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Symmetry-Breaking Corrections to Meson Decay Constants in the Heavy-Quark Effective Theory*

MATTHIAS NEUBERT†

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94309

ABSTRACT

Spin- and flavor-symmetry breaking corrections to decay constants of heavy mesons are analyzed in next-to-leading order in the $1/m_Q$ expansion. The general structure of these corrections is derived in an effective-field-theory approach. The subleading universal form factors, which parametrize the matrix elements of higher-dimensional operators in the effective theory, are estimated using QCD sum rules. The renormalization-group improvement of these low-energy parameters is discussed in detail. As an application, the spin-symmetry violating effects responsible for the vector–pseudoscalar mass difference and for the ratio of the corresponding decay constants, f_V/f_P, are calculated.

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1. Introduction

Over the last few years, the study of the properties of hadronic processes involving heavy quarks has become a very active field of research [1-31]. In the limit of very large quark masses, a number of exact relations can be derived despite of the presence of long-range strong interactions. The reason is that for heavy quarks QCD exhibits a spin-flavor symmetry which is only softly broken by terms of order Λ_{QCD}/m_Q [5]. This symmetry relates the hadronic matrix elements of heavy hadrons with different spin or flavor quantum numbers. It becomes explicit in an effective-field-theory formulation of QCD [8-10].

The phenomenological applications of this formalism are numerous [32-37]. In particular, it turns out that the description of current-induced processes like semi-leptonic decays of heavy mesons or baryons becomes very simple in the formal limit of infinite heavy-quark masses. The large set of hadronic form factors is then reduced to a small number of universal functions (the Isgur-Wise functions), which are independent of the heavy-quark masses [5]. They contain all long-distance dynamics relevant to the hadronic transition. This observation offers the exciting possibility to extract in a model-independent way some of the weak mixing angles from the measurement of decays of heavy hadrons, without limitations arising from the ignorance of long-distance dynamics [3,36,37].

Clearly, a thorough establishment of the heavy-quark expansion requires a careful analysis of symmetry-breaking corrections. Much attention has been devoted to this subject [11-20]. Already in leading order in the $1/m_Q$ expansion, the symmetry is violated by hard-gluon exchange. These effects allow for a perturbative treatment. The corresponding corrections have been calculated first in leading logarithmic approximation [4,11], and more recently in next-to-leading order in renromalization-group improved perturbation theory [17-20]. At subleading order in the $1/m_Q$ expansion, one is generally forced to introduce additional universal form factors. The structures that arise have been worked out for matrix elements between two heavy mesons [12] or Λ-baryons [14]. Some of the subleading form factors obey nontrivial constraints arising from the equations of motion. An additional complication results from the fact that higher-dimensional operators in the effective theory mix under renormalization [13,16]. The pattern of relations among matrix elements thus becomes considerably more complex than at leading order.

Many of the subtle issues related to the $1/m_Q$ expansion can already be studied in the simpler case of current matrix elements between a heavy meson and the vacuum. These matrix elements define meson decay constants, which are hadronic properties of primary theoretical and phenomenological interest. Following the analysis of Ref. 12, we derive in Sect. 2 the structure of $1/m_Q$ corrections in this case. It is shown that three additional universal parameters are induced at subleading order. Their behavior under the renormalization group is derived to one-loop order. In Sect. 3, we estimate these subleading form factors using QCD sum rules in the effective theory. At leading order in the $1/m_Q$ expansion, sum rules have recently been used to calculate the asymptotic value of the scaled pseudoscalar decay constant, $f_P\sqrt{m_P}$, and the Isgur-Wise form factor [28-30,35]. In this paper, we estimate the slope of the decay constants with respect to $1/m_Q$, as well as the spin-symmetry breaking effects responsible for the vector–pseudoscalar mass splitting and differences in f_V and f_P. The emphasis is to show that the sum rule technique can be extended to calculate form factors that appear in subleading order of the $1/m_Q$ expansion. In particular, we show that the constraints resulting from the equations of motion are respected. In Sect. 4, it is demonstrated that also the running of the universal form factors is correctly reproduced. Sect. 5 contains the conclusion.

2. Power Corrections to Decay Constants in the Heavy-Quark Effective Theory

A convenient framework for a systematic analysis of the behavior of hadronic matrix elements in the limit of large quark masses is provided by an effective-field-theory approach, the so-called heavy-quark effective theory [8]. It is based on the observation that, in the limit $m_Q \gg \Lambda_{QCD}$, the velocity v of a heavy quark is conserved with respect to soft processes. It is then possible to remove the mass-dependent piece of the momentum operator by the field redefinition

$$h_Q(v,x) = \exp(im_Q\not{v}v\cdot x)\,Q(x)\;, \tag{2.1}$$

such that

$$i\not{\partial}\,h_Q(v,x) = (\not{P} - m_Q\not{v})\,h_Q(v,x) \equiv \not{k}\,h_Q(v,x)\;, \tag{2.2}$$

where P is the total momentum of the heavy quark, and k denotes its residual "off-shell" momentum, which is of order Λ_{QCD}. The fields h_Q create and annihilate heavy quarks and antiquarks with velocity v. We shall furthermore project onto quark-states (as opposed to antiquarks) by imposing the condition $\not{v}\,h_Q = h_Q$.

Written in terms of these new fields, the renormalized effective Lagrangian is an infinite series of local operators with increasing canonical dimension, multiplied by powers of $1/m_Q$ [8-10]

$$\mathcal{L}_{eff} = \bar{h}_Q\left[iv\cdot D + \frac{(iD)^2}{2m_Q}\right]h_Q + \frac{Z_m g_s}{4m_Q}\,\bar{h}_Q\,\sigma_{\mu\nu}G^{\mu\nu}\,h_Q + \dots\;, \tag{2.3}$$

with $D_\mu = \partial_\mu - ig_sA_\mu$ being the gauge-covariant derivative. To leading order in the $1/m_Q$ expansion, this Lagrangian exhibits the spin and flavor symmetries for the heavy quarks. These symmetries are explicitly broken at subleading order, however. In particular, the spin symmetry is broken by the "magnetic interaction" operator involving the gluonic field-strength tensor $G_{\mu\nu}$. The ellipses in (2.3) stand for operators multiplied by $1/m_Q^2$, as well as for an operator whose matrix elements are of order $1/m_Q^2$ due to the equations of motion

$$iv\cdot D\, h_Q = \mathcal{O}\Big(\frac{1}{m_Q}\Big) . \tag{2.4}$$

In writing down (2.3) we have chosen a particular renormalization scheme by not including a residual mass term $\delta m\, \bar{h}_Q\, h_Q$ for the heavy quark [31], nor renormalization factors for the spin-symmetry conserving operators. In momentum space, the associated renormalized heavy quark propagator has a pole with unit residue at $v\cdot k + k^2/2m_Q = 0$, corresponding to $P^2 = m_Q^2$. In perturbation theory, therefore, the heavy-quark mass m_Q in (2.1) coincides with the so-called "physical" pole mass, which is a renormalization-group invariant quantity. This is in accordance with the interpretation of k as an "off-shell" momentum. The coefficient of the spin-symmetry breaking operator in (2.3) gets renormalized, however. In the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme, one finds in leading logarithmic approximation [13]

$$Z_m\Big(\frac{m_Q}{\mu}\Big) = \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-9/\beta} ; \quad \beta = 33 - 2n_f , \tag{2.5}$$

with n_f being the number of light-quark flavors.

Any current operator $J = \bar{q}\,\Gamma\, Q$ of the full theory can be expanded in terms of local operators of the effective theory. For the vector current, the result reads

$$\bar{q}\,\gamma_\mu\, Q \to C_0\Big(\frac{m_Q}{\mu}\Big)\, \bar{q}\,\gamma_\mu\, h_Q + C_1\Big(\frac{m_Q}{\mu}\Big)\, \bar{q}\, v_\mu\, h_Q + \frac{1}{m_Q}\sum_{i=1}^{6} B_i\Big(\frac{m_Q}{\mu}\Big)\, Q_i + \ldots . \tag{2.6}$$

In the limit $m_q = 0$, a convenient basis for the subleading operators is [31]

$$\begin{aligned}
Q_1 &= \bar{q}\,\gamma_\mu\, i\not{D}\, h_Q , & Q_4 &= \bar{q}\,(-iv\cdot \overleftarrow{D})\,\gamma_\mu\, h_Q , \\
Q_2 &= \bar{q}\, v_\mu\, i\not{D}\, h_Q , & Q_5 &= \bar{q}\,(-iv\cdot \overleftarrow{D})\, v_\mu\, h_Q , \\
Q_3 &= \bar{q}\, iD_\mu\, h_Q , & Q_6 &= \bar{q}\,(-i\overleftarrow{D}_\mu)\, h_Q .
\end{aligned} \tag{2.7}$$

The expansion of the axial vector current $\bar{q}\,\gamma_\mu\gamma_5\, Q$ is obtained by simply replacing $\bar{q}$ in (2.6) and (2.7) by $-\bar{q}\,\gamma_5$. The coefficients remain unchanged.

The effective current operators renormalize differently from their QCD counterparts. In particular, they have non-zero anomalous dimensions, such that matrix elements in the effective theory depend on the renormalization scheme. The short-distance coefficients C_i and B_i, which (in dimensional regularization) contain logarithms of m_Q/μ, ensure that the final results are independent of the renormalization procedure. At $\mu = m_Q$, they are obtained from the matching of QCD onto the effective theory. Their running below m_Q is determined by a renormalization-group equation. The coefficients C_i in (2.6) have been calculated to next-to-leading order in renormalization-group improved perturbation theory [4,17,18]. The coefficients B_i are known in leading logarithmic approximation only [16,31]. Then, in particular, $B_2 = B_3 = 0$. Without QCD corrections, $B_1 = 1/2$ and $B_i = 0$ otherwise.

The expansion of currents in terms of operators of the effective theory provides a separation of short- and long-distance phenomena. The short-distance physics associated with the large mass scale m_Q factorizes and can be treated perturbatively. Long-distance effects are m_Q-independent and are relevant only to hadronic matrix elements of local operators in the effective theory. These matrix elements are constrained by the heavy-quark symmetries and can be parametrized in terms of a few universal form factors. The number of independent form factors and the relations among matrix elements become most transparent in a compact trace-formalism [11,33]. To leading order in the $1/m_Q$ expansion, the matrix elements defining decay constants of heavy mesons are of the generic form [28]

$$\langle 0 \,|\, \bar q\,\Gamma\, h_Q \,|M(v)\rangle = \frac{F(\mu)}{2}\, \mathrm{Tr}\left\{\, \Gamma\, \mathcal{M}(v) \,\right\} \tag{2.8}$$

and are all related to a single universal low-energy parameter $F(\mu)$, which is independent of the heavy-quark mass. The Dirac structure Γ of the current is irrelevant. In the effective theory, a heavy meson is represented by its spin wave-function

$$\mathcal{M}(v) = \sqrt{m_M}\, \frac{(1+\not v)}{2} \begin{cases} -i\,\gamma_5 & ;\ J^P = 0^- \ , \\ \not\epsilon & ;\ J^P = 1^- \ , \end{cases} \tag{2.9}$$

which satisfies $\not v\, \mathcal{M}(v) = \mathcal{M}(v) = -\mathcal{M}(v)\,\not v$. The normalization in (2.8) is chosen such that, apart from QCD corrections, the universal parameter F is related to the decay constant of a heavy pseudoscalar meson P by $F = f_P\,\sqrt{m_P}$. This is the well-known scaling law which states that, up to logarithmic corrections, $f_P \propto 1/\sqrt{m_P}$.

At next-to-leading order in the heavy-quark expansion, one has to include the $1/m_Q$ corrections to the current [cf. (2.6)] as well as to the hadronic wave-function. The method is described in detail in Ref. 12. Concerning the matrix elements of the higher-dimensional operators Q_i in (2.7) we first note that, because of the field redefinition in (2.1), current operators in the effective theory carry the total external momentum minus $m_Q v$. Therefore,

$$\begin{aligned}\langle\, 0\,|\; i\partial_\alpha\,(\bar q\,\Gamma\, h_Q)\;|M(v)\rangle &= (m_M - m_Q)\, v_\alpha\;\langle\, 0\,|\;\bar q\,\Gamma\, h_Q\;|M(v)\rangle\\ &= \frac{\bar\Lambda}{2}\, F(\mu)\;\mathrm{Tr}\,\{\, v_\alpha\,\Gamma\,\mathcal{M}(v)\,\}\end{aligned} \tag{2.10}$$

in terms of the mass parameter $\bar\Lambda = m_M - m_Q$, which is a non-trivial observable of the effective theory [31]. Matrix elements of the operators Q_1, Q_2, and Q_3, which contain a covariant derivative acting on the heavy-quark field, have the general structure

$$\langle\, 0\,|\;\bar q\,\Gamma\, iD_\alpha\, h_Q\;|M(v)\rangle = \frac{1}{2}\;\mathrm{Tr}\,\{\,\big[F_1(\mu)\, v_\alpha + F_2(\mu)\,\gamma_\alpha\big]\,\Gamma\,\mathcal{M}(v)\,\}\;, \tag{2.11}$$

where $F_i(\mu)$ are new low-energy parameters. The equations of motion (2.4) imply $F_1(\mu) = F_2(\mu)$. We can furthermore relate $F_1(\mu)$ to $\bar\Lambda F(\mu)$. To this end, we set $\Gamma = \gamma^\alpha\,\hat\Gamma$ and use the equations of motion $i\not{D}\, q = 0$ for the light quark to rewrite $\bar q\,\gamma^\alpha\,\hat\Gamma\, iD_\alpha\, h_Q = i\partial^\alpha\,(\bar q\,\gamma_\alpha\,\hat\Gamma\, h_Q)$. From (2.10) and (2.11) it then follows that

$$F_1(\mu) = F_2(\mu) = -\frac{\bar\Lambda}{3}\, F(\mu)\;. \tag{2.12}$$

Matrix elements of Q_4, Q_5, and Q_6 can be evaluated along the same lines since

$$\bar q\,(-i\overleftarrow{D}_\alpha)\,\Gamma\, h_Q = \bar q\,\Gamma\,(iD_\alpha)\, h_Q - i\partial_\alpha\,(\bar q\,\Gamma\, h_Q)\;. \tag{2.13}$$

The $1/m_Q$ corrections to the hadronic wave function come from insertions of the subleading operators in the effective Lagrangian into matrix elements of the leading-order currents. They induce two additional universal parameters $G_1(\mu)$ and $G_2(\mu)$ defined by matrix elements of the time-ordered products

$$\begin{aligned}&\langle\, 0\,|\; i\int \mathrm{d}y\, T\{(\bar q\,\Gamma\, h_Q)_0, (\bar h_Q\,(iD)^2\, h_Q)_y\}\;|M(v)\rangle = F(\mu)\, G_1(\mu)\;\mathrm{Tr}\,\{\,\Gamma\,\mathcal{M}(v)\,\}\;,\\ &\langle\, 0\,|\; i\int \mathrm{d}y\, T\{(\bar q\,\Gamma\, h_Q)_0, \frac{g_s}{2}\,(\bar h_Q\,\sigma_{\mu\nu} G^{\mu\nu}\, h_Q)_y\}\;|M(v)\rangle\\ &= F(\mu)\, G_2(\mu)\;\mathrm{Tr}\,\{\,\sigma_{\mu\nu}\Gamma\frac{(1+\not{v})}{2}\sigma^{\mu\nu}\,\mathcal{M}(v)\,\} = 2d_\Gamma\, F(\mu)\, G_2(\mu)\;\mathrm{Tr}\,\{\,\Gamma\,\mathcal{M}(v)\,\}\;,\end{aligned} \tag{2.14}$$

with coefficients that are independent of the external states. In particular $d_V = -1$ and $d_A = 3$ for the vector and axial vector current, respectively.

Using the above relations, the matrix elements relevant to meson decay constants can be computed to subleading order in the $1/m_Q$ expansion. We find, to all orders in perturbation theory,

$$\begin{aligned}
\langle 0 |\, \bar{q}\,\Gamma\, Q\, | M(v) \rangle &= \frac{1}{2} C\Big(\frac{m_Q}{\mu}\Big)\, F(\mu)\; \mathrm{Tr}\,\{\, \Gamma\, \mathcal{M}(v)\,\} \\
&\times \Bigg\{ \big[1 + d_\Gamma\, c(m_Q)\big] \bigg(1 + \frac{1}{m_Q} \Big[G_1(\mu) + 2 d_\Gamma\, Z_m\Big(\frac{m_Q}{\mu}\Big)\, G_2(\mu) \Big] \bigg) \\
&\quad - \frac{\bar{\Lambda}}{6 m_Q} \Big[b\Big(\frac{m_Q}{\mu}\Big) + d_\Gamma\, B\Big(\frac{m_Q}{\mu}\Big) \Big] \Bigg\}
\end{aligned} \tag{2.15}$$

with QCD coefficients

$$\begin{aligned}
C &= C_0 + \frac{C_1}{4}\,, \quad c = \frac{C_1}{4C}\,, \\
B &= \frac{1}{2C}\, \big[4B_1 - 3B_2 - B_3 + 3B_5 + 2B_6 \big]\,, \\
b &= \frac{3}{2C}\, \big[-B_2 + B_3 + 4B_4 + B_5 + 2B_6 \big]\,.
\end{aligned} \tag{2.16}$$

We use capital letters for coefficients that were equal to one in the absence of QCD corrections, and small letters for those which are of order α_s. In next-to-leading order of renormalization-group improved perturbation theory the expressions for C and c are (in the $\overline{\mathrm{MS}}$ subtraction scheme) [17,18,20]

$$\begin{aligned}
C\Big(\frac{m_Q}{\mu}\Big) &= \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)} \right]^{6/\beta} \left\{ 1 + \frac{\alpha_s(m_Q)}{\pi} \Big(Z_{n_f} - \frac{1}{2} \Big) - \frac{\alpha_s(\mu)}{\pi}\, (Z_{n_f} + \delta_{\overline{\mathrm{MS}}}) \right\}\,, \\
c(m_Q) &= \frac{\alpha_s(m_Q)}{6\pi}\,,
\end{aligned} \tag{2.17}$$

where $\delta_{\overline{\mathrm{MS}}} = 2/3$ is a scheme-dependent constant, and the coefficient Z_{n_f} is defined in Ref. 28 ($Z_4 \simeq -0.894$). In leading logarithmic approximation, expressions for B and b can be derived from the results of Refs. 16 and 31. Allowing for a nonlogarithmic one-loop matching correction, we find

$$\begin{aligned}
B\Big(\frac{m_Q}{\mu}\Big) &= \frac{16}{9} \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)} \right]^{-9/\beta} - \frac{7}{9} + B_0\, \frac{\alpha_s}{\pi}\,, \\
b\Big(\frac{m_Q}{\mu}\Big) &= \frac{48}{\beta}\, \ln\left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)} \right] + b_0\, \frac{\alpha_s}{\pi}\,,
\end{aligned} \tag{2.18}$$

where B_0 and b_0 are again scheme-dependent. For later purpose,we have computed B_0 from one-loop matching of QCD and the effective theory. In the $\overline{\mathrm{MS}}$ subtraction scheme, the result is $B_0^{\overline{\mathrm{MS}}} = 35/9$.

It is convenient to rewrite (2.15) in terms of renormalization-group invariant form factors $\hat{F}(m_Q)$ and $\hat{G}_i(m_Q)$ which, to lowest order, coincide with the low-energy parameters F and G_i, *i.e.*

$$
\begin{aligned}
\langle\, 0\,|\,\bar{q}\,\Gamma\, Q\,|M(v)\rangle &= \frac{\hat{F}(m_Q)}{2}\,\big[1 + d_\Gamma\, c(m_Q)\big]\ \mathrm{Tr}\,\big\{\,\Gamma\,\mathcal{M}(v)\,\big\} \\
&\quad\times \left\{1 + \frac{\hat{G}_1(m_Q)}{m_Q} + \frac{2d_\Gamma}{m_Q}\left[\hat{G}_2(m_Q) - \frac{\bar{\Lambda}}{12}\right]\right\} .
\end{aligned}
\tag{2.19}
$$

In next-to-leading order of renormalization-group improved perturbation theory, we can neglect terms proportional to $c^2(m_Q)$ and find

$$
\begin{aligned}
\hat{F}(m_Q) &= C\Big(\frac{m_Q}{\mu}\Big)\, F(\mu)\,, \\
\hat{G}_1(m_Q) &= G_1(\mu) - \left[b\Big(\frac{m_Q}{\mu}\Big) - 3c(m_Q)\, B\Big(\frac{m_Q}{\mu}\Big)\right]\frac{\bar{\Lambda}}{6}\,, \\
\hat{G}_2(m_Q) &= Z_m\Big(\frac{m_Q}{\mu}\Big)\, G_2(\mu) - \left[\big[1 - 2c(m_Q)\big]\, B\Big(\frac{m_Q}{\mu}\Big) - c(m_Q)\, b\Big(\frac{m_Q}{\mu}\Big) - 1\right]\frac{\bar{\Lambda}}{12}\,.
\end{aligned}
\tag{2.20}
$$

From the fact that these expressions must be μ-independent one can deduce the scale-dependence of the universal parameters. To first order in α_s, we obtain

$$
\begin{aligned}
\mu\,\frac{\partial F}{\partial\mu} &= \frac{\alpha_s}{\pi}\, F\,, \\
\mu\,\frac{\partial G_1}{\partial\mu} &= -\frac{4}{3}\,\frac{\alpha_s}{\pi}\,\bar{\Lambda}\,, \\
\mu\,\frac{\partial G_2}{\partial\mu} &= -\frac{3}{2}\,\frac{\alpha_s}{\pi}\, G_2 + \frac{2}{9}\,\frac{\alpha_s}{\pi}\,\bar{\Lambda}\,.
\end{aligned}
\tag{2.21}
$$

These relations must be obeyed in any sensible calculation of the form factors which is sensitive to the μ-dependence.

As an application, we derive a relation for the ratio of the decay constants of a heavy vector meson V and a heavy pseudoscalar meson P, defined by

$$
\begin{aligned}
\langle\, 0\,|\,\bar{q}\,\gamma_\mu\gamma_5\, Q\,|P(v)\rangle &= i f_P m_P\, v_\mu\,, \\
\langle\, 0\,|\,\bar{q}\,\gamma_\mu\, Q\,|V(\epsilon, v)\rangle &= f_V m_V\, \epsilon_\mu\,.
\end{aligned}
\tag{2.22}
$$

From (2.19) it follows that

$$
\frac{f_V m_V^{1/2}}{f_P m_P^{1/2}} = \left(1 - \frac{2}{3}\,\frac{\alpha_s(m_Q)}{\pi}\right)\left\{1 - \frac{8}{m_Q}\left[\hat{G}_2(m_Q) - \frac{\bar{\Lambda}}{12}\right]\right\} .
\tag{2.23}
$$

The result involves the renormalized parameter $\hat{G}_2(m_Q)$, which arises from the spin-symmetry breaking "magnetic interaction" operator in the effective Lagrangian (2.3).

3. Subleading Form Factors from QCD Sum Rules

After this general discussion of the structure of $1/m_Q$ corrections to decay constants of heavy mesons, we now present a calculation of the subleading universal parameters $\bar{\Lambda}, G_1$, and G_2 using QCD sum rules in the effective theory. Throughout this section, we shall not consider QCD corrections. They are discussed in Sect. 4.

The application of the QCD sum rules developed by Shifman, Vainshtein and Zakharov [38] to the calculation of universal heavy-quark form factors has been recently worked out in Refs. 28–30. The idea is to study the analytic properties of correlators of heavy-quark currents in the effective theory. Consider, for instance, the two-point function

$$\Gamma = i \int \mathrm{d}x\, e^{ik\cdot x}\, \langle\, 0 \,|\, \mathcal{T}\big\{ \big[\bar{q}\,\Gamma_M\, h_Q(v)\big]_x , \big[\bar{h}_Q(v)\,\Gamma_M\, q\big]_0 \big\}\, |\, 0\,\rangle \ , \qquad (3.1)$$

where the currents interpolate the heavy meson M of interest. We choose

$$\Gamma_M = \begin{cases} -i\,\gamma_5 & ;\ J^P = 0^- \ , \\ \gamma_\mu - v_\mu & ;\ J^P = 1^- \ . \end{cases} \qquad (3.2)$$

According to (2.1) the total external momentum in (3.1) is $P = m_Q\, v + k$, and in QCD the correlator is an analytic function in

$$\omega_Q \equiv \frac{P^2 - m_Q^2}{m_Q} = 2\, v\cdot k + \frac{k^2}{m_Q} \qquad (3.3)$$

with a cut on the positive real axis starting at $P^2 = m_M^2$, corresponding to

$$\omega_Q^{pole} = \frac{m_M^2 - m_Q^2}{m_Q} \equiv \widetilde{\Lambda} = 2\bar{\Lambda} + \mathcal{O}\Big(\frac{1}{m_Q}\Big) \ . \qquad (3.4)$$

Note that for our particular choice of the dispersive variable ω_Q there is no left-hand cut in the complex ω_Q-plane.

The two-point function Γ can be written as a dispersion integral over a physical spectral function. Isolating the pole contribution, one obtains the phenomenological representation of the correlator in terms of hadronic states

$$\Gamma_{phen}(\omega_Q) = \Big(\sum_{\text{pol.}}\Big) \frac{\langle 0 |\, \bar q\, \Gamma_M\, h_Q\, | M(v)\rangle \langle M(v) |\, \bar h_Q\, \Gamma_M\, q\, | 0 \rangle}{m_Q\, (\widetilde\Lambda - \omega_Q - i\epsilon)} + \int\limits_{\omega > \widetilde\Lambda}^{\infty} \mathrm{d}\omega\, \frac{\rho_{phys}(\omega)}{\omega - \omega_Q - i\epsilon} + \text{subtractions}, \tag{3.5}$$

where one has to sum over polarizations if M is a vector meson. For the evaluation of the pole contribution, we use (2.8) and (2.14), as well as the relation

$$\Big(\sum_{\text{pol.}}\Big) \mathrm{Tr}\,\{\, \Gamma\, \mathcal{M}(v) \,\}\ \mathrm{Tr}\,\{\, \overline{\mathcal{M}}(v)\, \Gamma_M \,\} = -m_M\ \mathrm{Tr}\,\{\, \Gamma\, (\not v + 1)\, \Gamma_M \,\}\,, \tag{3.6}$$

which is valid for any matrix Γ. To subleading order in $1/m_Q$, we find

$$\Gamma_{pole}(\omega_Q) = -\frac{F^2}{4}\, \frac{\mathrm{Tr}\,\{\, \Gamma_M\, (\not v + 1)\, \Gamma_M \,\}}{(\widetilde\Lambda - \omega_Q - i\epsilon)} \left\{ 1 + \frac{2}{m_Q} \left[G_1 + \frac{\bar\Lambda}{2} + 2 d_\Gamma\, G_2 \right] \right\}. \tag{3.7}$$

For large negative values of ω_Q (*i.e.*, $\Lambda_{QCD} \ll -\omega_Q \ll m_Q$), the two-point function can be calculated in perturbation theory. As $(-\omega_Q)$ becomes smaller, however, nonperturbative effects start to be important. The idea of QCD sum rules is that, at the transition from the perturbative to the nonperturbative regime, these can be taken into account by including the leading power corrections in the operator product expansion of the correlator. These nonperturbative corrections are proportional to a small set of vacuum expectation values of local quark-gluon operators, the so-called condensates [38]. In the calculation of the two-point function Γ we use the Feynman rules of the effective theory [11] and include insertions of the subleading operators in the effective Lagrangian. The leading nonperturbative power corrections are proportional to the quark condensate (dimension $d = 3$), the gluon condensate ($d = 4$), and the mixed quark-gluon condensate ($d = 5$). In terms of the dispersive variable ω_Q defined in (3.3), the result reads

$$\begin{aligned} \Gamma_{th}(\omega_Q) = &-\frac{1}{4}\, \mathrm{Tr}\,\{\, \Gamma_M\, (\not v + 1)\, \Gamma_M \,\} \\ &\times \left\{ \frac{3}{8\pi^2} \int\limits_0^\infty \mathrm{d}\omega \frac{\omega^2}{\omega - \omega_Q - i\epsilon} \left[1 - \frac{3\omega}{2m_Q} \right] + \text{subtractions} \right. \\ &\left. + \frac{\langle\bar q q\rangle}{\omega_Q} + \frac{\langle\alpha_s G G\rangle}{24\pi\, m_Q\, \omega_Q} \left[1 - d_\Gamma \right] - \frac{g_s \langle \bar q \sigma_{\mu\nu} G^{\mu\nu} q\rangle}{2\omega_Q^3} \left[1 + d_\Gamma\, \frac{\omega_Q}{6 m_Q} \right] \right\}, \end{aligned} \tag{3.8}$$

with d_Γ as defined in (2.14). Note that there is no $1/m_Q$ correction to the quark condensate (apart from the k^2/m_Q term in ω_Q), and that the gluon condensate does not contribute in leading order of the $1/m_Q$ expansion [28]. Its contribution is tiny and will be neglected from here on.

The QCD sum rule is obtained by matching the phenomenological and theoretical expressions for the correlator. In doing this, one assumes quark-hadron duality to model the contributions of higher-resonance states in (3.5) by the perturbative continuum starting at a threshold energy ω_c. Furthermore, in order to improve the convergence and to reduce the importance of higher-resonance states, a Borel transformation $\omega_Q \to T$ is applied to both sides of the sum rule [38]. This yields to an exponential damping factor in the dispersion integral, and also eliminates subtraction terms in the dispersion relation. From the resulting Laplace sum rule, the parameters of the effective theory can be determined in a self-consistent way by requiring stability with respect to variations of T in a region where the theoretical calculation is reliable. Before presenting the result, it is convenient to redefine the Borel parameter T according to

$$\frac{1}{T} \to \frac{1}{T} - \frac{3}{2m_Q} \,. \tag{3.9}$$

On the phenomenological side, this adds $3\bar{\Lambda}/2$ to G_1. On the theoretical side, it absorbes the $1/m_Q$ corrections to the perturbative contribution. The final sum rule reads

$$\begin{aligned} &F^2 \left\{ 1 + \frac{2}{m_Q} \Big[(G_1 + 2\bar{\Lambda}) + 2d_\Gamma\, G_2 \Big] \right\} e^{-\widetilde{\Lambda}/T} \\ &= \frac{3}{8\pi^2} \int\limits_0^{\omega_c} \mathrm{d}\omega\, \omega^2\, e^{-\omega/T} - \langle\bar{q}q\rangle + \frac{g_s\langle\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{4T^2} \left\{ 1 - \left(1 + \frac{d_\Gamma}{9}\right) \frac{3T}{m_Q} \right\} . \end{aligned} \tag{3.10}$$

Let us first discuss the infinite-quark-mass limit of this expression [28-30,39]

$$F^2\, e^{-\widetilde{\Lambda}_0/T} = \frac{3}{8\pi^2} \int\limits_0^{\omega_0} \mathrm{d}\omega\, \omega^2\, e^{-\omega/T} - \langle\bar{q}q\rangle + \frac{g_s\langle\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{4T^2} \equiv K(T^{-1};\omega_0) \,. \tag{3.11}$$

By taking the derivative with respect to the inverse Borel parameter, one derives the sum rule for the asymptotic value of the mass parameter $\widetilde{\Lambda}_0$ [cf. (3.4)]

$$\widetilde{\Lambda}_0 = 2\bar{\Lambda} = -\frac{K'(T^{-1};\omega_0)}{K(T^{-1};\omega_0)} \,. \tag{3.12}$$

The aim is to optimize the value of the threshold energy ω_0 in such a way that the right-hand side of this equation becomes independent of T inside the so-called "sum rule window", where the calculation is reliable. For too small values of T, the power corrections blow up, *i.e.*, nonperturbative effects become dominant. We use the standard values of the vacuum condensates

$$\begin{aligned} \langle\bar{q}q\rangle &= -(230\ \text{MeV})^3 , \\ g_s\langle\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q\rangle &= 0.8\ \text{GeV}^2 \times \langle\bar{q}q\rangle , \end{aligned} \tag{3.13}$$

and require that the power corrections be less than 30% of the quark-loop contribution. This yields the lower limit $T \geq 0.6$ GeV. According to (3.11), the perturbative spectral density grows like ω^2, such that higher-resonance contributions are important even after the Borel improvement. This is a general feature of heavy-quark sum rules, which is unavoidable. In order to reduce the sensitivity to how well these contributions are approximated by duality, we require that the pole contribution of the heavy meson M give at least 30% of the quark loop. For typical threshold values $\omega_0 \simeq 2$ GeV, this implies $T \leq 1$ GeV. In Fig. 1 we show the behavior of $\bar{\Lambda}$ and F in this region. The stability is very good for values*

$$\begin{aligned} \omega_0 &\simeq 2.0 \pm 0.3\ \text{GeV} , \\ \bar{\Lambda} &\simeq 0.50 \pm 0.07\ \text{GeV} , \\ F &\simeq 0.30 \pm 0.05\ \text{GeV}^{3/2} , \end{aligned} \tag{3.14}$$

with correlated errors.

Let us now turn to the analysis of (3.10). The "source term" for $1/m_Q$ corrections on the theoretical side is proportional to the mixed condensate. It induces changes in the parameters ω_c and $\widetilde{\Lambda}$ with respect to their asymptotic values determined above

$$\begin{aligned} \omega_c &= \omega_0 \left\{ 1 + \frac{1}{m_Q} (\delta\omega_1 + d_\Gamma\, \delta\omega_2) \right\} , \\ \widetilde{\Lambda} &= 2\bar{\Lambda} \left\{ 1 + \frac{1}{m_Q} (\delta\Lambda_1 + d_\Gamma\, \delta\Lambda_2) \right\} . \end{aligned} \tag{3.15}$$

Inserting this ansatz into (3.10) and expanding in $1/m_Q$ leads to sum rules for the subleading parameters $\delta\Lambda_i$ and G_i [40]. We first discuss the spin-symmetry breaking corrections, which are proportional to the coefficient d_Γ. They obey the

* Here and in the following estimates the errors only reflect the variations under changes of the sum rule paramters. The intrinsic uncertainty of the sum rule approach may be somewhat larger, mainly due to the continuum model employed.

sum rules

$$
\begin{aligned}
\delta\Lambda_2 &= \frac{e^{2\bar\Lambda/T}}{24F^2\bar\Lambda}\left\{ g_s\langle\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle\left(1+\frac{2\bar\Lambda}{T}\right) + \frac{9\delta\omega_2}{2\pi^2}\,(\omega_0-2\bar\Lambda)\,\omega_0^3\,e^{-\omega_0/T}\right\} , \\
G_2 &= \frac{\bar\Lambda}{2T}\,\delta\Lambda_2 - \frac{e^{2\bar\Lambda/T}}{48F^2}\left\{ \frac{g_s\langle\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{T} - \frac{9\delta\omega_2}{2\pi^2}\,\omega_0^3\,e^{-\omega_0/T}\right\} .
\end{aligned}
\tag{3.16}
$$

These expressions involve the quantity $\delta\omega_2$, which has to be determined by requiring optimal stability of $\delta\Lambda_2$ inside the sum rule window. Using the central values for the parameters $\omega_0, \bar\Lambda$ and F of the leading-order sum rule, we find good stability for $\delta\omega_2 \simeq -(105\pm 20)$ MeV. The numerical evaluation of (3.16) in this region is shown in Fig. 2.

One can also analyze the sum rules analytically. The optimal value for $\delta\omega_2$ is determined by $(\mathrm{d}/\mathrm{d}T^{-1})\delta\Lambda_2 = 0$ for $T = T_0$, where $T_0 = 0.8$ GeV is the center of the sum rule window. The solution is then inserted back into (3.16). We find

$$
\begin{aligned}
\delta\Lambda_2 &= \frac{g_s\langle\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{12F^2(\omega_0-2\bar\Lambda)}\left\{1+\omega_0\left(\frac{1}{T_0}+\frac{1}{2\bar\Lambda}\right)\right\} e^{2\bar\Lambda/T_0} , \\
G_2 &= \frac{\bar\Lambda\, g_s\langle\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{12F^2(\omega_0-2\bar\Lambda)^2}\left\{1+\frac{\omega_0-\bar\Lambda}{T_0}+\frac{\omega_0(\omega_0-2\bar\Lambda)}{2T_0^2}\right\} e^{2\bar\Lambda/T_0} .
\end{aligned}
\tag{3.17}
$$

These equations show how the subleading corrections depend on the parameters $\omega_0, \bar\Lambda$ and F. It turns out that most of the numerical uncertainties associated with the theoretical errors in (3.14) cancel if one computes the products $\omega_0\,\delta\omega_2, \bar\Lambda\,\delta\Lambda_2$, and $F\,G_2$ which, according to (2.14) and (3.15), determine indeed the $1/m_Q$ corrections to $\omega_0, \bar\Lambda$, and F. Our final results are

$$
\begin{aligned}
\left(\frac{\omega_0}{2.0\ \mathrm{GeV}}\right)\delta\omega_2 &\simeq -(106\pm 20)\ \mathrm{MeV} , \\
\left(\frac{\bar\Lambda}{0.5\ \mathrm{GeV}}\right)\delta\Lambda_2 &\simeq -(173\pm 25)\ \mathrm{MeV} , \\
\left(\frac{F}{0.3\ \mathrm{GeV}^{3/2}}\right) G_2 &\simeq -(70\pm 10)\ \mathrm{MeV} .
\end{aligned}
\tag{3.18}
$$

This is in agreement with the numerical analysis in Fig. 2.

An interesting test of the value of $\delta\Lambda_2$ is provided by the calculation of the mass difference between a heavy vector and a pseudoscalar meson. From (3.4), one obtains in the $m_Q\to\infty$ limit

$$
m_V^2 - m_P^2 = -8\,\bar\Lambda\,\delta\Lambda_2 \simeq 0.69\pm 0.10\ \mathrm{GeV}^2 . \tag{3.19}
$$

This compares quite well with the mass splittings observed for B and D mesons, which are $m_{B^*}^2 - m_B^2 \simeq 0.48$ GeV^2 [41] and $m_{D^*}^2 - m_D^2 \simeq 0.55$ GeV^2 [42] with very small errors. Note, in particular, that the sign is unambiguously reproduced from our sum-rule analysis. This is an improvement over a recent analysis using standard QCD sum rules, where no definite prediction for the mass difference could be obtained [43].

Using the above value of G_2, an estimate of the ratio of vector to pseudoscalar decay constants can be obtained from (2.23). With $m_b = 4.8$ GeV, $m_c = 1.5$ GeV, and $\Lambda_{\overline{\rm MS}} = 0.25$ GeV (for $n_f = 4$) we find

$$\frac{f_{B^*}\sqrt{m_{B^*}}}{f_B\sqrt{m_B}} \simeq 1.14 \pm 0.03 \ , \quad \frac{f_{D^*}\sqrt{m_{D^*}}}{f_D\sqrt{m_D}} \simeq 1.49 \pm 0.08 \ . \tag{3.20}$$

We thus expect large spin-symmetry breaking effects in the case of charmed mesons. Radiative corrections will reduce these corrections slightly, as will be shown in Sect. 4.

Because of the structure of the sum rule (3.10), the spin-symmetry conserving corrections can be immediately related to the spin-symmetry violating ones

$$\begin{aligned} &\delta\omega_1 = 9\,\delta\omega_2 \ , \quad \delta\Lambda_1 = 9\,\delta\Lambda_2 \ , \\ &G_1 = 18\,G_2 - 2\,\bar\Lambda \simeq -(2.26 \pm 0.35) \text{ GeV} \ . \end{aligned} \tag{3.21}$$

In contrast to (3.18), these numbers are by no means small. Even for the b-quark, for instance, $G_1/m_b \simeq 0.5$. For charm the corrections are even larger than 100%, indicating a break-down of the $1/m_Q$ expansion. The presence of large finite-mass corrections to the decay constants f_P of pseudoscalar mesons is indeed a phenomenon well-known from lattice gauge theory [22-26] and QCD sum rules [28]. The corrections induced by (3.21) are even larger than those observed in these analyses, however. As an example, we compute the slope parameter c_P which describes the mass-dependence of f_P

$$f_P\,\sqrt{m_P} \equiv A_P \left\{ 1 + \frac{c_P}{m_Q} \right\} . \tag{3.22}$$

In terms of the subleading form factors, one finds from (2.19)

$$c_P = G_1 + 6G_2 - \frac{\bar\Lambda}{2} \simeq -(2.9 \pm 0.5) \text{ GeV} \ , \tag{3.23}$$

whereas recent lattice and sum rule computations indicate $c_P \simeq -1$ GeV [22,23,28]. It is important to notice, however, that these empirical results have not been obtained by directly studying matrix elements of higher-dimensional operators in the effective theory, but by fitting the mass-dependence observed in the the full theory, which includes all orders in $1/m_Q$, to (3.22). We are thus led to argue that higher-order corrections are important in these computations and mimic an *effective* $1/m_Q$ behavior in the region of the b- and c-quark masses. It is clear, for instance, that the effective value of $\delta\Lambda_1$ has to be much smaller than given in (3.21). Even for $\delta\Lambda_1 = 0$ one computes from (3.4), (3.15) and (3.18) $m_b \simeq 4.8$ GeV and $m_c \simeq 1.5$ GeV, which are very reasonable values for the pole masses of the heavy quarks. There is thus little room for additional corrections. One can estimate the effective values of G_1 and c_P by requiring stability of the sum rule (3.10) under the constraint $\delta\Lambda_1^{eff} = 0$. This leads to

$$G_1^{eff} \simeq -(0.5 \pm 0.2)\ \text{GeV}\ , \quad c_P^{eff} \simeq -(1.2 \pm 0.3)\ \text{GeV}\ . \tag{3.24}$$

The effective slope c_P^{eff} is in fact consistent with the empirically observed mass-dependence of f_P. It will be interesting to see if direct lattice computations of G_1 and c_P in terms of matrix elements of subleading operators will confirm the large values obtained from our sum-rule analysis in the effective theory.

Let us briefly also derive the sum rule for the parameters F_1 and F_2, which parametrize matrix elements of operators containing a covariant derivative acting on the heavy-quark field [cf. (2.11)]. The aim is to show how the constraint (2.12), which is a consequence of the equations of motion, is satisfied in the framework of QCD sum rules. We start from the two-point function

$$i \int \mathrm{d}x\, e^{ik\cdot x}\ \langle\, 0\,|\ \mathcal{T}\big\{\ \big[\bar q\,\Gamma\, iD_\mu\, h_Q(v)\big]_x,\ \big[\bar h_Q(v)\,\Gamma_M\, q\big]_0\ \big\}\ |\,0\,\rangle\ , \tag{3.25}$$

the pole contribution to which involves the matrix element (2.11). In the theoretical calculation we choose k and v parallel, *i.e.* $k_\mu = (v\cdot k)v_\mu$. On the phenomenological side, we use (3.6) to combine two traces into one. After applying the Borel operator, the resulting sum rule reads

$$\begin{aligned} &F\ \mathrm{Tr}\left\{(F_1\, v_\mu + F_2\, \gamma_\mu)\,\Gamma\,(\not v + 1)\,\Gamma_M\right\} e^{-\widetilde\Lambda_0/T} \\ &= -\frac{1}{6}\ \mathrm{Tr}\left\{(v_\mu + \gamma_\mu)\,\Gamma\,(\not v + 1)\,\Gamma_M\right\} \times \left\{ \frac{3}{8\pi^2} \int\limits_0^{\omega_0} \mathrm{d}\omega\,\omega^3\, e^{-\omega/T} - \frac{g_s\langle\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{2T} \right\} . \end{aligned} \tag{3.26}$$

The right-hand side is proportional to the derivative of the function $K = F^2 e^{-\widetilde\Lambda_0/T}$ defined in (3.11), and with $\widetilde\Lambda_0 = 2\bar\Lambda$ it follows that

$$F_1 = F_2 = \frac{1}{6F}\,K'(T^{-1};\omega_0) = -\frac{\bar\Lambda}{3}\,F\ , \tag{3.27}$$

which is indeed relation (2.12). QCD sum rules thus respect the equations of motion of the heavy-quark effective theory.

4. Renormalization Group Effects

We now refine the sum rule analysis of the previous section by including radiative corrections to the subleading form factors. We restrict ourselves to the computation of $G_2(\mu)$. Besides improving the numerical estimates obtained so far, the purpose is to show that QCD sum rules correctly reproduce the running of the low-energy parameters as derived in the effective theory.

We repeat the calculation of the two-point function defined in (3.1) including radiative corrections to the perturbative contribution and to the quark condensate. Since we restrict ourselves to spin-symmetry breaking effects, we only consider insertions of the "magnetic interaction" operator in (2.3). Let us first present the result of the perturbative calculation. In the $\overline{\mathrm{MS}}$ subtraction scheme, we find

$$
\begin{aligned}
&\left\{\left[1-\frac{\alpha_s}{\pi}\left(\ln\frac{\mu}{2\bar\Lambda}+\delta_{\overline{\mathrm{MS}}}\right)\right]F(\mu)\right\}^2\left\{1+\frac{4d_\Gamma}{m_Q}\,G_2(\mu)\right\}e^{-\bar\Lambda/T}\\
&=\frac{3}{8\pi^2}\int\limits_0^{\omega_c}\mathrm{d}\omega\,\omega^2\,e^{-\omega/T}\left\{1+\frac{2\alpha_s}{\pi}\left[\ln\frac{2\bar\Lambda}{\omega}+\frac{13}{6}+\frac{2\pi^2}{9}+d_\Gamma\,\frac{2\omega}{9m_Q}\left(\ln\frac{\mu}{\omega}+\frac{17}{12}\right)\right]\right\}\\
&+\ \text{condensates},
\end{aligned}
\tag{4.1}
$$

where $F(\mu)$ has been multiplied by a factor that cancels its μ- and scheme-dependence. To leading order in $1/m_Q$, the radiative corrections in the effective theory have been computed in Refs. 28, 30 and 44. The spin-symmetry breaking corrections proportional to α_s/m_Q are new. They arise from the diagrams depicted in Fig. 3. The calculation is outlined in the appendix. It is convenient to bring the scheme-dependent terms in (4.1) on the left-hand side. This can be achieved by a redefinition of the Borel parameter

$$
\frac{1}{T}\to\frac{1}{T}+\frac{d_\Gamma}{m_Q}\,\frac{4\alpha_s}{9\pi}\left(\ln\frac{\mu}{2\bar\Lambda}+\frac{17}{12}\right),
\tag{4.2}
$$

which, to the order we are working, does not affect the condensate contributions. The final sum rule becomes

$$
\begin{aligned}
&\hat F^2(2\bar\Lambda)\left\{1+\frac{4d_\Gamma}{m_Q}\,\hat G_2(2\bar\Lambda)\right\}e^{-\bar\Lambda/T}\\
&=\frac{3}{8\pi^2}\int\limits_0^{\omega_c}\mathrm{d}\omega\,\omega^2\,e^{-\omega/T}\left\{1+\frac{2\alpha_s}{\pi}\left[\ln\frac{2\bar\Lambda}{\omega}+\frac{13}{6}+\frac{2\pi^2}{9}+d_\Gamma\,\frac{2\omega}{9m_Q}\,\ln\frac{2\bar\Lambda}{\omega}\right]\right\}\\
&-\langle\bar qq\rangle(2\bar\Lambda)\left\{1+\frac{2\alpha_s}{3\pi}\left[1-d_\Gamma\,\frac{T}{m_Q}\right]\right\}+\frac{g_s\langle\bar q\sigma_{\mu\nu}G^{\mu\nu}q\rangle}{4T^2}\left\{1-\frac{d_\Gamma}{3}\,\frac{T}{m_Q}\right\}.
\end{aligned}
\tag{4.3}
$$

We have introduced the renormalization-group invariant parameters

$$\begin{aligned}\hat{F}(2\bar{\Lambda}) &= \left[1 - \frac{\alpha_s}{\pi}\left(\ln\frac{\mu}{2\bar{\Lambda}} + \delta_{\overline{\mathrm{MS}}}\right)\right] F(\mu) \,, \\ \hat{G}_2(2\bar{\Lambda}) &= G_2(\mu) - \frac{2\bar{\Lambda}}{9}\,\frac{\alpha_s}{\pi}\left(\ln\frac{\mu}{2\bar{\Lambda}} + \frac{17}{12}\right) .\end{aligned} \tag{4.4}$$

The choice of the reference scale $2\bar{\Lambda}$ is, of course, arbitrary. These quantities are related to the renormalized parameters $\hat{F}(m_Q)$ and $\hat{G}_2(m_Q)$ defined in (2.20) by evolution equations.* For $\hat{F}(m_Q)$, the complete next-to-leading order result is

$$\hat{F}(m_Q) = \left[\frac{\alpha_s(2\bar{\Lambda})}{\alpha_s(m_Q)}\right]^{6/\beta} \left\{1 + \frac{\alpha_s(m_Q) - \alpha_s(2\bar{\Lambda})}{\pi}\, Z_{n_f} - \frac{\alpha_s(m_Q)}{2\pi}\right\} \hat{F}(2\bar{\Lambda}) \,. \tag{4.5}$$

For $\hat{G}_2(m_Q)$, we use $B_0 = 35/9$ in (2.18) to obtain

$$\hat{G}_2(m_Q) = \left[\frac{\alpha_s(2\bar{\Lambda})}{\alpha_s(m_Q)}\right]^{-9/\beta} \hat{G}_2(2\bar{\Lambda}) + \left\{1 - \left[\frac{\alpha_s(2\bar{\Lambda})}{\alpha_s(m_Q)}\right]^{-9/\beta} + \frac{\alpha_s}{8\pi}\right\} \frac{4\bar{\Lambda}}{27} \,, \tag{4.6}$$

Since $\hat{G}_2(2\bar{\Lambda})$ is proportional to α_s or to the mixed condensate, it is sufficient to work with the leading logarithmic approximation for Z_m in (2.5).

The evaluation of (4.3) proceeds along the same lines as discussed in Sect. 3. Ignoring first $1/m_Q$ corrections, we find good stability inside the sum rule window for $\omega_0 \simeq 1.85 \pm 0.3$ GeV. In this region, the renormalized low-energy parameters are

$$\begin{aligned}\bar{\Lambda} &\simeq 0.49 \pm 0.07 \text{ GeV} \,, \\ F &\simeq 0.365 \pm 0.065 \text{ GeV}^{3/2} \,.\end{aligned} \tag{4.7}$$

Note that radiative corrections have increased the result for $\hat{F}$, as compared to F in (3.14), by 20%. The values of $\bar{\Lambda}$ and ω_0, on the other hand, remain almost unchanged, since these quantities are determined from ratios like (3.12), in which most of the radiative corrections cancel. Using (4.5) we can compute the so-called static limit of the decay constant of the B meson

$$f_B^{stat} = \frac{\hat{F}(m_b)}{\sqrt{m_B}} \simeq 192 \pm 35 \text{ MeV} \,. \tag{4.8}$$

This is slightly smaller than the value quoted in Ref. 28, which was based on a larger value of $\bar{\Lambda}$.

* When evaluating the evolution equations for $m_Q = m_b$, it is to be understood that the number n_f of light quarks changes as one crosses the charm threshold.

Due to the inclusion of radiative corrections, the analytical expressions for the spin-symmetry breaking corrections $\delta\Lambda_2$ and $\hat{G}_2(2\bar{\Lambda})$ differ from those given in (3.17). As an example, we present the result for $\delta\Lambda_2$

$$\begin{aligned}\delta\Lambda_2 = \frac{e^{2\bar{\Lambda}/T_0}}{12\hat{F}^2(2\bar{\Lambda})(\omega_0-2\bar{\Lambda})}\bigg\{ & g_s\langle\bar{q}\sigma_{\mu\nu}G^{\mu\nu}q\rangle\left[1+\omega_0\left(\frac{1}{T_0}+\frac{1}{2\bar{\Lambda}}\right)\right] \\ & -\frac{8\alpha_s}{\pi}T_0^2\,\langle\bar{q}q\rangle\left[\frac{\omega_0}{T_0}-1-\frac{\omega_0-2T_0}{2\bar{\Lambda}}\right] \\ & -\frac{2\alpha_s}{\pi^3}\int\limits_0^{\omega_0}\mathrm{d}\omega\,\omega^3(\omega_0-\omega)\Big(\frac{\omega}{2\bar{\Lambda}}-1\Big)\,\ln\frac{\omega}{2\bar{\Lambda}}\,e^{-\omega/T_0}\bigg\}\,.\end{aligned} \tag{4.9}$$

The expression for $\hat{G}_2(2\bar{\Lambda})$ is more complicated. We do not present it here. In the numerical evaluation we use the renormalized parameters from (4.7) together with $\alpha_s/\pi = 0.1$. We find [cf. (3.18) and (3.19)]

$$\begin{aligned} m_V^2 - m_P^2 &\simeq 0.46 \pm 0.08\ \mathrm{GeV}^2\,, \\ \hat{G}_2(2\bar{\Lambda}) &\simeq -(55\pm 8)\ \mathrm{MeV}\,. \end{aligned} \tag{4.10}$$

The vector–pseudoscalar mass splitting is now in excellent agreement with that observed for beauty mesons, $m_{B^*}^2 - m_B^2 \simeq 0.48\ \mathrm{GeV}^2$ [41].

The evolution of $\hat{G}_2(2\bar{\Lambda})$ up to the scale of the heavy quark yields a further reduction of this paramter due to the terms proportional to $\bar{\Lambda}$ in (4.6), which are induced by the renormalization group. We find the central values $\hat{G}_2(m_c) \simeq -44$ MeV and $\hat{G}_2(m_b) \simeq -26$ MeV. As a consequence, the spin-symmetry breaking effects in the ratio f_V/f_P are not quite as large as estimated in (3.20). Our final numbers are

$$\frac{f_{B^*}\sqrt{m_{B^*}}}{f_B\sqrt{m_B}} \simeq 1.07\pm 0.03\,, \quad \frac{f_{D^*}\sqrt{m_{D^*}}}{f_D\sqrt{m_D}} \simeq 1.36\pm 0.08\,. \tag{4.11}$$

The renormalization of $G_1(\mu)$ can be carried out in a similar way. In this case, we restrict ourselves to the leading logarithmic approximation and define the renormalized form factors [cf. (2.20)]

$$\begin{aligned} \hat{G}_1(2\bar{\Lambda}) &= G_1(\mu) + \frac{4\bar{\Lambda}}{3}\,\frac{\alpha_s}{\pi}\,\ln\frac{\mu}{2\bar{\Lambda}}\,, \\ \hat{G}_1(m_Q) &= \hat{G}_1(2\bar{\Lambda}) - \frac{8\bar{\Lambda}}{\beta}\,\ln\left[\frac{\alpha_s(2\bar{\Lambda})}{\alpha_s(m_Q)}\right]\,. \end{aligned} \tag{4.12}$$

In leading logarithmic order, $\hat{G}_1(2\bar{\Lambda})$ agrees with the lowest-order result G_1 in (3.21) except for the replacement of $\omega_0, \bar{\Lambda}$ and F by their renormalized values.

This gives $\hat{G}_1(2\bar{\Lambda}) \simeq -(2.0 \pm 0.3)$ GeV. The effect of the evolution from $\mu = 2\bar{\Lambda}$ up to the heavy-quark mass is rather moderate in this case. Even $\hat{G}_1(m_b)$ differs from $\hat{G}_1(2\bar{\Lambda})$ by less than 0.1 GeV. Finally, we note that in the slope parameter c_P defined in (3.23) the logarithms of $m_Q/\bar{\Lambda}$ cancel, such that this parameter is independent of m_Q to leading logarithmic order. Using $\hat{G}_i(2\bar{\Lambda})$ as given above, one finds $c_P \simeq -2.6$ GeV.

5. Conclusions

We have presented a detailed analysis of mesonic decay constants in subleading order of the $1/m_Q$ expansion for heavy quarks. The relevant matrix elements can be parametrized in terms of a leading-order low-energy parameter F, two subleading parameters G_1 and G_2, and the mass difference $\bar{\Lambda} = m_M - m_Q$, where m_Q is a generalization of the "physical" pole mass of the heavy quark. We have derived the general structure of the symmetry-breaking corrections using effective-field-theory techniques. The renormalization-group improvement of the low-energy parameters has been discussed in detail. Numerical values of the form factors have then been obtained from QCD sum rules in the effective theory. At the renormalization scale $\mu = 2\bar{\Lambda}$, the results are

$$\begin{aligned} \bar{\Lambda} &\simeq 0.50 \text{ GeV} , \\ \hat{F}(2\bar{\Lambda}) &\simeq 0.36 \text{ GeV}^{3/2} , \\ \hat{G}_1(2\bar{\Lambda}) &\simeq -2.0 \text{ GeV} , \\ \hat{G}_2(2\bar{\Lambda}) &\simeq -55 \text{ MeV} . \end{aligned}$$

$\bar{\Lambda}$ is the characteristic scale of low-energy parameters in the effective theory. For instance, $F \simeq \bar{\Lambda}^{3/2}$ with good accuracy. $G_2 \simeq -0.1\,\bar{\Lambda}$ is suppressed since this is a spin-symmetry violating form factor. In the framework of QCD sum rules, it only receives contributions from condensates of dimension $d \geq 5$, or from radiative corrections. On the other hand, the large value $G_1 \simeq -4\,\bar{\Lambda}$ is unexpected and leads to a break-down of the $1/m_Q$ expansion for decay constants of pseudoscalar mesons, already in the region below the b-quark mass. We have argued that higher-order terms in the $1/m_Q$ expansion partially compensate this effect and mimic an *effective* value which is significantly smaller, $G_1^{eff} \simeq -0.5$ GeV. It is important to emphasize that G_1 does not induce spin-symmetry-breaking effects, which therefore can be reliably computed. Including two-loop radiative corrections, we obtain for the vector–pseudoscalar mass splitting

$$m_V^2 - m_P^2 \simeq 0.46 \pm 0.08 \text{ GeV}^2$$

in excellent agreement with experiment. The symmetry-breaking effects to the ratio of decay constants f_V/f_P are estimated to be $\sim 7\%$ for beauty and $\sim 36\%$ for charmed mesons.

Besides obtaining these numerical results, the purpose of this paper is to present a consistent calculation of heavy-quark form factors at subleading order in the $1/m_Q$ expansion, which respects the equations of motion and correctly reproduces the running of the low-energies parameters. The application of the methods developed here to the calculation of the subleading form factors that describe transitions between two heavy mesons will be presented elsewhere [45].

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APPENDIX

We briefly outline the calculation of the two-loop diagram shown in Fig. 3 (a). In momentum space, the heavy-quark–gluon vertex denoted by a black square is given by $-(g_s/2m_Q)\sigma_{\mu\nu}k^\nu$, where k is the momentum of the incoming gluon. In D space-time dimensions, the diagram is proportional to the two-loop integral

$$\begin{aligned} I_{\alpha\beta\gamma}(v,\omega) &= \int \mathrm{d}^D s\, \mathrm{d}^D t\, \frac{s_\alpha\, t_\beta\, (s-t)_\gamma}{(\omega+2v\cdot s)\,(\omega+2v\cdot t)\, s^2\, t^2\, (s-t)^2} \\ &= \frac{1}{4}\,(v_\alpha\, g_{\beta\gamma} - v_\beta\, g_{\alpha\gamma})\, \hat I(\omega) \ , \end{aligned}$$

where we have used that $I_{\alpha\beta\gamma}$ is antisymmetric in α and β. Using standard reduction techniques and the integrals given in Ref. 18, we obtain

$$\begin{aligned} \hat I(\omega) &= \omega\left[\int \mathrm{d}^D s\, \frac{1}{(\omega+2v\cdot s)\, s^2}\right]^2 - \int \mathrm{d}^D s\, \mathrm{d}^D t\, \frac{1}{(\omega+2v\cdot s)\, t^2\, (s-t)^2} \\ &= \frac{\pi^D\, (-\omega)^{2D-5}}{(D-1)}\, \Gamma^2(D/2-1)\, \left[\Gamma^2(3-D) - \Gamma(5-2D)\right] \ . \end{aligned}$$

One then relates the imaginary part of this integral to that of the bare quark-loop.

The ratio of imaginary parts is proportional to

$$\frac{6\cos\left[\pi(D-4)\right]}{(D-1)}\left(\frac{\omega}{\sqrt{4\pi}\mu}\right)^{(D-4)}\Gamma(D/2-1)\left[\Gamma(3-D)-\frac{\Gamma(5-2D)}{\Gamma(3-D)}\right]$$
$$=\frac{1}{\hat{\epsilon}}+2\,\ln\frac{\omega}{\mu}-\frac{17}{6}+\mathcal{O}(D-4)\ ,$$

where $1/\hat{\epsilon}=2/(D-4)+\gamma_E-\ln 4\pi$.

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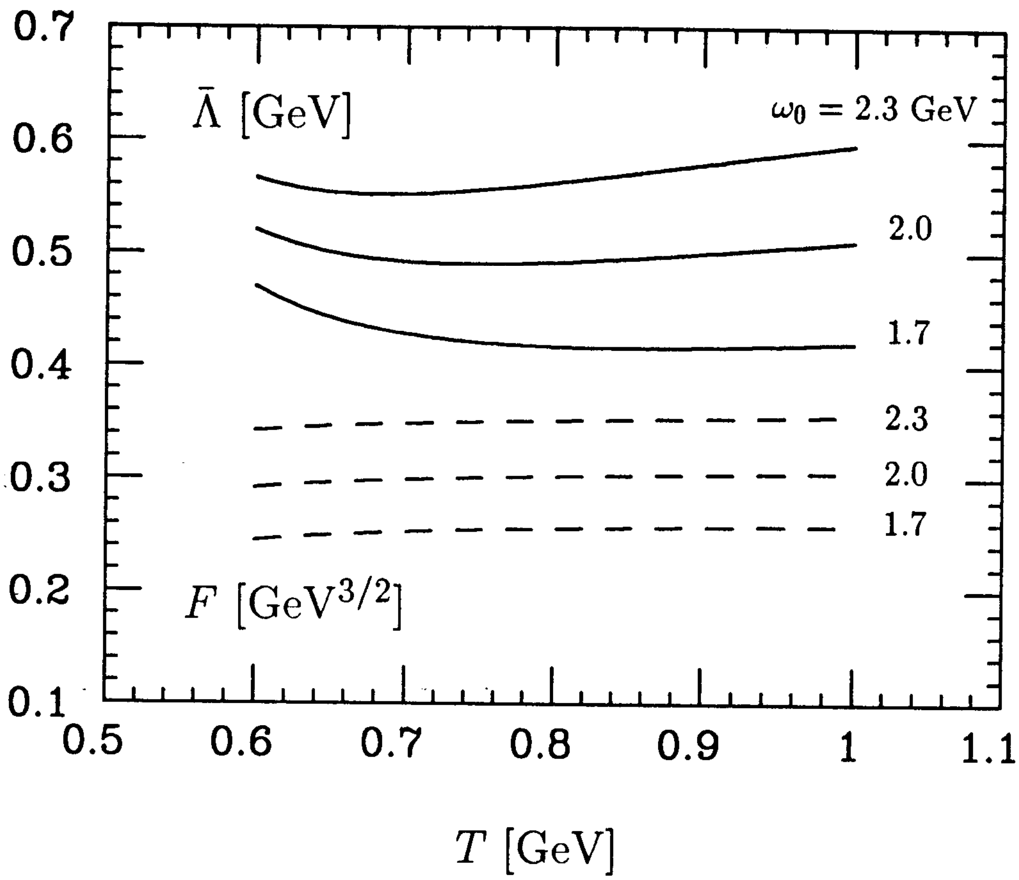
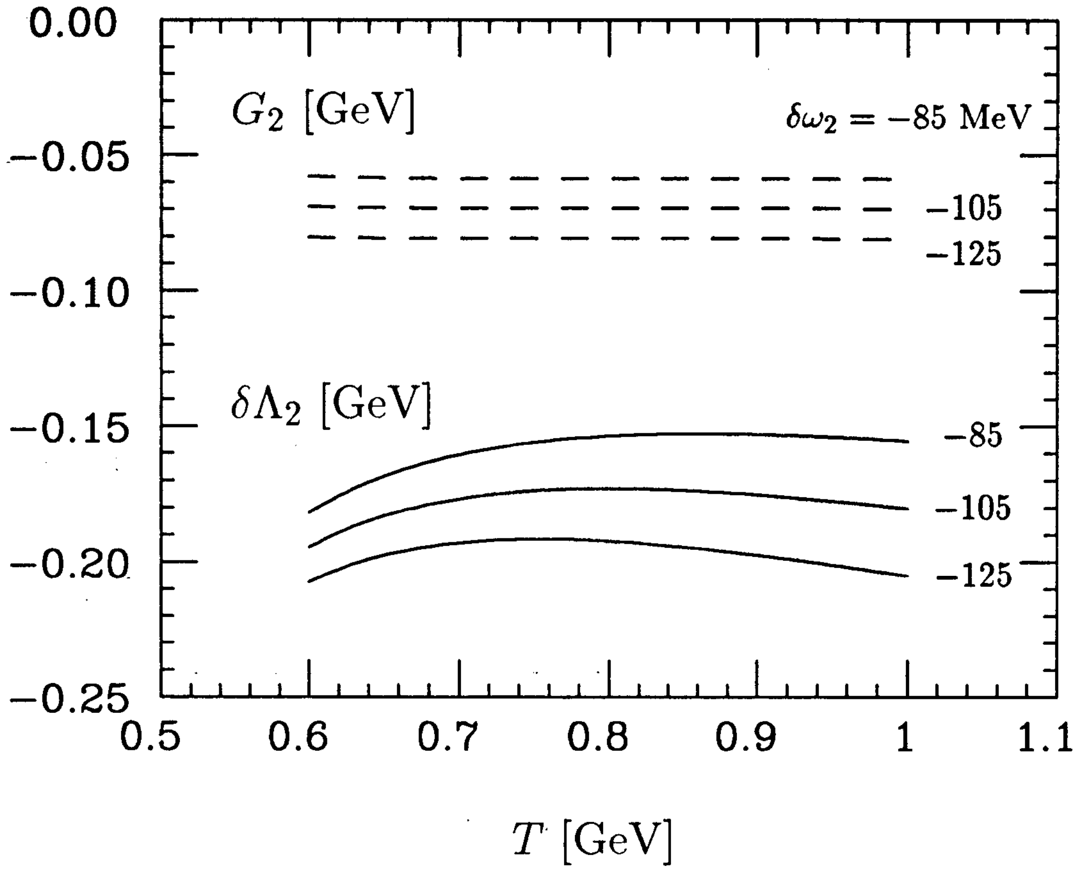
FIGURE CAPTIONS

1) Numerical evaluation of the sum rules (3.11) and (3.12) for different values of the threshold energy ω_0. The solid lines give $\bar\Lambda(T)$ in units GeV, the dashed ones $F(T)$ in units $\mathrm{GeV}^{3/2}$. In the computation of F we have used $\bar\Lambda = 0.57, 0.50, 0.43$ GeV for $\omega_0 = 2.3, 2.0, 1.7$ GeV, respectively.
2) Evaluation of the sum rules (3.16) for different values of $\delta\omega_2$. The solid lines refer to $\delta\Lambda_2(T)$, the dashed ones to $G_2(T)$, both in units GeV. In the computation of G_2 we have used $\delta\Lambda_2 = -155, -175, -195$ MeV for $\delta\omega_2 = -85, -105, -125$ MeV, respectively.
3) Feynman diagrams for the spin-symmetry breaking radiative corrections in (4.1). The heavy-quark propagators are represented by double lines. The black square denotes the heavy-quark–gluon vertex contained in the "magnetic interaction" operator in (2.3).



0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.5
0.6
0.7
0.8
0.9
1
1.1
$\bar{\Lambda}$ [GeV]
$\omega_0 = 2.3$ GeV
2.0
1.7
2.3
2.0
1.7
F [GeV$^{3/2}$]
T [GeV]

Fig. 1



0.00
−0.05
−0.10
−0.15
−0.20
−0.25
0.5
0.6
0.7
0.8
0.9
1
1.1
G_2 [GeV]
$\delta\omega_2 = -85$ MeV
−105
−125
$\delta\Lambda_2$ [GeV]
−85
−105
−125
T [GeV]

Fig. 2

(a) (b)

Fig. 3