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Symmetry in Perturbation Problems

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Abstract

The work is devoted to a new branch of application of continuous group's techniques in the investigation of nonlinear differential equations.

The principal stages of the development of perturbation theory of nonlinear differential equations are considered in short. It is shown that its characteristic features make it possible a fruitful usage of continuous group's techniques in problems of perturbation theory.

1. Introduction

The idea of introducing coordinate transformations to simplify the analytic expression of a general problem is a powerful one. Symmetry and differential equations have been close partners since the time of the founding masters, namely, Sophus Lie (1842–1899), and his disciples. To these days, symmetry has continued to play a strong role. The ideas of symmetry penetrated deep into various branches of science: mathematical physics, mechanics and so on.

The role of symmetry in perturbation problems of nonlinear mechanics, which was already used by many investigators since the 70-th years (J. Moser, G. Hori, A. Kamel, U. Kirchgraber), has been developed considerably in recent time to gain further understanding and development of such constructive and powerful methods as averaging and normal form methods.

The principal stages of the development of perturbation theory of nonlinear differential equations connected with the fundamental works by A. Poincaré, A.M. Lyapunov, Van der Pol, N.M. Krylov, N.N. Bogolyubov, and their followers are considered. The growing role of symmetry (and, accordingly, of continuous group's techniques) is shown by the example of perturbation theory's problems.

2. Short survey of perturbation theory

2.1. The basic problem

Let us consider the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + \varepsilon x^3, \end{aligned} \tag{1}$$

which is equivalent to Duffing's equation, ε is a small positive parameter.

When $\varepsilon = 0$, the system of *"zero approximation"* has the periodic solution

$$x = \cos t, \quad y = \sin t.$$

The main question is: has system (1) also a periodic solution when $\varepsilon \neq 0$ *and is sufficiently small?*

It is naturally to try to find a periodic solution of nonlinear system (1) as a series

$$\begin{aligned} x &= \cos t + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots, \\ y &= \sin t + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \cdots. \end{aligned} \tag{2}$$

Substitution of series (2) in equation (1) replaces the original system by an infinite sequence of simple systems of equations

$$\begin{aligned} &\dot{u}_1 = -v_1, \\ &\dot{v}_1 = u_1 + \cos^3 t, \\ &u_1(0) = 1, \quad v_1(0) = 0; \\ &\dot{u}_k = -v_k, \\ &\dot{v}_k = u_k + f_k(t, u_1(t), \ldots, u_{k-1}(t)), \\ &u_k(0) = 0, \quad v_k(0) = 0, \qquad k = 2, 3, \ldots. \end{aligned} \tag{3}$$

Let us solve system (3). Excepting the function v_1, one comes to the equation of the second order

$$\ddot{u}_1 + u_1 = \cos^3 t = \frac{4}{3}\cos t + \frac{1}{4}\cos 3t. \tag{4}$$

The general solution of (4) is

$$u_1 = A\cos t + B\sin t + \frac{3}{8} t \sin t - \frac{1}{9}\cos 3t.$$

Constants A and B are to be found to satisfy an initial value.

So, already the first member of series (2) has an addend $t \sin t$ and, as a result, the functions x, y are not periodic. It is easy to see that among subsequent members of series (2) there will be also terms of kind $t^n \sin t$. Such terms in celestial mechanics are called *"secular terms"*.

2.2. Principal stages of the development of perturbation theory

All the long-standing history of solution of nonlinear problems of form (1) (and more complicate ones) was connected with the construction of solutions as series (2) which do not contain secular terms. Here, let us go into three most important stages of the development of perturbation theory.

– Works by A. Poincaré and A.M. Lyapunov.

– Works by Van der Pol, N.M. Krylov and N.N. Bogoliubov.

– Group-theoretic methods in perturbation theory.

Works by A. Poincaré and A.M. Lyapunov

The problem formulated in section 2.1 was completely solved in works of the authors cited above. Really, they have received more general results, but there is no necessity of considering them here.

Theoretical bases for solving problem (1) are given by the following two theorems.

Let us formulate Poincaré's theorem suitably to the system of form (1). (The complete formulation see, for example, in [1], p.39.)

A. Poincaré's theorem. *The solutions of system (1) are analytic functions of the parameter ε, i.e., series converge when the absolute value of ε is sufficiently small and, hence, they are solutions of system (1), expanded in an infinite power series in the parameter ε.*

Therefore, the presentation of solutions of system (1) as series (2) is quite true. Its default is that the periodic solution of system (1) is expanded into series in nonperiodic functions.

Let us consider a more general system of the second order

$$\begin{aligned} \dot{x} &= -y + X(x, y), \\ \dot{y} &= x + Y(x, y), \end{aligned} \tag{5}$$

where $X(x, y), Y(x, y)$ are analytic functions.

A.M. Lyapunov's theorem. *If system (5) has the analytic first integral*

$$H(x, y) = x^2 + y^2 + R(x, y) = \mu \tag{6}$$

and μ is sufficiently small, then it has a family of solutions periodic in t.

The period of these functions tends to 2π when $\mu \to 0$. The solutions of system (5) are analytic functions of a quantity c, the initial deviation of variables x, y.

A.M. Lyapunov had proved also the inverse statement. Hence, the existence of integral (6) for system (5) is necessary and sufficient condition for the existence for system (5) in the neighborhood of the origin of the coordinate system of periodic solutions which depend upon an arbitrary constant c.

On making in (5) change of variables

$$x = \varepsilon \bar{x}, \qquad y = \varepsilon \bar{y},$$

one easy comes from system (5) to the one with a small parameter of the form (1).

System (1) has the first integral in elliptic functions. Hence, A.M. Lyapunov's theorem can be applied to it: there exists a periodic solution in the neighborhood of the origin of the coordinate system.

A.M. Lyapunov had gave an effective algorithm of construction of solutions of system (5) as series. The algorithm uses the change of variable

$$t = \tau(1 + c^2 h_2 + c^3 h_3 + \cdots),$$

where $h_2, h_3, \ldots$ are some constants which are to be find in the process of calculations.

A. Poincaré's method of defining the autoperiodic oscillations of equations of the form

$$\ddot{x} + \lambda y = \varepsilon F(x, \dot{x})$$

uses the change of variable

$$t = \frac{\tau}{\lambda}(1 + g_1\varepsilon + g_2\varepsilon^2 + \cdots),$$

where $g_1, g_2, \ldots$ are some constants which are to be find in the process of calculations.

In conclusion, let us note the typical features of A. Poincaré's and A.M. Lyapunov's methods.

- The creation of constructive algorithms of the producing of periodic solutions as series, which do not contain secular terms.
- The active transformation of the initial system: the introduction of arbitrary variables, which are to be find in the process of calculations.
- The base of the developed algorithms consists in the proof of analyticity of the series which represent a desired periodic solution.

For more details about questions touched here, see, for example, [1].

Researches by Van der Pol, N.M. Krylov and N.N. Bogolyubov

The next stage in perturbation theory is connected with names of the scientists cited above. A typical object of their investigations is the system of nonlinear equations

$$\begin{aligned} \dot{x} &= \varepsilon X(\varepsilon, x, y), \\ \dot{y} &= \omega(x) + \varepsilon Y(\varepsilon, x, y), \end{aligned} \tag{7}$$

where $x \in R^n, y \in R^1$, functions $X(\varepsilon, x, y), Y(\varepsilon, x, y)$ are supposed to be periodical in y of the period T.

One looks for solutions of system (7) as series

$$\begin{aligned} x &= \bar{x} + \varepsilon u_1(\bar{x}, \bar{y}) + \varepsilon^2 u_2(\bar{x}, \bar{y}) + \cdots, \\ y &= \bar{y} + \varepsilon v_1(\bar{x}, \bar{y}) + \varepsilon^2 v_2(\bar{x}, \bar{y}) + \cdots. \end{aligned} \tag{8}$$

Functions $u_i(\bar{x}, \bar{y})$, $v_i(\bar{x}, \bar{y})$ are unknown yet and are to be found in the process of solving the problem. Variables $\bar{x}(t), \bar{y}(t))$ must satisfy a system

$$\begin{aligned} \dot{\bar{x}} &= \varepsilon A_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}) + \cdots, \\ \dot{\bar{y}} &= \omega(\bar{x}) + \varepsilon B_1(\bar{x}) + \varepsilon^2 B_2(\bar{x}) + \cdots. \end{aligned} \tag{9}$$

Functions A, B in (9) are also unknown and are to be found in the process of calculations.

Therefore, there is a problem of transformation of the original system (7) to a new one (9) more simple for investigation. This transformation actively influences the system as it contains uncertain functions to be found.

An original algorithm close to the above described scheme for $n = 1$ was first suggested by the Dutch engineer Van der Pol in the 20-th years. His method had beautiful clearness and was convenient for design calculations. It very quickly became popular among engineers. But no proof of the method existed. That is why it was out of mathematics for a long time (like Heaviside's method).

In the 30-th years N.M. Krylov and N.N. Bogolyubov suggested the just cited above general scheme (7)–(9) for investigation of systems like (7). It started the creation of a rigorous theory of nonlinear oscillations developed in the subsequent decades.

N.N. Bogolyubov and his pupils also investigated systems

$$\dot{x} = \varepsilon X(\varepsilon, x, y),$$

which were called *standard form's systems*. They created the strictly proved *method of averaging*, which is successfully applied for investigation of nonlinear systems with a small parameter.

In conclusion, let us note typical features of the considered period of the theory of perturbations.

- The transformation of the original system to a simplified one. This transformation is active as it contains unknown functions to be found.
- The convergence of series of the form (8) is not investigated. Instead, the asymptotic nearness is investigated, i.e., the existence of relations

  $$x \to \bar{x}, y \to \bar{y}, \quad \text{when} \quad \varepsilon \to 0.$$

- The essential weakening of the demands on the analytic characteristics of the right-hand sides of (8).
- The essential extension of classes of the problems under consideration: searching for periodic solutions, limit cycles, the description of transition processes, resonances and so on.

One can find the detailed exposition of the questions touched here, for example, in [2], [3].

3. Group-theoretical methods in perturbation theory

3.1. Short survey

J. Moser [4] used the group-theoretic approach in the investigation of quasiperiodic solutions of nonlinear systems. Lie's rows and transformations in the perturbation problems were used by G. Hori [5], [6], A. Kamel [7], U. Kirchgraber and E. Steifel [8], U. Kirchgraber [9], Bogaevsky V.N., Povzner A.Ya. [10], and Zhuravlev V.F., Klimov D.N. [11].

Asymptotic methods of nonlinear mechanics developed by N.M. Krylov, N.N. Bogolyubov and Yu.A. Mitropolsky known as the KBM method (see, for example, Bogolyubov N.N. and Mitropolsky Yu.A. [2]) is a powerful tool for the investigation of nonlinear vibrations.

The further development of these methods took place due to work by Yu.A. Mitropolsky, A.K. Lopatin [12]–[14], A.K. Lopatin [15]. In their works, a new method was proposed for investigating systems of differential equations with small parameters. It was a further development of Bogolyubov's averaging method referred to by the authors as "the asymptotic decomposition method". The idea of a new approach originates from Bogolyubov's averaging method (see [2]) but its realization needs to use essentially a new apparatus – the theory of continuous transformation groups.

3.2. Generalization of Bogolyubov's averaging method through symmetry

The asymptotic decomposition method is based on the group-theoretic interpretation of the averaging method. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = \omega(x) + \varepsilon\tilde{\omega}(x), \tag{10}$$

where

$$\omega(x) = \mathrm{col}\,[\omega_1(x), \dots, \omega_n(x)], \quad \tilde{\omega}(x) = \mathrm{col}\,[\tilde{\omega}_1(x), \dots, \tilde{\omega}_n(x)].$$

The differential operator associated with the perturbed system (10) can be represented as

$$U_0 = U + \varepsilon\tilde{U},$$

where

$$U = \omega_1\frac{\partial}{\partial x_1} + \dots + \omega_n\frac{\partial}{\partial x_n}, \quad \tilde{U} = \tilde{\omega}_1\frac{\partial}{\partial x_1} + \dots + \tilde{\omega}_n\frac{\partial}{\partial x_n}.$$

By using a certain change of variables in the form of a series in ε

$$x = \varphi(\bar{x}, \varepsilon), \tag{11}$$

system (10) is transformed into a new system

$$\frac{d\bar{x}}{dt} = \omega(\bar{x}) + \sum_{\nu=1}^{\infty}\varepsilon^{\nu}b^{(\nu)}(\bar{x}), \tag{12}$$

which is referred to as *a centralized system*. For this system, $\bar{U}_0 = \bar{U} + \varepsilon\tilde{\bar{U}}$, where

$$\begin{aligned} \bar{U} &= \omega_1(\bar{x})\frac{\partial}{\partial \bar{x}_1} + \dots + \omega_n(\bar{x})\frac{\partial}{\partial \bar{x}_n}, \\ \tilde{\bar{U}} &= \sum_{\nu=1}^{\infty}\varepsilon^{\nu}N_{\nu}, \quad N_{\nu} = b_1^{(\nu)}(\bar{x})\frac{\partial}{\partial \bar{x}_1} + \dots + b_n^{(\nu)}(\bar{x})\frac{\partial}{\partial \bar{x}_n}. \end{aligned} \tag{13}$$

We impose a condition on the choice of transformations (11) saying that the centralized system (12) should be invariant with respect to the one-parameter transformation group

$$\bar{x} = e^{s\bar{\mathrm{U}}(\bar{x}_0)}\bar{x}_0, \tag{14}$$

where $\bar{x}_0$ is the vector of new variables. Therefore, after the change of variables (14), system (12) turns into

$$\frac{d\bar{x}_0}{dt} = \omega(\bar{x}_0) + \sum_{\nu=1}^{\infty}\varepsilon^{\nu}b^{(\nu)}(\bar{x}_0),$$

which coincides with the original one up to the notations. This means that we have the identities $[\bar{U}, N_\nu] \equiv 0$ for $\bar{U}, N_\nu, \nu = 1, 2, \dots$.

The essential point in realizing the above-mentioned scheme of the asymptotic decomposition algorithm is that transformations (11) are chosen in the form of a series

$$x = e^{\varepsilon \mathrm{S}}\bar{x}, \tag{15}$$

where

$$S = S_1 + \varepsilon S_2 + \cdots,$$

$$S_j = \gamma_{j1}(\bar{x})\frac{\partial}{\partial \bar{x}_1} + \cdots + \gamma_{jn}(\bar{x})\frac{\partial}{\partial \bar{x}_n}.$$

Coefficients of $S_j, \gamma_{j1}(\bar{x}), \ldots, \gamma_{jn}(\bar{x})$ are unknown functions. They should be determined by the recurrent sequence of operator equations

$$[U, S_\nu] = F_\nu. \tag{16}$$

The operator F_ν is a known function of U and $S_1, \ldots, S_{\nu-1}$ are obtained on previous steps ($\nu = 1, 2, \ldots$).

In the case where S depends upon ε, the Lie series (15) is called a *Lie transformation.* Thus, the application of a Lie transformation as a change of variables enables us to use the technique of continuous transformation groups.

From the theory of linear operators, it is known that the solvability of the nonhomogeneous operator equation (16) depends on the properties of solutions of the homogeneous equation

$$[U, S_\nu] = 0. \tag{17}$$

Operator (13) N_ν is a projection of the right-hand side of the equation onto the kernel of operator (17), which is determined from the condition of solvability in the sense of the nonhomogeneous equation

$$[U, S_\nu] = F_\nu - N_\nu, \quad \nu = 1, 2, \ldots . \tag{18}$$

Depending on the way for solving equations (16)–(18), various modifications of the algorithm of the method are obtained.

The above indicated result can be summarized as the following theorem.

Theorem 1. *There exists a formal change of variables as Lie's transform (15) which transforms the initial system (10) into the centralized one (12), invariant with respect to the one-parameter transformation group (14) generated by zero approximation system's vector field.*

One can find the detailed exposition of the questions touched here in [12]–[15].

4. Some examples

A further investigation of the structure of the centralized system (12) gives the possibility to receive some nontrivial conclusions. Let us illustrate it by the examples.

4.1. Example 1. Perturbed motion on $SO(2)$

Let us consider a system of the second order

$$\begin{aligned} \dot{\bar{x}} &= -\bar{y} + \varepsilon Q(\varepsilon, \bar{x}, \bar{y}), \\ \dot{\bar{y}} &= \bar{x} + \varepsilon R(\varepsilon, \bar{x}, \bar{y}), \end{aligned} \tag{19}$$

where $Q(\varepsilon, \bar{x}, \bar{y})$, $R(\varepsilon, \bar{x}, \bar{y})$ are the known analytical functions of variables $\varepsilon, \bar{x}, \bar{y}$.

When $\varepsilon = 0$, the structure of the solution of system (19) is quite simple: it is the movement on the circle of radius $R = \sqrt{\bar{x}^2 + \bar{y}^2}$ with the proportional angular velocity $\omega = 1$.

The following statement is true.

Theorem 2. *System (19) in the neighborhood of the point $\varepsilon = 0$ has a family of periodic solutions which depends upon an arbitrary constant if and only if it can be transformed by the analytic change of the variables*

$$\bar{x} = e^{\varepsilon S} u, \quad \bar{y} = e^{\varepsilon S} v,$$

where $S = S_1 + S_2 + \cdots$, S_j are known operators with the analytic coefficients, to the system

$$\dot{u} = -(1 + \varepsilon G_1(\varepsilon, u^2 + v^2))v,$$
$$\dot{v} = (1 + \varepsilon G_1(\varepsilon, u^2 + v^2))u,$$

where $G_1(\varepsilon, u^2 + v^2)$ is the known analytic function of $\varepsilon, u^2 + v^2$, which is invariant in respect to the following one-parameter transformation groups: $SO(2)$

$$\bar{u} = e^{\varepsilon U} u, \quad \bar{v} = e^{\varepsilon U} v, \quad U = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},$$

and the group defined via

$$\bar{u} = e^{\varepsilon W} u, \quad \bar{v} = e^{\varepsilon W} v, \quad W = f(u^2 + v^2)U,$$

where $f(\rho)$ is an arbitrary analytic function of $\rho = \sqrt{u^2 + v^2}$.

Theorem 2 generalizes the well-known result by A.M. Lyapunov about the existence of a family of periodic solutions in the neighborhood of the point $|\bar{x}| = 0, |\bar{y}| = 0$ of system (19) when $\varepsilon = 1$. (See A.M. Lyapunov's theorem above).

4.2. Example 2. Perturbed motion on $SO(3)$

Let us consider the system of the third order (in the spherical coordinates)

$$\begin{aligned}
\dot{\rho} &= \varepsilon F_1(\varepsilon, \rho, , \theta, \varphi), \\
\dot{\theta} &= \sin\varphi + \varepsilon F_2(\varepsilon, \rho, \theta, \varphi), \\
\dot{\varphi} &= -1 + \operatorname{ctg}\theta \cos\varphi + \varepsilon F_3(\varepsilon, \rho, \theta, \varphi),
\end{aligned} \tag{20}$$

where $F_j(\varepsilon, \rho, \theta, \varphi)$ are the known analytic functions of the variables $\varepsilon, \rho, \theta, \varphi$, $j = 1, 2, 3$.

The system of zero approximation, which is received from (20) if one supposes $\varepsilon = 0$, has a quite complicate structure. (See Fig.1.) The following statement is true.

Theorem 3. *System (20) in the neighborhood of the point $\varepsilon = 0$ has a family of solutions which depends upon an arbitrary constant and saves the topological structure of the system of zero approximation, if and only if it can be transformed by the analytic change of the variables*

$$\bar{\rho} = e^{\varepsilon S} \rho, \quad \bar{\theta} = e^{\varepsilon S} \theta, \quad \bar{\varphi} = e^{\varepsilon S} \varphi,$$

where $S = S_1 + S_2 + \cdots$, S_j are the known operators with analytic coefficients,

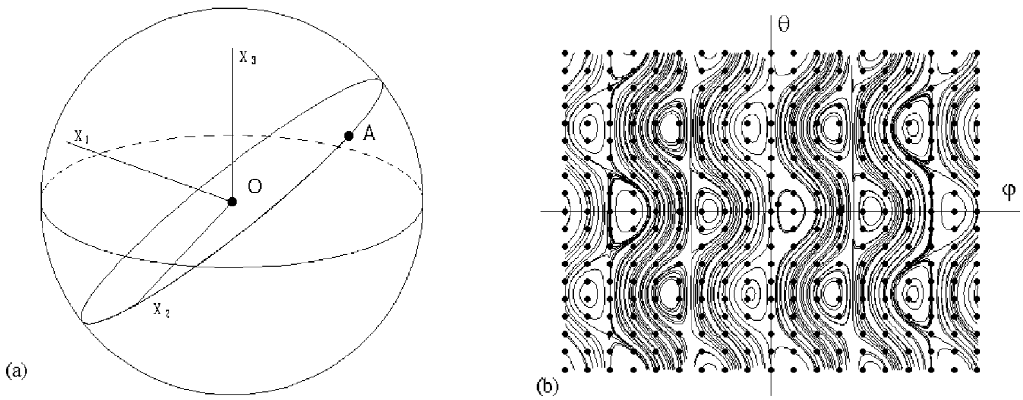

Fig. 1. (a) Solution for linear system of the movement of a point on a sphere.
(b) Solution in the phase plane for angle spherical variables, governing the movement of a point on a sphere.

to the system

$$\dot{\bar{\rho}} = 0,$$
$$\dot{\bar{\theta}} = \sin\bar{\varphi}(1 + \varepsilon G(\varepsilon, \bar{\rho}, \bar{\theta}, \bar{\varphi})),$$
$$\dot{\bar{\varphi}} = (-1 + \operatorname{ctg}\bar{\theta}\cos\bar{\varphi})(1 + \varepsilon G(\varepsilon, \bar{\rho}, \bar{\theta}, \bar{\varphi})),$$

where $G(\varepsilon, \bar{\rho}, \bar{\theta}, \bar{\varphi})$ *is the known analytic function of* $\varepsilon, \bar{\rho}, \bar{\theta}, \bar{\varphi}$,
which is invariant in respect to the one-parameter transformation groups:

$$\bar{\rho} = e^{\varepsilon U}\rho, \quad \bar{\theta} = e^{\varepsilon U}\theta, \quad \bar{\varphi} = e^{\varepsilon U}\varphi, \quad U = (-1 + \operatorname{ctg}\theta\cos\varphi)\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta}$$

and

$$\bar{\rho} = e^{\varepsilon W}\rho, \quad \bar{\theta} = e^{\varepsilon W}\theta, \quad \bar{\varphi} = e^{\varepsilon W}\varphi, \quad W = f(\rho, \theta, \varphi)U,$$

where $f(\rho, \theta, \varphi)$ *is an arbitrary analytic function of* ρ, θ, φ *which is an integral of the equation* $Uf = 0$.

References

[1] Moiseev N.N., Asymptotic Methods of Nonlinear Mechanics, Nauka, Moscow, 1981 (in Russian).

[2] Bogolyubov N.N. and Mitropolsky Yu.A., Asymptotic Methods in the Theory of Nonlinear Oscillations Nauka, Moscow, 1974; English transl., Gordon and Breach, New York, 1961.

[3] Mitropolsky Yu.A., Nonlinear Mechanics, Asymptotic Methods, Institute of Mathematics, Kyiv, 1995 (in Russian)

[4] Moser J., Convergent series-expansions for quasiperiodic motions, *Math. Annal.*, 1967, V.169, 1967, 136–176.

[5] Hori G., Theory of general perturbations with unspecified canonical variables, *J. Japan. Astron. Soc.*, 1966, V.18, N 4, 287—296.

[6] Hori G., Lie Transformations in Nonhamiltonian Systems, Lecture notes, Summer Institute in Orbital Mechanics, Univ. Texas, Austin, 1970.

[7] Kamel A., A perturbations methods in the theory of nonlinear oscillations, *Celest. Mech.*, 1970, V.3, 1970, N 1, 90—106.

[8] Kirchgraber U. and Stiefel E., Methoden der Analytischen Storungsrechnung und Anwedungen, Teubner, Anwedungen, Stuttgart, 1978 (in German).

[9] Kirchgraber U., On the Lie-series approach to the method of averaging, in: IX Mezhdunar. Konf. po Nelinein. Kolebanijam, Naukova Dumka, Kyiv, 1984, V.1, 173—178.

[10] Bogaevsky V.N., Povzner A.Ya., Algebraic Methods in Nonlinear Perturbation Theory, Nauka, Moscow, 1987 (in Russian).

[11] Zhuravlev V.F. and Klimov D.N., Applied Methods in the Oscillations Theory, Nauka, Moscow, 1988 (in Russian).

[12] Mitropolsky Yu.A. and Lopatin A.K., A Group-theoretic Approach in Asymptotic Methods of Nonlinear Mechanics, Naukova Dumka, Kyiv, 1988 (in Russian).

[13] Mitropolsky Yu.A. and Lopatin A.K., Bogolyubov's averaging and procedures of normalization in nonlinear mechanics I,II,III, *Ukr. Mat. Zhurn.*, 1994, V.42, N 9, 1171–1188, N 11, 1509–1526, N 12, 1627–1646.

[14] Mitropolsky Yu.A. and Lopatin A.K., Nonlinear Mechanics, Groups and Symmetry, Kluwer Academic Publisher, Dordrecht, Boston, London, 1995.

[15] Lopatin A.K., Normalization and averaging on compact Lie groups in nonlinear mechanics, *Ukr. Mat. Zh.*, 1997, V.49, N 1, 47–67.