SYMMETRY IN THE VANISHING OF EXT OVER GORENSTEIN RINGS

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Abstract

We investigate symmetry in the vanishing of Ext for finitely generated modules over local Gorenstein rings. In particular, we define a class of local Gorenstein rings, which we call AB rings, and show that for finitely generated modules M and N over an AB ring R, $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $\text{Ext}_R^i(N, M) = 0$ for all $i \gg 0$.

Introduction

Let R be a local Gorenstein ring and let M and N denote finitely generated R-modules. This paper is concerned with the relation between the vanishing of all higher $\text{Ext}_R(M, N)$ and the vanishing of all higher $\text{Ext}_R(N, M)$. As a means of investigation we concentrate on the more natural duality between the vanishing of all higher $\text{Ext}_R(M, N)$ and the vanishing of all higher Tor modules where either M or N is replaced by its dual $M^*(:= \text{Hom}_R(M, R))$ or N^*.

Our interest in this topic came about in part from the following striking result proved recently by Avramov and Buchweitz [2, Thm. III]. Suppose M and N are finitely generated modules over a complete intersection R. Then the following are equivalent:

(1) $\text{Tor}_i^R(M, N) = 0$ for all $i \gg 0$

(2) $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$

(3) $\text{Ext}_R^i(N, M) = 0$ for all $i \gg 0$.

Their proof relies heavily on the use of certain affine algebraic sets associated to M and N, called support varieties. In their paper [2], Avramov and Buchweitz raise the question of what class of rings satisfy these equivalences for all finitely generated modules M and N. They point out this class lies somewhere between

complete intersections and local Gorenstein rings, but mention that they do not know whether this class is equal to either complete intersections or Gorenstein rings. In this paper we introduce a class of local Gorenstein rings, which we call *AB rings*, and prove that AB rings satisfy the property that for all finitely generated modules M and N, the following are equivalent (Theorem 4.1):

(1) $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$

(2) $\operatorname{Ext}_R^i(N, M) = 0$ for all $i \gg 0$.

Regular local rings are AB rings, and if R is an AB ring, then $R/(x_1, \ldots, x_c)$ is also an AB ring whenever $x_1, \ldots, x_c$ is a regular sequence (see Proposition 3.3). This implies that complete intersections are AB rings. Even when restricted to the case of a complete intersection, our proof of the above equivalence avoids the use of support varieties, and in some ways is more direct than the methods of [2]. We also prove that local Gorenstein rings of minimal possible multiplicity are AB rings, for the strong reason that over such rings (except when the embedding dimension is 2) all large $\operatorname{Ext}_R(M, N)$ vanish if and only if either M or N has finite projective dimension. See Theorem 3.6 for a precise statement. These rings are not complete intersections in general, so that in particular the class of AB rings is strictly larger than that of complete intersections. In Theorem 3.8 and Proposition 4.4 we show that there also exist AB rings R which are not complete intersections and over which there are finitely generated modules M and N such that $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > \dim R$ even though both M and N have infinite projective dimension over R.

An AB ring R is a local Gorenstein ring defined by the property that there is a constant C, depending only on the ring, such that if $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$, then $Ext_R^i(M, N) = 0$ for all $i > C$. As far as we know *every* Gorenstein ring is an AB ring; we have been unable to find an example which is not. The name 'AB' stands for both Auslander-Bridger and Avramov-Buchweitz.

The paper is organized as follows. In Section 1 we give some preliminary and straightforward results concerning the relationship of Ext and Tor. In Section 2 we prove a basic result concerning what holds over an arbitrary local Gorenstein ring. Specifically, if M and N are finitely generated maximal Cohen-Macaulay modules over a local Gorenstein ring R, then the following are equivalent:

(1) $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$,

(2) $\operatorname{Ext}_R^i(M, N^*) = 0$ for all $i \gg 0$, and

(3) $\operatorname{Ext}_R^i(N, M^*) = 0$ for all $i \gg 0$.

Section 3 introduces AB rings, details their basic properties, and gives the main examples. In Section 4 we prove the main theorem of symmetry in the vanishing of Ext over AB rings. Section 5 contains some independent observations concerning what the vanishing of Ext means. In particular we relate the vanishing of d consecutive Ext modules (d being the dimension of the ring) to the Cohen-Macaulay property of a related tensor product. We include some questions in a final section.

1. Preliminaries

In this section we set notation and discuss some basic facts which will be used throughout the paper.

Unless otherwise stated, we will assume R to be a local Gorenstein ring. Also, M and N will denote finitely generated R-modules. For an R-module M we let M^* denote its dual $\mathrm{Hom}_R(M, R)$. If M is maximal Cohen-Macaulay then it is also reflexive, meaning $M^{**} \simeq M$ (assuming R is Gorenstein).

By a *complete intersection* we mean a local ring whose completion with respect to the maximal ideal is the quotient of a regular local ring by a regular sequence.

For a local ring R, we let embdim R denote its *embedding dimension*.

Syzygies and Conversions for Ext *and* Tor

Suppose M is an R-module. Then for $i \geq 0$ we let M_i denote image f_i, where f_i is the ith differential in a minimal free resolution

$$\mathbf{F}: \quad \cdots \rightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0$$

of M. These M_i are the *non-negative syzygies* of M. They are unique up to isomorphism, or if one considers a non-minimal resolution any two are stably isomorphic.

Now suppose that M is a maximal Cohen-Macaulay R-module. Let

$$\mathbf{G}: \quad \cdots \rightarrow G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \rightarrow M^* \rightarrow 0$$

a minimal free resolution of its dual M^*. Since M^* is maximal Cohen-Macaulay, the dual sequence

$$\mathbf{G}^*: \quad 0 \rightarrow M^{**} \rightarrow G_0^* \xrightarrow{g_1^*} G_1^* \xrightarrow{g_2^*} G_2^* \rightarrow \cdots$$

is exact. Using the fact that M is reflexive, we can splice $\mathbf{F}$ and $\mathbf{G}^*$ together, getting the doubly infinite long exact sequence

$$\mathsf{C}(M): \quad \cdots \rightarrow \underset{2}{F_2} \xrightarrow{f_2} \underset{1}{F_1} \xrightarrow{f_1} \underset{0}{F_0} \rightarrow \underset{-1}{G_0^*} \xrightarrow{g_1^*} \underset{-2}{G_1^*} \xrightarrow{g_2^*} \underset{-3}{G_2^*} \rightarrow \cdots.$$

(Note the degree convention.) For $i \leq -1$ we set $M_i := \text{image}(g_{-i}^*)$. These are the *negative syzygies* of M. They are unique up to isomorphism. Note that M_i is again maximal Cohen-Macaulay for all i when M is.

We now list some properties of the long exact sequences $\mathsf{C}(M)$.

LEMMA 1.1. *Let M be a finitely generated maximal Cohen-Macaulay R-module, and let N be a finitely generated R-module.*

(1) $\mathsf{C}(M)^* \simeq \mathsf{C}(M^*)$

(2) $(M_i)^* \simeq (M^*)_{-i}$ *for all i.*

(3) $\text{Tor}_i^R(M, N) \simeq \text{H}_i(\mathsf{C}(M) \otimes N)$ *for $i \geq 1$.*

(4) $\text{Ext}_R^i(M, N) \simeq \text{H}_{-i-1}(\mathsf{C}(M)^* \otimes N)$ *for $i \geq 1$.*

(5) *For fixed $t \geq 3$ and for $1 \leq i \leq t-2$ we have*

   (i) $\text{Ext}_R^i(M_{-t}, N) \simeq \text{Tor}_{t-i-1}^R(M^*, N)$

   and

   (ii) $\text{Tor}_i^R(M_{-t}, N) \simeq \text{Ext}_R^{t-i-1}(M^*, N)$.

PROOF. Conditions (1)–(3) are straightforward. Condition (5) follows easily from (1)–(4), so only Condition (4) needs some explanation. The critical fact we need to show is that for any complex of free R-modules $\mathbf{F}$, $\text{Hom}_R(\mathbf{F}, N)$ and $\text{Hom}_R(\mathbf{F}, R) \otimes_R N$ are isomorphic as complexes: write

$$\mathbf{F}: \quad \cdots \to F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \to \cdots,$$

where the F_i are free R-modules. The natural maps $h_i : \text{Hom}_R(F_i, R) \otimes_R N \to \text{Hom}_R(F_i, N)$ given by $f \otimes n \mapsto \{a \mapsto f(a)n\}$ are isomorphisms since F_i is free. It is easy to check that the diagram

$$
\begin{array}{ccc}
\text{Hom}_R(F_i, R) \otimes_R N & \xrightarrow{f_{i+1}^* \otimes N} & \text{Hom}_R(F_{i+1}, R) \otimes_R N \\
\downarrow{h_i} & & \downarrow{h_{i+1}} \\
\text{Hom}_R(F_i, N) & \xrightarrow{\text{Hom}(f_{i+1}, N)} & \text{Hom}_R(F_{i+1}, N)
\end{array}
$$

is commutative, and this establishes our fact.

Suppose M and N are R-modules with M maximal Cohen-Macaulay. Then $\text{Ext}^i(M, R) = 0$ for all $i \geq 1$, and so by shifting along the short exact sequences $0 \to N_{n+1} \to G_n \to N_n \to 0$ (with G_n free) we obtain isomorphisms

$$(1.2) \qquad \qquad \text{Ext}_R^i(M, N) \simeq \text{Ext}_R^{i+n}(M, N_n)$$

for $i \geq 1$ and $n \geq 0$.

The Change of Rings Long Exact Sequences of Ext *and* Tor

Suppose that S is a commutative ring, x is a non-zerodivisor of S and $R :=$ $S/(x)$. Let M and N be R-modules. Then we have the *change of rings long exact sequence of* Ext [6, 11.65]

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$(1.3) \qquad \mathrm{Ext}^1_R(M, N) \leftarrow \mathrm{Ext}^2_S(M, N) \leftarrow \mathrm{Ext}^2_R(M, N) \leftarrow$$

$$\mathrm{Ext}^0_R(M, N) \leftarrow \mathrm{Ext}^1_S(M, N) \leftarrow \mathrm{Ext}^1_R(M, N) \leftarrow 0,$$

and the *change of rings long exact sequence of* Tor [6, 11.64]

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$(1.4) \qquad \mathrm{Tor}^R_1(M, N) \rightarrow \mathrm{Tor}^S_2(M, N) \rightarrow \mathrm{Tor}^R_2(M, N) \rightarrow$$

$$\mathrm{Tor}^R_0(M, N) \rightarrow \mathrm{Tor}^S_1(M, N) \rightarrow \mathrm{Tor}^R_1(M, N) \rightarrow 0.$$

2. Vanishing of Ext and Tor over Arbitrary Local Gorenstein Rings

In this section we prove what type of duality between the vanishing of Ext and Tor holds over arbitrary local Gorenstein rings. It is possible an even stronger result is true, as we discuss in Section 4, but the main result of this section is what is true 'on the surface'. In particular, Theorem 2.1 states that one can flip the arguments in vanishing Ext modules 'up to duals'.

THEOREM 2.1. *Let R be a local Gorenstein ring, and let M and N be finitely generated maximal Cohen-Macaulay R-modules. Then the following are equivalent:*

(1) $\mathrm{Tor}^R_i(M, N) = 0$ *for all $i \gg 0$,*

(2) $\mathrm{Ext}^i_R(M, N^*) = 0$ *for all $i \gg 0$, and*

(3) $\mathrm{Ext}^i_R(N, M^*) = 0$ *for all $i \gg 0$.*

PROOF. Suppose we have shown that (1) and (2) are equivalent. By replacing (1) by the equivalent condition that $\mathrm{Tor}^R_i(N, M) = 0$ for all $i \gg 0$, we see then that (1) is equivalent to (3). Hence it suffices to prove (1) and (2) are equivalent, and for this we only need to assume that N is maximal Cohen-Macaulay.

We induce upon the dimension of R, say d. If $d = 0$, then $\mathrm{Ext}^i_R(M, N^*)$ is the Matlis dual of $\mathrm{Tor}^R_i(M, N)$, so the result is immediate in this case.

Now assume that $d > 0$. Choose $x \in R$ a non-zerodivisor on M_d, N, N^*, and R. We have

$$\operatorname{Tor}_i^R(M, N) = 0 \qquad \text{for all } i \gg 0;$$

$$\Longleftrightarrow \quad \operatorname{Tor}_i^R(M_d, N) = 0 \qquad \text{for all } i \gg 0;$$

and from the long exact sequence of Tor coming from the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ and Nakayama's lemma,

$$\Longleftrightarrow \quad \operatorname{Tor}_i^R(M_d, N/xN) = 0 \qquad \text{for all } i \gg 0;$$

and by the standard isomorphisms $\operatorname{Tor}_i^R(M_d, N/xN) \simeq \operatorname{Tor}_i^{R/(x)}(M_d/xM_d, N/xN)$,

$$\Longleftrightarrow \quad \operatorname{Tor}_i^{R/(x)}(M_d/xM_d, N/xN) = 0 \qquad \text{for all } i \gg 0;$$

by the induction hypothesis,

$$\Longleftrightarrow \quad \operatorname{Ext}_{R/(x)}^i(M_d/xM_d, (N/xN)^*) = 0 \qquad \text{for all } i \gg 0;$$

and now since N is maximal Cohen-Macaulay and R is Gorenstein, $N^*/xN^* \simeq (N/xN)^*$ (where the second module is $\operatorname{Hom}_{R/(x)}(N/xN, R/(x))$), thus,

$$\Longleftrightarrow \quad \operatorname{Ext}_{R/(x)}^i(M_d/xM_d, N^*/xN^*) = 0 \qquad \text{for all } i \gg 0;$$

by the isomorphisms [5] $\operatorname{Ext}_R^i(M_d, N^*/xN^*) \simeq \operatorname{Ext}_{R/(x)}^i(M_d/xM_d, N^*/xN^*)$,

$$\Longleftrightarrow \quad \operatorname{Ext}_R^i(M_d, N^*/xN^*) = 0 \qquad \text{for all } i \gg 0;$$

and now from the long exact sequence of Ext coming from the short exact sequence $0 \to N^* \xrightarrow{x} N^* \to N^*/xN^* \to 0$ and Nakayama's lemma,

$$\Longleftrightarrow \quad \operatorname{Ext}_R^i(M_d, N^*) = 0 \qquad \text{for all } i \gg 0;$$

$$\Longleftrightarrow \quad \operatorname{Ext}_R^i(M, N^*) = 0 \qquad \text{for all } i \gg 0.$$

REMARK 2.2. Suppose M and N are maximal Cohen-Macaulay modules over the local Gorenstein ring R. Then

$$\operatorname{Ext}_R^i(M, N) \simeq \operatorname{Ext}_R^i(N^*, M^*).$$

This isomorphism can be seen as follows: suppose that $i = 1$. As M and N are maximal Cohen-Macaulay, they are reflexive, so short exact sequences $0 \to N \to T \to M \to 0$ dualize to short exact sequences $0 \to M^* \to T^* \to N^* \to 0$ and vice-versa. The Yoneda definition of Ext^1 then gives the isomorphism.

For $i > 1$ we have

$$\begin{aligned}
\mathrm{Ext}^i_R(M, N) &\simeq \mathrm{Ext}^1_R(M_{i-1}, N) \\
&\simeq \mathrm{Ext}^1_R(N^*, M^*_{i-1}) && \text{by the } i = 1 \text{ case} \\
&\simeq \mathrm{Ext}^1_R(N^*, (M^*)_{-i+1}) && \text{by (2) of Lemma 1.1} \\
&\simeq \mathrm{Ext}^i_R(N^*, M^*) && \text{by (1.2).}
\end{aligned}$$

Presumably, one can directly prove this remark using the Yoneda definition of Ext and the fact that both M and N are maximal Cohen-Macaulay.

Below is an example showing that the hypothesis that N is maximal Cohen-Macaulay in the equivalence of (1) and (2) in Theorem 2.1 cannot be dropped.

EXAMPLE 2.3.   Let R be the 3-dimensional hypersurface $k[[W, X, Y, Z]]/(WX - YZ)$, and set $M := k$ and $N := \mathrm{coker}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$. Then $\mathrm{pd}_R N = 1$ (but $\mathrm{pd}_R N^* = \infty$), so we have $\mathrm{Tor}^R_i(M, N) = 0$ for all $i \gg 0$. However, $\mathrm{Ext}^4_R(M, N^*) \neq 0$. By what is shown in the next section, R is an AB ring. If it were the case that $\mathrm{Ext}^i_R(M, N^*)$ is zero for all $i \gg 0$ then, as R is AB, Proposition 3.2 shows then that $\mathrm{Ext}^i_R(M, N^*) = 0$ for all $i > \dim R$, which would be a contradiction.

3. AB Rings

Let R be a commutative ring. We define the Ext-*index* of R to be

$$\sup\{\, n \mid \mathrm{Ext}^i_R(M, N) = 0 \text{ for all } i > n \text{ and } \mathrm{Ext}^n_R(M, N) \neq 0 \,\},$$

where the sup is taken over all pairs of finitely generated R-modules (M, N) with $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \gg 0$.

DEFINITION 3.1.   If R is a local Gorenstein ring of finite Ext-index, we say that R is an *AB ring*.

We will prove that all complete intersections are AB rings. More generally, it is obvious that R is an AB ring if $\hat{R}$ is (where $\hat{R}$ is the completion of R), and we show (3.3) that if R is an AB ring and $x_1, \ldots, x_c$ is a regular sequence, then $R/(x_1, \ldots, x_c)$ is also an AB ring. The class of AB rings also includes local Gorenstein rings of 'minimal' multiplicity $\mathrm{embdim}(R) - \dim(R) + 2$ (see 3.6).

PROPOSITION 3.2.   *Suppose that R is an AB ring. Then the* Ext-*index of R equals* $\dim R$.

PROOF. Let n denote the Ext-index of R and d the dimension of R. Let $x_1, \ldots, x_d$ be a maximal R-sequence. Set $M := R/(x_1, \ldots, x_d)$ and $N = k$, the residue field of R. Then $\operatorname{Ext}_R^i(M, N) = 0$ for $i > d$ and $\operatorname{Ext}_R^d(M, N) \simeq k \neq 0$. Hence $n \geq d$.

Suppose that $n > d$. There exists a pair of finitely generated R-modules (M, N) such that $\operatorname{Ext}_R^i(M, N) = 0$ for $i > n$ and $\operatorname{Ext}_R^n(M, N) \neq 0$. We have the isomorphisms $\operatorname{Ext}_R^{i+1}((M_d)_{-d-1}, N) \simeq \operatorname{Ext}_R^{i-d}(M_d, N) \simeq \operatorname{Ext}_R^i(M, N)$ for $i > d$. Hence $\operatorname{Ext}^i((M_d)_{-d-1}, N) = 0$ for $i > n + 1$ and $\operatorname{Ext}^{n+1}((M_d)_{-d-1}, N) \neq 0$, which contradicts the definition of n. Therefore $n = d$.

We of course have the dual notion of Tor-*index*. If R is local Gorenstein with finite Tor-index, then it is also equal to dim R, by an argument analogous to that of 3.2.

Another related property of rings we are interested in is the following. We say that $\operatorname{Ext}_R^*(M, N)$ has a *gap of length t* if for some $n \geq 0$, $\operatorname{Ext}_R^i(M, N) = 0$ for $n + 1 \leq i \leq n + t$, but $\operatorname{Ext}_R^n(M, N)$ and $\operatorname{Ext}_R^{n+t+1}(M, N)$ are both nonzero. We have the analogous notion of gap for $\operatorname{Tor}_*^R(M, N)$. (We allow gaps of length 0.) We set

$$\operatorname{Ext-gap}(R) := \sup\left\{ t \in \mathsf{N} \;\middle|\; \begin{array}{l} \operatorname{Ext}_R^*(M, N) \text{ has a gap of length } t \\ \text{for finite } R\text{-modules } M \text{ and } N \end{array} \right\},$$

and

$$\operatorname{Tor-gap}(R) := \sup\left\{ t \in \mathsf{N} \;\middle|\; \begin{array}{l} \operatorname{Tor}_*^R(M, N) \text{ has a gap of length } t \\ \text{for finite } R\text{-modules } M \text{ and } N \end{array} \right\}.$$

We say that R is Ext-*bounded* if it has finite Ext-gap. Similarly, we say R is Tor-*bounded* if it has finite Tor-gap.

We list some elementary properties involving finite Ext-index, Tor-index, Ext-boundedness and Tor-boundedness for local Gorenstein rings.

PROPOSITION 3.3. *Let x be a non-zerodivisor of the d-dimensional local Gorenstein ring R. Then*

(1) *R is an AB ring if and only if $R/(x)$ is an AB ring.*

(2) *R has finite* Tor-*index if and only if $R/(x)$ does.*

(3) *R is* Ext-*bounded if and only if $R/(x)$ is.*

(4) *R is* Tor-*bounded if and only if $R/(x)$ is.*

PROOF. (1). Suppose that R is an AB ring. Let M and N be finitely generated $R/(x)$-modules such that $\operatorname{Ext}_{R/(x)}^i(M, N) = 0$ for all $i \gg 0$. By the change

of rings long exact sequence of Ext (1.3) we conclude that $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \gg 0$, and so $\mathrm{Ext}^i_R(M, N) = 0$ for all $i > d$. Looking at (1.3) again, we see that $\mathrm{Ext}^i_{R/(x)}(M, N) \simeq \mathrm{Ext}^{i+2}_{R/(x)}(M, N)$ for $i > d - 1$. But as $\mathrm{Ext}^i_{R/(x)}(M, N) = 0$ for all $i \gg 0$, we have $\mathrm{Ext}^i_{R/(x)}(M, N) = 0$ for all $i > d - 1$. Hence $R/(x)$ is an AB ring.

Now suppose that $R/(x)$ is an AB ring, and let M and N be finitely generated R-modules such that $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \gg 0$. We have the isomorphisms

$$\mathrm{Ext}^i_R(M, N) \simeq \mathrm{Ext}^{i-d}_R(M_d, N) \simeq \mathrm{Ext}^i_R(M_d, N_d)$$

which are valid for $i > d$, the second one being that of (1.2). The short exact sequence $0 \to N_d \xrightarrow{x} N_d \to N_d/xN_d \to 0$ gives rise to the the long exact sequence of Ext
(3.3.1)
$$\cdots \to \mathrm{Ext}^i_R(M_d, N_d) \xrightarrow{x} \mathrm{Ext}^i_R(M_d, N_d) \to \mathrm{Ext}^i_R(M_d, N_d/xN_d) \to \cdots.$$

Since $\mathrm{Ext}^i_R(M_d, N_d) = 0$ for all $i \gg 0$, we see that $\mathrm{Ext}^i_R(M_d, N_d/xN_d) = 0$ for all $i \gg 0$. We have the isomorphisms [5]

$$(3.3.2) \qquad \mathrm{Ext}^i_{R/(x)}(M_d/xM_d, N_d/xN_d) \simeq \mathrm{Ext}^i_R(M_d, N_d/xN_d)$$

for all $i \geq 0$. Hence $\mathrm{Ext}^i_{R/(x)}(M_d/xM_d, N_d/xN_d) = 0$ for all $i \gg 0$, which means that $\mathrm{Ext}^i_{R/(x)}(M_d/xM_d, N_d/xN_d) = 0$ for all $i > d - 1$, since $R/(x)$ is an AB ring. Therefore $\mathrm{Ext}^i_R(M_d, N_d/xN_d) = 0$ for all $i > d - 1$. By (3.3.1) and Nakayama's Lemma, we conclude that $\mathrm{Ext}^i_R(M_d, N_d) = 0$ for all $i > d - 1$, and so $\mathrm{Ext}^i_R(M, N) = 0$ for all $i > d$. Therefore R is an AB ring.

The proof of (2) is exactly analogous to the proof of (1), using (1.4) and a long exact sequence of Tor this time.

(3). Assume that $e := \mathrm{Ext}\text{-gap}(R) < \infty$. Let M and N be finitely generated $R/(x)$-modules such that $\mathrm{Ext}^i_{R/(x)}(M, N) = 0$ for $n \leq i \leq n + e + 1$, some $n \geq 1$. The change of rings long exact sequence of Ext (1.3) shows that $\mathrm{Ext}^i_R(M, N) = 0$ for $n + 1 \leq i \leq n + e + 1$. Since $\mathrm{Ext}\text{-gap}(R) = e$ we have $\mathrm{Ext}^i_R(M, N) = 0$ for all $i > n$. Another look at (1.3) shows that $\mathrm{Ext}^i_{R/(x)}(M, N) \simeq \mathrm{Ext}^{i+2}_{R/(x)}(M, N)$ for all $i > n - 1$. Since $\mathrm{Ext}^i_{R/(x)}(M, N) = 0$ for $i = n, n + 1$, we see then that $\mathrm{Ext}^i_{R/(x)}(M, N) = 0$ for $i > n - 1$ Hence $\mathrm{Ext}\text{-gap}(R/(x)) \leq e + 1$.

Now assume that $e := \mathrm{Ext}\text{-gap}(R/(x)) < \infty$. Suppose that M and N are finitely generated R-modules with $\mathrm{Ext}^i_R(M, N) = 0$ for $n \leq i \leq n + d + e + 1$, some $n \geq 1$. We have $\mathrm{Ext}^i_R(M_d, N_d) \simeq \mathrm{Ext}^i_R(M, N) = 0$ for $n + d \leq i \leq n + d + e + 1$. Therefore, from (3.3.1), we get $\mathrm{Ext}^i_R(M_d, N_d/xN_d) = 0$ for $n + d \leq$

$i \leq n+d+e$. Equivalently, $\mathrm{Ext}^i_{R/(x)}(M_d/xM_d, N_d/xN_d) = 0$ for $n+d \leq i \leq n+d+e$. Since Ext-gap$(R/(x)) = e$, $\mathrm{Ext}^i_{R/(x)}(M_d/xM_d, N_d/xN_d) = 0$ for all $i \geq n+d$, which implies, by (3.3.1) and Nakayama's lemma, $\mathrm{Ext}^i_R(M_d, N_d) = 0$ for all $i \geq n + d$, which means $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \geq n$. Therefore Ext-gap$(R) \leq d + e + 1$.

The proof of (4) is similar to the proof of (3).

THEOREM 3.4. *Assume that R is a local Gorenstein ring. Then*

(1) *R is an AB ring if and only if it has finite* Tor*-index;*

(2) *R is* Ext*-bounded if and only if it is* Tor*-bounded;*

(3) *if R is* Ext*-bounded, then it is an AB ring.*

PROOF. We first prove (1). Choose a maximal regular sequence in R and let I be the ideal generated by this sequence. Proposition 3.3 states that R is an AB ring if and only if R/I is an AB ring, and R has finite Tor-index if and only if R/I has finite Tor-index. Hence it suffices to prove (1) in case R is 0-dimensional. In this case $\mathrm{Ext}^i_R(M, N)$ is the Matlis dual of $\mathrm{Tor}^R_i(M, N^*)$ so that the vanishing of one implies the vanishing of the other. This proves (1).

Statement (2) is proved in a similar manner, using Proposition 3.3.

We prove (3). Assume R is Ext-bounded. We prove that R has finite Tor-index. Let d denote the dimension of R and $e := \mathrm{Ext\text{-}gap}(R)$, and suppose that for finite R-modules M and N, $\mathrm{Tor}^R_i(M, N) = 0$ for all $i \gg 0$. Let $b = d - \mathrm{depth}\, M$ so that M_b is maximal Cohen-Macaulay. Choose n largest such that $\mathrm{Tor}^R_n(M_b, N) \neq 0$. Using $t = e + n + 3$ in (5)(i) of Lemma 1.1, we have $\mathrm{Ext}^i_R((M^*_b)_{-e-n-3}, N) \simeq \mathrm{Tor}^R_{e+n+2-i}(M_b, N) = 0$ for $1 \leq i \leq e + 1$. Hence we have a gap of zero Ext larger than e. Therefore $\mathrm{Ext}^i_R((M^*_b)_{-e-n-3}, N) = 0$ for all $i \geq 1$, which forces $n = 0$. Thus $\mathrm{Tor}^R_i(M, N) = 0$ for all $i > d$.

The following Corollary is an almost immediate consequence of Proposition 3.4, as regular local rings are clearly Ext-bounded.

COROLLARY 3.5. *Let R be a local Gorenstein ring. If R is a complete intersection, then R is* Ext*-bounded. In particular, R is an AB ring.*

PROOF. Since $R \hookrightarrow \hat{R}$ is a faithfully flat extension, $\mathrm{Ext}^i_R(M, N) = 0$ if and only if $\mathrm{Ext}^i_{\hat{R}}(\hat{M}, \hat{N}) = 0$ and so R is an AB ring if $\hat{R}$ is. Therefore we may without loss of generality assume that $R = S/(x_1, \ldots, x_c)$ where S is a regular local ring and $x_1, \ldots, x_c$ is an S-regular sequence. By Proposition 3.3 it suffices to prove that S is Ext-bounded. But this is trivial as every finitely generated module over S has projective dimension $\leq \dim S$, so that $\mathrm{Ext}^i_R(M, N) = 0$ for $i > \dim S$ and Ext-gaps can occur of length no longer than $\dim S - 2$.

It could be that *all* local Gorenstein rings are AB rings; we have no counter-example. The class of AB rings is strictly bigger than the class of complete intersections as the next theorem proves, albeit for rather strong reasons. (See, however, Theorem 3.8 and Propostion 4.4.)

THEOREM 3.6. *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring with multiplicity equal to* embdim $R - \dim R + 2$. *Assume that* embdim $R > \dim R + 2$ *(so that R is not a complete intersection). Then for finitely generated R-modules M and N, $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if either M or N has finite projective dimension. In particular, R is an AB ring.*

PROOF. We induce on $d := \dim R$.

$d = 0$. In this case by duality we have that $\operatorname{Tor}_i^R(M, N^*) = 0$ for all $i \gg 0$. Replace M by a high enough syzygy such that $\operatorname{Tor}_3^R(M, N^*) = \operatorname{Tor}_4^R(M, N^*) = \operatorname{Tor}_5^R(M, N^*) = 0$. Now the following lemma, 3.7, says that either M or N is free.

$d > 0$. By replacing M and N by syzygies we can assume they are both maximal Cohen-Macaulay. Choose a minimal generator x of the maximal ideal of R such that the multiplicity of $R/(x)$ is embdim $R/(x) - \dim R/(x) + 2$, and x is a non-zerodivisor on both M and N. Once again we use the fact that $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}_{R/(x)}^i(M/xM, N/xN) = 0$ for all $i \gg 0$. By induction either M/xM or N/xN has finite projective dimension over $R/(x)$. But then either M or N has finite projective dimension over R.

LEMMA 3.7. *Let $(R, \mathfrak{m}, k)$ be a 0-dimensional local Gorenstein ring with multiplicity* embdim $R + 2$. *Assume* embdim $R > 2$ *(so that R is not a complete intersection). Let M and N be finitely generated R-modules. Then $\operatorname{Tor}_R^3(M, N) = \operatorname{Tor}_R^4(M, N) = \operatorname{Tor}_R^5(M, N) = 0$ if and only if either M or N is free.*

PROOF. Let n denote the embedding dimension of R. Assume that M is not free. If k is a summand of M_i for any $0 \le i \le 4$, then we get right away that N is free, since $\operatorname{Tor}_1^R(M_i, N) = \operatorname{Tor}_{i+1}^R(M, N) = 0$ would then imply $\operatorname{Tor}_1^R(k, N) = 0$. Therefore assume k is not a summand of M_i for all $0 \le i \le 4$. Replace M by its first syzygy. Then as $M \subseteq \mathfrak{m}F$, for F a free module, we have $\mathfrak{m}^2 M = 0$ (since $\mathfrak{m}^3 = 0$). Let b_i denote the ith Betti number of M and $s := \dim_k \mathfrak{m}M$. Then, as in Lescot's paper [3, Lemma 3.3], M is 3-exceptional and $b_1 = nb_0 - s$, $b_2 = b_0(n^2 - 1) - sn$ and $b_3 = b_0(n^3 - 2n) - s(n^2 - 1)$.

Now suppose that N is also not free. Also replace N by its first syzygy, so that $\mathfrak{m}^2 N = 0$. Write $\mathfrak{m}N \simeq k^d$. We have a short exact sequence $0 \to k^d \to N \to k^c \to 0$, where c is the minimal number of generators of N. Applying $M \otimes_R \_$ to this short exact sequence and using the fact that $\operatorname{Tor}_R^1(M, N) = \operatorname{Tor}_R^2(M, N) = \operatorname{Tor}_R^3(M, N) = 0$ we get $cb_2 = db_1$ and $cb_3 = db_2$. Letting

$\alpha := d/c$ we can write these equations as $b_2 = \alpha b_1$ and $b_3 = \alpha b_2 = \alpha^2 b_1$. Substituting for the b_i we get

$$b_0(n^2 - 1) - sn = \alpha(nb_0 - s)$$
$$b_0(n^3 - 2n) - s(n^2 - 1) = \alpha^2(nb_0 - s).$$

After rearranging we arrive at

$$b_0(n^2 - \alpha n - 1) = s(n - \alpha)$$
$$b_0(n^3 - \alpha^2 n - 2n) = s(n^2 - \alpha^2 - 1).$$

Now cross multiplying, cancelling off the $b_0 s$ terms, and simplifying we are left with the condition $\alpha^2 - n\alpha + 1 = 0$. This says that $\alpha \in \mathbf{Q}$ is algebraic over $\mathbf{Z}$. Hence α is an integer, and the only choice is $\alpha = 1$. But then $n = 2$, which is a contradiction. Hence N must be free.

There also exist AB rings which are not complete intersections and which have multiplicity $>$ embdim $R -$ dim $R + 2$:

THEOREM 3.8. *Suppose that* $(R, \mathfrak{m}_R, k)$ *and* $(S, \mathfrak{m}_S, k)$ *are local rings essentially of finite type over the same field* k. *Set* $A := (R \otimes_k S)_P$, *where* $P := \mathfrak{m}_R \otimes_k S + R \otimes_k \mathfrak{m}_S$. *If both* R *and* S *are Gorenstein then* A *is Gorenstein, and the multiplicity of* A *is* $>$ embdim $A -$ dim $A + 2$. *If* R *is an AB ring and* S *is the quotient of a regular local ring by a regular sequence (so that* S *is a complete intersection), then* A *is an AB ring;* A *is not a complete intersection if* R *is not.*

PROOF. Suppose that R and S are Gorenstein. Obviously A is Noetherian, being a localization of a finitely generated k-algebra. By applying [7] we have, moreover, that it is Gorenstein. Proposition 4.3 of the next section shows that there exist finitely generated A-modules M and N both of infinite projective dimension over A such that $\mathrm{Ext}^i_A(M, N) = 0$ for all $i > \dim A$. It follows from Theorem 3.6 that the multiplicity of A is larger than embdim $A -$ dim $A + 2$.

Suppose now that R is an AB ring and that S is the quotient of the regular local ring $(T, \mathfrak{m}_T, k)$ by the T-regular sequence $\mathbf{x} := x_1, \ldots, x_c$. Clearly A is isomorphic to $(R \otimes_k T)_{P'}/(1 \otimes \mathbf{x})$ where $P' := \mathfrak{m}_R \otimes_k T + R \otimes_k \mathfrak{m}_T$ and $1 \otimes \mathbf{x}$ is the regular sequence $1 \otimes x_1, \ldots, 1 \otimes x_c$. Hence by Proposition 3.3, A is an AB ring if and only if $(R \otimes_k T)_{P'}$ is an AB ring. But now going modulo $1 \otimes y_1, \ldots, 1 \otimes y_d$ where $d := \dim T$ and $y_1, \ldots, y_d$ a regular system of parameters of T shows, again by Proposition 3.3, that $(R \otimes_k T)_{P'}$ is an AB ring if and only if R is an AB ring.

Recall that a local ring $(A, \mathfrak{m}, k)$ is a complete intersection precisely when

$$\dim_k H_1(A) = \text{embdim } A - \dim A,$$

where $H_1(A)$ is the first Koszul homology module on a system of generators for the maximal ideal of A. In the current situation we have $\dim_k H_1(A) = \dim_k H_1(R) + \dim_k H_1(S)$, $\text{embdim } A = \text{embdim } R + \text{embdim } S$, and $\dim A = \dim R + \dim S$. Hence if R is not a complete intersection, then $\dim_k H_1(R) > \text{embdim } R - \dim R$, so that $\dim_k H_1(A) > \text{embdim } A - \dim A$.

4. Vanishing of Ext and Tor over AB Rings

In this section we prove that AB rings are a class which gives the duality of vanishing Ext discussed in the introduction. Our main theorem states:

THEOREM 4.1. *Suppose that R is an AB ring, and let M and N be finitely generated R-modules. Then*

$$\text{Ext}^i_R(M, N) = 0 \quad \text{for all } i \gg 0$$

if and only if

$$\text{Ext}^i_R(N, M) = 0 \quad \text{for all } i \gg 0.$$

PROOF. First assume the theorem is true if both M and N are maximal Cohen-Macaulay. For the general case, take syzygies M_m and N_n ($m, n \geq 0$) of M and N, respectively, which are maximal Cohen-Macaulay. We have $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ if and only if $\text{Ext}^i_R(M_m, N) = 0$ for all $i \gg 0$ and by (1.2) this is equivalent to $\text{Ext}^i_R(M_m, N_n) = 0$ for all $i \gg 0$. Thus $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ if and only if $\text{Ext}^i_R(M_m, N_n) = 0$ for all $i \gg 0$, and so the theorem holds generally.

Now suppose that M and N are maximal Cohen-Macaulay and $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$. Then for all $t \geq 1$, $\text{Ext}^i_R(M_{-t}, N) = 0$ for all $i \gg 0$. Since R is an AB ring, it follows from Proposition 3.2 that for all $t \geq 1$ and all $i > d := \dim(R)$, $\text{Ext}^i_R(M_{-t}, N) = 0$. However, $\text{Ext}^i_R(M_{-t}, N) \simeq \text{Ext}^1_R(M_{i-t-1}, N)$. Hence for all $t \geq 1$ and all $i > d$, $\text{Ext}^1_R(M_{i-t+1}, N) = 0$. By varying i and t, we obtain that $\text{Ext}^1_R(M_{-t}, N) = 0$ for all $t \geq 1$. Therefore by (5)(i) of Lemma 1.1, $\text{Tor}^R_{t-2}(M^*, N) = \text{Tor}^R_{t-2}(N, M^*) = 0$ for all $t \geq 3$. Applying Theorem 2.1 then shows that $\text{Ext}^i_R(N, M) = 0$ for all $i \gg 0$.

As an immediate corollary, we have an analogue of Theorem 2.1

COROLLARY 4.2. *Suppose that R is an AB ring, and let M and N be finitely generated maximal Cohen-Macaulay R-modules. Then the following are equivalent:*

(1) $\text{Tor}^R_i(M, N) = 0$ *for all $i \gg 0$,*

(2) $\text{Ext}^i_R(M^*, N) = 0$ *for all $i \gg 0$, and*

(3) $\mathrm{Ext}_R^i(N^*, M) = 0$ *for all $i \gg 0$.*

PROOF. Due to the natural symmetry in Tor, it suffices to prove just the equivalence between (1) and (2), and for this we only need to assume that M is maximal Cohen-Macaulay.

We have

$$\mathrm{Tor}_i^R(M, N) = 0 \text{ for all } i \gg 0 \Longleftrightarrow \mathrm{Tor}_i^R(N, M) = 0 \text{ for all } i \gg 0$$

$$\Longleftrightarrow \mathrm{Ext}_R^i(N, M^*) = 0 \text{ for all } i \gg 0 \quad \text{by 2.1}$$

$$\Longleftrightarrow \mathrm{Ext}_R^i(M^*, N) = 0 \text{ for all } i \gg 0 \quad \text{by 4.1}$$

If we strengthen the condition on the symmetry in the vanishing of Ext we get an equivalent definition of AB ring.

PROPOSITION 4.3. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Then the following are equivalent:*

(1) *R is an AB ring;*

(2) *For finitely generated R-modules M and N, $\mathrm{Ext}_R^i(M, N) = 0$ for $i > d$ if and only if $\mathrm{Ext}_R^i(N, M) = 0$ for $i > d$.*

PROOF. (1) $\Longrightarrow$ (2). This is simply Theorem 4.1 coupled with Proposition 3.2.

For (2) $\Longrightarrow$ (1), we get right away that R is Gorenstein from the hypothesis with $M := R$ and $N := k$.

For the remainder of the proof we will use the following fact. If M and N are finitely generated R-modules, then $\mathrm{Ext}_R^i(M, N) \simeq \mathrm{Ext}_R^{i+n}(M, N_n)$ for $i > d$ and $n \geq 0$. This follows from the isomorphisms $\mathrm{Ext}_R^i(M, N) \simeq \mathrm{Ext}_R^{i-d}(M_d, N) \simeq \mathrm{Ext}^{i-d+n}(M_d, N_n) \simeq \mathrm{Ext}_R^{i+n}(M, N_n)$ for $i > d$ and $n \geq 0$, where the middle isomorphism is formula (1.2) of Section 1.

Suppose that $\mathrm{Ext}_R^i(M, N) = 0$ for all $i > d + n$, some fixed $n \geq 1$. Then we have

$$\mathrm{Ext}_R^i(M_n, N) = 0 \qquad \text{for } i > d;$$

$$\Longrightarrow \quad \mathrm{Ext}_R^i(N, M_n) = 0 \qquad \text{for } i > d \quad \text{by hypothesis;}$$

$$\Longrightarrow \quad \mathrm{Ext}_R^{i+n}(N, M_n) = 0 \qquad \text{for } i > d;$$

$$\Longrightarrow \quad \mathrm{Ext}_R^i(N, M) = 0 \qquad \text{for } i > d \quad \text{by the isomorphism above;}$$

$$\Longrightarrow \quad \mathrm{Ext}_R^i(M, N) = 0 \qquad \text{for } i > d \quad \text{by hypothesis.}$$

Our final proposition in this section is an observation that there are circumstances other than where one module has finite projective dimension or where the ring is a complete intersection in which all large Ext modules vanish.

PROPOSITION-EXAMPLE 4.4. *Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be two local Gorenstein rings essentially of finite type over the same field k, and let M_R be a finitely generated R-module and N_S a finitely generated S-module. Set $A := (R \otimes_k S)_P$ where $P := (\mathfrak{m}_R \otimes_k S + R \otimes_k \mathfrak{m}_S)$, $M := (M_R \otimes_k S)_P$ and $N := (R \otimes_k N_S)_P$. Then A is a local Gorenstein ring, M has finite projective dimension over A if and only if M_R has finite projective dimension over R and similarly for N, and $\mathrm{Ext}^i_A(M, N) = 0$ for all $i > \dim A$.*

PROOF. We remark again that A is also Gorenstein by [7].

For the second statement, we induce on $d := \dim A = \dim R + \dim S$. Suppose that A has dimension 0. In this case duality yields $\mathrm{Ext}^i_A(M, N)^* \simeq \mathrm{Tor}^A_i(M, N^*)$. Thus it suffices to prove $\mathrm{Tor}^A_i(M, N^*) = 0$ for all $i > 0$.

We first claim that $\mathrm{Hom}_{R \otimes_k S}(M_R \otimes_k S, R \otimes_k S) \simeq \mathrm{Hom}_R(M_R, R) \otimes_k S$. To see this, note that these modules are naturally isomorphic if M_R is free R-module. In general, let $R^m \overset{\rho}{\to} R^n \to M_R \to 0$ be a presentation of M_R over R. Let $A' := R \otimes_k S$ and $M' := M_R \otimes_k S$. This yields a presentation $(A')^m \to (A')^n \to M' \to 0$ of M' over A'. We obtain a commutative diagram

$$0 \to \quad \mathrm{Hom}_{A'}(M', A') \quad \to \quad \mathrm{Hom}_{A'}((A')^n, A') \quad \to \quad \mathrm{Hom}_{A'}((A')^m, A')$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \to \mathrm{Hom}_R(M_R, R) \otimes_k S \to \mathrm{Hom}_R(R^n, R) \otimes_k S \to \mathrm{Hom}_R(R^m, R) \otimes_k S,$$

where the first row is exact and the vertical arrows are isomorphisms. To establish the claim we only need to know that the bottom row is exact, but this follows from the fact that $0 \to \mathrm{Hom}_R(M_R, R) \to \mathrm{Hom}_R(R^n, R) \to \mathrm{Hom}_R(R^m, R)$ is an exact sequence of k-modules and S is flat as a k-module.

Localizing the isomorphism in the claim above at P, we see that the A-module $M^* := \mathrm{Hom}_A(M, A)$ comes from the R-module $\mathrm{Hom}_R(M_R, R)$. Similarly $N^* := \mathrm{Hom}_A(N, A)$ comes from the S-module $\mathrm{Hom}_S(N_S, S)$. Hence there is no distinction between proving $\mathrm{Tor}^A_i(M, N^*) = 0$ for all $i > 0$ and proving $\mathrm{Tor}^A_i(M, N) = 0$ for all $i > 0$. We will prove the latter.

Let $(\mathbf{F}, f)$ be an R-free resolution of M_R. Then $\mathbf{F}$ is an exact sequence of k modules, and since S is flat as a k-module, $\mathbf{F} \otimes_k S$ is an exact sequence, of $R \otimes_k S$-modules. Thus $(\mathbf{F} \otimes_k S)_P$ is an A-free resolution of M. To show that $\mathrm{Tor}^A_i(M, N) = 0$ for all $i > 0$ we will simply show that the complex $(\mathbf{F} \otimes_k S)_P \otimes_A N$ is acyclic (meaning the homology is zero except in degree zero).

For all i we have a commutative diagram

$$
\begin{array}{ccc}
(F_i \otimes_k S) \otimes_{R \otimes_k S} (R \otimes_k N_S) & \xrightarrow{(f_i \otimes S) \otimes (R \otimes N_S)} & (F_{i-1} \otimes_k S) \otimes_{R \otimes_k S} (R \otimes_k N_S) \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
F_i \otimes_k N_S & \xrightarrow{\quad f_i \otimes N_S \quad} & F_{i-1} \otimes_k N_S,
\end{array}
$$

where the vertical arrows are the natural isomorphisms. Hence $(\mathbf{F} \otimes_k S) \otimes_{R \otimes_k S} (R \otimes_k N_S)$ and $\mathbf{F} \otimes_k N_S$ are isomorphic complexes of $R \otimes_k S$-modules. Since N_S is flat as a k-module, the latter is acyclic, and therefore so is $(\mathbf{F} \otimes_k R_S) \otimes_{R \otimes_k S} (R \otimes_k N_S)$. Finally, localizing at P we get that $(\mathbf{F} \otimes_k S)_P \otimes_A N$ is acyclic, and this finishes the proof in the $d = 0$ case.

Now without loss of generality assume that $\dim R > 0$. From the discussion above we know that $M_1 \simeq ((M_R)_1 \otimes_k S)_P$. Let x be a non-zerodivisor on both $(M_R)_1$ and R. Then $x \otimes 1$ is a non-zerodivisor on M_1, A and N, and we have

$$
A/(x \otimes 1) \simeq (R/(x) \otimes_k S)_P,
$$

$$
M_1/(x \otimes 1)M_1 \simeq ((M_R)_1/x(M_R)_1 \otimes_k S)_P
$$

and

$$
N/(x \otimes 1)N \simeq (R/(x) \otimes_k N_S)_P.
$$

Hence by induction we have that

$$
\mathrm{Ext}^i_{A/(x \otimes 1)}(M_1/(x \otimes 1)M_1, N/(x \otimes 1)N) = 0
$$

for all $i > d - 1$. Now (3.3.1) and (3.3.2) show that $\mathrm{Ext}^i_A(M_1, N) = 0$ for all $i > d - 1$, which means that $\mathrm{Ext}^i_A(M, N) = 0$ for all $i > d$.

5. What does the vanishing of Ext mean?

Many of the results in this section are closely related to the work of Auslander and Bridger. See [1], and the writeup [4] of the contents of [1]. However, none of the results below is explicitly in these works, and we found they gave us a better understanding of what the vanishing of Ext means.

The natural maps $M^* \otimes_R N \to \mathrm{Hom}_R(M, N)$ and $M \otimes_R N^* \to \mathrm{Hom}_R(M, N)^*$

Assume that M is maximal Cohen-Macaulay. From the short exact sequence $0 \to M_1 \to F \to M \to 0$ we get the dual short exact sequence $0 \to M^* \to F^* \to M_1^* \to 0$, and these yield a commutative diagram

$$
\begin{array}{ccccccc}
M^* \otimes_R N & \xrightarrow{\alpha} & F^* \otimes_R N & \longrightarrow & M_1^* \otimes_R N & \longrightarrow & 0 \\
\downarrow{f_0} & & \downarrow{g} & & \downarrow{f_1} & & \\
0 \longrightarrow \mathrm{Hom}_R(M, N) & \longrightarrow & \mathrm{Hom}_R(F, N) & \xrightarrow{\beta} & \mathrm{Hom}_R(M_1, N), & &
\end{array}
$$

where the vertical arrows are the natural maps $M^* \otimes_R N \to \operatorname{Hom}_R(M, N)$ given by $\phi \otimes n \mapsto \{m \mapsto \phi(m)n\}$. Note that g is an isomorphism (since F is free), $\ker \alpha \simeq \operatorname{Tor}_1^R(M_1^*, N)$ and $\operatorname{coker} \beta \simeq \operatorname{Ext}_R^1(M, N)$. From this diagram one easily deduces the following three facts.

(1) $\ker f_1 \simeq \operatorname{coker} f_0$,

(2) $\operatorname{Tor}_1^R(M_1^*, N) \simeq \ker f_0$, and

(3) $\operatorname{Ext}_R^1(M, N) \simeq \operatorname{coker} f_1$.

Building a diagram as above for each of the exact sequences $0 \to M_{i+1} \to F_i \to M_i \to 0$ and using the corresponding three facts as above, we see that we have exact sequences

$$0 \to \operatorname{Ext}_R^1(M_{i-2}, N) \to M_i^* \otimes_R N$$
$$\to \operatorname{Hom}_R(M_i, N) \to \operatorname{Ext}_R^1(M_{i-1}, N) \to 0,$$

and

$$0 \to \operatorname{Tor}_1^R(M_{i+1}^*, N) \to M_i^* \otimes_R N$$
$$\to \operatorname{Hom}_R(M_i, N) \to \operatorname{Tor}_1^R(M_{i+2}^*, N) \to 0.$$

For $i \geq 2$ the first exact sequence can be written as

$$(5.1) \quad 0 \to \operatorname{Ext}_R^{i-1}(M, N) \to M_i^* \otimes_R N$$
$$\to \operatorname{Hom}_R(M_i, N) \to \operatorname{Ext}_R^i(M, N) \to 0.$$

An immediate observation is

PROPOSITION 5.2. *Let R be a local Gorenstein ring, and let M and N be finitely generated R-modules with M maximal Cohen-Macaulay. Then $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if the natural maps $M_i^* \otimes_R N \to \operatorname{Hom}_R(M_i, N)$ are isomorphisms for all $i \gg 0$.*

Note also that building exact sequences (5.1) for arbitrarily large negative syzygies of M, and then splicing the resulting exact sequences together, we obtain a doubly infinite long exact sequence

$$(5.3) \quad \cdots \to M_i^* \otimes_R N \to \operatorname{Hom}_R(M_i, N)$$
$$\to M_{i+i}^* \otimes_R N \to \operatorname{Hom}_R(M_{i+1}, N) \to \cdots.$$

Now suppose that N is maximal Cohen-Macaulay and that $\operatorname{Ext}_R^1(M, N) = 0$. From the short exact sequence $0 \to M_1 \to F \to M \to 0$ we get the short

exact sequence $0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(F, N) \to \operatorname{Hom}_R(M_1, N) \to 0$, and a commutative diagram

$$
\begin{array}{ccccccc}
M_1 \otimes_R N^* & \xrightarrow{\gamma} & F \otimes_R N^* & \longrightarrow & M \otimes_R N^* & \longrightarrow & 0 \\
\downarrow{h_1} & & \downarrow{g'} & & \downarrow{h_0} & & \\
0 \longrightarrow \operatorname{Hom}_R(M_1, N)^* & \longrightarrow & \operatorname{Hom}_R(F, N)^* & \xrightarrow{\delta} & \operatorname{Hom}_R(M, N)^*, & &
\end{array}
$$

where the vertical arrows are the natural maps $M \otimes_R N^* \to \operatorname{Hom}_R(M, N)^*$ given by $m \otimes \phi \mapsto \{\psi \mapsto \phi(\psi(m))\}$. Note that g' is an isomorphism (since F is free), $\ker \gamma \simeq \operatorname{Tor}_1^R(M, N^*)$ and $\operatorname{coker} \delta \simeq \operatorname{Ext}_R^1(\operatorname{Hom}_R(M_1, N), R)$. Regarding this diagram, we have the following three facts.

(1) $\ker h_0 \simeq \operatorname{coker} h_1$,

(2) $\operatorname{Tor}_1^R(M, N^*) \simeq \ker h_1$, and

(3) $\operatorname{Ext}_R^1(\operatorname{Hom}_R(M_1, N), R) \simeq \operatorname{coker} h_0$.

Now assume that $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 0$, equivalently $\operatorname{Ext}_R^1(M_i, N) = 0$ for all $i \geq 0$. Constructing such a diagram as above for each of the short exact sequences $0 \to M_{i+1} \to F_i \to M_i \to 0$ and using the corresponding three facts as above we obtain exact sequences

$$
(5.4) \quad 0 \to \operatorname{Tor}_i^R(M, N^*) \to M_i \otimes_R N^*
$$
$$
\xrightarrow{h_i} \operatorname{Hom}_R(M_i, N)^* \to \operatorname{Tor}_{i-1}^R(M, N^*) \to 0
$$

for $i \geq 2$. From Theorem 2.1 we know that $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Tor}_i^R(M, N^*) = 0$ for all $i \gg 0$. Hence

PROPOSITION 5.5. *Let R be a local Gorenstein ring, and let M and N be finitely generated R-modules with N maximal Cohen-Macaulay. Then $\operatorname{Ext}^i(M, N) = 0$ for all $i \gg 0$ implies the natural maps $M_i \otimes_R N^* \to \operatorname{Hom}_R(M_i, N)^*$ are isomorphisms for all $i \gg 0$.*

Theorem 5.9 below contains a similar result.

Ext *and Stable* Hom

Recall that the *stable* Hom, $\underline{\operatorname{Hom}}_R(M, N)$, is the cokernel of the natural map $M^* \otimes_R N \to \operatorname{Hom}_R(M, N)$. Equivalently, it is the quotient of $\operatorname{Hom}_R(M, N)$ by maps $f : M \to N$ which factor through a free module. Stable Homs offer

a convenient way of interpreting the vanishing of all higher $\text{Ext}_R^i(M, N)$: from (5.1) (and the exact sequence involving Ext preceding it) we see that

$$(5.6) \qquad\qquad \text{Ext}_R^i(M, N) \simeq \underline{\text{Hom}}_R(M_i, N)$$

for $i \geq 1$. Hence we may record the following as a corollary of 5.2.

COROLLARY 5.7. *Let R be a local Gorenstein ring, and let M and N be finitely generated R-modules with M maximal Cohen-Macaulay. Then $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if for all $i \gg 0$ every map $M_i \to N$ factors through a free module.*

The next Proposition allows us to shift among the stable Homs with ease, which often can clarify basic vanishing results concerning Ext. (*cf.* Remark 2.2 and (1.2).)

PROPOSITION 5.8. *Let M and N be finitely generated maximal Cohen-Macaulay modules over the local Gorenstein ring R. Then*

(1) $\underline{\text{Hom}}_R(M, N) \simeq \underline{\text{Hom}}_R(N^*, M^*)$.

(2) $\underline{\text{Hom}}_R(M, N) \simeq \underline{\text{Hom}}_R(M_t, N_t)$ *for all $t \in \mathsf{Z}$.*

PROOF. (1). The isomorphism is induced by the obvious mapping $\text{Hom}_R(, R) : \text{Hom}_R(M, N) \to \text{Hom}_R(N^*, M^*)$. The fact that the induced map on stable Homs is an isomorphism is straightforward (since M and N are reflexive) provided it is well-defined. But this is clear since if $f \in \text{Hom}_R(M, N)$ factors through a free module F, then $\text{Hom}_R(f, R)$ factors through F^*.

(2). It is enough to prove (2) in the case $t = 1$. Given a map $f \in \text{Hom}_R(M, N)$ we get a map $f_1 \in \text{Hom}_R(M_1, N_1)$ by completing the diagram

$$
(5.8.1) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & F & \xrightarrow{\ \epsilon\ } & M & \longrightarrow & 0 \\
& & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & N_1 & \longrightarrow & G & \xrightarrow{\ \epsilon'\ } & N & \longrightarrow & 0.
\end{array}
$$

Define $\Phi : \underline{\text{Hom}}_R(M, N) \to \underline{\text{Hom}}_R(M_1, N_1)$ by $\Phi(\bar{f}) = \bar{f}_1$.

We first show that $\Phi(\bar{f})$ is determined independent of the choice of chain map $\{f_0, f_1\}$. Suppose that $g_0 : F \to G$ and $g_1 : M_1 \to N_1$ are two other maps making the diagram (5.8.1) commute and such that $\epsilon' g_0 = f\epsilon$. Then we have the standard homotopy $h : F \to N_1$ such that $f_1 - g_1 = h(M_1 \hookrightarrow F)$. That is, $f_1 - g_1$ factors through a free module, so that $\bar{f}_1 = \bar{g}_1$ in $\underline{\text{Hom}}_R(M_1, N_1)$.

Next we show Φ is well-defined. Suppose that $f \in \mathrm{Hom}_R(M, N)$ factors through a free module H. Then $f_1 = 0$ completes the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & F & \overset{\epsilon}{\longrightarrow} & M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & H & \longrightarrow & H & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_1 & \longrightarrow & G & \overset{\epsilon'}{\longrightarrow} & N & \longrightarrow & 0.
\end{array}
$$

Hence, $\Phi(\bar{f}) = 0$, as desired.

In order to show that Φ is an isomorphism we exhibit its inverse. Let g be in $\mathrm{Hom}_R(M_1, N_1)$. We dualize and complete the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \longrightarrow & F^* & \longrightarrow & M_1^* & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle (g^*)_1} & & \uparrow{\scriptstyle (g^*)_0} & & \uparrow{\scriptstyle g^*} & & \\
0 & \longrightarrow & N^* & \longrightarrow & G^* & \longrightarrow & N_1^* & \longrightarrow & 0.
\end{array}
$$

Define $\Psi : \underline{\mathrm{Hom}}_R(M_1, N_1) \to \underline{\mathrm{Hom}}_R(M, N)$ by $\Psi(\bar{g}) = \overline{(g^*)_1^*}$. It's not hard to see that Φ and Ψ are inverses of one another (since M and N are reflexive).

Vanishing Ext *and Cohen-Macaulayness*

We end with a theorem which shows the relationship between the vanishing of d consecutive Ext modules and the Cohen-Macaulayness of a certain tensor product.

THEOREM 5.9. *Let R be a d-dimensional local Gorenstein ring, and let M and N be maximal Cohen-Macaulay modules. Consider the following two conditions.*

(1) *$M^* \otimes_R N$ is maximal Cohen-Macaulay,*

(2) *$\mathrm{Ext}_R^1(N, M) = \cdots = \mathrm{Ext}_R^d(N, M) = 0$.*

Then (2) implies (1). If we assume that $\mathrm{Ext}_R^1(N, M), \ldots, \mathrm{Ext}_R^d(N, M)$ have finite length, then (1) implies (2). Furthermore, if (1) holds then $\mathrm{Hom}_R(N, M)$ is maximal Cohen-Macaulay and $M^* \otimes_R N \simeq \mathrm{Hom}_R(N, M)^*$. If (1) holds and R is also integrally closed, then $M^* \otimes_R N \simeq \mathrm{Hom}_R(M, N)$.

PROOF. We first prove that if (1) holds then the R-module $\mathrm{Hom}_R(N, M)$ is maximal Cohen-Macaulay and $M^* \otimes_R N \simeq \mathrm{Hom}_R(N, M)^*$. Note that as both M and N are reflexive, the natural map $\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(M^*, N^*)$ is an

isomorphism. Since $M^* \otimes_R N$ is maximal Cohen-Macaulay so is $(M^* \otimes_R N)^*$, and we have

$$(M^* \otimes_R N)^* = \operatorname{Hom}_R(M^* \otimes_R N, R)$$
$$\simeq \operatorname{Hom}_R(M^*, N^*) \qquad \text{by Hom-tensor adjointness}$$
$$(5.9.1) \qquad\qquad \simeq \operatorname{Hom}_R(N, M)$$

Hence $\operatorname{Hom}_R(N, M)$ is maximal Cohen-Macaulay and $M^* \otimes_R N \simeq (M^* \otimes_R N)^{**} \simeq \operatorname{Hom}_R(N, M)^*$.

We prove $(1) \Longrightarrow (2)$ under the assumption that $\operatorname{Ext}^1_R(N, M), \ldots, \operatorname{Ext}^d_R(N, M)$ have finite length. We induce on d. The case in which $d = 0$ is vacuous.

$d = 1$. Since $\operatorname{Ext}^1_R(N, M)$ has finite length and M and N are maximal Cohen-Macaulay, we can choose a non-zerodivisor $x \in R$ such that x is a non-zerodivisor on both M and N and $x \operatorname{Ext}^1_R(N, M) = 0$. For any R-module X we let $\overline{X}$ denote X/xX, and we let * indicate Hom into either R or $\overline{R}$ depending on the module in question. We have the short exact sequence $0 \to M \xrightarrow{x} M \to \overline{M} \to 0$, which yields the exact sequence

$$0 \to \operatorname{Hom}_R(N, M) \xrightarrow{x} \operatorname{Hom}_R(N, M)$$
$$\to \operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{M}) \to \operatorname{Ext}^1_R(N, M) \to 0.$$

Hence $\operatorname{length}(\operatorname{Hom}_{\overline{R}}(\overline{N},\overline{M})) = \operatorname{length}(\overline{\operatorname{Hom}_R(N,M)})+\operatorname{length}(\operatorname{Ext}^1_R(N,M))$. Note that for any maximal Cohen-Macaulay R-module X, $\overline{X^*} \simeq \overline{X}^*$. We have

$$\operatorname{length}(\operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{M})) = \operatorname{length}((\overline{M}^* \otimes_{\overline{R}} \overline{N})^*) \qquad \text{by (5.9.1)}$$
$$= \operatorname{length}(\overline{M}^* \otimes_{\overline{R}} \overline{N}) \qquad \text{since } \overline{R} \text{ is 0-dimensional}$$
$$= \operatorname{length}(\overline{M^*} \otimes_{\overline{R}} \overline{N}) \qquad \text{since } \overline{M}^* \simeq \overline{M^*}$$
$$= \operatorname{length}(\overline{M^* \otimes_R N})$$
$$= \operatorname{length}(\overline{\operatorname{Hom}_R(N, M)^*}) \qquad \text{by (5.9.1)}$$
$$= \operatorname{length}(\overline{\operatorname{Hom}_R(N, M)}^*)$$
$$= \operatorname{length}(\overline{\operatorname{Hom}_R(N, M)}) \qquad \text{since } \overline{R} \text{ is 0-dimensional.}$$

Therefore

$$\operatorname{length}(\overline{\operatorname{Hom}_R(N, M)}) = \operatorname{length}(\overline{\operatorname{Hom}_R(N, M)}) + \operatorname{length}(\operatorname{Ext}^1_R(N, M)),$$

and so $\operatorname{Ext}^1_R(N, M) = 0$.

$d > 1$. Choose a parameter $x \in \cap_{i=1}^d \operatorname{ann}_R \operatorname{Ext}^i_R(N, M)$. We have $\overline{M}^* \otimes_{\overline{R}} \overline{N} \simeq \overline{M^* \otimes_R N}$ is maximal Cohen-Macaulay. The short exact sequence $0 \to$

$M \xrightarrow{x} M \to \overline{M} \to 0$ yields exact sequences

(5.9.2) $0 \to \operatorname{Ext}^i_R(N, M) \to \operatorname{Ext}^i_R(N, \overline{M}) \to \operatorname{Ext}^{i+1}_R(N, M) \to 0$

for $i = 1, \ldots, d-1$. Hence $\operatorname{Ext}^i_R(N, \overline{M})$ have finite length for $1 \le i \le d-1$. Since $\operatorname{Ext}^i_R(N, \overline{M}) \simeq \operatorname{Ext}^i_{\overline{R}}(\overline{N}, \overline{M})$ for all i [5], it follows by induction that $\operatorname{Ext}^1_{\overline{R}}(\overline{N}, \overline{M}) = \cdots = \operatorname{Ext}^{d-1}_{\overline{R}}(\overline{N}, \overline{M}) = 0$. Now the exact sequences (5.9.2), for $i = 1, \ldots, d-1$, and the fact that $\operatorname{Ext}^i_R(N, \overline{M}) \simeq \operatorname{Ext}^i_{\overline{R}}(\overline{N}, \overline{M})$ for all $i \ge 1$ gives $\operatorname{Ext}^i_R(N, M) = 0$ for $i = 1, \ldots, d$.

Assume (2). Note then that $\operatorname{Hom}_R(N, M)$ is maximal Cohen-Macaulay: let $\mathbf{F} : \cdots \to F_1 \to F_0 \to N \to 0$ be an R-free resolution of N. Applying $\operatorname{Hom}_R(\ , M)$ and using our hypothesis we get the exact sequence

$$0 \to \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(F_0, M) \to \cdots \to \operatorname{Hom}_R(F_{d+1}, M).$$

Each $\operatorname{Hom}_R(F_i, M)$ is maximal Cohen-Macaulay since M is. By counting depths along this exact sequence we get the desired conclusion.

For $(2) \Longrightarrow (1)$ we again induce on d. The case $d = 0$ is trivial.

$d = 1$. Let x be a non-zerodivisor on both M and N. The short exact sequence $0 \to M \xrightarrow{x} M \to \overline{M} \to 0$ and our hypothesis yield the short exact sequence

$$0 \to \operatorname{Hom}_R(N, M) \xrightarrow{x} \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{M}) \to 0.$$

Therefore

(5.9.3) $\overline{\operatorname{Hom}_R(N, M)} \simeq \operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{M})$

Consider the natural map $M^* \otimes_R N \xrightarrow{h} \operatorname{Hom}_R(N, M)^*$. We have a commutative diagram

$$
\begin{array}{ccc}
\overline{M^* \otimes_R N} & \xrightarrow{\overline{h}} & \overline{\operatorname{Hom}_R(N, M)^*} \\
\simeq \downarrow & & \simeq \downarrow \\
\overline{M}^* \otimes_{\overline{R}} \overline{N} & \xrightarrow{\simeq} & \operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{M})^*
\end{array}
$$

where the right vertical arrow comes from (5.9.3) and the bottom arrow is the isomorphism of (5.9.1). Thus h is an isomorphism modulo x. By Nakayama's lemma, h must be onto. Now we have a short exact sequence

$$0 \to K \to M^* \otimes_R N \xrightarrow{h} \operatorname{Hom}_R(N, M)^* \to 0.$$

The fact that $\mathrm{Hom}_R(N, M)$ is maximal Cohen-Macaulay implies that $\overline{K} = 0$, and therefore $K = 0$. Thus h is an isomorphism and $M^* \otimes_R N$ is maximal Cohen-Macaulay.

$d > 1$. For x a non-zerodivisor on M and N, the hypothesis yields $\mathrm{Ext}^1_{\overline{R}}(\overline{N}, \overline{M}) = \cdots = \mathrm{Ext}^{d-1}_{\overline{R}}(\overline{N}, \overline{M}) = 0$, so by induction $\overline{M}^* \otimes_{\overline{R}} \overline{N}$ is maximal Cohen-Macaulay, which means so is $M^* \otimes_R N$.

Finally suppose that R is integrally closed. There is always a natural map from $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(N, M)^*$ obtained by composition, and this map is an isomorphism if either N or M is free. Since M and N are maximal Cohen-Macaulay modules and R_P is regular if the height of P is at most one, it follows that this natural map is an isomorphism in codimension one. It is a standard result that a map between reflexive modules which is an isomorphism in codimension one must itself be an isomorphism. Hence, as both $\mathrm{Hom}_R(M, N)$ and $\mathrm{Hom}_R(N, M)^*$ are reflexive, the natural map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(N, M)^*$ is an isomorphism. The stated isomorphism of $M^* \otimes_R N$ with $\mathrm{Hom}_R(M, N)$ follows from the above paragraph.

6. Questions

This work leaves quite a few questions unresolved. We list a few for further study. Perhaps the most intriguing is

    1. Are all local Gorenstein rings AB rings?

Some other interesting questions are:

    2. Let R and S be AB rings which are essentially of finite type over the same field k. Is $R \otimes_k S$ locally an AB ring if neither R nor S is a complete intersection?

    3. Are localizations of AB rings AB rings?

    4. Are AB rings Ext-bounded?

NOTE ADDED IN PROOF. This question has recently been answered in the negative by the second author and L. M. Şeqa in 'Nonvanishing cohomology and classes of Gorenstein rings', Preprint, available at http://www.arxiv.org/abs/math/0306001.

REFERENCES

1. Auslander, M., and Bridger, M., *Stable Module Theory*, Mem. Amer. Math. Soc. 94 (1969).
2. Avramov, L. L., and Buchweitz, R-.O., *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000), 285–318.
3. Lescot, J., *Asymptotic properties of Betti numbers of modules over certain rings*, J. Pure Appl. Algebra 38 (1985), 287–298.

4. Maşek, V., *Gorenstein dimension and torsion of modules over commutative Noetherian rings*, Comm. Algebra (2000), 5783–5812.
5. Rees, D., *A theorem of homological algebra*, Proc. Cambridge Philos. Soc. 52 (1956), 605–610.
6. Rotman, J., *An Introduction to Homological Algebra*, Academic Press, New York, 1979.
7. Watanabe, K. I., Ishikawa, T., Tachibana, S., and Otsuka, K., *On tensor products of Gorenstein rings*, J. Math. Kyoto Univ. 9 (1969), 413–423.

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