

***Uniqueness of ground states
for quasilinear elliptic equations in the exponential case***

Patrizia Pucci¹ & James Serrin

We consider ground states of the quasilinear equation

$$(1.1) \quad \operatorname{div}(A(|Du|)Du) + f(u) = 0 \quad \text{in } \mathbb{R}^n, \quad n \geq 2,$$

that is, C^1 solutions of (1.1) such that

$$(1.2) \quad u \geq 0, \quad u \not\equiv 0; \quad u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Two particular model operators A motivate this work, first

$$(1.3) \quad A(t) \equiv 1, \quad n = 2,$$

the Laplace case, and second

$$(1.4) \quad A(t) = t^{m-2}, \quad t > 0, \quad m \geq n,$$

the case of the degenerate Laplacian operator. When $n = m$ it is well known that radially symmetric ground states can exist in the case (1.4), even for functions $f(u)$ with exponential growth as $u \rightarrow \infty$, that is

$$f(u) = O(e^{\beta u^{n/(n-1)}}) \quad \text{as } u \rightarrow \infty,$$

where β is an appropriate positive constant, see [1, 2, 4, 6, 7].

The question of uniqueness of ground states for (1.1) when $f(u)$ has such exponential behavior has not previously been treated; the purpose of this paper is to make a beginning on this problem.

Appropriate assumptions on the operator A will be given in the next section. On the other hand, for the nonlinearity f we consider the specific family of functions

$$(1.5) \quad f(u) = -\alpha u + (e^u - 1 - u), \quad \alpha > 0,$$

or, more generally,

$$(1.6) \quad f(u) = u^{p-1} \{-\alpha u^p + (e^{u^p} - 1 - u^p)\}, \quad \alpha > 0, \quad \frac{1}{2} < p \leq 1,$$

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of which (1.5) is the special case $p = 1$. For the family (1.6) it is not hard to show (see Section 2) that f is in $C[0, \infty) \cap C^1(0, \infty)$ and that there exists $a > 0$ such that $f(a) = 0$ and

$$f(u) < 0 \quad \text{for } 0 < u < a, \quad f(u) > 0 \quad \text{for } a < u < \infty.$$

In consequence we are able to apply the principal result of [9] to obtain uniqueness of the corresponding radial ground states of (1.1). Our main conclusions will be given in the next section.

In recent work, e.g. [8, 10], the question whether all ground states of (1.1) are radially symmetric has been considered. In particular, when the operator A is regular on the entire interval $[0, \infty)$, as for example for the Laplacian case (1.3), it is known for wide classes of functions f , including the family (1.6), that ground states are radially symmetric with respect to an appropriate origin $O \in \mathbb{R}^n$, see [10]. In such cases the a priori assumption of radial symmetry is then automatically satisfied, and we find for $n = 2$ that ground states of (1.1), (1.6) are unique up to translations, see Theorem 2.

§2. The main result.

We consider the quasilinear equation (1.1), in which the operator A is of class $C^1(0, \infty)$. Let

$$\Omega(t) = tA(t), \quad G(t) = \int_0^t \Omega(s) ds, \quad t > 0,$$

and suppose that the following specific conditions are satisfied:

- (1) $\Omega'(t) > 0$ for $t > 0$, $\Omega(t) \rightarrow 0$ as $t \rightarrow 0$,
- (2) $t\Omega(t)/G(t)$ is (non-strictly) increasing in $(0, \infty)$;

see [9, Section 1].

We define the critical constant

$$(2.1) \quad m = \lim_{t \rightarrow 0} \frac{t\Omega(t)}{G(t)}.$$

It is clear that $m \geq 1$, since $G(t) < t\Omega(t)$ for $t > 0$ by (1).

The operator A in (1.4) obviously satisfies conditions (1), (2), where the exponent m there coincides with the constant m given in (2.1).

Theorem 1. *Suppose that the operator A obeys conditions (1), (2), and let the critical constant m satisfy*

$$m \geq n.$$

Assume that f in (1.1) is of the form (1.6). Then radially symmetric ground states of (1.1) are unique up to translations.

Proof. By [9, Theorem 1 and Lemma 2.1] it is enough to prove for the family (1.6) that

$$(2.2) \quad \frac{d}{du} \left[\frac{F(u)}{f(u)} \right] \geq 0 \quad \text{for } u > 0, \quad u \neq a,$$

where

$$F(u) = \int_0^u f(s) ds = \frac{1}{p} \left\{ e^{u^p} - 1 - u^p - \frac{(\alpha + 1)}{2} u^{2p} \right\}$$

and a is the unique positive zero of f .

In fact, to see that f has exactly one positive zero a , we note that, in the new variable $v = u^p$,

$$f = f(u(v)) = v^{(p-1)/p} [e^v - 1 - (\alpha + 1)v].$$

It is thus enough to show that $e^v - 1 - (\alpha + 1)v$ has only one zero in $(0, \infty)$, which follows by trivial calculus. Indeed one finds $a > [\log(\alpha + 1)]^{1/p}$.

Since

$$\left[\frac{F}{f} \right]' = \frac{f^2 - f'F}{f^2}, \quad ' = \frac{d}{du},$$

to show (2.2) it is enough to verify that

$$(2.3) \quad f^2 - f'F \geq 0 \quad \text{for } u > 0.$$

Again in the v variable, we find

$$f^2 - f'F = \frac{1}{p} v^{(p-2)/p} [(1-p)(e^v - 1)^2 + v(e^v - 1)\{\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2\} + v^2\{\mathbb{D} + \mathbb{E}v\}],$$

where

$$\begin{aligned} \mathbb{A} &= (2p-1)(\alpha+1) - 1, & \mathbb{B} &= -\frac{1}{2}(3p+1)\alpha - \frac{1}{2}(p+1), \\ \mathbb{C} &= \frac{1}{2}p(\alpha+1), & \mathbb{D} &= (1-2p)\alpha + 1 - p, & \mathbb{E} &= \frac{1}{2}(\alpha+1)(\alpha+p+1). \end{aligned}$$

Let φ be the function in brackets in the previous formula, namely

$$\varphi(v, \alpha; p) = (1-p)(e^v - 1)^2 + v(e^v - 1)\{\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2\} + v^2\{\mathbb{D} + \mathbb{E}v\}.$$

The required condition (2.3) then follows if we show that

$$(2.4) \quad \varphi(v, \alpha; p) > 0 \quad \text{for } v, \alpha \in (0, \infty) \text{ and } p \in [\frac{1}{2}, 1].$$

Note moreover that $\varphi(v, \alpha; p)$ is *linear* in p , so that it is enough to prove (2.4) at the endpoints $p = 1/2$ and $p = 1$.

Case 1. $p = 1/2$. Here

$$(2.5) \quad \begin{aligned} \varphi(v, \alpha; 1/2) &= \frac{1}{2}(e^v - 1)^2 + v(e^v - 1)\{\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2\} + v^2\{\mathbb{D} + \mathbb{E}v\}, \\ \mathbb{A} &= -1, \quad \mathbb{B} = -\frac{1}{4}(5\alpha + 3), \quad \mathbb{C} = \frac{1}{4}(\alpha + 1), \\ \mathbb{D} &= \frac{1}{2}, \quad \mathbb{E} = \frac{1}{4}(2\alpha^2 + 5\alpha + 3). \end{aligned}$$

We first show

$$(2.6) \quad \varphi(v, \alpha; 1/2) > 0 \quad \text{for } v \geq 5; \quad \varphi(0, \alpha; 1/2) = 0 \quad \text{for } \alpha > 0.$$

Indeed

$$\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2 = \begin{cases} -1 & \text{when } v = 0 \\ 3/2 & \text{when } v = 5 \end{cases}.$$

Therefore, since $\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2$ is quadratic in v , with $\mathbb{C} > 0$, we get $\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2 \geq 3/2$ for $v \geq 5$, $\alpha > 0$. Now (2.6) follows from (2.5).

Next, observe that

$$\frac{1}{2}(e^v - 1)^2 + v(e^v - 1)\mathbb{A} + \mathbb{D}v^2 = \frac{1}{2}(e^v - 1 - v)^2 > 0.$$

Therefore, since \mathbb{B} is linear in α and \mathbb{E} quadratic, and also $e^v - 1 \leq \frac{1}{5}(e^5 - 1)v$ for $v \in [0, 5]$, it is clear from (2.5) that there is $\alpha_1 > 0$ such that

$$(2.7) \quad \varphi(v, \alpha; 1/2) > 0 \quad \text{if } 0 < v \leq 5 \text{ and } \alpha \geq \alpha_1.$$

Also, by a straightforward calculation we obtain

$$\begin{aligned} \varphi(v, 0; 1/2) &= \frac{1}{8}v^5 + \left(e^v - 1 - v - \frac{v^2}{2} \right) \\ &\quad \cdot \left\{ -\frac{1}{4}v^2 + \frac{1}{3}v^3 + \frac{1}{2}\left(e^v - 1 - v - \frac{v^2}{2} - \frac{v^3}{6} \right) \right\}. \\ &= \frac{1}{12}v^5 + \text{Positive function (of order } v^6 \text{ at } v = 0). \end{aligned}$$

Therefore

$$(2.8) \quad \varphi(v, 0; 1/2) > 0 \quad \text{for all } v > 0.$$

From (2.6)–(2.8) it follows that if $\varphi(v, \alpha; 1/2)$ takes negative values in $(0, \infty) \times (0, \infty)$, then it attains a *negative minimum* at some point (v_0, α_0) , with $0 < v_0 < 5$ and $0 < \alpha_0 < \alpha_1$. Moreover, by direct calculation we find

$$(2.9) \quad \begin{aligned} \frac{\partial \varphi}{\partial v}(v, \alpha; 1/2) &= (e^v - 1)^2 + v(e^v - 1)\{(\mathbb{A} + 2\mathbb{B}) + (\mathbb{B} + 3\mathbb{C})v + \mathbb{C}v^2\} \\ &\quad + (\mathbb{B} + 3\mathbb{E})v^2 + \mathbb{C}v^3. \end{aligned}$$

$$(2.10) \quad \frac{\partial \varphi}{\partial \alpha}(v, \alpha; 1/2) = \frac{1}{4}v^2\{(e^v - 1)(v - 5) + (4\alpha + 5)v\}.$$

We claim that at the minimum point (v_0, α_0) there holds

$$(2.11) \quad \varphi(v_0, \alpha_0; 1/2) = \frac{v_0^2}{4(5-v_0)^2} [32\alpha_0^2 - 2\alpha_0(25\alpha_0 + 22)v_0 + 4(5\alpha_0^2 + 8\alpha_0 + 3)v_0^2 - 2(\alpha_0 + 1)^2v_0^3].$$

Indeed, since (v_0, α_0) is a critical point, by (2.10) and the fact that $0 < v_0 < 5$ there follows

$$(2.12) \quad e^{v_0} - 1 = \frac{4\alpha_0 + 5}{5 - v_0} v_0.$$

The claim (2.11) now arises by eliminating $e^{v_0} - 1$ from the main formula (2.5).

By (2.9), we also have at (v_0, α_0)

$$(e^{v_0} - 1)^2 = -v_0(e^{v_0} - 1)\{(\mathbb{A} + 2\mathbb{B}) + (\mathbb{B} + 3\mathbb{C})v_0 + \mathbb{C}v_0^2\} - (\mathbb{B} + 3\mathbb{E})v_0^2 - \mathbb{C}v_0^3.$$

Eliminating $(e^{v_0} - 1)^2$ from the main formula (2.5) and then using (2.12) once more, we get

$$(2.13) \quad \varphi(v_0, \alpha_0; 1/2) = \frac{v_0^2}{4} [(70\alpha_0^2 + 108\alpha_0 + 75) - (28\alpha_0^2 + 44\alpha_0 + 23)v_0 + (2\alpha_0^2 + 3\alpha_0 + 1)v_0^2].$$

Define

$$\psi(v, \alpha) = 70\alpha^2 + 108\alpha + 75 - (28\alpha^2 + 44\alpha + 23)v + (2\alpha^2 + 3\alpha + 1)v^2.$$

Clearly

$$\begin{aligned} \psi(v, 0) &= 75 - 23v + v^2 > 0 && \text{for } v \in [0, 3.9], \\ \frac{\partial\psi}{\partial\alpha}(v, 0) &= 108 - 44v + 3v^2 > 0 && \text{for } v \in [0, 3.1], \\ \frac{\partial^2\psi}{\partial\alpha^2}(v, \alpha) &= 4[35 - 14v + v^2] > 0 && \text{for } v \in [0, 3.2] \text{ and all } \alpha > 0. \end{aligned}$$

Consequently, by integration with respect to α from 0 to any $\alpha > 0$, we get $\psi(v, \alpha) > 0$ on $[0, 3] \times [0, \infty)$. But $\psi(v_0, \alpha_0) = 2v_0^{-2}\varphi(v_0, \alpha_0; 1/2) < 0$, so that

$$(2.14) \quad 3 < v_0 < 5.$$

Next define

$$\omega(v, \alpha) = 32\alpha^2 - 2\alpha(25\alpha + 22)v + 4(5\alpha^2 + 8\alpha + 3)v^2 - 2(\alpha + 1)^2v^3.$$

Then

$$\begin{aligned}\omega(v, 0) &= 2v^2(6 - v) > 0 && \text{for } v \in (0, 6), \\ \frac{\partial\omega}{\partial\alpha}(v, 0) &= -4v(11 - 8v + v^2) > 0 && \text{for } v \in [1.8, 6.2], \\ \frac{\partial^2\omega}{\partial\alpha^2}(v, \alpha) &= 4(1 - v)(v^2 - 9v + 16) > 0 && \text{for } v \in [2.5, 6.5] \text{ and all } \alpha > 0.\end{aligned}$$

Hence, by integration with respect to α from 0 to $\alpha > 0$, we get $\omega(v, \alpha) > 0$ on $[2.5, 6) \times (0, \infty)$. Since $\omega(v_0, \alpha_0) = 4(5 - v_0)^2 v_0^{-2} \varphi(v_0, \alpha_0; 1/2) < 0$ by (2.11) and since $v_0 < 5$, this implies

$$0 < v_0 < 2.5,$$

which contradicts (2.14). Therefore $\varphi(v, \alpha; 1/2)$ cannot take negative values. This completes the proof of Case 1.

Case 2. $p = 1$. Here $v = u$ and

$$(2.15) \quad \begin{aligned}\varphi(v, \alpha; 1) &= v(e^v - 1)\{\mathbb{A} + \mathbb{B}v + \mathbb{C}v^2\} + v^2\{\mathbb{D} + \mathbb{E}v\}, \\ \mathbb{A} &= \alpha, \quad \mathbb{B} = -2\alpha + 1, \quad \mathbb{C} = \frac{1}{2}(\alpha + 1), \\ \mathbb{D} &= -\alpha, \quad \mathbb{E} = \frac{1}{2}(\alpha + 1)(\alpha + 2).\end{aligned}$$

By Taylor's expansion and use of the formula (2.15) for the coefficients, $\mathbb{A}, \dots, \mathbb{E}$, one finds

$$\begin{aligned}\varphi(v, \alpha; 1) &= \frac{1}{2}\alpha^2 v^3 - \frac{1}{3}\alpha v^4 + \sum_{k=2}^{\infty} v^{k+3} \left\{ \frac{\alpha}{(k+2)!} - \frac{2\alpha+1}{(k+1)!} + \frac{\alpha+1}{2k!} \right\} \\ &= \frac{1}{2}v^3 \left\{ \alpha^2 - \frac{2}{3}\alpha v + \sum_{k=2}^{\infty} \frac{v^k}{(k+2)!} [(k^2 - k - 4)\alpha + k^2 + k - 2] \right\}.\end{aligned}$$

For all $k \geq 3$ the coefficients $(k^2 - k - 4)\alpha + k^2 + k - 2$ are non-negative. Hence, dropping all terms with $k \geq 4$, we obtain

$$(2.16) \quad \varphi(v, \alpha; 1) \geq \frac{1}{2}v^3 \left\{ \alpha^2 - \frac{2}{3}\alpha v + \frac{1}{12}(2 - \alpha)v^2 + \frac{1}{60}(\alpha + 5)v^3 \right\}.$$

By (2.16), to prove the assertion it is enough to show that

$$(2.17) \quad 60\alpha^2 - 40\alpha v + 5(2 - \alpha)v^2 + (\alpha + 5)v^3 > 0 \quad \text{on } (0, \infty) \times (0, \infty).$$

By Cauchy's inequality $40\alpha v \leq 20\alpha^2 + 20v^2$. We are thus led to consider the function

$$(2.18) \quad \psi(v, \alpha) = 40\alpha^2 - 5(\alpha + 2)v^2 + (\alpha + 5)v^3 \quad v > 0, \quad \alpha > 0.$$

For fixed α the minimum of $\psi(\cdot, \alpha)$ is attained at

$$v_\alpha = \frac{10}{3} \frac{\alpha + 2}{\alpha + 5}.$$

Hence

$$\psi(v, \alpha) \geq \psi(v_\alpha, \alpha) = \frac{20}{27} \frac{1}{(\alpha + 5)^2} \{54\alpha^2(\alpha + 5)^2 - 25(\alpha + 2)^3\}.$$

The expression in braces is the quartic function

$$Q(\alpha) = 54\alpha^4 + 515\alpha^3 + 1200\alpha^2 - 300\alpha - 200.$$

Observe that $Q(2/3) > 0$, $Q'(2/3) > 0$ and $Q''(\alpha) > 0$ on $(0, \infty)$. Consequently, $Q(\alpha) > 0$ for all $\alpha \geq 2/3$. It is now evident that $\psi(v, \cdot) > 0$ for $\alpha \geq 2/3$, and so (2.17) holds in the subset $(0, \infty) \times [2/3, \infty)$.

On the other hand,

$$60\alpha^2 - 40\alpha v + 5(2 - \alpha)v^2 + (\alpha + 5)v^3 \geq 5\{12\alpha^2 - 8\alpha v + (2 - \alpha)v^2\}.$$

The quadratic function on the right side has minimum value

$$\frac{20\alpha^2}{2 - \alpha}(2 - 3\alpha) > 0$$

for $\alpha < 2/3$. Hence (2.17) holds on the entire set $(0, \infty) \times (0, \infty)$, as required.

This completes the proof of Case 2, and in turn of the theorem.

It is surprising that the proof of Case 1 is so tricky. On the other hand, the function $\varphi(\cdot, \cdot; 1/2)$ in (2.5) is very close to zero when (v, α) is in $[0, 1]^2$. Indeed

$$\begin{aligned} \varphi(0.01, 0.01; 1/2) &= 21 \times 10^{-12}, & \varphi(0.1, 0.1; 1/2) &= 22 \times 10^{-7}, \\ \varphi(1, 1/2; 1/2) &= 0.039, & \varphi(1/2, 1; 1/2) &= 0.040, & \varphi(1, 1; 1/2) &= 0.181. \end{aligned}$$

Instead of (1.6), another natural choice for the nonlinearity $f(u)$ in (1.1) would be

$$f(u) = -\alpha u + (e^{u^p} - 1 - u^p), \quad \alpha > 0, \quad \frac{1}{2} < p \leq 1.$$

For this family, however, the verification of (2.2) does not seem to be obvious.

Since in Theorem 1 we treat radially symmetric ground states $u = u(r)$ of (1.1), it is clear that the dimension n may be taken as any real number greater than 1. In fact, it is precisely this point of view which was used in Theorem 1 of [9]. With this interpretation the condition $m \geq n$ in Theorem 1 then allows values $m < 2$ when $n < 2$.

When the operator A is smooth on the entire interval $[0, \infty)$, the result of Theorem 1 can be improved. In particular we have the following

Theorem 2. *Let $n = 2$. Suppose that A is of class $C^{1,1}[0, \infty)$, that $\Omega'(t) > 0$ for $t \geq 0$, and that condition (2) is satisfied. Also let f in (1.1) have the form (1.6). Then ground states of (1.1) having connected support are radially symmetric, and unique up to translations.*

When $p = 1$ in (1.6), the condition of connected support can be omitted.

Proof. First observe that A satisfies conditions (1), (2). Moreover $\Omega(t) = t\Omega'(0) + o(t)$ as $t \rightarrow 0$, with $\Omega'(0) > 0$. Hence from the definition (2.1) and use of condition (2) we get $m = 2$.

The family (1.6) has the property that $f'(u) < 0$ for $0 < u < \delta$ and some $\delta > 0$; indeed, it is not hard to see that (using the variable $v = u^p$)

$$f'(u) \leq v^{2(p-1)/p} e^v - 1 - \frac{2p-1}{p} \alpha < 0$$

for $v \in (0, \eta)$, $\eta = \log(1 + \frac{2p-1}{p} \alpha) > 0$ since $p > 1/2$. Thus we can take $\delta = \eta^{1/p}$. Hence f is nonincreasing for u near zero (note also that $f'(0) = -\infty$ when $1/2 < p < 1$), and by Theorem 1 of [10] any corresponding ground state u of (1.1) having connected support is radially symmetric about some origin $O = O(u) \in \mathbb{R}^n$. The first part of Theorem 2 now follows from Theorem 1, since for $n = 2$ the required condition $m \geq n$ is fulfilled.

To obtain the second part of the theorem, note that when $p = 1$ the function f in (1.6), see also (1.5), is uniformly Lipschitz continuous on $[0, \delta]$ for any $\delta > 0$. Hence by Corollary 1.1 of [10] any ground state is radially symmetric and the result follows from Theorem 1 above.

Similar results can be proved for singular operators A such as the m -Laplacian. We omit the details since the required conditions are somewhat complicated to state (see in particular [5, 10]).

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