell \geq 2$, and then γ_0, the *bifurcation point*, is determined uniquely by $\mu, \overline{\sigma}, \widetilde{\sigma}$ and ℓ; we shall actually prefer to consider γ_0 as a function of R_0, since we view the solutions (1.7) as a non-radial branch bifurcating from the radial solution with radius R_0.

Let us explain the relation between this paper and general bifurcation theory. In the abstract setting of bifurcation theory one considers the problem

$$(1.8) \qquad F(\lambda, u) = 0$$

where u varies in a Banach space X and λ varies in $\mathbb{R}$. Assume for simplicity that

$$F(\lambda, 0) \equiv 0 \text{ for all } \lambda \in \mathbb{R}.$$

A bifurcation point $(\lambda_c, 0)$ is a point for which there exists a one- or multiple-parameter branch of nontrivial solutions

$$\lambda = \lambda(\varepsilon), \ u = u(\varepsilon)$$

of (1.8) with $\lambda(0) = \lambda_c, u(0) = 0$. The Liapounov-Schmidt procedure reduces the construction of a bifurcation branch from the infinite dimensional Banach space X to a finite dimensional space:

Suppose $F(\lambda, u)$ maps $u \in X$ into a Banach space Y such that $X \subset Y$, and F is smooth. Set

$$L_0 = \frac{\partial F(\lambda_c, 0)}{\partial u}$$

and assume that L_0 is a Fredholm operator with index zero. Denote its null space by $\mathcal{N}$ (we assume that $n = \dim \mathcal{N} < \infty$) and its range by $\mathcal{R}_0$. Let P be the projection of Y into $\mathcal{N}$, so that $Q = I - P$ is the projection of Y into $\mathcal{R}_0$. If we decompose u into $Pu + Qu$ and project (1.8) into $\mathcal{N}$ and $\mathcal{R}_0$, we get the equations

$$(1.9) \qquad K(\lambda, v, \psi) \equiv QF(\lambda, v + \psi) = 0 \ ,$$

$$(1.10) \qquad PF(\lambda, v + \psi) = 0$$

where $v = Pu$ and $\psi = Qu$. This set of equations is equivalent to the single equation (1.8). Noting that $K(\lambda_c, 0, 0) = 0$ and $\partial K(\lambda_c, 0, 0)/\partial \psi = QL_0$ is an isomorphism from QX to QY, we can apply the implicit function theorem to (1.9) and thus solve for $\psi = \psi(\lambda, v)$. We then substitute ψ into (1.10) and get the Liapounov-Schmidt bifurcation equation

$$(1.11) \qquad PF(\lambda, v + \psi(\lambda, v)) = 0,$$

which is a system of n equations; for more details see, for instance, [14], [15].

The above scheme cannot be applied to the problem (1.1)–(1.5) since the spaces X, Y will vary in ε in a way which is unknown in advance (as it will depend on the free boundary). But even if we transform the problem to one in a fixed domain, as we shall do later on in this paper, we still cannot simply apply the above scheme, since there are no implicit function theorems that can be used to solve the resulting equation (1.9) for our problem. Thus, in a sense, our work establishes a new implicit function theorem in a space of analytic functions. The null set $\mathcal{N}$ is our case will consist of the linear space generated by $\cos(\ell\theta)$, and the orthogonality of Pu to Qu is realized by the condition

$$(1.12) \qquad \int_0^{2\pi} f_j(\theta) \cos \ell\theta d\theta = 0 \ \text{ for all } \ j \geq 2.$$

Mode 1 solutions (containing multiples of $\cos(\theta)$ in their Fourier series representations) trivially arise from infinitesimal translations of the disc $r = R_0$; we shall

exclude them by imposing the conditions

$$(1.13) \qquad \int_0^{2\pi} \sigma_j(r,\theta) \cos \theta d\theta = \int_0^{2\pi} p_j(r,\theta) \cos \theta d\theta = 0,$$

$$\int_0^{2\pi} f_j(\theta) \cos \theta d\theta = 0 \text{ for all } j \geq 1.$$

We shall prove that there exists, for each $\ell \geq 2$, a unique bifurcation branch of solutions (1.7) satisfying the orthogonality conditions (1.12), (1.13) and the additional condition that

$$(1.14) \qquad f_j(\theta), \sigma_j(r,\theta), p_j(r,\theta) \quad \text{are even functions in } \theta.$$

The solutions are jointly analytic in the spatial variables and in ε. As we will show (see Remark 5.1) the orthogonality condition (1.12) implies that

$$(1.15) \qquad \gamma_{2n+1} = 0 \text{ for all integers } n \geq 0;$$

the other coefficients γ_{2n} $(n \geq 1)$ are uniquely determined by consistency conditions that need to be satisfied by the coefficients of $\cos \ell\theta$ in the ε^{2n+1} approximation.

The techniques developed in this paper depend, in part, on expanding solutions of (1.1) in a circle in terms of the modified Bessel functions $I_n(r)$. In §2 we introduce these Bessel functions, recall some known facts and prove new relations that will be needed in this paper. In §3 we compute the radial solution of (1.1)–(1.5). In §4 we consider the linearization of (1.1)–(1.5) about the radial solution (1.6) and prove that it has a nontrivial solution if and only if $\gamma = \gamma_0$ is a solution of a bifurcation equation (to be introduced later on).

In §5 we establish a formal expansion (1.7), deriving for each power ε^n a system of equations for the coefficients $\sigma_n, p_n, f_n, \gamma_{n-1}$. The system has a unique solution subject to the orthogonality conditions (1.12), (1.13). The system is obtained by expressing the solution of the differential equations in terms of Bessel functions, and then expanding the boundary conditions, for the solution, in Taylor's series about $r = R_0$. Although this approach seems very natural, we were surprised to discover that it does not lead to a convergence proof by the standard majorization method. Indeed, in §6 we explain why a convergence proof must exploit some "hidden" cancellations in the recursive relations that determine $\sigma_n, p_n, f_n, \gamma_{n-1}$. We also outline in §6 another less direct and somewhat more complicated approach for deriving formal expansions. It is this approach that we shall pursue for the rest of the paper, and we briefly describe it below.

In §7 we transform the free boundary problem into a problem in the disc $\{r' < 1\}$ by a change of variables

$$r' = \frac{r}{R_0 + f(\theta, \epsilon)}$$

and set

$$(1.16) \qquad \widehat{\sigma}_\varepsilon(r,\theta) = \sigma_\varepsilon(r(R_0 + f(\theta,\varepsilon)),\theta), \quad \hat{p}_\varepsilon(r,\vartheta) = p_\varepsilon(r(R_0 + f(\theta,\varepsilon)),\theta).$$

Then $\widehat{\sigma}_\varepsilon, \widehat{p}_\varepsilon$ satisfy elliptic equations with coefficients that depend on $f, f_\theta, f_{\theta\theta}$. As in §5 we derive in §8 a formal expression

$$
\begin{aligned}
\widehat{\sigma}_\varepsilon(r,\theta) &= \widehat{\sigma}_0(r) + \varepsilon\widehat{\sigma}_1(r,\theta) + \varepsilon^2\widehat{\sigma}_2(r,\theta) + \cdots, \\
\widehat{p}_\varepsilon(r,\theta) &= \widehat{p}_0(r) + \varepsilon\widehat{p}_1(r,\theta) + \varepsilon^2\widehat{p}_2(r,\theta) + \cdots, \\
f(\theta,\varepsilon) &= \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \cdots, \\
\gamma &= \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \cdots.
\end{aligned}
$$

(1.17)

However, from the recursive formulas for this expansion we are, in fact, able to deduce estimates that establish the convergence of the series to an analytic solution for (x, y) in a neighborhood of $\{r \leq 1\}$ and $|\varepsilon|$ small; this is done in §§9–11. In §11 we also show that $\sigma_\varepsilon, p_\varepsilon$ defined in terms of $\widehat{\sigma}_\varepsilon, \widehat{p}_\varepsilon$, by (1.16) together with $f(\theta,\varepsilon)$ and $\gamma = \gamma(\varepsilon)$ form a non-radial analytic solution of (1.1)–(1.5) jointly in the spatial variables and in ε for $X = (x, y)$ in $|X| \leq R_0 + \delta_0, |\varepsilon| \leq \varepsilon_0$ and the expansion (1.7) is convergent for $0 < r \leq R_0 + \delta_0, |\varepsilon| \leq \varepsilon_0$, where δ_0, ε_0 are small positive constants.

Finally, in §12, we give an example of a (fixed-)boundary value problem for which our method proves joint analyticity in (x, y, ε).

2. BESSEL FUNCTIONS

In the sequel we shall use the modified Bessel functions $I_m(r)$ for $m \geq 0$ and $r \geq 0$. Recall that $I_m(r)$ satisfies the differential equation

$$
\text{(2.1)} \qquad I_m''(r) + \frac{1}{r}I_m'(r) - \left(1 + \frac{m^2}{r^2}\right)I_m(r) = 0
$$

and is given by

$$
\text{(2.2)} \qquad I_m(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{m+2k}}{k!\,\Gamma(m+k+1)}.
$$

Furthermore,

$$
\text{(2.3)} \qquad I_m'(r) + \frac{m}{r}I_m(r) = I_{m-1}(r), \quad m \geq 1,
$$

$$
\text{(2.4)} \qquad I_m'(r) - \frac{m}{r}I_m(r) = I_{m+1}(r),
$$

so that

$$
\text{(2.5)} \qquad I_{m-1}(r) - I_{m+1}(r) = \frac{2m}{r}I_m(r), \quad m \geq 1,
$$

and

$$
\text{(2.6)} \qquad r^{m+1}I_m(r) = \frac{d}{dr}\left(r^{m+1}I_{m+1}(r)\right),
$$

$$
\text{(2.7)} \qquad I_m(r) = \left(\frac{1}{2\pi r}\right)^{1/2} e^r \left[1 - \frac{4m^2-1}{8r} + O\left(\frac{1}{r^2}\right)\right] \quad \text{if} \quad r \to \infty,
$$

$$
\text{(2.8)} \qquad I_m(r)I_n(r) = \sum_{k=0}^{\infty} \frac{\Gamma(m+n+2k+1)(r/2)^{m+n+2k}}{k!\,\Gamma(m+k+1)\Gamma(n+k+1)\Gamma(m+n+k+1)}.
$$

In particular,

$$
I_0'(r) = I_1(r), \qquad rI_0(r) = \frac{d}{dr}\left(rI_1(r)\right).
$$

Lemma 2.1. *The following inequalities hold:*

$$(2.9) \qquad\qquad I_{m-1}(r)I_{m+1}(r) < I_m^2(r) \qquad if \quad r > 0,$$

$$(2.10) \qquad I_{m-1}(r)I_{m+1}(r) > I_m^2(r) - \frac{2}{r}I_m(r)I_{m+1}(r) \qquad if \quad r > 0.$$

Proof. Using (2.8) we obtain

$$
\begin{aligned}
&I_{m-1}(r)I_{m+1}(r) - I_m^2(r) \\
&\quad = \sum_{k=0}^{\infty} \frac{(r/2)^{2m+2k}\ \Gamma(2m+2k+1)}{k!\ \Gamma(2m+k+1)} \left[\frac{1}{\Gamma(m+k)\Gamma(m+k+2)} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - \frac{1}{(\Gamma(m+k+1))^2} \right],
\end{aligned}
$$

$$
\frac{2}{r}I_m(r)I_{m+1}(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{2m+2k}\ \Gamma(2m+2k+2)}{k!\ \Gamma(m+k+1)\Gamma(m+k+2)\Gamma(2m+k+2)}.
$$

To prove (2.9) it suffices to show that

$$
\frac{1}{\Gamma(m+k)\Gamma(m+k+2)} < \frac{1}{\Gamma(m+k+1)^2} \qquad if \quad k \geq 1
$$

or

$$
\frac{1}{m+k+1} < \frac{1}{m+k},
$$

which is obvious.

To prove (2.10) it suffices to show that

$$
\begin{aligned}
&\frac{1}{\Gamma(m+k)\Gamma(m+k+2)} - \frac{1}{(\Gamma(m+k+1))^2} \\
&\qquad > -\frac{(2m+2k+1)}{\Gamma(m+k+1)\Gamma(m+k+2)(2m+k+1)}
\end{aligned}
$$

if $k \geq 1$ (equality holds if $k = 0$), or

$$
\begin{aligned}
&\frac{1}{\Gamma(m+k)\Gamma(m+k+1)(m+k+1)} \\
&\quad + \frac{(2m+2k+1)}{\Gamma(m+k+1)\Gamma(m+k)(m+k+1)(m+k)(2m+k+1)} > \frac{1}{\Gamma(m+k+1)^2}.
\end{aligned}
$$

But this reduces to

$$
\frac{1}{m+k+1} + \frac{(2m+2k+1)}{(m+k+1)(m+k)(2m+k+1)} > \frac{1}{m+k}\ ,
$$

or

$$
\frac{m+k}{m+k+1} + \frac{(2m+2k+1)}{(m+k+1)(2m+k+1)} > 1\ ,
$$

that is,

$$
\frac{2m+2k+1}{(m+k+1)(2m+k+1)} > \frac{1}{m+k+1}
$$

which is obviously true.   □

Corollary 2.2. *If $\ell \geq 2$, then*

$$(2.11) \qquad I_2(r) > \frac{I_1(r)I_{\ell+1}(r)}{I_\ell(r)}.$$

Indeed, by (2.9),

$$\frac{I_{\ell+1}(r)}{I_\ell(r)} < \frac{I_\ell(r)}{I_{\ell-1}(r)} < \cdots < \frac{I_3(r)}{I_2(r)} < \frac{I_2(r)}{I_1(r)} < \frac{I_1(r)}{I_0(r)}.$$

Lemma 2.3. *For any $r > 0$, the function*

$$(2.12) \qquad f(m) = \frac{I_2(r) - I_1(r)I_{m+1}(r)/I_m(r)}{(m-1)m(m+1)}, \qquad m > 1,$$

satisfies

$$f(m+1) < f(m).$$

Proof. Since

$$f(m) = \frac{I_2 I_m - I_1 I_{m+1}}{I_m(m-1)m(m+1)}$$

we need to show that

$$\frac{I_2 I_{m+1} - I_1 I_{m+2}}{I_{m+1}(m+2)} < \frac{I_2 I_m - I_1 I_{m+1}}{I_m(m-1)},$$

or

$$(m-1)I_m(I_2 I_{m+1} - I_1 I_{m+2}) < (m+2)I_{m+1}(I_2 I_m - I_1 I_{m+1}).$$

This can be written in the form

$$(m-1)I_m I_1 I_{m+2} > (m-1)I_{m+1}^2 I_1 - 3I_{m+1}(I_2 I_m - I_1 I_{m+1})$$

or

$$3I_{m+1}(I_2 I_m - I_1 I_{m+1}) > I_1(m-1)(I_{m+1}^2 - I_m I_{m+2}).$$

Since, by (2.10),

$$I_{m+1}^2 - I_m I_{m+2} < \frac{2}{r}\, I_{m+1} I_{m+2},$$

it suffices to show that

$$(2.13) \qquad 3I_{m+1}(I_2 I_m - I_1 I_{m+1}) > I_1(m-1)\frac{2}{r}\, I_{m+1} I_{m+2}.$$

From (2.8)

$$(2.14) \qquad \begin{aligned}
I_2 I_m &- I_1 I_{m+1} \\
&= \sum_{k=0}^{\infty} \frac{(r/2)^{m+2+k}}{k!}\left[\frac{(m+2+2k)!}{(m+k)!\,(2+k)!\,(m+2+k)!} \right. \\
&\qquad\qquad\qquad\qquad \left. - \frac{(m+2+2k)!}{(m+1+k)!\,(1+k)!\,(m+2+k)!} \right] \\
&= \sum_{k=0}^{\infty} \frac{(r/2)^{m+2+2k}}{k!}\frac{(m+2+2k)!}{(m+2+k)!}\frac{1}{(2+k)!\,(m+1+k)!}(m-1)
\end{aligned}$$

and

$$\frac{2}{r} \, I_1 I_{m+2} = \sum_{k=0}^{\infty} \frac{(r/2)^{m+2+2k}}{k!} \frac{(m+3+2k)!}{(m+2+k)! \, (1+k)! \, (m+3+k)!} \ .$$

Therefore (2.13) will follow provided we have

$$\frac{3(m+2+2k)! \, (m-1)}{(m+2+k)! \, (2+k)! \, (m+1+k)!} > \frac{(m-1)(m+3+2k)!}{(m+2+k)! \, (1+k)! \, (m+3+k)!}$$

for all $k \geq 0$, that is, provided

$$(2.15) \qquad 3(m+k+3)(m+k+2) > (m+3+2k)(2+k) \qquad \forall \, k \geq 0.$$

But since

$$\begin{aligned}
(m+3+2k)(2+k) &= (m+3+k)(2+k) + k(2+k) \\
&< (m+3+k)(m+k+2) + (m+3+k)(m+2+k) \\
&= 2(m+k+3)(m+k+2),
\end{aligned}$$

(2.15) is indeed valid. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 2.4. *The function*

$$(2.16) \qquad\qquad G_m(r) = r \left(\frac{I_2(r)}{I_1(r)} - \frac{I_{m+1}(r)}{I_m(r)} \right) \qquad (m \geq 2)$$

satisfies

$$(2.17) \qquad\qquad\qquad G_m'(r) > 0 \qquad \text{for all } r > 0.$$

Proof. Using (2.4) we can write

$$G_m(r) = r \left(\frac{I_1'}{I_1} - \frac{I_m'}{I_m} \right) + m - 1$$

so that we need to show that the function

$$(2.18) \qquad\qquad\qquad F_m(r) = r \left(\frac{I_1'}{I_1} - \frac{I_m'}{I_m} \right)$$

satisfies

$$(2.19) \qquad\qquad\qquad F_m'(r) > 0 \qquad \text{for all } \quad r > 0.$$

Introducing the functions

$$U_m(r) = r D_m(r) \qquad \text{where} \qquad D_m(r) = \frac{I_m'}{I_m} \ ,$$

(2.19) becomes

$$(2.20) \qquad\qquad \frac{d}{dr} \left(U_1(r) - U_m(r) \right) > 0 \qquad \text{for all} \quad r > 0.$$

By (2.1)

$$D_m' + D_m^2 + \frac{1}{r} D_m = 1 + \frac{m^2}{r^2}$$

so that

$$(2.21) \qquad\qquad\qquad r U_m' + U_m^2 = r^2 + m^2$$

and, by differentiation,

$$(2.22) \qquad\qquad rU_m'' + U_m' + 2U_mU_m' = 2r.$$

Using (2.2) we easily compute that

$$(2.23) \qquad U_m(r) = m + \frac{r^2}{2(m+1)} + O(r^3) \qquad \text{as} \quad r \to 0.$$

In order to prove (2.20) we first need to prove that

$$(2.24) \qquad\qquad U_m'(r) > 0 \qquad \text{for all} \quad r > 0$$

and

$$(2.25) \qquad\qquad U_1(r) - U_m(r) < 0 \qquad \text{for all} \quad r > 0.$$

Note that these inequalities are true if r is near 0, by (2.23). Suppose first that (2.24) is not valid for all $r > 0$. Then there is a smallest $r = \bar{r}$ such that

$$U_m'(\bar{r}) = 0$$

and then, of course, also $U_m''(\bar{r}) \leq 0$. Substituting these relations into (2.22), we get a contradiction.

Similarly, if (2.25) does not hold for all $r > 0$, then there is a smallest $r = \bar{\bar{r}}$ such that

$$U_1\left(\bar{\bar{r}}\right) - U_m\left(\bar{\bar{r}}\right) = 0;$$

then also $(U_1 - U_m)'\left(\bar{\bar{r}}\right) \geq 0$. Substituting these relations into (2.21), we again get a contradiction.

We shall now prove (2.20) by the same method. The inequality clearly holds for r small, by (2.23). Hence if (2.20) is not true, then there is a smallest $r = r_0$ for which

$$(2.26) \qquad\qquad U_1'(r_0) - U_m'(r_0) = 0,$$

and then also $U_1''(r_0) - U_m''(r_0) \leq 0$. Substituting these relations into (2.22) we get

$$U_mU_m' \leq U_1U_1' \qquad \text{at} \quad r = r_0$$

or, since $U_1'(r_0) = U_m'(r_0) > 0$ by (2.24),

$$U_m(r_0) \leq U_1(r_0),$$

which is a contradiction to (2.25).

Lemma 2.5. *Let*

$$g_{1m}(r) = \frac{1}{r^2} \int_0^r \left[m^2 I_1(s) I_m(s) - 2s I_0(s) I_m(s) \right] ds,$$

$$g_{2m}(r) = -\frac{(m-2)}{r} I_1(r) I_m(r).$$

Then

$$(2.27) \qquad g_{1m}(r) + g_{2m}(r) + \left[I_2(r) I_m(r) - I_1(r) I_{m+1}(r) \right] = 0$$

for any $m \geq 1$.

Proof. Using (2.8) we get

$$
\begin{aligned}
g_{1m}(r) &= \sum_{k=0}^{\infty} \frac{(m+2k)!}{k!(m+k)!(k+1)!} \frac{(r/2)^{m+2k}}{(m+k+1)!} \frac{1}{(m+2+2k)2} \\
&\quad \times \left[m^2(m+2k+1) - 4(k+1)(m+k+1) \right], \\
g_{2m}(r) &= -\sum_{k=0}^{\infty} \frac{(m+1+2k)!(r/2)^{m+2k}(m-2)}{k!(m+k)!(k+1)!(m+k+1)!2}.
\end{aligned}
$$

Hence

$$
g_{1m}(r) + g_{2m}(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{m+2k}}{k!(m+k)!(k+1)!(m+k+1)!} \frac{(m+2k)!}{(m+2+2k)2} [\cdots]
$$

where

$$
\begin{aligned}
[\cdots] &= m^2(m+2k+1) - 4(k+1)(m+k+1) \\
&\quad -(m-2)(m+1+2k)(m+2+2k) \\
&= -2k(m-1)(m+2k+2).
\end{aligned}
$$

We conclude, after changing k into $k+1$, that

$$
g_{1m}(r) + g_{2m}(r) = -\sum_{k=0}^{\infty} \frac{(r/2)^{m+2k+2}}{(k+1)!(m+k+1)!} \frac{(m+2k+2)!(k+1)(m-1)}{(k+2)!(m+k+2)!}
$$

from which (2.27) follows by using (2.14). $\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.6. *For any positive integers $m, \ell, m \neq \ell$,*

$$
(2.28) \qquad g_{1m}(r) + g_{2m}(r) + \frac{m(m^2-1)}{\ell(\ell^2-1)} \frac{I_m(r)}{I_\ell(r)} \left[I_2(r)I_\ell(r) - I_1(r)I_{\ell+1}(r) \right] \neq 0
$$

for all $r > 0$.

Proof. By Lemma 2.5 the left-hand side of (2.28) is equal to

$$
\begin{aligned}
&- (I_2 I_m - I_1 I_{m+1}) + \frac{m(m^2-1)I_m}{\ell(\ell^2-1)I_\ell} (I_2 I_\ell - I_1 I_{\ell+1}) \\
&= m(m^2-1)I_m \left\{ -\frac{I_2 - I_1 I_{m+1}/I_m}{m(m^2-1)} + \frac{I_2 - I_1 I_{\ell+1}/I_\ell}{\ell(\ell^2-1)} \right\}
\end{aligned}
$$

which is $\neq 0$ by Lemma 2.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. THE RADIAL SOLUTION

The radial stationary solution to (1.2), (1.3), (1.4) is given by

$$
(3.1) \qquad\qquad\qquad \sigma_0(r) = \bar{\sigma} \frac{I_0(r)}{I_0(R)} , \qquad r < R,
$$

$$
(3.2) \qquad\qquad p_0(r) = -\mu\, \sigma_0(r) + A + \frac{\mu}{4}\, \tilde{\sigma} r^2 , \qquad r < R
$$

(since $\Delta(p_0 + \mu\sigma_0)$ must be equal to $\mu\tilde{\sigma}$), where A is a constant determined by (1.5):

$$
-\mu\,\bar{\sigma} + A + \frac{\mu\tilde{\sigma}}{4}R^2 = \gamma\frac{1}{R}
$$

and R is determined by the condition (1.4) or, equivalently, by

$$\int_{r=R} \frac{\partial p_0}{\partial n} = 0,$$

i.e.,

$$0 = \int_{r<R} \Delta p_0 = -\mu \int_{r<R} (\sigma_0 - \tilde{\sigma}).$$

This can be written more explicitly as

$$\frac{2\pi\bar{\sigma}}{I_0(R)} \int_0^R I_0(r)rdr = \tilde{\sigma}\pi R^2.$$

Since

$$\begin{aligned}
\int_0^R I_0(r)rdr &= \int_0^R \frac{d}{dr}\left(rI_1(r)\right) dr \qquad \text{(by (2.6))} \\
&= R\, I_1(R) \ ,
\end{aligned}$$

we get

$$\frac{2\pi\bar{\sigma}}{I_0(R)} R\, I_1(R) = \tilde{\sigma}\pi R^2 \ .$$

Setting

(3.3)
$$\Lambda = \frac{\tilde{\sigma}}{2\bar{\sigma}}$$

we conclude that the radius R is determined by

(3.4)
$$\frac{I_1(R)}{I_0(R)} = \Lambda R.$$

Theorem 3.1. *If $0 < \Lambda < \frac{1}{2}$, then there exists a unique solution $R = R_0$ to (3.4).*

Proof. By (2.5)

$$\frac{1}{R} I_1(R) = \frac{1}{2}\left(I_0(R) - I_2(R)\right)$$

so that (3.4) is equivalent to

$$\left(\frac{1}{2} - \Lambda\right) I_0(R) = \frac{1}{2} I_2(R) \ ,$$

or

(3.5)
$$\frac{I_2(R)}{I_0(R)} = 1 - 2\Lambda \qquad (0 < 1 - 2\Lambda < 1).$$

Consider the function

$$f(R) = \frac{I_2(R)}{I_0(R)} \ .$$

By (2.2) and (2.7),

$$\begin{aligned}
f(R) &\to 0 \qquad \text{if} \quad R \to 0 \ , \\
f(R) &\to 1 \qquad \text{if} \quad R \to \infty \ .
\end{aligned}$$

Therefore it suffices to show that $f(R)$ is strictly monotone increasing. But

$$f' = \frac{1}{I_0^2}(I_2'I_0 - I_0'I_2) = \frac{1}{I_0^2}\left[\left(I_1 - \frac{2}{R} I_2\right) I_0 - I_1 I_2\right]$$

so that

$$
\begin{aligned}
I_0^2 f' &= (I_0 - I_2)I_1 - \frac{2}{R} I_2 I_0 \\
&= \frac{2}{R} I_1^2 - \frac{2}{R} I_2 I_0 > 0 \qquad \text{by (2.9)} . \quad \square
\end{aligned}
$$

From now on we shall always assume that

$$
(3.6) \qquad\qquad 0 < \tilde{\sigma} < \frac{1}{2}\bar{\sigma}
$$

so that there exists a unique radial solution to (1.1)–(1.5).

Remark 3.1. Our model can be extended to transient situations by setting [10]

$$
\begin{aligned}
c\frac{\partial \sigma}{\partial t} &= \Delta \sigma - \sigma \ \ \text{in} \ \ \Omega(t), \\
\Delta p &= -\mu(\sigma - \tilde{\sigma}) \ \ \text{in} \ \ \Omega(t), \\
\frac{\partial p}{\partial n} &= -V_n \ \ \text{in} \ \ \partial \Omega(t)
\end{aligned}
$$

with (1.3), (1.5) on $\partial \Omega(t)$; here V_n denotes the velocity of the free boundary $\partial \Omega(t)$. This problem was considered in [10] for 3-dimensional domains $\Omega(t)$ and solutions which are radially symmetric, i.e, $\sigma = \sigma(r,\theta), p = p(r,\theta), \Omega(t) = \{r < R(t)\}$. It was proved that if $0 < \Lambda < \frac{1}{3}$, then there exists a unique stationary radial solution $(\sigma_0(r), p_0(r), R_0)$, and it is globally asymptotically stable with respect to the radial solutions of the time-dependent problem, provided c is sufficiently small; furthermore, $|R(t) - R_0| \le Be^{-\delta t}$ for all $t > 0$ where B, δ are positive constants. The same result (with similar proof) is valid for the present 2-dimensional problem.

For future reference we compute

$$
(3.7) \qquad\qquad \alpha = \frac{\partial \sigma_0}{\partial r}(R_0) = \bar{\sigma} \frac{I_1(R_0)}{I_0(R_0)} ,
$$

and

$$
(3.8) \qquad\qquad \beta = -\frac{\partial^2 p_0}{\partial r^2}(R_0) = \mu(\bar{\sigma} - \tilde{\sigma}) = \mu\bar{\sigma}(1 - 2\Lambda)
$$

or, by (3.5),

$$
(3.9) \qquad\qquad \beta = \mu\bar{\sigma}\frac{I_2(R_0)}{I_0(R_0)} .
$$

4. The linearized problem

In this section we determine the branching points. As explained in the Introduction it will be convenient to consider the parameters $\bar{\sigma}, \tilde{\sigma}, \mu$ as fixed, and γ as variable. We shall denote the radius R of a stationary solution by R_0, and determine the parameter $\gamma = \gamma_0$ for which the linearized problem (about the stationary solution) has a nontrivial solution.

The radial solution is then

$$
\Big(\sigma_0(r), p_0(r), R_0\Big)
$$

and we take a perturbation of the free boundary of the form

$$
(4.1) \qquad\qquad r = R_0 + \varepsilon f_1(\theta)
$$

where $f_1(\theta)$ is of mode $\ell, \ell \geq 2$, namely,

$$f_1(\theta) = N_1 \cos \ell\theta + N_2 \sin \ell\theta$$

where N_1, N_2 are constants. By rotating the coordinate system we may assume, without loss of generality, that $N_2 = 0$. Further, by rescaling ε, we can choose $N_1 = 1$ so that

$$(4.2) \qquad f_1(\theta) = \cos \ell\theta , \qquad \ell \geq 2 .$$

Correspondingly, we take

$$(4.3) \qquad \begin{aligned} \sigma(r,\theta) &= \sigma_0(r) + \varepsilon\sigma_1(r)f_1(\theta) , \\ p(r,\theta) - \frac{\gamma_1}{R_0} &= p_0(r) + \varepsilon p_1(r) f_1(\theta) \end{aligned}$$

where the constant γ_1/R_0 arises from (1.5). We easily find that

$$(4.4) \qquad \Delta(\sigma_1 f_1) - \sigma_1 f_1 = 0 \quad \text{if } r < R_0 ,$$

$$(4.5) \qquad \Delta(p_1 f_1) = -\mu(\sigma_1 f_1) \quad \text{if } r < R_0 ,$$

$$(4.6) \qquad \sigma_1(R_0) + \alpha = 0 ,$$

$$(4.7) \qquad \frac{\partial p_1}{\partial r}(R_0) - \beta = 0$$

where α, β are defined in (3.7)–(3.9), and

$$(4.8) \qquad p_1(R_0) - \frac{\gamma_0}{R_0^2}(\ell^2 - 1) = 0 .$$

The last formula is obtained by using the general formula for the curvature of a curve $r = g(\theta)$,

$$(4.9) \qquad \kappa(g) = \frac{2(g')^2 - gg'' + g^2}{[g^2 + (g')^2]^{3/2}} , \quad r = g(\theta)$$

which gives, to first order in ε ,

$$\kappa(R_0 + \varepsilon f_1) = \frac{1}{R_0} - \frac{\varepsilon}{R_0^2}(f_1'' + f_1) = \frac{1}{R_0} + \frac{\varepsilon}{R_0^2}(\ell^2 - 1)f_1 .$$

Using this in (1.5), we easily derive the relation (4.8).

Solving for σ_1 from (4.4), (4.6), we get

$$(4.10) \qquad \sigma_1(r) = -\alpha\frac{I_\ell(r)}{I_\ell(R_0)} .$$

Further, since the function

$$M = \mu\,\sigma_1 + p_1$$

satisfies

$$M'' + \frac{1}{r}\,M' - \frac{\ell^2}{r^2}\,M = 0 ,$$

and thus is equal to Br^ℓ, B constant, we have

$$(4.11) \qquad p_1(r) = Br^\ell + \mu\alpha\,\frac{I_\ell(r)}{I_\ell(R_0)} .$$

The constant B is determined by (4.8),

$$(4.12) \qquad BR_0^\ell = -\mu\alpha + \frac{\gamma_0}{R_0^2}(\ell^2 - 1) ,$$

and, finally, (4.7) becomes

$$
\begin{aligned}
\beta &= \ell B R_0^{\ell-1} + \mu\,\alpha\,\frac{I_\ell'(R_0)}{I_\ell(R_0)} \\
&= \frac{\ell}{R_0}\Big[-\mu\alpha + \frac{\gamma_0}{R_0^2}(\ell^2-1)\Big] + \mu\alpha\,\frac{I_\ell'(R_0)}{I_\ell(R_0)} \\
&= \frac{\gamma_0}{R_0^3}\,\ell(\ell^2-1) + \mu\alpha\Big[\frac{I_\ell'(R_0)}{I_\ell(R_0)} - \frac{\ell}{R_0}\Big]\ .
\end{aligned}
$$

Using (2.4) we get

$$
\beta = \frac{\gamma_0}{R_0^3}\,\ell(\ell^2-1) + \mu\alpha\,\frac{I_{\ell+1}(R_0)}{I_\ell(R_0)}
$$

which, by (3.7), (3.9), is equivalent to

$$
(4.13)\qquad I_2(R_0) = \ell(\ell^2-1)\frac{I_0(R_0)}{\mu\bar\sigma\,R_0^3}\,\gamma_0 + \frac{I_1(R_0)\,I_{\ell+1}(R_0)}{I_\ell(R_0)}\ .
$$

This is the *bifurcation equation* or the *eigenvalue equation* which determines γ_0 as a function of R_0 (and $\mu, \bar\sigma, \tilde\sigma$); we refer to γ_0 as the *branching point*.

Theorem 4.1. *For any $R_0 > 0$ there exists a unique solution γ_0 of* (4.13).

Proof. By Corollary 2.2

$$
I_2(R_0) = \frac{I_1(R_0)\,I_{\ell+1}(R_0)}{I_\ell(R_0)} + q
$$

where q is a positive number. Hence the unique solution of (4.13) is given by

$$
\ell(\ell^2-1)\frac{I_0(R_0)}{\mu\bar\sigma R_0^3}\,\gamma_0 = q\ .\quad\square
$$

Remark 4.1. For $\ell = 1$, (4.13) holds identically for all γ_0, R_0. The reason is that any ε-translation of the radial solution represents an ε-perturbation of mode $\ell = 1$ when viewed from the original system of coordinates. More precisely, the radial solution (with radius R_0) with respect to the center $x_1 = \varepsilon, x_2 = 0$, when written in polar coordinates centered at $x_1 = 0, x_2 = 0$, has the form

$$
r = f(\theta, \varepsilon) = R_0 + \varepsilon R_0 \cos\theta + O(\varepsilon^2)\ .
$$

Later on we shall construct bifurcation branches with free boundary

$$
r = R_0 + \varepsilon\cos\ell\theta + \sum_{n\geq 2}\varepsilon^n\lambda_n(\theta)\ ,
$$

uniquely determined by the orthogonality conditions $\int_0^{2\pi}\lambda_n(\theta)\cos\ell\theta\,d\theta = 0$. For $\ell \geq 2$ such a curve is not radially symmetric if $|\varepsilon|$ is small enough. However, for $\ell = 1$ this curve may correspond to the curve obtained above by ε-translation of a radial solution. Since we are interested in symmetry-breaking bifurcations, we restrict ourselves to $\ell \geq 2$.

From (3.3), (3.4) we have

$$
\bar\sigma = \frac{\tilde\sigma}{2\Lambda} = \frac{\tilde\sigma}{2}\,R_0\,\frac{I_0(R_0)}{I_1(R_0)}
$$

so that the eigenvalue equation (4.13) can be written in the form

$$
(4.14) \qquad H(R_0) \equiv R_0^4 \left[\frac{I_2(R_0)}{I_1(R_0)} - \frac{I_{\ell+1}(R_0)}{I_\ell(R_0)} \right] = N
$$

where

$$
N = \frac{1}{\mu\tilde{\sigma}}\, 2\ell(\ell^2 - 1)\gamma_0 \ .
$$

By Lemma 2.4, $H(R_0)$ is strictly increasing in R_0, and in fact $H'(R_0) > 0$, so that for each γ_0 there is a unique R_0 satisfying (4.13). From (2.2) we easily deduce that $H(R_0) \to 0$ if $R_0 \to 0$, and from (2.7) we deduce that $H(R_0) \to \infty$ if $R_0 \to \infty$. Hence we have

Theorem 4.2. *The function* $\gamma_0 = \gamma(R_0)$ *determined by the eigenvalue equation* (4.13) *satisfies*

$$
(4.15) \qquad\qquad\qquad \gamma'(R_0) > 0 \ ,
$$
$$
(4.16) \qquad\qquad \lim_{R_0 \to 0} \gamma(R_0) = 0 \ , \qquad \lim_{R_0 \to \infty} \gamma(R_0) = \infty \ .
$$

In §8 we shall encounter linear systems of two equations with coefficient matrices A_m, where

$$
(4.17)
$$
$$
\det A_m = \mu\tilde{\sigma}R_0 + \frac{\mu}{I_m(R_0)} \int_0^1 I_m(R_0\tau)g_m(\tau)d\tau - m\left(\mu\frac{\tilde{\sigma}}{2}R_0 - \frac{\gamma_0}{R_0^2}(m^2 - 1) \right)
$$

with

$$
(4.18) \qquad g_m(\tau) = \left[m^2 I_1(R_0\tau) - 2R_0\tau I_0(R_0\tau) \right] \frac{\bar{\sigma}}{I_0(R_0)}.
$$

Using the relation $\tilde{\sigma}/\bar{\sigma} = 2I_1(R_0)/(R_0 I_0(R_0))$ (see (3.3), (3.4)) we can write

$$
\det A_m = \frac{\mu\bar{\sigma}R_0}{I_0(R_0)I_m(R_0)} \Bigg\{ -\frac{(m-2)}{R_0}I_1(R_0)I_m(R_0)
$$
$$
+ \frac{1}{R_0^2}\int_0^{R_0} \left[m^2 I_1(\tau) - 2\tau I_0(\tau) \right] I_m(\tau)d\tau + \frac{\gamma_0 m(m^2-1)}{\mu\bar{\sigma}R_0^3}I_0(R_0)I_m(R_0) \Bigg\}.
$$

Substituting γ_0 from (4.13) and using Lemma 2.5 and Corollary 2.6, we get

Lemma 4.3. *If* $\det A_m$ *is defined by* (4.17), (4.18), *then*

$$
\det A_\ell = 0, \quad and \quad \det A_m \neq 0 \quad for\ all \quad m \neq \ell.
$$

5. FORMAL EXPANSION

In this section we derive the formal expansion (1.7) of the bifurcation branch near $\gamma = \gamma_0$ in the following form:

$$(5.1) \quad \sigma = \sigma(r, \theta, \varepsilon) = \sum_{n \geq 0} \sigma_n(r, \theta) \varepsilon^n = \sigma_0(r) + \sum_{n \geq 1} \Big(\sum_{k \geq 0} \sigma_{nk}(\theta)(r - R_0)^k \Big) \varepsilon^n \ ,$$

$$(5.2) \quad p = p(r, \theta, \varepsilon) = \sum_{n \geq 0} p_n(r, \theta) \varepsilon^n = p_0(r) + \sum_{n \geq 1} \Big(\sum_{k \geq 0} p_{nk}(\theta)(r - R_0)^k \Big) \varepsilon^n \ ,$$

$$(5.3) \quad r = R_0 + f(\theta, \varepsilon) = R_0 + \varepsilon \cos \ell\theta + \sum_{n \geq 2} f_n(\theta) \, \varepsilon^n \ ,$$

$$(5.4) \quad \gamma = \gamma(\varepsilon) = \gamma_0 + \sum_{n \geq 1} \gamma_n \varepsilon^n$$

where the radius R_0 and the integer ℓ are arbitrarily fixed. The functions σ and p must satisfy the equations (1.2) and (1.3) subject to the boundary conditions

$$\sigma\Big(R_0 + f(\theta, \varepsilon), \theta, \varepsilon\Big) = \bar{\sigma} \ , \qquad \frac{\partial p}{\partial n_\varepsilon}\Big(R_0 + f(\theta, \varepsilon), \theta, \varepsilon\Big) = 0$$

and

$$p\Big(R_0 + f(\theta, \varepsilon), \theta, \varepsilon\Big) = \gamma(\varepsilon)\kappa_\varepsilon$$

where n_ε and κ_ε denote the unit normal vector and curvature of the boundary $\{r = R_0 + f(\theta, \varepsilon)\}$, respectively.

Recall that the curvature $\kappa(g)$ of a curve $r = g(\theta)$ is given by (4.9) and note that

$$\frac{\partial}{\partial n} = \frac{1}{(g^2 + (g')^2)^{1/2}} \left(g\frac{\partial}{\partial r} - \frac{g'}{g}\frac{\partial}{\partial \theta} \right)$$

along the curve $r = g(\theta)$. Hence the boundary conditions (1.3)– (1.5) can be written in the form

$$(5.5) \quad \sum_{n \geq 1} \varepsilon^n \sum_{k \geq 0} \sigma_{nk}(\theta) \Big(\sum_{i \geq 1} f_i(\theta)\varepsilon^i \Big)^k + \sum_{k \geq 1} \sigma_{0k} \Big(\sum_{i \geq 1} f_i(\theta)\varepsilon^i \Big)^k = 0 \ ;$$

$$(R_0 + f(\theta, \varepsilon))^2 \frac{\partial p}{\partial r}(R_0 + f(\theta, \varepsilon), \theta, \varepsilon) - f'(\theta, \varepsilon) \frac{\partial p}{\partial \theta}(R_0 + f(\theta, \varepsilon), \theta, \varepsilon) = 0$$

or

$$\Big(R_0 + \sum_{i \geq 1} f_i \varepsilon^i \Big)^2 \left\{ \sum_{n \geq 1} \varepsilon^n \sum_{k \geq 1} k \, p_{nk}(\theta) \Big(\sum_{i \geq 1} f_i \varepsilon^i \Big)^{k-1} + \sum_{k \geq 2} k p_{0k} \Big(\sum_{i \geq 1} f_i \varepsilon^i \Big)^{k-1} \right\}$$

$$(5.6) \quad - \Big(\sum_{i \geq 1} f_i' \varepsilon^i \Big) \sum_{n \geq 1} \varepsilon^n \left[\sum_{k \geq 0} p_{nk}'(\theta) \Big(\sum_{i \geq 1} f_i \varepsilon^i \Big)^k \right] = 0 \ ;$$

and

$$\big[(R_0 + f(\theta, \varepsilon))^2 + (f'(\theta, \varepsilon))^2\big]^{3/2} p(R_0 + f(\theta, \varepsilon), \theta, \varepsilon)$$
$$= \gamma(\varepsilon) \big[(R_0 + f(\theta, \varepsilon))^2 + 2(f'(\theta, \varepsilon))^2 - (R_0 + f(\theta, \varepsilon)) f''(\theta, \varepsilon)\big] \ ,$$

or

$$(5.7) \qquad \left[\left(R_0 + \sum_{i \geq 1} f_i \varepsilon^i \right)^2 + \left(\sum_{i \geq 1} f_i' \varepsilon^i \right)^2 \right]^{3/2} \sum_{n \geq 0} \varepsilon^n \left[\sum_{k \geq 0} p_{nk} \left(\sum_{i \geq 1} f_i \varepsilon^i \right)^k \right]$$

$$= \left(\sum_{i \geq 0} \gamma_i \varepsilon^i \right) \left[\left(R_0 + \sum_{i \geq 1} f_i \varepsilon^i \right)^2 + 2 \left(\sum_{i \geq 1} f_i' \varepsilon^i \right)^2 - \left(R_0 + \sum_{i \geq 1} f_i \varepsilon^i \right) \left(\sum_{i \geq 1} f_i'' \varepsilon^i \right) \right] .$$

Comparing the coefficients of ε^n we get

$$(5.8) \qquad \sigma_{n0} + \sigma_{01} \, f_n(\theta) = F^1(\theta) \qquad (\sigma_{01} = \alpha; \text{ cf. } (3.7))$$

where

$$F^1(\theta) = -\frac{1}{n!} \partial_\varepsilon^n \Big|_{\varepsilon=0} \left[\sum_{m=1}^{n-1} \varepsilon^m \sum_{k \geq 0} \sigma_{mk}(\theta) \left(\sum_{i \geq 1} f_i \varepsilon^i \right)^k + \sum_{k \geq 2} \sigma_{0k} \left(\sum_{i \geq 1} f_i \varepsilon^i \right)^k \right] ,$$

$$(5.9) \qquad p_{n1} + 2p_{02} f_n = F^2(\theta) \qquad (2p_{02} = -\beta; \text{ cf. } (3.9))$$

where

$$F^2(\theta) = -\frac{1}{R_0^2} \frac{1}{n!} \partial_\varepsilon^n \Big|_{\varepsilon=0} \Big[\left(R_0 + \sum_{i \geq 1} f_i \varepsilon^i \right)^2 \sum_{m=1}^{n-1} \varepsilon^m \left[\sum_{k \geq 1} k p_{mk} \left(\sum_{i \geq 1} f_i \varepsilon^i \right)^{k-1} \right.$$

$$+ \left. \sum_{k \geq 3} k p_{0k} \left(\sum_{i \geq 1} f_i \varepsilon^i \right) \right] - \left(\sum_{i \geq 1} f_i' \varepsilon^i \right) \sum_{m=1}^{n-1} \varepsilon^m \sum_{k \geq 0} p_{mk}' \left(\sum_{i \geq 1} f_i \varepsilon^i \right)^k \Big] ,$$

and, writing $p_{n-1,0} = \left(p_{n-1,0} - \frac{\gamma_{n-1}}{R_0} \right) + \frac{\gamma_{n-1}}{R_0}$ in (5.7),

$$(5.10) \qquad \begin{aligned} &\frac{\gamma_{n-1}}{R_0} \frac{3}{R_0} f_1 + p_{n0} + \frac{3}{R_0} f_n p_{00} = \frac{\gamma_n}{R_0} + \frac{\gamma_0}{R_0^3} (2R_0 f_n - R_0 f_n'') \\ &\qquad + \frac{\gamma_{n-1}}{R_0^3} (2R_0 f_1 - R_0 f_1'') + F^3(\theta) \qquad \left(p_{00} = \frac{\gamma_0}{R_0} , p_{01} = 0 \right) \end{aligned}$$

where

$$F_3(\theta) = \frac{1}{R_0^3} \frac{1}{n!} \partial_\varepsilon^n \Big|_{\varepsilon=0} \Big\{ \left(\sum_{i=1}^{n-2} \gamma_i \varepsilon^i \right) \Big[\left(R_0 + \sum_{i=1}^{n-1} f_i \varepsilon^i \right)^2 + 2 \left(\sum_{i=1}^{n-1} f_i' \varepsilon^i \right)^2$$

$$- \left(R_0 + \sum_{i=1}^{n-1} f_i \varepsilon^i \right) \left(\sum_{i=1}^{n-1} f_i'' \varepsilon^i \right) \Big] - \Big[\left(R_0 + \sum_{i=1}^{n-1} f_i \varepsilon^i \right)^2 + \left(\sum_{i=1}^{n-1} f_i' \varepsilon^i \right)^2 \Big]^{3/2}$$

$$\times \Big[\sum_{m=0}^{n-2} \varepsilon^m \sum_{k \geq 0} p_{mk} \left(\sum_{i \geq 1} f_i \varepsilon^i \right)^k + \varepsilon^{n-1} \left(\left(p_{n-1,0} - \frac{\gamma_{n-1}}{R_0} \right) + p_{n-1,1} f_1 \varepsilon \right) \Big] \Big\} .$$

Thus, (σ_n, p_n, f_n) will satisfy

$$(5.11) \qquad \Delta \sigma_n - \sigma_n = 0 , \quad \Delta p_n = -\mu \sigma_n \qquad \text{in} \quad r < R_0$$

and, from (5.8)–(5.10),

$$\sigma_n(R_0, \theta) + \alpha f_n(\theta) = F^1(\theta),$$

(5.12)
$$\frac{\partial p_n}{\partial r}(R_0, \theta) - \beta f_n(\theta) = F^2(\theta),$$

$$p_n(R_0, \theta) + \frac{\gamma_0}{R_0^2}(f_n + f_n'') = \frac{\gamma_n}{R_0} + \frac{\gamma_{n-1}}{R_0^2}(\ell^2 - 1)\cos\ell\theta + F^3(\theta).$$

We shall inductively show that functions $\sigma_n(r, \theta)$, $p_n(r, \theta) - \gamma_n/R_0$ and $f_n(\theta)$ together with numbers γ_{n-1} can be chosen so as to satisfy (5.11)–(5.12) for each n and this choice is unique under the orthogonality condition (1.12), (1.13) and the condition (1.14).

Thus we assume that for $m < n, f_m, p_{m0} - \gamma_m/R_0, \sigma_{m0}$, and $p_{mk}, \sigma_{mk}(k \geq 1)$ have been determined and prove that the functions $f_n, p_{n0} - \gamma_n/R_0, \sigma_{n0}, p_{nh}(h \geq 1)$, $\sigma_{nh}(h \geq 1)$ and the constant γ_{n-1} are uniquely determined (subject to the orthogonality conditions (1.12), (1.13) and the condition (1.14)). Note that once γ_{n-1} is determined, so is also $p_{n-1,0}$ as the difference $p_{n-1,0} - \gamma_{n-1}/R_0$ was already determined inductively.

The inductive proof follows from the following general lemma.

Lemma 5.1. *Let* $F^1 \in C^{2+\alpha}$, $F^2 \in C^{1+\alpha}$, $F^3 \in C^\alpha$ *be even functions of* θ *satisfying*

(5.13)
$$\int_0^{2\pi} F^i(\theta) \cos\theta \, d\theta = 0,$$

and consider the problem

$$\Delta\sigma - \sigma = 0, \qquad \Delta p = -\mu\sigma \qquad in \quad r < R_0$$

with boundary conditions

$$\sigma + \alpha f = F^1, \qquad \frac{\partial p}{\partial r} - \beta f = F^2,$$

$$p + \frac{\gamma_0}{R_0^2}(f + f'') = F^3 + \frac{\gamma}{R_0^2}(\ell^2 - 1)\cos\ell\theta.$$

Then there exists a unique solution (σ, p, f, γ) *whose components are even functions of* θ *and such that*

(5.14)
$$\int_0^{2\pi} f(\theta) \cos m\theta \, d\theta = 0 \qquad for \quad m = 1, \ell,$$

$$\int_0^{2\pi} \sigma(r, \theta) \cos\theta \, d\theta = \int_0^{2\pi} p(r, \theta) \cos\theta \, d\theta = 0;$$

moreover,

$$|\sigma|_{2+\alpha} + |p|_{2+\alpha} + |f|_{2+\alpha} \leq C(|F^1|_{2+\alpha} + |F^2|_{1+\alpha} + |F^3|_\alpha) \quad and$$

(5.15)
$$|\gamma| \leq C \sum_{i=1}^3 \left| \int_0^{2\pi} F^i(\theta) \cos\ell\theta \, d\theta \right|.$$

Here $|\sigma|_{2+\alpha}$ and $|p|_{2+\alpha}$ are the $C^{2+\alpha}$ norms in the disc $r < R_0$, whereas $|f|_{2+\alpha}$ is the $C^{2+\alpha}$ norm with respect to θ.

Proof. We first establish uniqueness. Since the F^i and σ, p, f are even functions of θ, it follows from (5.13) and (5.14) that

$$\sigma = \sum_{m \neq 1} A_m \cos m\theta \cdot I_m(r) ,$$

$$p = \sum_{m \neq 1} \left[B_m \cos m\theta \cdot r^m - \mu A_m \cos m\theta \cdot I_m(r) \right] ,$$

$$f = \sum_{m \neq 1} \tau_m \cos m\theta$$

and

$$F^i = \sum_{m \neq 1} F_m^i \cos m\theta ;$$

we do not impose as yet the orthogonality condition $\int_0^{2\pi} f(\theta) \cos \ell\theta d\theta = 0$.

The boundary conditions then yield, for $m \neq \ell$,

$$A_m I_m(R_0) + \alpha\tau_m = F_m^1 ,$$

(5.16)
$$-\mu A_m I'_m(R_0) + B_m R_0^{m-1} m - \beta\tau_m = F_m^2 ,$$

$$-\mu A_m I_m(R_0) + B_m R_0^m - \frac{\gamma_0}{R_0^2}(m^2 - 1)\tau_m = F_m^3 ,$$

or

$$T_m \begin{pmatrix} A_m \\ B_m \\ \tau_m \end{pmatrix} = \begin{pmatrix} F_m^1 \\ F_m^2 \\ F_m^3 \end{pmatrix}$$

where T_m is given by

(5.17)
$$T_m = \begin{pmatrix} I_m(R_0) & 0 & \alpha \\ -\mu I'_m(R_0) & mR_0^{m-1} & -\beta \\ -\mu I_m(R_0) & R_0^m & -\frac{\gamma_0}{R_0^2}(m^2 - 1) \end{pmatrix} .$$

Lemma 5.2. *There holds*

(5.18)
$$\det T_1 = 0 , \quad \det T_\ell = 0$$

and

(5.19)
$$\det T_m \neq 0 \quad if \quad m \neq 1 , \ m \neq \ell;$$

furthermore,

(5.20)
$$|\det T_m| \geq C_1 R_0^m m^3 I_m(R_0) \quad \forall \ m \neq 1, \ell$$

where C_1 is a positive constant independent of m.

Proof. Adding $(-m/R_0) \times$ (third row) to the second row we get (cf. (2.4))

$$-R_0^{-m} \det T_m = \begin{vmatrix} I_m(R_0) & \alpha \\ -\mu I_{m+1}(R_0) & -\beta + \frac{\gamma_0}{R_0^3} m(m^2 - 1) \end{vmatrix}$$

$$= I_m(R_0) \left\{ -\beta + \frac{\gamma_0}{R_0^3} m(m^2 - 1) + \mu\alpha \frac{I_{m+1}(R_0)}{I_m(R_0)} \right\} .$$

By (3.7), (3.9) the function in braces is equal to

$$\frac{\mu\bar{\sigma}}{I_0(R_0)}\Big\{m(m^2-1)\frac{I_0(R_0)}{\mu\bar{\sigma}R_0^3}\gamma_0 + \frac{I_1(R_0)\,I_{m+1}(R_0)}{I_m(R_0)} - I_2(R_0)\Big\}\;,$$

and by (4.13) it is equal to zero if $m = \ell$; it is clearly also equal to zero if $m = 1$. Note also that the expression in braces is $\neq 0$ if $m = 0$ since $I_1^2/I_0 \neq I_2$ (by Lemma 2.1), and so $\det T_0 \neq 0$.

Recalling the definition of $f(m)$ in (2.12) we can write, for $m \neq 0, 1$,

$$|\det T_m| = R_0^m\,I_m(R_0)\,\frac{\mu\bar{\sigma}}{I_0(R_0)}\,m(m^2-1)\Big|f(m) - \frac{\gamma_0\,I_0(R_0)}{\mu\bar{\sigma}R_0^3}\Big|\;.$$

Using Lemma 2.3 we conclude that (5.19) holds. Furthermore, since $I_{m+1}(r) < \frac{r}{2}\,I_m(r)$, $f(m) \to 0$ if $m \to \infty$ and, consequently, (5.20) is valid. $\qquad\square$

Returning to the proof of Lemma 5.1 we note that from the last two equations in (5.16) it follows that

$$\Big[-\mu\,I_m'(R_0)R_0 + \mu\,I_m(R_0)m\Big]A_m + \Big[m(m^2-1)\frac{\gamma}{R_0^2} - \beta R_0\Big]\tau_m = F_m^2 R_0 - mF_m^3$$

and, by (2.4), the coefficient of A_m is equal to $-\mu\,R_0 I_{m+1}(R_0)$. Solving this equation together with the first equation in (5.16) we find that, for $m \geq 1$,

$$\begin{aligned}
|A_m| &\leq \frac{R_0^{m-1}}{|\det T_m|}\Big|\Big[m(m^2-1)\frac{\gamma}{R_0^2} - \beta R_0\Big]F_m^1 - \alpha(F_m^2 R_0 - mF_m^3)\Big| \\
&\leq \frac{C}{|I_m|}\Big[|F_m^1| + \frac{|F_m^2|}{m^3} + \frac{|F_m^3|}{m^2}\Big]\;, \\
|\tau_m| &\leq C\Big[|F_m^1|\frac{|I_{m+1}|}{|I_m|m^3} + \frac{|F_m^2|}{m^3} + \frac{|F_m^3|}{m^2}\Big]\;.
\end{aligned}$$

Finally, the third equation in (5.16) gives

$$\begin{aligned}
|B_m| &\leq \frac{C}{R_0^m}\big[\,|A_m|I_m + m^2|\tau_m| + |F_m^3|\,\big] \\
&\leq \frac{C}{R_0^m}\Big[|F_m^1| + |F_m^1|\frac{I_{m+1}}{I_m m} + \frac{|F_m^2|}{m} + |F_m^3|\Big]\;.
\end{aligned}$$

Using the relation [16, p. 225]

$$(5.21)\qquad I_m(z) = i^{-m}\sqrt{\frac{1}{2\pi m}}\Big(\frac{eiz}{2m}\Big)^m\Big(1+O\Big(\frac{1}{m}\Big)\Big)\qquad m \to \infty\;,$$

if we write

$$\sigma = \sum_{m\neq 1}\tilde{A}_m\cos m\theta\,\frac{I_m(r)}{I_m(R_0)}\;,$$

$$(5.22)\qquad p = \sum_{m\neq 1}\Big[\tilde{B}_m\cos m\theta\Big(\frac{r}{R_0}\Big)^m - \mu\,\tilde{A}_m\cos m\theta\,\frac{I_m(r)}{I_m(R_0)}\Big]\;,$$

$$f = \sum_{m\neq 1}\tilde{\tau}_m\cos m\theta,$$

then, for $m \neq \ell$,

$$|\tilde{A}_m| \leq C\Big[|F_m^1| + \frac{|F_m^2|}{m^3} + \frac{|F_m^3|}{m^2}\Big] ,$$

(5.23)
$$|\tilde{B}_m| \leq C[|F_m^1| + \frac{F_m^2}{m} + |F_m^3|] ,$$

$$|\tilde{\tau}_m| \leq C\Big[\frac{|F_m^1|}{m^4} + \frac{|F_m^2|}{m^3} + \frac{|F_m^3|}{m^2}\Big] .$$

In case $m = \ell$ we have $\det T_\ell = 0$ and the solvability condition becomes

$$F_\ell^1\Big[\ell(\ell^2 - 1)\frac{\gamma_0}{R_0^2} - \beta R_0\Big] = \alpha\Big[F_\ell^2 R_0 - \ell F_\ell^3 - \ell(\ell^2 - 1)R_0^{-2}\gamma\Big] .$$

This defines γ uniquely, and also establishes the last estimate in (5.15). However, the system for $(A_\ell, B_\ell, \tau_\ell)$ has infinitely many solutions, obtained by taking a special solution and adding a multiple of $\cos \ell\theta$. By imposing the orthogonality condition

$$\int_0^{2\pi} f(\theta) \cos \ell\theta \, d\theta = 0$$

we get uniqueness. The orthogonality condition means that $\tilde{\tau}_\ell = 0$, and then

$$A_\ell = \frac{F_\ell^1}{I_\ell(R_0)} , \qquad B_\ell = (F_\ell^3 + \mu F_\ell^1)/R_0^\ell ,$$

or

(5.24)
$$\tilde{A}_\ell = F_\ell^1 , \qquad \tilde{B}_\ell = F_\ell^3 + \mu F_\ell^1 , \qquad \tilde{\tau}_\ell = 0 .$$

The above proof actually establishes also existence in the L^2 framework. Indeed, with $\tilde{A}_m, \tilde{B}_m, \tilde{\tau}_m$ defined as above, the estimates (5.23) show that (σ, p, f) form a solution in a weak sense, and

(5.25)
$$\int_0^{2\pi} \Big[|\sigma(R_0, \theta)|^2 + |p(R_0, \theta)|^2\Big] d\theta \leq C \int_0^{2\pi} \sum (F^i)^2 \, d\theta .$$

We shall now establish the $(2 + \alpha)$-estimates asserted in (5.15).

By Schauder's estimate

(5.26)
$$|\sigma|_{2+\alpha} + |p|_{2+\alpha} \leq C\Big(|F^1|_{2+\alpha} + |F^2|_{1+\alpha} + |f|_{2+\alpha} + |p|_0\Big) .$$

On the other hand, from the third boundary condition for the system satisfied by (σ, p, f, γ) we deduce that

(5.27)
$$f(\theta) = \int_0^\theta \sin(\theta - s) h(s)ds + a \cos \theta + b \sin \theta$$

where

(5.28)
$$h(s) = \Big[F^3(s) + \frac{\gamma}{R_0^2} (\ell^2 - 1) \cos \ell s - p(R_0, s) \Big]\frac{R_0^2}{\gamma_0}.$$

Since

$$\int_0^{2\pi} h(s) \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds = 0 ,$$

the requirement that

$$\int_0^{2\pi} f(\theta) \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} d\theta = 0$$

gives

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2\pi} \int_0^{2\pi} s\, h(s) \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds \ .$$

Using this in (5.27) we can estimate f:

$$|f|_{2+\alpha} \le C|h|_\alpha \le C\Big[|F^3|_\alpha + |\gamma| + |p(R_0, \cdot)|_\alpha\Big] \ .$$

Recalling the estimate (5.15) on γ and using also the inequality

$$(5.29) \qquad |p(R_0, \cdot)|_\alpha \le \varepsilon|p|_{2+\alpha} + C|p(R_0, \cdot)|_{L^2} \qquad \forall\, \varepsilon > 0 \ ,$$

and (5.25), we conclude that

$$|f|_{2+\alpha} \le C\Big[|F^3|_\alpha + |F^1|_0 + |F^2|_0 + \varepsilon|p|_{2+\alpha}\Big] \ .$$

If we substitute this into (5.26) and again use (5.25), we obtain the assertion (5.15).
□

Remark 5.1. From the second inequality in (5.15) it follows that $\gamma_{2k+1} = 0$ for all integers $k \ge 0$. Indeed, since $n = 2k + 2 = 2(k + 1)$ is even, the right-hand sides $F^i(\theta)$ in (5.12) for this value of n do not contain terms with wavenumber ℓ.

6. Two approaches for proving convergence

In §5 we determined an asymptotic series for the solution of (1.1)–(1.5), whose successive terms are defined as the solutions to the problems (5.11)–(5.12). As we said, a natural approach to a proof of convergence of the series suggests itself, namely the recursive estimation of the terms with the help of Lemma 5.1. This, of course, requires the estimation of the homogeneous terms F^i at every step of the recursion and, from §5, we know that these functions can be expressed as sums

$$(6.1) \qquad \sum_q F_q^i$$

where each F_q^i is a product of the unknowns corresponding to previous stages of the recursive procedure and can therefore be estimated. Thus, at the n-th stage, the solution (σ_n, p_n, f_n) can be written as

$$(6.2) \qquad (\sigma_n, p_n, f_n) = \sum_q (\sigma_n^q, p_n^q, f_n^q)$$

where $(\sigma_n^q, p_n^q, f_n^q)$ solves (5.11)–(5.12) with F^i replaced by F_q^i. Lemma 5.1 then would give, for instance,

$$(6.3) \qquad |\sigma_n|_{2+\alpha} \le \sum_q |\sigma_n^q|_{2+\alpha} \le C \sum_q \Big(|F_1^q|_{2+\alpha} + |F_2^q|_{1+\alpha} + |F_3^q|_\alpha\Big)$$

and similar estimates for $|p_n|_{2+\alpha}$ and $|f_n|_{2+\alpha}$. Unfortunately, the recursive bounds on (σ_n, p_n, f_n) that result from this approach do not imply the convergence of the series. Indeed, the need to expand F^i as in (6.1) to derive a bound for use in (6.3) precludes this procedure from exploiting substantial cancellations that are present in (6.1). We illustrate this surprising phenomenon with a simple example.

Consider the elliptic problem (a "scattering problem" at zero frequency) for a function $u = u(x, y)$:

$$(6.4) \qquad \Delta u = 0 \quad \text{in } y > \varepsilon f(x) \ ,$$

$$(6.5) \qquad u = H(x) \quad \text{on } y = \varepsilon f(x)$$

where $f(x), H(x)$ are analytic functions, say

$$H(x) = \sum_{s=-F}^{F} h_s e^{isx} \ ,$$

$$(6.6) \qquad f(x) = 2\cos x = e^{ix} + e^{-ix} \ ;$$

we seek a solution which is 2π-periodic in x. By extending the problem to complex values of ε it was shown in [4] that there exists an analytic solution $u(x, y, \varepsilon)$ for $y \geq 0, |\varepsilon|$ small, which remains bounded as $y \to \infty$ (see also [3]). We can then write

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n(x, y)$$

where the series is convergent for small $|\varepsilon|$. Let us now proceed to determine the coefficients u_n, and to estimate them, by the method of §5. Expanding the relation

$$u(x, \varepsilon f(x), \varepsilon) = H(x)$$

about $\varepsilon = 0$, by Taylor's series, we obtain

$$(6.7) \qquad u_n(x, 0) + \sum_{k=0}^{n-1} \frac{f^{n-k}}{(n-k)!} \partial_y^{n-k} u_k(x, 0) = 0 \ , \ n \geq 1 \ ,$$

$$u_0(x, 0) = H(x).$$

Writing

$$(6.8) \qquad \frac{f(x)^k}{k!} = \sum_{r=-k}^{k} c_{k,r} e^{irx} \ ,$$

$$(6.9) \qquad u_m(x, y) = \sum_{r=-F-m}^{F+m} d_{m,r} e^{irx - |r|y} \ ,$$

we find that

$$u_0(x, y) = \sum_{s=-F}^{F} h_s e^{isx - |s|y}$$

and

$$\sum_{r=-F-n}^{F+n} d_{n,r} e^{irx} + \sum_{k=0}^{n-1} \left(\sum_{s=-(n-k)}^{n-k} c_{n-k,s} e^{isx} \right) \sum_{q=-(F+k)}^{F+k} d_{k,q} (-|q|)^{n-k} e^{iqx} \ ,$$

so that

$$(6.10) \qquad d_{n,r} + \sum_{k=0}^{n-1} \sum_{q=-(F+n-1)}^{F+n-1} c_{n-k,r-q} d_{k,q} (-|q|)^{n-k} = 0 \ , \ n \geq 1 \ .$$

This equation is analogous to a modal form of (5.11)–(5.12). Let us take, in particular, $r = F + n$ and set $D_n = d_{n,F+n}$. Note that if $c_{n-k,F+n-q}d_{k,q} \neq 0$, then $F + n - q \leq n - k$ so that $F + k \leq q$. Since also $q \leq k + F$, we have $q = k + F$ and

$$c_{n-k,F+n-q} = c_{n-k,n-k} = \frac{1}{(n-k)!} \quad \text{by } (6.6), (6.8).$$

Thus (6.10) gives

$$(6.11) \qquad D_n = -\sum_{k=0}^{n-1} \frac{D_k}{(n-k)!} (-(k+F))^{n-k} \ ,$$

whereas $D_0 = h_F$.

Multiplying (6.11) by x^n and summing over n, we get

$$\sum_{k \geq 0} x^k D_k \sum_{n \geq k} \frac{x^{n-k}}{(n-k)!} (-(k+F))^{n-k} = h_F \ ,$$

or

$$\sum_{k \geq 0} x^k D_k e^{-(k+F)x} = h_F \ .$$

Hence

$$\sum_{k \geq 0} x^k D_k e^{-kx} = h_F e^{Fx} \ .$$

Setting $D(z) = \sum_{k \geq 0} D_k z^k$ it follows that

$$(6.12) \qquad D(xe^{-x}) = h_F e^{Fx} \ .$$

Now introduce (by the implicit function theorem) the analytic function $\lambda(\xi)$ for $|\xi|$ small by

$$\lambda(xe^{-x}) = x \ .$$

Then we can rewrite (6.12) in the form

$$D(y) = h_F e^{F\lambda(y)} \ , \quad |y| \text{ small,}$$

and since the right-hand side is analytic, we deduce that

$$(6.13) \qquad \left| D_n \right| \leq C_0 H^n \quad \text{for } n \geq 0$$

where C_0, H are positive constants. This estimate is of course also a consequence of the analyticity result in [4] obtained by working with complex ϵ.

On the other hand, the analog of (6.3) in this case corresponds to the majorizing recursion

$$(6.14) \qquad \Delta_n = \sum_{k=0}^{n-1} \frac{\Delta_k}{(n-k)!} (k+F)^{n-k}$$

obtained from (6.11) by taking absolute values and using the triangle inequality. We shall prove

$$(6.15) \qquad \text{limsup}_{k \to \infty} |\Delta_k|^{1/k} = \infty \ ;$$

this will show that the estimate (6.13) (implied by analyticity) cannot be obtained in this manner. We take for simplicity $F = 1$. Then, for any $M > 1$,

$$\Delta_{nM} \geq \Delta_{n(M-1)} \frac{(n(M-1)+1)^n}{n!} \geq \frac{M^n}{c^n} \Delta_{n(M-1)}$$

where c is a positive constant independent of M, n. By repeated application we get

$$\Delta_{nM} \geq \frac{(M!)^n}{c^{nM}} \Delta_n .$$

Then also

$$\Delta_{M^n} \geq \frac{(M!)^{M^{n-1}}}{c^{M^n}} \Delta_{M^{n-1}} \geq \cdots \geq \frac{(M!)^{M^{n-1}+\cdots+M}}{c^{M^n+\cdots+M^2}} \Delta_M ,$$

and this is true for any M, n. For any $c_1 > c$ we can then choose M large enough so that

$$\Delta_{M^n} \geq \frac{(M!)^{M^{n-1}}}{(c_1)^{M^n}} \Delta_M .$$

Consequently,

$$(\Delta_{M^n})^{1/M^n} \geq \frac{(M!)^{1/M}}{c_1} (\Delta_M)^{1/M^n} \geq \frac{M}{c_2} \quad \text{if} \ \ n \to \infty ,$$

for any constant $c_2, c_2 > ec_1$. Since M is arbitrary, (6.15) follows.

The above example indicates that a proof of analyticity based on the recursive relations (6.7) must be extremely delicate, for one must take into account subtle cancellations that should occur in (6.10). The same phenomenon is expected of course of the much more complicated free boundary problem (1.1)–(1.5). We shall therefore use another approach. Let us describe it using again the simple example (6.4), (6.5).

We make a change of variables

(6.16) $$z = y - \varepsilon f(x)$$

which flattens the boundary, and introduce the function

(6.17) $$v(x, z, \varepsilon) = u(x, y, \varepsilon).$$

Then v satisfies the elliptic equation

(6.18) $$\mathcal{L}_\varepsilon v \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} - 2\varepsilon f' \frac{\partial^2 v}{\partial x \partial z} + \varepsilon^2 (f')^2 \frac{\partial^2 v}{\partial z^2} - \varepsilon f'' \frac{\partial v}{\partial z} = 0 \quad \text{in} \ \ z > 0,$$

(6.19) $$v(x, 0, \varepsilon) = H(x) .$$

We can now determine a formal series

(6.20) $$v(x, z, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n v_n(x, z)$$

by applying $D_\varepsilon^n|_{\varepsilon=0}$ to (6.18) successively. The function $v_n \equiv \frac{1}{n!} D_\varepsilon^n v|_{\varepsilon=0}$ solves a system

(6.21) $$\mathcal{L}_0 v_n = F_n \quad \text{in} \ \ z > 0,$$

(6.22) $$v_n = 0 \quad \text{if} \ \ n \geq 1, \quad v_0 = H(x) \quad \text{on} \ \ z = 0$$

where F_n depends on the v_m with $m < n$. Using elliptic estimates we may then try to prove by induction on n that

$$(6.23) \qquad ||D_x^k D_\varepsilon^n v|_{\varepsilon=0}|| \leq C_0 A^k C^n (n+k)! \quad \forall k \geq 0,$$

where C_0, A, C are some positive constants and the norm $|| \ ||$ is appropriately defined. Once (6.23) has been established for all n, we can then use the differential equation for v to estimate also all the z-derivatives:

$$||D_z^j D_x^k D_\varepsilon^n v|_{\varepsilon=0}|| \leq C_0 B^j A^k C^n (n+k+j)!,$$

and this will establish the convergence in (6.20) and the analyticity of $v(x, z, \varepsilon)$ in (x, z, ε) for $-\infty < x < \infty, z \geq 0$ and $|\varepsilon|$ small.

This approach was used in [9] to prove analyticity up to the boundary for elliptic equations with boundary which is independent of ε. As we shall see in the following sections, this approach can be applied to the free boundary problem to establish analyticity of the solution σ, p, f in the *new variables*; once this has been done, we can go back to the original variables and deduce the analyticity of σ, p, f in the original variables. The "subtle cancellations" alluded to above in connection with the first approach to proving analyticity has thus been accomplished by the change to the new variables. Although we shall not be needing Lemma 5.1 in this second approach, some of the arguments used in the proof of the lemma will be useful.

7. CHANGE OF VARIABLES

The new approach, to be used throughout the rest of the paper, is based on transforming the free boundary problem (1.1)–(1.5) to a problem in the disc $\{r < 1\}$ by means of the change of variable

$$(7.1) \qquad r = \frac{r'}{R_0 + f(\theta, \varepsilon)} \ ;$$

of course $f(\theta, \varepsilon)$ is *unknown*. We define new functions $\widehat{\sigma}(r, \theta, \varepsilon), \widehat{p}(r, \theta, \varepsilon)$ by

$$(7.2) \qquad \widehat{\sigma}(r, \theta, \varepsilon) = \sigma(r', \theta, \varepsilon), \quad \widehat{p}(r, \theta, \varepsilon) = p(r', \theta, \varepsilon).$$

Then $\widehat{\sigma}, \widehat{p}$ satisfy the following system:

$$(7.3) \qquad \mathcal{L}(D)\widehat{\sigma} - (R_0 + f)^2 \widehat{\sigma} = 0 \quad \text{if} \ \ r < 1,$$

$$(7.4) \qquad \mathcal{L}(D)\widehat{p} + (R_0 + f)^2 \mu(\widehat{\sigma} - \tilde{\sigma}) = 0 \quad \text{if} \ \ r < 1 \ ,$$

$$(7.5) \qquad \widehat{\sigma} = \bar{\sigma} \quad \text{on} \ \ r = 1 \ ,$$

$$(7.6) \qquad (R_0 + f)\frac{\partial \widehat{p}}{\partial r} + \frac{(f_\theta)^2}{(R_0 + f)}\frac{\partial \widehat{p}}{\partial r} - f_\theta \frac{\partial \widehat{p}}{\partial \theta} = 0 \ \ \text{on} \ \ r = 1,$$

$$(7.7) \quad ((R_0 + f)^2 + f'^2)^{3/2}\widehat{p} = \gamma[(R_0 + f)^2 + 2(f_\theta)^2 - (R_0 + f)f_{\theta\theta}] \ \ \text{on} \ \ r = 1 \ ,$$

where

$$\mathcal{L}(D) = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{2f_\theta}{(R_0 + f)}\frac{1}{r}\frac{\partial^2}{\partial \theta \partial r}$$

$$(7.8) \qquad - \frac{f_{\theta\theta}}{(R_0 + f)}\frac{1}{r}\frac{\partial}{\partial r} + 2\frac{(f_\theta)^2}{(R_0 + f)^2}\frac{1}{r}\frac{\partial}{\partial r} + \frac{(f_\theta)^2}{(R_0 + f)^2}\frac{\partial^2}{\partial r^2}.$$

Our approach to establishing branches of analytic solutions is to apply D_ε^n ($n = 1, 2, \cdots$) to the system (7.3)–(7.7) at $\varepsilon = 0$ in order to get recursive relations for $D_\varepsilon^n \widehat{\sigma}, D_\varepsilon^n \widehat{p}, D_\varepsilon^n f, \gamma_{n-1}$ (at $\varepsilon = 0$) and then use a lemma analogous to Lemma 5.1 to estimate these derivatives. The right-hand sides that we shall obtain (and which

are assumed known by the inductive assumption) are much simpler than the right-hand sides that we had to deal with in §5; this is a critical fact, which will enable us to derive estimates on the derivatives that imply convergence and analyticity.

Taking $\varepsilon = 0$ in the above system we get

$$\widehat{\sigma} = S_0(r) = \bar{\sigma}\frac{I_0(R_0 r)}{I_0(R_0)} \ ,$$

$$\widehat{p} = P_0(r) = -\mu S_0(r) + A + \frac{\mu\tilde{\sigma}}{4}R_0^2 r^2, \ P_0(1) = \frac{\gamma_0}{R_0}$$

where γ_0 is the value of γ at $\varepsilon = 0$. If we linearize (7.2)–(7.7) about (S_0, P_0) by taking

$$\widehat{\sigma} = S_0 + \varepsilon S_1, \quad \widehat{p} = P_0 + \varepsilon P_1$$

where $r = 1 + \varepsilon \cos \ell\theta$ is the free boundary, we find that (S_1, P_1) is a nontrivial solution if and only if γ_0 is given by (4.13). Indeed, if

$$S_1 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[\sigma_0(r(R_0 + \varepsilon \cos \ell\theta)) + \varepsilon\sigma_1(r(R_0 + \varepsilon \cos \ell\theta), \theta)] \ ,$$

$$P_1 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[p_0(r(R_0 + \varepsilon \cos \ell\theta)) + \varepsilon p_1(r(R_0 + \varepsilon \cos \ell\theta), \theta]$$

and γ_0 is as in (4.13), then (S_1, P_1) is a solution of the linearized problem. Conversely, if (S_1, P_1) is a solution of the linearized problem corresponding to $f_1 = \cos \ell\theta$ and some $\gamma = \gamma'$, then

$$\sigma_1 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big[S_0(\frac{r}{R_0 + \varepsilon \cos \ell\theta}) + \varepsilon S_1(\frac{r}{R_0 + \varepsilon \cos \ell\theta}, \theta)\Big] \ ,$$

$$p_1 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big[P_0(\frac{r}{R_0 + \varepsilon \cos \ell\theta}) + \varepsilon P_1(\frac{r}{R_0 + \varepsilon \cos \ell\theta}, \theta)\Big]$$

form a solution of the linearized problem constructed in §4 and, consequently, by uniqueness, γ' must coincide with γ_0 as defined in (4.13).

We intend to prove that

$$(7.9) \qquad\qquad \widehat{\sigma}(r, \theta, \varepsilon) = S_0(r) + \sum_{n=1}^{\infty} \varepsilon^n S_n(r, \theta) \ ,$$

$$(7.10) \qquad\qquad \widehat{p}(r, \theta, \varepsilon) = P_0(r) + \sum_{n=1}^{\infty} \varepsilon^n P_n(r, \theta) \ ,$$

$$(7.11) \qquad\qquad f(\theta, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n f_n(\theta) \ ,$$

$$(7.12) \qquad \gamma = \gamma_0 + \gamma_2\varepsilon^2 + \gamma_4\varepsilon^n + \cdots \text{ (recall that } \gamma_{2k+1} = 0; \text{ see Remark 5.1)}$$

where the series converge for $r \leq 1 + \delta_0, |\varepsilon| \leq \varepsilon_0$ and define analytic functions in (x, y, ε) in the set $\{r \leq 1 + \delta_0 \text{ and } |\varepsilon| \leq \varepsilon_0\}$, for some $\varepsilon_0 > 0, \delta_0 > 0$.

To do this we shall first derive recursive formulas for

$$(7.13) \qquad S_m = D_\varepsilon^m \widehat{\sigma}|_{\varepsilon=0}, \ P_m = D_\varepsilon^m \widehat{p}|_{\varepsilon=0}, \ \wedge_m = D_\varepsilon^m f|_\varepsilon = 0 \ :$$

we assume that

$$(7.14) \qquad\qquad S_m, P_m - \gamma_m/R_0, \wedge_m \text{ and } \gamma_{m-1}$$

have already been determined for all $m < n$, and we shall proceed to determine these for $m = n$.

To this end we differentiate the system (7.3)–(7.7) n times with respect to ε and set $\varepsilon = 0$. Using the relations

$$\frac{\partial S_0}{\partial \theta} = \frac{\partial P_0}{\partial \theta} = 0 \ , \ \ f_\theta = 0 \ \text{ at } \ \varepsilon = 0 \ , \ \ P_0(1) = \gamma_0/R_0$$

we find that

$$(7.15) \qquad \Delta S_n - \frac{\wedge_{n,\theta\theta}}{R_0} \frac{1}{r} \frac{\partial S_0}{\partial r} - 2R_0 S_0 \wedge_n - R_0^2 S_n = F_n^1 \ , \ \ r < 1 \ ,$$

$$(7.16) \qquad \Delta P_n - \frac{\wedge_{n,\theta\theta}}{R_0} \frac{1}{r} \frac{\partial P_0}{\partial r} + 2R_0 \mu (S_0 - \tilde{\sigma}) \wedge_n + \mu R_0^2 S_n = F_n^2 \ , \ \ r < 1 \ ,$$

$$(7.17) \qquad\qquad\qquad\qquad S_n = 0 \ \ \text{on} \ \ r = 1 \ ,$$

$$(7.18) \qquad\qquad\qquad\qquad \frac{\partial P_n}{\partial r} = F_n^3 \ \ \text{on} \ \ r = 1 \ ,$$

$$(7.19) \qquad P_n + \frac{\gamma_0}{R_0^2}(\wedge_n + \wedge_{n,\theta\theta}) = \frac{\gamma_{n-1}}{R_0^2}(\ell^2 - 1)\cos\ell\theta + \frac{\gamma_n}{R_0} + F_n^4 \ \ \text{on} \ \ r = 1$$

where the F_n^j depend only on the functions (7.14) for $m < n$.

Note that the term $(\gamma_{n-1}/R_0)(\ell^2 - 1)\cos\ell\theta$ appears on the right-hand side of (7.19) in precisely the same way that it appeared in the third equation of (5.12); F_n^4 includes the term $P_{n-1} - \gamma_{n-1}/R_0$.

In the next section we prove an analog of Lemma 5.1 for the system (7.15)–(7.19), but with Sobolev norms instead of Hölder norms. This lemma will enable us, in §10, to prove convergence and joint analyticity (in (x, y) and ε) of the series (7.9)–(7.12).

8. A FUNDAMENTAL LEMMA

Set $B = \{r < 1\}, \partial B = \{r = 1\}$. We shall need the following calculus type lemma:

Lemma 8.1. *Let u be any function in $H^2(B)$ which is even in θ and satisfies*

$$(8.1) \qquad\qquad\qquad \int_0^{2\pi} u\cos\theta d\theta = 0 \ ,$$

$$(8.2) \qquad\qquad\qquad u|_{\partial B} = 0 \ .$$

Then

$$\frac{1}{C}||u||_{H^2(B)} \le ||u||_{L^2(B)} + ||\frac{1}{r}\frac{\partial u}{\partial r}||_{L^2(B)} + ||\frac{1}{r^2}\frac{\partial u}{\partial \theta}||_{L^2(B)}$$

$$(8.3) \qquad + ||\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}||_{L^2(B)} + ||\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}||_{L^2(B)} + ||\frac{\partial^2 u}{\partial r^2}||_{L^2(B)} \le C||u||_{H^2(B)}$$

where C is a universal constant.

Proof. Setting $f = \Delta u$ we can write

$$u = \sum_{n \ne 1} u_n(r)\cos n\theta \ ,$$

$$f = \sum_{n \ne 1} f_n(r)\cos n\theta$$

where

$$\frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r} - \frac{n^2}{r^2}u_n = f_n(r) \ , \quad u_n(1) = 0.$$

It follows that

$$(8.4) \qquad u_n(r) = r^n \int_1^r \frac{d\rho}{\rho^{2n+1}} \int_0^\rho \tau^{n+1} f_n(\tau) d\tau.$$

Consider first the case $n \geq 2$. Changing the order of integration we get

$$
\begin{aligned}
u_n(r) &= -r^n \int_0^1 \tau^{n+1} f_n(\tau) d\tau \int_{max(\tau,r)}^1 \rho^{-2n-1} d\rho \\
&= -r^n \int_0^r \tau^{n+1} f_n(\tau) \frac{(1 - r^{-2n})}{(-2n)} d\tau - r^n \int_r^1 \tau^{n+1} f_n(\tau) \frac{(1 - \tau^{-2n})}{(-2n)} d\tau \\
&= \frac{r^n}{2n} \int_0^1 \tau^{n+1} f_n(\tau) d\tau - \int_0^r \frac{1}{2n} r^{-n} \tau^{n+1} f_n(\tau) d\tau - \int_r^1 \frac{r^n}{2n} \tau^{-n+1} f_n(\tau) d\tau \\
&= \frac{1}{2n}(I_1 - I_2 - I_3).
\end{aligned}
$$

We have

$$
\begin{aligned}
\left(\int_0^1 \left(\frac{I_2}{r^2} \right)^2 r dr \right)^{1/2} &= \left(\int_0^1 \left(\int_0^r r^{-n} \tau^{n+1} f_n(\tau) d\tau \right)^2 \frac{1}{r^4} r dr \right)^{1/2} \\
&= \left(\int_0^1 \left(\int_0^1 u^{n+1} f_n(ru) du \right)^2 r dr \right)^{1/2} \quad (\tau = ru) \\
&\leq \int_0^1 u^{n+1} \left(\int_0^1 r(f_n(ru))^2 dr \right)^{1/2} du
\end{aligned}
$$

by the generalized Minkowski inequality. Substituting $r = \tau/u$ in the inner integral, we get

$$(8.5)$$
$$\left(\int_0^1 (\frac{I_2}{r^2})^2 r dr \right)^{1/2} \leq \int_0^1 u^n \left(\int_0^u f_n^2(\tau) \tau d\tau \right)^{1/2} du \leq \frac{1}{n+1} \left(\int_0^1 f_n^2(\tau) \tau d\tau \right)^{1/2}.$$

Similarly,

$$
\begin{aligned}
\left(\int_0^1 \left(\frac{I_3}{r^2} \right)^2 r dr \right)^{1/2} &= \left(\int_0^1 \left(\int_r^1 r^n \tau^{-n+1} f_n(\tau) d\tau \right)^2 \frac{1}{r^4} r dr \right)^{1/2} \\
&= \left(\int_0^1 \left(\int_r^1 u^{n-1} f_n(\frac{r}{u}) \frac{du}{u^2} \right)^2 r dr \right)^{1/2} \quad (\tau = \frac{r}{u}) \\
&\leq \int_0^1 u^{n-3} \left(\int_0^u (f_n(\frac{r}{u}))^2 r dr \right)^{1/2} du \\
&\leq \int_0^1 u^{n-3} \left(\int_0^1 u^2 f_n(\tau)^2 \tau d\tau \right)^{1/2} du \quad (r = \tau u) \\
&\leq \left(\int_0^1 u^{n-2} du \right) \left(\int_0^1 f_n(\tau)^2 \tau d\tau \right)^{1/2} \\
&= \frac{1}{n-1} \left(\int_0^1 f_n(\tau)^2 \tau d\tau \right)^{1/2}.
\end{aligned}
$$

Since $I_1 \leq I_2 + I_3$, we conclude that

$$(8.6) \qquad \left[\int_0^1 \frac{1}{r^4} \left(u_n(r) \right)^2 r dr \right]^{1/2} \leq \frac{4}{2(n-1)n} \left[\int_0^1 f_n(\tau)^2 \tau d\tau \right]^{1/2}.$$

Consider next the case $n = 0$. We then have

$$u_0(\tau) = \int_0^r \tau f_0(\tau)d\tau \cdot \log r + \int_r^1 \tau f_0(\tau) \log \tau d\tau$$

so that

$$(8.7) \qquad \frac{\partial u_0}{\partial r} = \frac{1}{r}\int_0^r \tau f_0(\tau)d\tau .$$

We can now proceed to estimate the L^2 norm of $\frac{1}{r}\frac{\partial u}{\partial r}$. Note first that

$$(8.8) \qquad \frac{1}{r}\frac{\partial u_n}{\partial r} = n\frac{u_n}{r^2} + r^{-n-2}\int_0^r \tau^{n+1}f_n(\tau)d\tau = n\frac{u_n}{r^2} + \frac{I_2(r)}{r^2} .$$

Hence, by (8.6) and (8.5),

$$\Big(\int_0^1 (\frac{1}{r}\frac{\partial u_n}{\partial r})^2 rdr \Big)^{1/2} \leq \frac{C}{n}\Big(\int_0^1 f_n(\tau)^2 \tau d\tau \Big)^{1/2} .$$

It easily follows that

$$||\frac{1}{r}\frac{\partial u}{\partial r}||_{L^2(R)} + ||\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}||_{L^2(B)} \leq C||f||_{L^2(B)}.$$

From (8.6) we also deduce that

$$||\frac{1}{r^2}\frac{\partial u}{\partial\theta}||_{L^2(B)} + ||\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}||_{L^2(B)} \leq C||f||_{L^2(R)} .$$

Finally,

$$||\frac{\partial^2 u}{\partial r^2}||_{L^2(B)} = ||\Delta u - \frac{2}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}||_{L^2(B)} \leq C||f||_{L^2(B)} . \quad \square$$

Remark 8.1. The condition (8.1) cannot be dropped. Indeed, the function $u = (r - r^3)\cos\theta$ satisfies (8.2) but not (8.1), and $\frac{1}{r}u_r$ is not in $L^2(B)$.

We now state a fundamental lemma, an analog to Lemma 5.1, for the system (7.15)–(7.19).

Lemma 8.2. *Consider the problem*

$$(8.9) \qquad \Delta S - \frac{1}{R_0}\wedge_{\theta\theta}\frac{1}{r}\frac{\partial S_0}{\partial r} - 2R_0 S_0 \wedge -R_0^2 S = F^1 \quad in \quad B ,$$

$$(8.10) \qquad \Delta P - \frac{1}{R_0}\wedge_{\theta\theta}\frac{1}{r}\frac{\partial P_0}{\partial r} + 2R_0\mu(S_0 - \tilde\sigma)\wedge +\mu R_0^2 S = F^2 \quad in \quad B ,$$

$$(8.11) \qquad\qquad\qquad S = 0 \quad on \quad \partial B ,$$

$$(8.12) \qquad\qquad\qquad \frac{\partial P}{\partial r} = F^3 \quad on \quad \partial B ,$$

$$(8.13) \qquad P + \frac{\gamma_0}{R_0^2}(\wedge + \wedge_{\theta\theta}) = \frac{\gamma}{R_0^2}(\ell^2 - 1)\cos\ell\theta + F^4 \quad on \quad \partial B$$

where $\ell \geq 2$, $F^1 \in L^2(B)$, $F^2 \in L^2(B)$, $F^3 \in H^{1/2}(\partial B)$, $F^4 \in H^{3/2}(\partial B)$, F^j *is even in* θ, *and*

$$(8.14) \qquad\qquad \int_0^{2\pi} F^j \cos\theta d\theta = 0 \quad (1 \leq j \leq 4) .$$

Set

$$(8.15) \qquad ||F|| = ||F^1||_{L^2(B)} + ||F^2||_{L^2(B)} + ||F^3||_{H^{1/2}(\partial B)} + ||F^4||_{L^2(\partial B)} .$$

Then there exists a unique solution $(S, P, \wedge, \gamma)$ of (8.9)–(8.13) with S, P in $H^2(B)$, $\wedge$ in $H^2(\partial B)$ such that $S, P, \wedge$ are even in θ and

$$(8.16) \qquad \int_0^{2\pi} \binom{S}{P} \cos \theta d\theta = 0 \ ,$$

$$(8.17) \qquad \int_0^{2\pi} \wedge(\theta) \cos m\theta \, d\theta = 0 \quad for \quad m = 1, \ell \ ;$$

furthermore,

$$(8.18) \qquad ||S||_{H^2(B)} \leq C||F|| \ ,$$

$$(8.19) \qquad ||P||_{H^2(B)} \leq C||F|| \ ,$$

$$(8.20) \qquad || \wedge ||_{H^2(\partial B)} \leq C||F|| \ ,$$

$$|\gamma| \leq C\Big\{ \Big[\int_0^1 \Big| \int_0^{2\pi} d\theta F^1 \cos \ell\theta \Big|^2 r dr \Big]^{1/2} + \Big[\int_0^1 \Big| \int_0^{2\pi} d\theta F^2 \cos \ell\theta \Big|^2 r dr \Big]^{1/2}$$

$$(8.21) \qquad + \Big| \int_0^{2\pi} d\theta F^3(\theta) \cos \ell\theta \Big| + \Big| \int_0^{2\pi} d\theta F^4(\theta) \cos \ell\theta \Big| \Big\} \ ,$$

$$(8.22) \qquad \begin{aligned} ||\frac{1}{r}\frac{\partial S}{\partial r}||_{L^2(B)} + ||\frac{1}{r^2}\frac{\partial S}{\partial \theta}||_{L^2(B)} + ||\frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2}||_{L^2(B)} \\ + ||\frac{1}{r}\frac{\partial^2 S}{\partial r \partial \theta}||_{L^2(B)} + ||\frac{\partial^2 S}{\partial r^2}||_{L^2(B)} \leq C||F|| \ , \end{aligned}$$

$$(8.23) \qquad \begin{aligned} ||\frac{1}{r}\frac{\partial P}{\partial r}||_{L^2(B)} + ||\frac{1}{r^2}\frac{\partial P}{\partial \theta}||_{L^2(B)} + ||\frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2}||_{L^2(B)} \\ + ||\frac{1}{r}\frac{\partial^2 P}{\partial r \partial \theta}||_{L^2(B)} + ||\frac{\partial^2 P}{\partial r^2}||_{L^2(B)} \leq C(||F|| + ||F^4||_{H^{3/2}(\partial B)}), \end{aligned}$$

and

$$(8.24) \qquad || \wedge ||_{H^{7/2}(\partial B)} \leq C(||F|| + ||F^4||_{H^{3/2}(\partial B)});$$

the constant C in the above estimates is independent of the F^i.

Proof. The assumptions of the lemma imply that

$$(8.25) \qquad F^j = \sum_{m \neq 1} F_m^j \cos m\theta$$

and that any solution of (8.9)–(8.13) can be written in the form

$$S = \sum_{m \neq 1} T_m(r) \cos m\theta \ ,$$

$$P = \sum_{m \neq 1} Q_m(r) \cos m\theta \ ,$$

$$\wedge = \sum_{m \neq 1} \Gamma_m \cos m\theta \ ;$$

we do not as yet impose the orthogonality condition $\int_0^{2\pi} \wedge(\theta) \cos \ell\theta d\theta = 0$.

We then obtain the system of differential equations

(8.26)
$$\frac{\partial^2}{\partial r^2}T_m + \frac{1}{r}\frac{\partial}{\partial r}T_m - \frac{m^2}{r^2}T_m - R_0^2 T_m$$
$$+ \Big(\frac{m^2}{rR_0}\bar{\sigma}\frac{R_0 I_1(R_0 r)}{I_0(R_0)} - 2R_0\bar{\sigma}\frac{I_0(R_0 r)}{I_0(R_0)}\Big)\Gamma_m = F_m^1 \ \text{ in } B,$$

(8.27)
$$\frac{\partial^2}{\partial r^2}Q_m + \frac{1}{r}\frac{\partial}{\partial r}Q_m - \frac{m^2}{r^2}Q_m + \mu R_0^2 T_m$$
$$+ \Big\{ - \frac{m^2}{rR_0}\mu\bar{\sigma}\frac{R_0 I_1(R_0 r)}{I_0(R_0)} + \frac{m^2}{rR_0}\frac{\mu\tilde{\sigma}R_0^2 r}{2}$$
$$+ 2R_0\mu\bar{\sigma}\frac{I_0(R_0 r)}{I_0(R_0)} - 2R_0\mu\tilde{\sigma}\Big\}\Gamma_m = F_m^2 \ \text{ in } B,$$

with the boundary conditions

(8.28)
$$T_m = 0 \quad \text{on} \quad \partial B ,$$

(8.29)
$$\frac{\partial Q_m}{\partial r} = F_m^3 \quad \text{on} \ \partial B ,$$

(8.30)
$$Q_m - \frac{\gamma_0}{R_0^2}(m^2 - 1)\Gamma_m = \frac{\delta_0}{R_0^2}(\ell^2 - 1)\delta_{m\ell} + F_m^4 \quad \text{on} \ \partial B .$$

The function
$$w = \mu T_m + Q_m$$

satisfies
$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{m^2}{r^2}w = M_m \Gamma_m + (\mu F_m^1 + F_m^2)$$

where $M_m = -\frac{1}{2}\mu\tilde{\sigma}(m^2 - 4)R_0$. The general solution of this equation which is regular in B is

$$\mu\frac{\tilde{\sigma}}{2}R_0 r^2 \Gamma_m + K_m r^m + r^m \int_1^r \frac{ds}{s^{2m+1}} \int_0^s \tau^{m+1}(\mu F_m^1 + F_m^2)d\tau$$

where K_m is an arbitrary constant. Hence

(8.31)
$$Q_m(r) = -\mu T_m(r) + \mu\frac{\tilde{\sigma}}{2}R_0 r^2 \Gamma_m + K_m r^m$$
$$+ r^m \int_1^r \frac{ds}{s^{2m+1}} \int_0^s \tau^{m+1}(\mu F_m^1(\tau) + F_m^2(\tau))d\tau .$$

Next we calculate $T_m(r)$. Setting

(8.32)
$$g_m(r) = [m^2 I_1(R_0 r) - 2R_0 r I_0(R_0 r)]\frac{\bar{\sigma}}{I_0(R_0)}$$

and trying a solution of the form

$$T_m(r) = I_m(R_0 r)v(r) ,$$

we get

$$\frac{1}{rI_m(R_0 r)}(r(I_m(R_0 r))^2 v')' = -\frac{g_m(r)}{r}\Gamma_m + F_m^1 .$$

Hence

$$
\begin{aligned}
T_m(r) = {}& -I_m(R_0 r) \int_1^r \frac{ds}{sI_m(R_0 s)^2} \int_0^s I_m(R_0 \tau) g_m(\tau) d\tau \cdot \Gamma_m \\
& + I_m(R_0 r) \int_1^r \frac{ds}{sI_m(R_0 s)^2} \int_0^s \tau I_m(R_0 \tau) F_m^1(\tau) d\tau \ ;
\end{aligned}
\tag{8.33}
$$

notice that $T_m(1) = 0$. Thus it remains to choose the constants K_m, Γ_m in such a way that the boundary conditions (8.29) and (8.30) are satisfied. Using (8.31) and (8.33), these conditions become

$$
A_m \begin{pmatrix} \Gamma_m \\ K_m \end{pmatrix} = \begin{pmatrix} \widetilde{F}_m^1 \\ \widetilde{F}_m^2 \end{pmatrix}
\tag{8.34}
$$

where

$$
A_m = \begin{pmatrix} \mu \tilde{\sigma} R_0 + \frac{\mu}{I_m(R_0)} \int_0^1 I_m(R_0 \tau) g_m(\tau) d\tau & m \\ \mu \frac{\tilde{\sigma}}{2} R_0 - \frac{\gamma_0}{R_0^2}(m^2 - 1) & 1 \end{pmatrix}
$$

and

$$
\begin{aligned}
\widetilde{F}_m^1 &= \frac{\mu}{I_m(R_0)} \int_0^1 \tau I_m(R_0 \tau) F_m^1(\tau) d\tau - \int_0^1 \tau^{m+1} \Big(\mu F_m^1(\tau) + F_m^2(\tau) \Big) d\tau + F_m^3 \ , \\
\widetilde{F}_m^2 &= \frac{\gamma}{R_0^2}(\ell^2 - 1)\delta_{m\ell} + F_m^4 \ .
\end{aligned}
$$

From Lemma 4.3 we see that $\det A_m \neq 0$ if $m \neq \ell$ and therefore the system (8.34) has a unique solution if $m \neq \ell$ (note that $\widetilde{F}_m^2 = F_m^4$ if $m \neq \ell$). On the other hand, if $m = \ell$, then $\det A_\ell = 0, \delta_{m\ell} = 1$ and a solution exists if and only if the first row of the augmented matrix is a multiple of the second row; the multiple is necessarily ℓ. We thus get the consistency condition $\ell \widetilde{F}_\ell^2 = \widetilde{F}_\ell^1$ or,

$$
\frac{\ell(\ell^2 - 1)}{R_0^2} \gamma = \widetilde{F}_\ell^1 - \ell F_\ell^4 \ .
$$

This determines γ uniquely, and also proves the estimate (8.21).

There is an infinite number of solutions $(T_\ell, Q_\ell, \Gamma_\ell)$, obtained from one solution by adding a multiple of $\cos \ell \theta$ to $\wedge$. We determine a unique solution by requiring that $\Gamma_\ell = 0$, i.e.,

$$
\int_0^{2\pi} \wedge(\theta) \cos \ell \theta d\theta = 0 \ .
$$

We proceed to derive the estimates (8.18)–(8.20). By elliptic estimates (using (8.9) and (8.11))

$$
||S||_{H^2(B)} \leq C \Big(||F^1||_{L^2(B)} + || \wedge ||_{H^2(\theta B)} \Big) \ .
\tag{8.35}
$$

Similarly, from (8.10) and (8.12),

$$
\begin{aligned}
||P||_{H^2(B)} \leq C \Big(||F^2||_{L^2(B)} &+ ||F^3||_{H^{1/2}(\partial B)} + ||S||_{L^2(B)} \\
&+ || \wedge ||_{H^2(\partial B)} + ||P||_{L^2(B)} \Big) \ .
\end{aligned}
\tag{8.36}
$$

From (8.13) and the trace theorem [2] we also have

$$
|| \wedge ||_{H^2(\partial B)} \leq C \Big(||P||_{H^1(B)} + ||F^4||_{L^2(\partial B)} + |\gamma| \Big) \ .
\tag{8.37}
$$

We claim that there is a constant C_0 independent of the F^i such that

$$(8.38) \qquad\qquad ||P||_{H^1(B)} \le C_0 ||F|| .$$

Indeed, otherwise there is a sequence $(F_j^1, F_j^2, F_j^3, F_j^4) \equiv F_j$ and corresponding solutions $(S_j, P_j, \wedge_j, \gamma_j)$ such that

$$1 = ||P_j||_{H^1(B)} \ge j||F_j|| .$$

From (8.21), (8.35)–(8.37) we deduce that for a subsequence, $S_j \rightharpoonup \widetilde{S}, P_j \rightharpoonup \widetilde{P}$ weakly in $H^2(B), \wedge_j \rightharpoonup \widetilde{\wedge}$ weakly in $H^2(B)$ and $\gamma_j \to \widetilde{\gamma}$, whereas $F_j^k \to 0$ $(1 \le k \le 4)$ in the norm of (8.15). It follows that $(\widetilde{S}, \widetilde{P}, \widetilde{\wedge}, \widetilde{\gamma})$ is a solution of the homogeneous system (8.9)–(8.13) and, by uniqueness, $\widetilde{S} \equiv 0, \widetilde{P} \equiv 0$. But

$$||\widetilde{P}||_{H^1(B)} = \lim ||P_j||_{H^1(B)} = 1 ,$$

a contradiction.

Having proved (8.38), we use it in (8.37) to conclude the proof of (8.20). Next, substituting (8.20) into (8.35), (8.36), the inequalities (8.18), (8.19) follow.

Observe that the estimate (8.22) follows from (8.18) and Lemma 8.1, and the estimate (8.24) follows from (8.13) and (8.19). Thus it remains to prove (8.23). Since $P(1, \theta) \not\equiv 0$ this estimate does not follow directly from Lemma 8.1. However, the part of P which comes from the first and last terms on the right-hand side of (8.31) are estimated by (8.22) and Lemma 8.1. Thus it remains to estimate

$$||\frac{1}{r} \frac{\partial \widetilde{P}}{\partial r}||_{L^2(B)}, \quad ||\frac{1}{r} \frac{\partial^2 \widetilde{P}}{\partial r \partial \theta}||_{L^2(B)}$$

$$(8.39) \qquad ||\frac{1}{r^2} \frac{\partial \widetilde{P}}{\partial \theta}||_{L^2(B)}, \quad ||\frac{1}{r^2} \frac{\partial^2 \widetilde{P}}{\partial \theta^2}||_{L^2(B)}, \quad ||\frac{\partial^2 \widetilde{P}}{\partial r^2}||_{L^2(B)}$$

where

$$\widetilde{P} = \mu \frac{\tilde{\sigma}}{2} R_0 r^2 \sum_{m \ne 1} \Gamma_m \cos m\theta + \sum_{m \ne 1} K_m r^m \cos m\theta \equiv P_1 + P_2.$$

The estimates for P_1 follow immediately from the inequality

$$(8.40) \qquad\qquad \sum (m^{7/2} \Gamma_m)^2 \le C|| \wedge ||_{H^{7/2}(\partial B)}$$

and (8.24).

Next from the second equation of (8.34) we get

$$|K_m| \le Cm^2 |\Gamma_m| + C\gamma \delta_{ml} + C|F_m^4|$$

and therefore

$$||\frac{1}{r^2} \frac{\partial^2 P_2}{\partial \theta^2}||_{L^2(B)} \le C\Big(\sum_{m>1} m^3 |K_m|^2 \Big) \le C\Big(\gamma + || \wedge ||_{H^{7/2}(\partial B)} + ||F^4||_{H^{3/2}(\partial B)} \Big) .$$

The remaining partial derivatives of P_2 which occur in (8.39) (with $\widetilde{P}$ replaced by P_2) are estimated in the same way. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We shall need, in addition to Lemma 8.2, its counterpart for odd functions of θ.

Lemma 8.3. *Consider the system* (8.9)–(8.13) *with* $\cos \ell\theta$ *replaced by* $\sin \ell\theta$ *in* (8.13) *and* F^j *as in Lemma 8.2 but odd functions in* θ, *satisfying*

$$\int_0^{2\pi} F^j \sin \theta d\theta = 0 \quad (1 \le j \le 4)$$

instead of (8.14). *Then there exists a unique solution* $(S, P, \wedge, \gamma)$ *with* S, P *in* $H^2(B), \wedge$ *in* $H^2(\partial B)$ *such that* $S, P, \wedge$ *are odd in* θ *and*

$$\int_0^{2\pi} \binom{S}{P} \sin \theta d\theta = 0 \ ,$$

$$\int_0^{2\pi} \wedge(\theta) \sin m\theta d\theta = 0 \quad \text{for} \ \ m = 1, \ \ell \ ,$$

and the estimates (8.18)–(8.24) *hold with* $\cos \ell\theta$, *in* (8.21), *replaced by* $\sin \ell\theta$.

9. An auxiliary lemma

We shall need the following known estimates [2]:

$$||f||_{L^\infty(\partial B)} \le C||f||_{H^s(\partial B)},$$

(9.1) $$||fg||_{H^{1/2}(\partial B)} \le C||f||_{H^s(\partial B)}||g||_{H^{1/2}(\partial B)} \ \text{ and }$$

$$||fg||_{H^s(\partial B)} \le C||f||_{H^s(\partial B)}||g||_{H^s(\partial B)} \ ,$$

where $s > \frac{1}{2}$ and C is a constant depending only on s. We shall also need the inequality

(9.2) $$\binom{a}{b} \binom{c}{d} \le \binom{a+c}{b+d}$$

which follows by induction on a $(a \ge b)$ using the relation

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b} \ .$$

Lemma 9.1. *Let*

$$F(x) = \sum_{n\ge 1} \alpha_n x^n, \quad \lambda(\theta, \varepsilon) = \sum_{n\ge 1} \lambda_n(\theta)\varepsilon^n$$

be such that $|\alpha_n| \le C^n$ *and, for some* $s > \frac{1}{2}$,

(9.3) $$||D_\theta^k \lambda_n||_{H^s(\partial B)} \le C_0 A^{k-2} H^{n-2} \frac{(k+n)!}{(k+n)^3 n!} \quad \forall k, n$$

where C_0, A, H *are positive constants, and* $A > 1, H > 1$. *Set*

(9.4) $$F(\lambda(\theta, \varepsilon)) = \sum_{n\ge 1} F_n(\theta)\varepsilon^n \ .$$

Then

(9.5) $$||D_\theta^k F_n(\theta)||_{H^s(\partial B)} \le C_0 K A^{k-2} H^{n-2} \frac{(k+n)!}{(k+n)^3 n!}$$

for all n, k, *provided* $H \ge 2CC_0 S$, *where* S *is a universal constant and* K *depends only on* C, C_0.

The lemma is similar to one proved in [9]. In (9.3), (9.5) we use the following convention:

$$A^{-j} = H^{-j} = 1 \quad \text{if} \ \ j \ge 0 \ .$$

Proof. We shall abbreviate $||\cdots||_{H^s(\partial B)}$ by $||\cdots||_s$. Consider first the case $F(x) = x^m$. We write

$$(9.6) \qquad \left(\sum_{n\geq 1}\lambda_n(\theta)\varepsilon^n\right)^m = \sum_{n\geq m}\varphi_n^m(\theta)\varepsilon^n$$

and proceed to prove by induction on m that

$$(9.7) \qquad ||D_\theta^k\varphi_n^m||_s \leq S^{m-1}C_0^m A^{k-2}H^{n-1-m}\frac{(k+n)!}{(k+n)^3 n!} \; .$$

From (9.3) this holds for $m = 1$. To go from m to $m+1$ we shall make use of the relation

$$\varphi_n^{m+1}(\theta) = \sum_{j=m}^{n-1}\varphi_j^m(\theta)\lambda_{n-j}(\theta) \; .$$

Using (9.1) we see that

$$\begin{aligned}
||D_\theta^k\varphi_n^{m+1}||_s &\leq \sum_{j=m}^{n-1}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}||D_\theta^l\varphi_j^m||_s||D_\theta^{k-l}\lambda_{n-j}||_s \\
&\leq \sum_{j=m}^{n-1}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}A^{l-2}H^{j-1-m}C_0^m S^{m-1}\frac{(j+l)!}{(j+l)^3 j!} \\
&\quad \times C_0 A^{k-l-2}H^{n-j-2}\frac{(n-j+k-l)!}{(n-j+k-l)^3(n-j)!} \; .
\end{aligned}$$

Using (9.2) we get the bound

$$C_0^{m+1}S^m A^{k-2}H^{n-1-(m+1)}\frac{(k+n)!}{(k+n)^3 n!}$$

where

$$S \geq \sum_{l=0}^{k}\sum_{j=1}^{n-1}\frac{(k+n)^3}{(j+l)^3(k+n-j-l)^3}$$

is a universal constant. This completes the proof of (9.7).

Consider now the general case of $F(x)$, and write

$$\begin{aligned}
F(\lambda(\theta,\varepsilon)) &= \sum_{m\geq 1}\alpha_m(\lambda(\theta,\varepsilon))^m \\
&= \sum_{m\geq 1}\alpha_m\sum_{n\geq m}\varphi_n^m(\theta)\varepsilon^n = \sum_{n\geq 1}\left(\sum_{m=1}^{n}\alpha_m\varphi_n^m(\theta)\right)\varepsilon^n \; ,
\end{aligned}$$

so that, by (9.4),

$$F_n(\theta) = \sum_{m=1}^{n}\alpha_m\varphi_n^m(\theta) \; .$$

Using (9.7) we get

$$\|D_\theta^k F_n\|_s \leq \sum_{m=1}^n |\alpha_m| A^{k-2} H^{n-1-m} C_0^m S^{m-1} \frac{(k+n)!}{(k+n)^3 n!}$$

$$\leq \frac{(k+n)!}{(k+n)^3 n!} A^{k-2} \left[H^{n-2} \sum_{m=1}^{n-1} \frac{|\alpha_m| C_0^m S^{m-1}}{H^{m-1}} + |\alpha_n| C_0^n S^{n-1} \right]$$

$$\leq \frac{(k+n)!}{(k+n)^3 n!} A^{k-2} H^{n-2} C_0 K$$

provided $H \geq 2CC_0 S$, where K depends only on C_0, C. $\qquad\square$

10. CONVERGENCE

In §§7 and 8 we determined a formal solution (7.9)–(7.12) to the system (7.3)–(7.7), showing that the system (7.15)–(7.19) has a unique solution (even in θ) subject to the orthogonality conditions

$$(10.1) \qquad \int_0^{2\pi} f_j(\theta) \cos \ell\theta d\theta = 0 \quad (j \geq 2) .$$

The modes that appear in (7.15)–(7.19) are $\cos m\ell\theta$ where m is a nonnegative integer. Using Lemmas 8.2 and 8.3 we shall prove the following theorem.

Theorem 10.1. *The series (7.9)–(7.12) are uniformly convergent for $\varepsilon' < r \leq 1 + \delta_0, |\varepsilon| \leq \varepsilon_0$ where δ_0 and ε_0 are positive numbers and ε' is arbitrarily small positive number; furthermore, the functions $\widehat{\sigma}(r, \theta, \varepsilon), \widehat{p}(r, \theta, \varepsilon)$ are analytic in (r, θ, ε) for $0 < r \leq 1 + \delta_0, |\varepsilon| \leq \varepsilon_0$ and the function $f(\theta, \varepsilon)$ is analytic in (θ, ε) for $0 \leq \theta \leq 2\pi, |\varepsilon| \leq \varepsilon_0$.*

The proof is given in this and the next section.

To prove the theorem we introduce the norm

$$\|u\|_{H_r^2(B)} = \|u\|_{L^2(B)} + \|\frac{1}{r} \frac{\partial u}{\partial r}\|_{L^2(B)} + \|\frac{1}{r^2} \frac{\partial u}{\partial \theta}\|_{L^2(B)}$$

$$+ \|\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}\|_{L^2(B)} + \|\frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta}\|_{L^2(B)} + \|\frac{\partial^2 u}{\partial r^2}\|_{L^2(B)}$$

and derive the following estimates:

$$(10.2) \qquad \|\partial_\theta^j S_m\|_{H_r^2(B)} + \|\partial_\theta^j \left(P_m - \frac{\gamma_m}{R_0}\right)\|_{H_r^2(B)} \leq C_0 A^{j-2} H^{m-2} \frac{(j+m)!}{(j+m)^3 m!} ,$$

$$(10.3) \qquad \|\partial_\theta^j f_m\|_{H^{7/2}(\partial B)} \leq C_0 A^{j-2} H^{m-2} \frac{(j+m)!}{(j+m)^3 m!} ,$$

$$(10.4) \qquad |\gamma_{m-1}| \leq C_0 \frac{H^{m-2}}{m^3}$$

for all $m \geq 0, j \geq 0, m + j > 0$ in (10.2), (10.3) and $m \geq 1$ in (10.4).

The proof is by induction on m. We assume that (10.2)–(10.4) hold for all $m < n$, and proceed to prove these inequalities for $m = n$. For simplicity we take $n \geq 3$; the proof for $n = 0, 1, 2$ can be established from the explicit formulas for S_m, P_m, f_m in §8 .

Lemma 10.2. *If* (10.2)–(10.4) *hold for all* $m < n$ *and* $j \geq 0$ *then, for all* $k \geq 0$,

$$(10.5) \qquad ||\partial_\theta^k F_n^i||_{L^2(B)} \leq C_0 M A^{k-2} H^{n-3} \frac{(k+n)!}{(k+n)^3 n!} \quad (i = 1, 2) ,$$

$$(10.6) \qquad ||\partial_\theta^k F_n^3||_{H^{1/2}(\partial B)} + ||\partial_\theta^k F_n^4||_{H^{3/2}(\partial B)} \leq C_0 M A^{k-2} H^{n-3} \frac{(k+n)!}{(k+n)^3 n!}$$

where M *is a constant independent of* A, H *and* k, n.

Suppose the lemma is true. Then we can apply Lemma 8.2 to conclude that (10.2)–(10.4) hold for $m = n, j = 0$ with H^{n-2} replaced by CMH^{n-3}. The same estimates hold for $j = 1$, using Lemma 8.3. For general j we use either Lemma 8.2 (if j is even) or Lemma 8.3 (if j is odd). Note that the term

$$\frac{\gamma_{n-1}}{R_0^2}(\ell^2 - 1) \cos \ell\theta$$

in (7.19) is replaced, for j even, by

$$\frac{\gamma_{n-1}}{R_0^2}(\ell^2 - 1)\ell^j(-1)^{j/2} \cos \ell\theta ,$$

but this does not affect the estimates; the same is true if j is odd (with $\cos \ell\theta$ replaced by $(-1)^{\frac{j+1}{2}} \sin \ell\theta$). Thus, choosing $H > MC$, we conclude that in order to prove (10.2)–(10.4) it suffices to prove Lemma 10.2.

Proof of Lemma 10.2. We introduce auxiliary functions which will appear in the expressions for the F_n^i. First,

$$(10.7) \qquad -\frac{1}{R_0 + f} = -\frac{1}{R_0} + \sum_{n \geq 1} \frac{(-1)^n f^n}{R_0^n} \equiv \sum_{n \geq 0} G_n(\theta)\varepsilon^n ,$$

$$(10.8) \qquad \log(R_0 + f) = \log R_0 + \sum_{n \geq 1} (-1)^{n-1} \frac{f^n}{nR_0^n} \equiv \sum_{n \geq 0} L_n(\theta)\varepsilon^n .$$

By Lemma 9.1 and the inductive assumption (10.3),

$$(10.9) \quad ||D_\theta^k G_m||_{H^{7/2}(\partial B)} + ||D_\theta^k L_m||_{H^{7/2}(\partial B)} \leq K C_0 \frac{(k+m)!}{(k+m)^3 m!} A^{k-2} H^{m-2} .$$

We have

$$(10.10) \qquad \frac{f_\theta}{R_0 + f} = \sum_{n \geq 1} L_n'(\theta)\varepsilon^n ,$$

$$(10.11) \qquad \left(\frac{f_\theta}{R_0 + f}\right)^2 = \sum_{n \geq 2} \left(\sum_{k=1}^{n-1} L_{n-k}' L_k'\right)\varepsilon^n \equiv \sum_{n \geq 2} D_n(\theta)\varepsilon^n$$

and by (9.7) (with $m = 2$)

$$(10.12) \qquad ||D_\theta^k D_m||_{H^{5/2}(\partial B)} \leq K C_0 \frac{(k+m)!}{(k+m)^3 m!} A^{k-2} H^{m-3} .$$

Next

$$(10.13) \qquad \frac{f_{\theta\theta}}{R_0 + f} = (\log(R_0 + f))'' + \left(\frac{f_\theta}{R_0 + f}\right)^2 = \sum_{n \geq 1} (L_n'' + D_n)\varepsilon^n .$$

Writing

$$f^2 = \sum_{n \geq 2} Q_n(\theta)\varepsilon^n$$

we have (again by (9.7) with $m = 2$)

(10.14) $$||D_\theta^k Q_m||_{H^{7/2}(\partial B)} \leq KC_0 \frac{(k+m)!}{(k+m)^3 m!} A^{k-2} H^{m-3} .$$

The estimate (10.9) holds for all $m < n$ and $k \geq 0$, while the estimates (10.12), (10.14) are valid for $m \leq n$ and $k \geq 0$.

We now write the F_n^i explicitly from (7.3)–(7.7):

$$F_n^1 = \sum_{k=0}^{n-1} \left\{ 2L'_{n-k} \frac{1}{r} \frac{\partial^2 S_k}{\partial\theta\partial r} - D_{n-k}\left(\frac{2}{r}\frac{\partial S_k}{\partial r} + \frac{\partial^2 S_k}{\partial r^2}\right) \right.$$

(10.15) $$\left. + \left(L''_{n-k} + D_{n-k} - \frac{f''_n}{R_0}\delta_{k,0}\right)\frac{1}{r}\frac{\partial S_k}{\partial r} + Q_{n-k}S_k + 2R_0 f_{n-k}(S_k - S_0\delta_{k,0}) \right\} .$$

Similarly,

$$F_n^2 = \sum_{k=0}^{n-1} \left\{ 2L'_{n-k} \frac{1}{r} \frac{\partial^2 P_k}{\partial\theta\partial r} - D_{n-k}\left(\frac{2}{r}\frac{\partial P_k}{\partial r} + \frac{\partial^2 P_k}{\partial r^2}\right) \right.$$

(10.16)
$$+ \left(L''_{n-k} + D_{n-k} - \frac{f''_n}{R_0}\delta_{k,0}\right)\frac{1}{r}\frac{\partial P_k}{\partial r} - Q_{n-k}(S_k - \tilde\sigma\delta_{k,0})$$

$$\left. - 2R_0 f_{n-k}(S_k - S_0\delta_{k,0}) \right\} .$$

Finally, F_n^3 and F_n^4 can be written in the form

(10.17) $$F_n^3 = \sum_{k=1}^{n-1} V_{n-k} \frac{\partial P_k}{\partial\theta} ,$$

(10.18) $$F_n^4 = \sum_{k=1}^{n-2} \gamma_k U_{n-k}$$

where

(10.19) $$||D_\theta^k V_m||_{H^{5/2}(\partial B)} + ||D_\theta^k U_m||_{H^{3/2}(\partial B)} \leq KC_0 \frac{(k+m)!}{(k+m)^3 m!} A^{k-2} H^{m-2} .$$

Indeed, this follows by applying Lemma 9.1 to estimate

$$\partial_\varepsilon^k\Big|_{\varepsilon=0} \frac{f_\theta(R_0 + f)}{(f_\theta^2 + (R_0 + f)^2)} \quad \text{and}$$

$$\partial_\varepsilon^k\Big|_{\varepsilon=0} \frac{(R_0 + f)^2 + 2f_\theta^2 - (R_0 + f)f_{\theta\theta}}{\left((R_0 + f)^2 + f_\theta^2\right)^{3/2}}$$

in $H^{5/2}(\partial B)$ and $H^{3/2}(\partial B)$ norms, respectively (recall that we have $H^{2+3/2}(\partial B)$ estimate on f).

Using (10.4) (with $m < n$) and (10.19), we easily obtain

$$(10.20) \qquad ||\partial_\theta^k F_n^4||_{H^{3/2}(\partial B)} \leq MC_0 \frac{(k+n)!}{(k+n)^3 n!} A^{k-2} H^{n-3} .$$

Next

$$||\partial_\theta^k F_n^3||_{H^{1/2}(\partial B)} \leq \sum_{l=1}^{n-1} ||\partial_\theta^k \Big(V_{n-l} \frac{\partial P_l}{\partial \theta}\Big)||_{H^{1/2}(\partial B)}$$

$$\leq \sum_{l=1}^{n-1} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} ||\partial_\theta^{k-m} V_{n-l} \cdot \partial_\theta^m \frac{\partial P_l}{\partial \theta}||_{H^{1/2}(\partial B)}$$

$$\leq C \sum_{l=1}^{n-1} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} ||\partial_\theta^{k-m} V_{n-l}||_{H^{5/2}(\partial B)} ||\partial_\theta^m \Big(P_l - \frac{\gamma_l}{R_0}\Big)||_{H^2(B)}$$

where we used (9.1) and the trace theorem. Using (10.19) and the inductive assumption we find that the right-hand side is bounded by

$$C \sum_{l=1}^{n-1} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} KC_0 \frac{(n-l+k-m)!}{(n-l+k-m)^3(n-l)!} A^{k-m-2} H^{n-l-2}$$

$$\times C_0 \frac{(l+m)!}{(l+m)^3 l!} A^{m-2} H^{l-2} .$$

Thus

$$||\partial_\theta^k F_n^3||_{H^{1/2}(\partial B)} \leq KCC_0^2 A^{k-2} H^{n-3} \frac{(k+n)!}{(k+n)^3 n!} J$$

where

$$J = \sum_{l,m} \frac{k!n!}{(k+n)!} \frac{(l+m)!}{m!l!} \frac{(n-l+k-m)!}{(k-m)!(n-l)!} \frac{(k+n)^3}{(l+m)^3(k+n-(m+l))^3}.$$

Using (9.2) we conclude that J is bounded by a constant.

We next need to estimate F_n^2. We can write

$$\begin{aligned}
F_n^2 &= \sum_{l=1}^{n-1} \Big\{ 2L'_{n-l} \frac{1}{r} \frac{\partial^2 P_l}{\partial r \partial \theta} - D_{n-l}\Big(\frac{2}{r} \frac{\partial P_l}{\partial r} + \frac{\partial^2 P_l}{\partial r^2}\Big) \\
&\quad + \Big(L''_{n-l} + D_{n-l}\Big)\frac{1}{r} \frac{\partial P_l}{\partial r} - Q_{n-l} S_l - 2R_0 f_{n-l} S_l \Big\} \\
&\quad - D_n \Big(\frac{2}{r} \frac{\partial P_0}{\partial r} + \frac{\partial^2 P_0}{\partial r^2}\Big) + \Big(E''_n + D_n\Big)\frac{1}{r} \frac{\partial P_0}{\partial r} - Q_n(S_0 - \tilde{\sigma})
\end{aligned}$$

where $E_n \equiv L_n - \frac{f_n}{R_0}$ involves only the f_m with $m < n$. Using (9.1) and the trace

theorem we get

$$||\partial_\theta^k F_n^2||_{L^2(B)} \le C \sum_{l=1}^{n-1} \sum_{m=0}^k \frac{k!}{m!(k-m)!} \Big\{ ||\partial_\theta^{k-m} L_{n-l}||_{H^{2+3/2}(\partial B)} ||\frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}\partial_\theta^m P_l||_{L^2(B)}$$

$$+ ||\partial_\theta^{k-m} D_{n-l}||_{H^{1+3/2}} \Big(||\frac{1}{r}\frac{\partial}{\partial r}\partial_\theta^m P_l||_{L^2(B)} + ||\frac{\partial^2}{\partial r^2}\partial_\theta^m P_l||_{L^2(B)} \Big)$$

$$+ \Big(||\partial_\theta^{k-m} L_{n-l}||_{H^{2+3/2}(\partial B)} + ||\partial_\theta^{k-m} D_{n-l}||_{H^{1+3/2}} \Big) ||\frac{1}{r}\frac{\partial}{\partial r}\partial_\theta^m P_l||_{L^2(B)}$$

$$+ ||\partial_\theta^{k-m} Q_{n-l}||_{H^{2+2/3}(\partial B)} ||\partial_\theta^m S_l||_{L^2(B)} + ||\partial_\theta^{k-m} f_{n-l}||_{H^{2+2/3}(\partial B)} ||\partial_\theta^m S_l||_{L^2(B)} \Big\}$$

$$+ C||\partial_\theta^k D_n||_{H^{1+3/2}(\partial B)} \Big(||\frac{1}{r}\frac{\partial P_0}{\partial r}||_{L^2(B)} + ||\frac{\partial^2 P_0}{\partial r^2}||_{L^2(B)} \Big)$$

$$+ C\Big(||\partial_\theta^k E_n||_{H^{2+3/2}(\partial B)} + ||\partial_\theta^k D_n||_{H^{1+3/2}(\partial B)} \Big) ||\frac{1}{r}\frac{\partial P_0}{\partial r}||_{L^2(B)}$$

$$+ C||\partial_\theta^k Q_n||_{H^{2+3/2}(\partial B)} (||S_0||_{L^2(B)} + 1) .$$

By the inductive assumption and (10.9), (10.12) (10.14) we find that the sum of the above terms is bounded by

$$C \sum_{\ell=1}^{n-1} \sum_{m=0}^k \frac{k!}{m!(k-m)!} KC_0 \frac{(n-l+k-m)!}{(n-l+k-m)^3(n-l)!} C_0 \frac{(m+l)!}{(m+l)^3 l!}$$

$$\times A^{k-m-2} H^{n-l-2} A^{m-2} H^{l-2} + CKC_0^2 \frac{(k+n)!}{(k+n)^3 n!} A^{k-2} H^{n-3}$$

$$\le CKC_0^2 S A^{k-2} H^{n-3} \frac{(k+n)!}{(k+n)^3 n!} .$$

Similarly, one can estimate $||\partial_\theta^k F_n^1||_{L^2(B)}$ and together with the above estimates on F_n^3, F_n^4, the proof of Lemma 10.2 is complete. $\qquad\square$

11. CONVERGENCE (CONTINUED)

From the estimates of Lemma 10.2 we already conclude that $f(\theta, \varepsilon)$ is analytic in (θ, ε) and $\hat{\sigma}(r, \theta, \varepsilon), \hat{p}(r, \theta, \varepsilon)$ are analytic in (θ, ε) for $0 \le \theta \le 2\pi, |\varepsilon| \le \varepsilon_0$. In this section we want to extend the estimates to all r-derivatives so as to conclude the joint analyticity of $\hat{\sigma}, \hat{p}$ in r, θ, ε up to $r = 1$ (and actually to $r \le 1 + \delta_0$ for some small positive δ_0).

We shall prove the following:

Lemma 11.1. *For all $k \ge 0, n \ge 0, h > 0$*

(11.1)

$$||r^h \partial_r^h \partial_\theta^k S_n||_{L^2(B)} + ||r^h \frac{1}{r}\frac{\partial}{\partial r}\partial_r^h \partial_\theta^k S_n||_{L^2(B)} + ||r^h \frac{1}{r^2}\frac{\partial}{\partial \theta}\partial_r^h \partial_\theta^k S_n||_{L^2(B)}$$

$$+ ||r^h \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\partial_r^h \partial_\theta^k S_n||_{L^2(B)} + ||r^h \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}\partial_r^h \partial_\theta^k S_n||_{L^2(B)}$$

$$+ ||r^h \frac{\partial^2}{\partial r^2}\partial_r^h \partial_\theta^k S_n||_{L^2(B)} \le C_0 \frac{(k+h+n)!}{(k+h+n)^3 n!} B^h A^{k-2} H^{n-2} ;$$

the same estimates hold for P_n.

Proof. From (7.15) we have

$$
(11.2) \quad
\begin{aligned}
\frac{\partial^2}{\partial r^2}\partial_r^j\partial_\theta^k S_n &= \partial_r^j\partial_\theta^k F_n^1 - \partial_r^j\Big(\frac{1}{r}\partial_\theta^k S_n\Big) - \partial_r^j\Big(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\partial_\theta^k S_n\Big) \\
&\quad + \Big(\partial_\theta^k \wedge_{n,\theta\theta}\Big)\frac{1}{R_0}\partial_r^j\Big(\frac{1}{r}\partial_r S_0\Big) + R_0^2\partial_r^j\partial_\theta^k S_n + 2\Big(\partial_\theta^k f_n\Big)R_0\partial_r^j S_0 \ .
\end{aligned}
$$

We assume inductively that (11.1) and its analog for P_n hold for all h smaller than j and proceed to prove it for j; for $j = 0$ this follows from (10.2), (10.3). We shall consider in detail the term

$$
J_1 = r^j\partial_r^j\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\partial_\theta^k S_n
$$

and a typical term from $\partial_r^j\partial_\theta^k F_n^1$ (cf. (10.15)), namely,

$$
J_2 = r^j\partial_r^j\partial_\theta^k\sum_{t=1}^{n-1}L'_{n-t}\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}S_t \ ;
$$

all the other terms on the right-hand side of (11.2) can be estimated in a similar way.

We can write

$$
\begin{aligned}
J_1 &= r^j\sum_{s=0}^{j}(-1)^{j-s}r^{-2-(j-s)}(j-s+1)!\frac{j!}{s!(j-s)!}\partial_\theta^2\partial_\theta^k\partial_r^s S_n \\
&= \sum_{s=1}^{j}\frac{j!}{s!(j-s)!}(j-s+1)!(-1)^{j-s}r^{s-1}\frac{1}{r}\partial_r\partial_\theta\partial_\theta^{k+1}\partial_r^{s-1}S_n \\
&\quad + (-1)^j(j+1)!\frac{1}{r^2}\partial_\theta^2\partial_\theta^k S_n \ .
\end{aligned}
$$

The last term is bounded in $L^2(B)$ by

$$
\begin{aligned}
C_0 A^{k-2}&H^{n-2}\frac{(k+n)!(j+1)!}{(k+n)^3 n!} \\
&\le C_0 A^{k-2}H^{n-2}B^j\frac{(k+j+n)!}{(k+j+n)^3 n!}\Big[\frac{(k+n)!(j+1)!}{(k+n+j)!}\frac{(k+n+j)^3}{(k+n)^3}B^{-j}\Big] \\
&\le C_0 A^{k-2}H^{n-2}B^j\frac{(k+j+n)!}{(k+j+n)^3 n!}\Big[(j+1)^4 B^{-j}\Big] \ ,
\end{aligned}
$$

whereas the $L^2(B)$ norm of the sum is bounded by

$$
\begin{aligned}
\sum_{s=1}^{j}&\frac{j!}{s!(j-s)!}(j-s+1)!C_0 A^{k-1}B^{s-1}H^{n-2}\frac{(k+s+n)!}{(k+s+n)^3 n!} \\
&\le C_0 H^{n-2}B^j A^{k-2}\frac{(k+n+j)!}{(k+n+j)^3 n!}\frac{A}{B} \\
&\quad\times\sum_{s=1}^{j}B^{-(j-s)}\frac{(k+s+n)!}{(k+n+j)!}\frac{(k+n+j)^3}{(k+n+s)^3}(j-s+1)\frac{j!}{s!} \ .
\end{aligned}
$$

By (9.2)

$$
\frac{(k+s+n)!}{(k+n+j)!}\frac{j!}{s!} = \binom{j}{s}\Big/\binom{k+n+j}{k+n+s} \le 1
$$

so that the expression in the last sum is less than

$$\sum_{s=1}^{j}(1+j-s)\frac{(k+n+j)^3}{(k+n+s)^3}B^{-(j-s)} \le C\sum_{s=1}^{j}(1+(j-s))^4 B^{-(j-s)} \ .$$

Taking B large enough we get the bound

$$||J_1||_{L^2(B)} \le C_0 H^{n-2}B^j A^{k-2}\frac{(k+j+n)!}{(k+j+n)^3 n!} \ .$$

To estimate J_2 we write

$$J_2 = \sum_{t=1}^{n-1}\sum_{p=0}^{k}\sum_{s=0}^{j}\frac{k!}{p!(k-p)!}\frac{j!}{s!(j-s)!}\partial_\theta^{k-p}\partial_\theta L_{n-t}(-1)^{j-s}r^j r^{-(j-s)}(j-s)!$$

$$\times \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\partial_\theta^p\partial_r^s S_t$$

$$= \sum_{t=1}^{n-1}\sum_{p=0}^{k}\sum_{s=0}^{j-1}\frac{k!}{p!(k-p)!}\frac{j!}{s!(j-s)!}\partial_\theta^{k-p}\partial_\theta L_{n-t}(-1)^{j-s}r^s(j-s)!\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\partial_\theta^p\partial_r^s S_t$$

$$+ \sum_{t=1}^{n-1}\sum_{p=0}^{k}\frac{k!}{p!(k-p)!}\partial_\theta^{k-p}\partial_\theta L_{n-t}r^{j-1}\frac{\partial^2}{\partial r^2}\partial_\theta^{p+1}\partial_r^{j-1}S_t \ .$$

The $L^2(B)$ norm of the last sum can be estimated by

$$\sum_{t,p}\frac{k!}{p!(k-p)!}KC_0\frac{(k-p+n-t)!}{(k-p+n-t)^3(n-t)!}A^{k-p-2}H^{n-t-2}$$

$$\times C_0\frac{(p+j+t)!}{(p+j+t)^3 t!}B^{j-1}A^{p-1}H^{t-2}$$

$$\le KC_0^2 A^{k-2}H^{n-3}B^j\frac{(k+n+j)!}{(k+n+j)^3 n!}\frac{A}{B}$$

$$\times \sum_{t,p}\frac{(k+n+j)^3}{\Big((k+n+j)-(p+j+t)\Big)^3(p+j+t)^3}\frac{\binom{n}{t}}{\binom{k+n+j}{p+t+j}}$$

and, since $\binom{n}{t} \le \binom{k+n+j}{p+t+j}$ (by (9.2)),

$$\sum_{t,p}\frac{(k+n+j)^3}{\Big((k+n+j)-(p+j+t)\Big)^3(p+j+t)^3}\frac{\binom{n}{t}}{\binom{k+n+j}{p+t+j}}$$

$$\le \sum_{t,p}\frac{(k+n+j)^3}{\Big((k+n+j)-(p+j+t)\Big)^3(p+j+t)^3} < \infty \ .$$

Similarly, the $L^2(B)$ norm of the first sum in the expression for J_2 is bounded by

$$\sum_{t,p,s} \frac{k!}{p!(k-p)!} \frac{j!}{s!(j-s)!} (j-s)! K C_0 \frac{(k-p+n-t)!}{(k-p+n-t)^3(n-t)!} \times A^{k-p-2}H^{n-t-2}$$

$$\times C_0 \frac{(p+s+t)!}{(p+s+t)^3 t!} B^s A^{p-2} H^{t-2}$$

$$\leq K C_0^2 A^{k-2} H^{n-3} B^j \frac{(k+n+j)!}{(k+n+j)^3 n!} \sum_{s=0}^{j-1} \sum_{t,p} B^{-(j-s)}$$

$$\times \left[\frac{(k-p+n-t)!(p+t+s)!}{(k+n+s)!} \frac{(k+n+s)!}{(k+n+j)!} (j-s)! \binom{j}{s} \binom{n}{t} \binom{k}{p} \right]$$

$$\times \frac{(k+n+j)^3}{(k+n-p-t)^3(p+s+t)^3}$$

and, by (9.2), the expression in brackets is bounded by

$$\frac{(k-p+n-t)!(p+t+s)!}{(k+n+s)!} \frac{(k+n+s)!}{(k+n+j)!} \binom{n+k+j}{p+t+s} (j-s)!$$

$$= \frac{(k+n-(p+t))!(j-s)!}{(n+k+j-(p+t+s))!} \leq 1 \Big/ \binom{n+k+j-(t+p+s)}{j-s} \leq 1 .$$

Consequently, after choosing B large enough, we get the same bound as for J_1.

As mentioned above, all other terms can be estimated in a similar way.

From Theorem 10.1 and Lemma 11.1 we deduce the convergence of the series (7.9)–(7.12) and the analyticity of $\widehat{\sigma}(r,\theta,\varepsilon), \widehat{p}(r,\theta,\varepsilon)$ in (r,θ,ε) in the region $0 < r \leq 1 + \delta_0, 0 \leq \theta \leq 2\pi, |\varepsilon| \leq \varepsilon_0$ for some $\delta_0 > 0, \varepsilon_0 > 0$.

From the transformation (7.1), (7.2) it follows that $\sigma(x,\varepsilon)$ and $p(x,\varepsilon)$ are analytic functions jointly in $x = (x_1, x_2)$ and ε for $\eta < |x| \leq R_0 + \delta_0, |\varepsilon| \leq \varepsilon_0$ where η is any positive number. Since $\sigma(x,\varepsilon)$ is bounded in a neighborhood of $x = 0$, the singularity at $x = 0$ is removable and we can represent σ in terms of the fundamental solution $G(x) = \frac{1}{2\pi} K_0(|x|)$ where $K_0(z)$ is the modified Bessel function of the third kind and order 0,

$$\sigma(x,\varepsilon) = -\int_{|y|=\eta_0} \frac{\partial G}{\partial n}(x-y)\sigma(y,\varepsilon)dS_y + \int_{|y|=\eta_0} G(x-y)\frac{\partial}{\partial n}\sigma(y,\varepsilon)dS_y$$

for $|x| < \eta_0$ where $\eta_0 > \eta$. From this representation we deduce the analyticity of $\sigma(x,\varepsilon)$ in (x,ε) for $|x| \leq \eta, |\varepsilon| \leq \varepsilon_0$.

Finally, since the function $w = p + \mu\sigma + \mu\frac{(x^2+y^2)}{4}\tilde{\sigma}$ is harmonic, a similar argument to the one above (now with the fundamental solution $H(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$) implies the analyticity of w in a neighborhood of the origin. The corresponding analyticity for p can then be derived from the representation $p = w - \mu\sigma - \mu(x^2 + y^2)\frac{\tilde{\sigma}}{4}$.

We summarize:

Theorem 11.2. *For any $R_0 > 0$, and $f_1(\theta) = \cos\ell\theta$ ($\ell \geq 2$), there exists a family of solutions, even in $\theta, (\sigma, p, f, \gamma)$ of (1.1)–(1.5) of the form (1.7), for all $|\varepsilon| \leq \varepsilon_0$, where ε_0 is a positive number; the series are convergent for $|\varepsilon| \leq \varepsilon_0$ and $|x| \leq R_0 + \delta_0$ (for some positive δ_0) and are analytic functions in (x, y, ε) for $\sqrt{x^2 + y^2} \leq R_0 +$*

$\delta_0, |\varepsilon| \leq \varepsilon_0$. *For each ε, the solution is unique under the orthogonality conditions:*

$$\int_0^{2\pi} f_j(\theta) \cos(m\theta)d\theta = 0 \quad for \quad m = 1, \ell \quad and \quad j \geq 2 \ ,$$

$$\int_0^{2\pi} \sigma \cos\theta d\theta = \int_0^{2\pi} p \cos\theta d\theta = 0 \quad \forall \ 0 < r \leq R_0 \ .$$

12. A FURTHER APPLICATION OF THE METHOD

In §6 we motivated our approach to the proof of convergence (given in §§7–11) by the example (6.4), (6.5). We now show that this method actually works for this example. For simplicity we assume (as in [4]) that

(12.1) $f(x)$ and $H(x)$ are 2π-periodic.

We seek a solution of (6.18), (6.19) which is 2π-periodic in x and remains bounded as $z \to \infty$.

Equation (6.21) can be written in more detail in the form

(12.2) $\Delta v_n = -(f')^2 \partial_z^2 v_{n-2} + 2f' \partial_x \partial_z v_{n-1} + f'' \partial_z v_{n-1} \equiv F_n \quad \text{if} \quad n \geq 0 \ .$

Equation (12.2) is analogous to the system (7.15)–(7.19).

We need to show that (12.2) with boundary condition $v_n(x, 0) = 0$ $(n \geq 1)$ has a unique 2π-periodic (in x) solution, and to derive H^2 bounds on v_n. For this purpose we write

$$f(x) = \sum c_q e^{iqx}$$

and expand formally

$$v_n(z) = \sum_r d_{n,r}(z) e^{irx} \ .$$

Substituting these expressions into (12.2) we get, after equating the coefficients of e^{irx},

$$d''_{n,r} - r^2 d_{nr} = F_{nr}$$

where F_{nr} depends only on the $d_{m,s}$ for $m < n$. From this relation we deduce uniqueness.

To prove existence and analyticity, we establish a result similar to Lemma 8.1, with weighted norms

$$||k||_\lambda^2 = \frac{1}{2\pi} \int_0^\infty \Big(\int_0^{2\pi} |k(x,z)|^2 dx \Big) e^{\lambda z} dz \ .$$

Lemma 12.1. *Let $g(x,z)$ be a function 2π-periodic in x with $||g||_\lambda < \infty$ $(\lambda > 0)$ and consider the problem*

(12.3) $\Delta w = g \quad in \quad z > 0 \ ,$

(12.4) $w = 0 \quad on \quad z = 0 \ .$

Then there exists a unique solution w which is 2π-periodic in x such that

(12.5) $||\partial_z w||_\lambda + ||\partial_z \partial_x w||_\lambda + ||\partial_z^2 w||_\lambda \leq C||g||_\lambda \ .$

Proof. We expand

$$g(x,z) = \sum_r g_r(z)e^{irx} ,$$

$$w(x,z) = \sum_r w_r(z)e^{irx} ,$$

so that (12.3), (12.4) reduce to

$$w_r'' - r^2 w_r = g_r(z), \quad w_r(0) = 0$$

and

$$\int_0^\infty |g_r(z)|^2 e^{\lambda z} dz < \infty .$$

The only solution to this problem is given by

$$w_r(z) = -\frac{e^{|r|z}}{2|r|} \int_z^\infty e^{-|r|s} g_r(s) ds + \frac{e^{-|r|z}}{2|r|} \int_0^z g_r(s) e^{|r|s} ds$$

$$(12.6) \qquad + \int_0^\infty g_r(s) e^{-|r|s} \frac{e^{-|r|z}}{2|r|} ds + c_r \left(e^{|r|z} - e^{-|r|z} \right) \quad \text{if} \quad r \geq 1 ,$$

$$w_0(z) = -\int_0^z sg(s) ds - \int_z^\infty zg(s) ds + c_0 z$$

where the c_r are constants. The assumption of finite norm (in particular, $||\partial_z(w_r(z)e^{irx})||_\lambda < \infty$) implies that $c_r = 0$ if $r \geq 0$. Let us now prove that

$$(12.7) \qquad \left(\int_0^\infty |w_r'(z)|^2 e^{\lambda z} dz \right)^{1/2} \leq C \left(\int_0^\infty |g_r(s)|^2 e^{\lambda s} ds \right)^{1/2} .$$

From (12.6) we get

$$w_r'(z) = -\frac{1}{2} e^{|r|z} \int_z^\infty e^{-|r|s} g_r(s) ds - \frac{1}{2} e^{-|r|z} \int_0^z g_r(s) e^{|r|s} ds$$

$$-\frac{1}{2} \int_0^\infty g_r(s) e^{-|r|s} e^{-|r|z} ds , \quad r \geq 1 ,$$

$$w_0'(z) = -\int_z^\infty g_0(s) ds .$$

Consider the first term in $w_r'(z), r \geq 1$:

$$\left(\int_0^\infty \left| e^{|r|z} \int_z^\infty e^{-|r|s} g_r(s) ds \right|^2 e^{\lambda z} dz \right)^{1/2}$$

$$= \left(\int_0^\infty \left| \int_0^\infty e^{-|r|u} g_r(u+z) du \right|^2 e^{\lambda z} dz \right)^{1/2}$$

$$\leq \int_0^\infty e^{-|r|u} \left(\int_0^\infty g_r(u+z)^2 e^{\lambda z} dz \right)^{1/2} du \quad \text{(by Minkowski's inequality)}$$

$$= \int_0^\infty e^{-|r|u} \left(\int_u^\infty g_r(s)^2 e^{\lambda(s-u)} ds \right)^{1/2} du \leq \frac{1}{|r| + \lambda/2} \left(\int_0^\infty g_r(s)^2 e^{\lambda s} ds \right)^{1/2} .$$

Similarly, one can estimate the other terms in w'_r. To deal with w'_0 we estimate

$$
\left(\int_0^\infty |w'_0(z)|^2 e^{\lambda z} dz \right)^{1/2} = \left(\int_0^\infty \left(\int_z^\infty g_0(s) ds \right)^2 e^{\lambda z} dz \right)^{1/2}
$$

$$
= \left(\int_0^\infty \left(\int_0^\infty g_0(u+z) du \right)^2 e^{\lambda z} dz \right)^{1/2}
$$

$$
\leq \int_0^\infty du \left(\int_0^\infty g_0(u+z)^2 e^{\lambda z} dz \right)^{1/2} = \int_0^\infty du \left(\int_u^\infty g_0(s)^2 e^{\lambda z} ds \right)^{1/2} e^{-\lambda u/2}
$$

$$
\leq \frac{2}{\lambda} \left(\int_0^\infty g_0(s)^2 e^{\lambda s} ds \right)^{1/2} .
$$

Thus, we conclude that

$$
||\partial_z w||_\lambda \leq C ||g||_\lambda .
$$

The other norms

$$
||\partial_z \partial_x w||_\lambda , \quad ||\partial_z^2 w||_\lambda
$$

can be estimated in the same way.   □

Having proved Lemma 12.1 we now follow the procedure of §§10, 11 and establish the analyticity of $v(x, z, \varepsilon)$. Then, by a change of variables, we conclude the existence of 2π-periodic (in x) solution $u(x, y, \varepsilon)$ of (6.4), (6.5) which is analytic in (x, y, ε) for $-\infty < x < \infty, -\delta_0 \leq y < \infty, |\varepsilon| \leq \varepsilon_0$, where δ_0, ε_0 are small positive constants.

REFERENCES

1. J.A. Adam, General aspects of modeling tumor growth and immune response, in *A Survey of Models for Tumor-Immune System Dynamics*, J.A. Adam and N. Bellomo, eds., Birkhäuser, Boston (1996), 15–87.
2. R.A. Adams, *Sobolev Spaces*, Academic Press, New York (1975). MR **56:**9247
3. O. Bruno and P. Laurence, *Existence of three-dimensional toroidal MHD equilibria with non-constant pressure*, Comm. Pure Appl. Math., 49 (1996), 717–764. MR **97g:**35137
4. O. Bruno and F. Reitich, *Solution of the boundary value problem for the Helmholz equation via variation of the boundary into the complex domain*, Proc. Roy. Soc. Edinburgh, Sec. A 122 (1992), 317–340. MR **94e:**35131

5. H.M. Byrne, *The importance of intercellular adhesion in the development of carcinomas*, IMA J. Math. Appl. Med. and Biol., 14 (1997), 305–323.
6. H.M. Byrne, *The effect of time delays on the dynamics of avascular tumor growth*, Math. Biosciences, 144 (1997), 83–117. MR **98h:**92014
7. H.M. Byrne and M.A.J. Chaplain, *Growth of necrotic tumors in the presence and absence of inhibitors*, Math. Biosciences, 135 (1996), 187–216.
8. M.A.J. Chaplain, *The development of a spatial pattern in a model for cancer growth*, in *Experimental and Theoretical Advances in Biological Pattern Formation*, H.G. Othmer, P.K. Maini and J.D. Murray eds., Plenum Press (1993), 45–59. MR **22:**9739
9. A. Friedman, *On the regularity of solutions of nonlinear elliptic and parabolic systems of partial differential equations*, J. Math. Mech., 7 (1958), 43–60.
10. A. Friedman and F. Reitich, *Analysis of a mathematical model for the growth of tumors*, J. Math. Biol., 38 (1999), 262–284. CMP 99:11
11. A. Friedman and F. Reitich, *Nonlinear stability of a quasi-static Stefan problem with surface tension: a continuation approach*, to appear.
12. H.P. Greenspan, *On the growth and stability of cell cultures and solid tumors*, Theor. Biol., 56 (1976), 229–242. MR **55:**2183
13. D.L.S. McElwain and L.E. Morris, *Apoptosis as a volume loss mechanism in mathematical models of solid tumor growth*, Math. Biosciences, 39 (1978), 147–157.
14. D.H Sattinger, *Group-Theoretic Methods in Bifurcation Theory, Lecture Notes in Mathematics*, 762, Springer-Verlag, Berlin (1979). MR **81e:**58022
15. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York (1983). MR **84d:**35002
16. G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press (1944). MR **6:**64a

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: `friedman@math.umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: `reitich@math.umn.edu`