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SYMMETRY OF SOLUTIONS TO SOME SYSTEMS OF INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we study some systems of integral equations, including those related to Hardy-Littlewood-Sobolev (HLS) inequalities. We prove that, under some integrability conditions, the positive regular solutions to the systems are radially symmetric and monotone about some point. In particular, we established the radial symmetry of the solutions to the Euler-Lagrange equations associated with the classical and weighted Hardy-Littlewood-Sobolev inequality.

1. Introduction

Let R^n be the n-dimensional Euclidean space, and let α be a real number satisfying $0 < \alpha < n$. Consider the integral equation:

$$u(x) = \int_{R^n} |x-y|^{\alpha-n} u(y)^{\frac{n+\alpha}{n-\alpha}} dy, \;\; u > 0 \text{ in } R^n. \tag{1.1}$$

It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequality. In [14], Lieb classified the maximizers of the functional, and thus obtained the best constant in the HLS inequality. He then posed the classification of all the critical points of the functional, i.e. the solutions of the integral equation (1.1), as an open problem.

This integral equation is closely related to the following well-known family of semi-linear partial differential equations

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}, \;\; u > 0, \quad \text{in } R^n. \tag{1.2}$$

In the special case $n \geq 3$ and $\alpha = 2$, (1.2) becomes

$$-\Delta u = u^{(n+2)/(n-2)}, \; u > 0, \;\; \text{in } R^n. \tag{1.3}$$

Solutions to (1.3) were studied by Gidas, Ni, and Nirenberg [12] and classified under some decay condition at infinity.

Later, Caffarelli, Gidas, and Spruck [5] removed the decay condition and obtained the same result. Then Chen and Li [7], and Li [13] simplified their proof. Recently, Wei and Xu [19] generalized this result to the solutions of the more general equation (1.2) with α being any even numbers between 0 and n.

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After this, Chen, Li, and Ou [8] solved Lieb's open problem by using the method of moving planes and proved that every positive solution $u(x)$ of (1.1) is radially symmetric and decreasing about some point x_o. They also showed the equivalence between the integral equation (1.1) and the differential equation (1.2), and therefore classified all the solutions of the semi-linear differential equation (1.2).

Furthermore, another paper from Chen, Li, an Ou [9] discussed the actual system of integral equations that maximize the constant in the Hardy-Littlewood-Sobolev inequality. They presented and proved:

Theorem. *Let the pair (u, v) be a solution of the system of integral equations:*

$$\left\{ \begin{array}{l} u(x) = \int_{R^n} |x-y|^{\alpha-n} v^q(y)dy, \\ v(x) = \int_{R^n} |x-y|^{\alpha-n} u^p(y)dy \end{array} \right. \tag{1.4}$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$; $p,\ q \geq 1,\ 0 < \alpha < n$.

Assume that $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$. Then u and v are radially symmetric and decreasing about some point x_o.

In this paper, we will study the system of integral equations which is in a more general form:

$$\mathbf{U(x)} = \int_{R^n} \mathbf{L}(|x|, |x-y|)\mathbf{H}(|y|)\mathbf{F}(\mathbf{U}(y))dy. \tag{1.5}$$

Here,

$$\mathbf{U}(x) = \{u_1(x), u_2(x), \ldots, u_m(x)\}, \quad \mathbf{F}(z) = \{f_1(z), f_2(z), \ldots, f_m(z)\},$$
$$\mathbf{L}(|x|, |x-y|) = \{l_1(|x|, |x-y|), l_2(|x|, |x-y|), \ldots, l_m(|x|, |x-y|)\},$$
$$\mathbf{H}(|x|) = \{h_1(|x|), h_2(|x|), \ldots, h_m(|x|)\} \quad \text{where } x, y \in R^n,\ z \in R^m.$$

This system covers the Euler-Lagrange system from not only the classical HLS inequality, which has been studied by many researchers, but also the double weighted HLS inequality (see section 2 for more details).

Before presenting the theorem, we first give a definition.

Definition 1.1. We say that $f_1, f_2, \ldots, f_m$ are essentially related if

$$\sum_{l=1}^{l_o} f_{i_l}(u_1, u_2, \ldots, u_m) \neq \sum_{l=1}^{l_o} f_{i_l}(v_1, v_2, \ldots, v_m) \tag{1.6}$$

provided that $u_i \leq v_i$ for $i = 1, \ldots, m$ and $u_j < v_j$ for $j \in S$, where $S = \{1, \ldots, m\} \backslash \{i_1, \ldots, i_{l_o}\}$.

This definition is a simple way to say that the system (1.5) can't be divided into two or more independent subsystems. In this paper, we say that system (1.5) is essentially related when $f_1, f_2, \ldots, f_m$ are essentially related.

To be able to prove that the solutions $u_1, ..., u_n$ are radially symmetric around a common center x_o, this kind of assumption is necessary.

Theorem 1.2. *Let $\boldsymbol{U}(x)$ be the positive regular solutions of the essentially related system (1.5), $0 < \alpha_i < n$, and $\beta_j > 0$ for $i, j = 1, \ldots, m$. Assume that:*

(1) $l_i(s,t) \leq \frac{C}{t^{n-\alpha_i}}$, $\frac{\partial f_i}{\partial u_j}(u) \leq C_{i,j}|u|^{\beta_j}$ *and* $l_i(s_1, t_1) < l_i(s_2, t_2)$ *provided* $s_1 < s_2$ *and* $t_1 < t_2$*;*
(2) $l_i(s,t)$ *and* $h_i(t)$ *are positive nonincreasing in* s, t*;* $f_i(z) \geq 0$ *and is nondecreasing in all variables for* $i = 1, 2, \ldots, m$*;*

(3) $h_i(|y|)(u_j)^{\beta_j} \in L^{p_{i,j}}$ *where* $p_{i,j} = \frac{nq_iq_j}{q_jq_i\alpha_i + n(q_j - q_i)}$, $q_i > \frac{n}{n-\alpha_i}$, $q_j > \frac{n}{n-\alpha_j}$ *for any* $i,\ j = 1, 2, ...m$;

Then all $u_i(x)$ *are symmetric and decreasing about some point* x_o.

Remark 1.3. In most cases, the center is at the origin. For example, in the case that $\mathbf{H}$ is not constant, $\mathbf{L} \neq 0$, and $\mathbf{f}(u) \neq 0$ for $\mathbf{u} \neq 0$, the center is at the origin.

Remark 1.4. In many applications, most of the $C_{i,j}$ are zero, i.e., the systems related to the classical HLS inequality and the weighted HLS inequality.

Remark 1.5. This theorem is a generalization of the results in [9] and [8] about the classification of nonnegative solutions.

In particular, the result of Theorem 1.2 holds for the single integral equation:

$$u(x) = \int_{R^n} L(|x|, |x-y|)H(|y|)f[u(y)]dy. \tag{1.7}$$

We present it as the following:

Theorem 1.6. *Let* $u(x)$ *be a positive regular solution of* (1.7) *and* $0 < \alpha < n$. *Assume that* $L(s,t)$ *and* $H(t)$ *are nonnegative and nonincreasing,* $f(s)$ *is nonnegative nondecreasing,* $L(s,t) \leq \frac{C}{t^{n-\alpha}}$, $f'(u) \leq C|u|^{\beta}$, $\beta > 0$ *and* $u(x)^{\beta}H(|y|) \in L^{\frac{n}{\alpha}}(R^n)$. *Then* $u(x)$ *is radially symmetric and decreasing about some point* x_o.

The following is the system of Euler-Lagrange equations for the weighted HLS inequality:

$$\begin{cases} u(x) = \frac{1}{|x|^{\alpha}} \int_{R^n} \frac{v(y)^q}{|y|^{\beta}|x-y|^{\lambda}} dy, \\ v(x) = \frac{1}{|x|^{\beta}} \int_{R^n} \frac{u(y)^p}{|y|^{\alpha}|x-y|^{\lambda}} dy \end{cases} \tag{1.8}$$

where $0 < p, q < \infty$, $0 < \lambda < n$, $\beta \geq 0$, $0 \leq \frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$, and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$.

Theorem 1.7. *Let the pair (u, v) be a positive solution of system* (1.8) *and* $p, q \geq 1, pq \neq 1$. *Then* u *and* v *are radially symmetric and decreasing about some point* x_o.

In section 2, we present some background knowledge about Hardy-Littlewood-Sobolev inequalities. In section 3, we present the proof of Theorem 1.7, which is similar to but much simpler than the proof of Theorem 1.2 . In section 4, we prove Theorems 1.2 and 1.6.

2. Hardy-Littlewood-Sobolev inequalities

2.1. **The classical Hardy-Littlewood-Sobolev inequality.** Let $0 < \alpha < n$, $s, r > 1$ such that $\frac{1}{r} + \frac{1}{s} = \frac{n+\alpha}{n}$. The well-known Hardy-Littlewood-Sobolev inequality states that:

$$\int_{R^n} \int_{R^n} f(x)|x-y|^{\alpha-n} g(y) dx dy \leq C(n, s, \alpha)||f||_r ||g||_s \tag{2.1}$$

for any $f \in L^r(R^n)$ and $g \in L^s(R^n)$.

Let $Tg(x) = \int_{R^n} |x-y|^{\alpha-n} g(y) dy$. Then the above inequality implies that:

$$\langle f, Tg \rangle = \langle Tf, g \rangle \leq C(n, s, \alpha) \|f\|_r \|g\|_s .$$

Consequently, $\|Tg\|_p = \mathrm{Sup}_{\|f\|_r=1}\langle f, Tg\rangle \ \leq C(n,s,\alpha)\|g\|_s$, where

$$\begin{cases} \frac{1}{p}+\frac{1}{r}=1 \\ \frac{1}{r}+\frac{1}{s}=\frac{n+\alpha}{n} \end{cases}.$$

Solving for p, $\quad p=\frac{ns}{n-s\alpha}$.

Thus, $\|Tg\|_{\frac{ns}{n-s}} \leq C(n,s,\alpha)\|g\|_s$, where $1<s<\frac{n}{\alpha}$. Similarly,

$$\|Tg\|_p \leq C(n,p,\alpha)\|g\|_{\frac{np}{n+\alpha p}}, \text{ where } \frac{n}{n-\alpha}<p<\infty. \tag{2.2}$$

To find the best constant $C=C(n,s,\alpha)$ in (2.1), one can maximize the functional

$$J(f,g)=\int_{R^n}\int_{R^n} f(x)|x-y|^{\alpha-n}g(y)dxdy \tag{2.3}$$

under the constraints: $\|f\|_r=\|g\|_s=1$.

Letting $I(f) = \int_{R^n} f(x)^r dx$, by the Lagrange multiplier, $D_f J(f,g)(v) = \lambda_1 DI(f)(v)$, which gives the first equation:

$$\int_{R^n} |x-y|^{\alpha-n} g(y)dy=\lambda_1 r f(x)^{r-1}. \tag{2.4}$$

If we multiply both sides by $f(x)$ and then integrate, we obtain $\lambda_1 r=\frac{1}{J(f,g)}$.

Similarly, for $I(g)=\int_{R^n} g(x)^s dx$, we have

$$\int_{R^n} |x-y|^{\alpha-n} f(y)dy=\lambda_2 s g(x)^{s-1} \text{ and } \lambda_2 s=\frac{1}{J(f,g)}. \tag{2.5}$$

Let $u=c_1 f^{r-1}$, $\quad v=c_2\, g^{s-1}$, $\quad p=\frac{1}{r-1}$ and $q=\frac{1}{s-1}$. By a proper choice of constants c_1 and c_2, (2.4) and (2.5) turn into the system of Euler-Lagrange equations for the HLS inequality:

$$\begin{cases} u(x)=\int_{R^n}|x-y|^{\alpha-n}v^q(y)dy, \\ v(x)=\int_{R^n}|x-y|^{\alpha-n}u^p(y)dy \end{cases} \tag{2.6}$$

with $0<p<\infty$, $0<q<\infty$, $\frac{1}{p+1}+\frac{1}{q+1}=\frac{n-\alpha}{n}$, $u\in L^{p+1}$ and $v\in L^{q+1}$.

Later in section 4, we will see how our Theorem 1.2 applies to this system.

2.2. **The weighted Hardy-Littlewood-Sobolev inequality.** Let $1<s,r<\infty$, $0<\lambda<n$, $\alpha+\beta\geq 0$, $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}=2$ and $1-\frac{1}{r}-\frac{\lambda}{n}\leq\frac{\alpha}{n}<1-\frac{1}{r}$. Then the weighted HLS inequality states

$$\left|\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy\right| \leq C_{\alpha,\beta,s,\lambda,n}\|f\|_r\|g\|_s. \tag{2.7}$$

We can also write the weighted HLS inequality in another form. Let $Tg(x)=\int_{R^n}\frac{g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dy$. Then

$$\|Tg(x)\|_p=\mathrm{Sup}_{\|f\|_r=1}\langle Tg(x), f(x)\rangle \leq C\|g(y)\|_s \tag{2.8}$$

where $1+\frac{1}{p}=\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}$, $1<s,p<\infty$, $\alpha+\beta\geq 0$, $0<\lambda<n$ and $\frac{1}{p}-\frac{\lambda}{n}<\frac{\alpha}{n}<\frac{1}{p}$.

Similarly, the corresponding system of Euler-Lagrange equations is:

$$\begin{cases} \lambda_1 r f(x)^{r-1}=\frac{1}{|x|^\alpha}\int_{R^n}\frac{g(y)}{|y|^\beta|x-y|^\lambda}dy, \\ \lambda_2 s g(x)^{s-1}=\frac{1}{|x|^\beta}\int_{R^n}\frac{f(y)}{|y|^\alpha|x-y|^\lambda}dy \end{cases} \tag{2.9}$$

where $\lambda_1 r=\lambda_2 s=J(f,g)$.

Let $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$. When $pq \neq 1$, (2.9) turns into

$$\begin{cases} u(x) = \frac{1}{|x|^\alpha} \int_{R^n} \frac{v(y)^q}{|y|^\beta |x-y|^\lambda} dy, \\ v(x) = \frac{1}{|x|^\beta} \int_{R^n} \frac{u(y)^p}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{2.10}$$

where $0 < p, q < \infty$, $0 < \lambda < n$, $\alpha + \beta \geq 0$, $\frac{1}{p+1} - \frac{\lambda}{n} < \frac{\alpha}{n} < \frac{1}{p+1}$ and $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda+\alpha+\beta}{n}$.

3. On the system related to the weighted HLS inequality

In this section, we present the proof of Theorem 1.7. We prove a lemma first. For a given real number κ, define

$$\Sigma_\kappa = \{x = (x_1, \ldots, x_n) \mid x_1 \geq \kappa\},\ x^\kappa = (2\kappa - x_1, x_2, \ldots, x_n), u_\kappa(x) = u(x^\kappa).$$

Lemma 3.1. *For any solution $u(x)$ of (1.7), we have*

$$u_\kappa(x) - u(x) = \int_{\Sigma_\kappa} [L(|x|, |x-y|) - L(|x^\kappa|, |x^\kappa - y|)] H(|y^\kappa|)[f(u_\kappa) - f(u)]dy + E(x, \kappa)$$

where

$$\begin{aligned} E(x,\kappa) = &- \int_{\Sigma_\kappa} [L(|x|, |x-y|) - L(|x^\kappa|, |x^\kappa - y|)][H(|y|) - H(|y^\kappa|)] f(u) dy \\ &- \int_{\Sigma_\kappa} [L(|x|, |x-y|) - L(|x^\kappa|, |x-y|)] H(|y^\kappa|) f(u_\kappa) dy \\ &- \int_{\Sigma_\kappa} [L(|x|, |x^\kappa - y|) - L(|x^\kappa|, |x^\kappa - y|)] H(|y^\kappa|) f(u_\kappa) dy. \end{aligned}$$

Proof. Since $|x - y^\kappa| = |x^\kappa - y|$, we have

$$u(x) = \int_{\Sigma_\kappa} L(|x|, |x-y|) H(|y|) f(u) dy + \int_{\Sigma_\kappa} L(|x|, |x^\kappa - y|) H(|y^\kappa|) f(u_\kappa) dy,$$

$$u_\kappa(x) = \int_{\Sigma_\kappa} L(|x^\kappa|, |x^\kappa - y|) H(|y|) f(u) dy + \int_{\Sigma_\kappa} L(|x^\kappa|, |x-y|) H(|y^\kappa|) f(u_\kappa) dy.$$

Then, it is easy to verify the lemma. □

To prove Theorem 1.7, we compare $u(x)$ with $u_\kappa(x)$ and $v(x)$ with $v_\kappa(x)$ on Σ_κ. The proof consists of two steps. In step 1, we show there exists a real number $R < 0$ such that for $\kappa \leq R$ and $x \in \Sigma_\kappa$, we have

$$u_\kappa(x) \leq u(x) \quad \text{and} \quad v_\kappa(x) \leq v(x) \quad \text{a.e.} \tag{3.1}$$

Thus, we can start moving the plane from $\kappa \leq R$ to the right as long as (3.1) holds. In step 2, we show that if the plane stops at $x_1 = \kappa_o$ for some $\kappa_o < 0$, then $u(x)$ and $v(x)$ must be symmetric and monotone about the plane $x_1 = \kappa_o$; otherwise, we can move the plane all the way to $x_1 = 0$. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and decreasing about some point.

Proof. Step 1. Define

$$\Sigma_\kappa^u = \{x \in \Sigma_\kappa \mid u(x) < u_\kappa(x)\} \text{ and } \Sigma_\kappa^v = \{x \in \Sigma_\kappa \mid v(x) < v_\kappa(x)\}.$$

Similar to the calculation of Lemma 3.1, we can obtain

$$u_\kappa(x) - u(x) \le C \int_{\Sigma_\kappa^v} \frac{1}{|x|^\alpha |x-y|^\lambda |y|^\beta} v_\kappa^{q-1}(y)[v_\kappa(y) - v(y)]dy.$$

By the weighted Hardy-Littlewood-Sobolev inequality (2.8),

$$\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} \le C \left\| v_\kappa^{q-1}(v_\kappa - v) \right\|_{L^{(q+1)/q}(\Sigma_\kappa^v)}. \tag{3.2}$$

Then by the Hölder inequality,

$$\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} \le C \|v_\kappa\|^{q-1}_{L^{q+1}(\Sigma_\kappa^v)} \|(v_\kappa - v)\|_{L^{q+1}(\Sigma_\kappa^v)}. \tag{3.3}$$

Similarly, we have

$$\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)} \le C \|u_\kappa\|^{p-1}_{L^{p+1}(\Sigma_\kappa^u)} \|(u_\kappa - u)\|_{L^{p+1}(\Sigma_\kappa^u)}. \tag{3.4}$$

Combining (3.3) and (3.4),

$$\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} \le C \|v_\kappa\|^{q-1}_{L^{q+1}(\Sigma_\kappa^v)} \|u_\kappa\|^{p-1}_{L^{p+1}(\Sigma_\kappa^u)} \|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)}.$$

Since $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, we can choose a sufficiently large $|R|$ with $R < 0$ such that for $\kappa \le R < 0$,

$$C \|v_\kappa\|^{q-1}_{L^{q+1}(\Sigma_\kappa^v)} \|u_\kappa\|^{p-1}_{L^{p+1}(\Sigma_\kappa^u)} \le C \|v\|^{q-1}_{L^{q+1}(\Sigma_\kappa^C)} \|u\|^{p-1}_{L^{p+1}(\Sigma_\kappa^C)} \le \frac{1}{2}$$

where Σ_κ^C is the complement of Σ_κ in R^n.

So, $\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} \le \frac{1}{2}\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)}$. Similarly,

$$\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)} \le \frac{1}{2}\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)}.$$

These imply that $\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} = 0$ and $\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)} = 0$. Therefore, the measure of Σ_κ^u and Σ_κ^v must be zero, i.e. (3.1).

Step 2. We now move $x_1 = \kappa$ to the right as long as (3.1) holds. Suppose that at a point $\kappa_o < 0$, we have, on Σ_{κ_o},

$$u(x) \ge u_{\kappa_o}(x) \text{ and } v(x) \ge v_{\kappa_o}(x), \text{ but } u(x) \not\equiv u_{\kappa_o}(x) \text{ or } v(x) \not\equiv v_{\kappa_o}(x).$$

Then the plane can be moved further to the right. More precisely, there exists an ϵ such that

$$u(x) \ge u_\kappa(x) \;\text{ and }\; v(x) \ge v_\kappa(x) \text{ on } \Sigma_\kappa \text{ for all } \kappa \in [\kappa_o, \kappa_o + \epsilon). \tag{3.5}$$

In the case that $v(x) \not\equiv v_{\kappa_o}(x)$ on Σ_{κ_o}, similarly as Lemma 3.1 indicates, we have in fact $u(x) > u_{\kappa_o}(x)$ in the interior of Σ_{κ_o}. Let

$$\Phi^u_{\kappa_o} = \{x \in \Sigma_{\kappa_o} | \, u(x) \le u_{\kappa_o}(x)\} \quad \text{and} \quad \Phi^v_{\kappa_o} = \{x \in \Sigma_{\kappa_o} | \, v(x) \le v_{\kappa_o}(x)\}.$$

Then, obviously $\Phi^u_{\kappa_o}$ has measure zero and $\limsup_{\kappa \to \kappa_o} \Sigma^u_\kappa \subset \Phi^u_{\kappa_o}$. The same is true for that of v. Let D^* be the reflection of the set D about the plane $x_1 = \kappa$.

By (3.3) and (3.4),

$$\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} \le C \|v_\kappa\|^{q-1}_{L^{q+1}(\Sigma_\kappa^v)} \|u_\kappa\|^{p-1}_{L^{p+1}(\Sigma_\kappa^u)} \|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)}.$$

The integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$ ensure that one can choose ϵ sufficiently small, so that for all κ in $[\kappa_o, \kappa_o + \epsilon)$,

$$C \|v_\kappa\|^{q-1}_{L^{q+1}(\Sigma_\kappa^v)} \|u_\kappa\|^{p-1}_{L^{p+1}(\Sigma_\kappa^u)} = C\|u\|^{p-1}_{L^{p+1}((\Sigma_\kappa^u)^*)} \|v\|^{q-1}_{L^{q+1}((\Sigma_\kappa^v)^*)} \le \frac{1}{2}.$$

So, $\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} \leq \frac{1}{2}\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)}$. Similarly, we can obtain

$$\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)} \leq \frac{1}{2}\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)}.$$

These imply $\|u_\kappa - u\|_{L^{p+1}(\Sigma_\kappa^u)} = 0$ and $\|v_\kappa - v\|_{L^{q+1}(\Sigma_\kappa^v)} = 0$ for all κ in $[\kappa_o, \kappa_o + \epsilon)$. Therefore the measure of Σ_κ^u and Σ_κ^v must be zero. This verifies (3.5), and therefore completes the proof of the theorem. □

4. Proof of the main theorem

In this section, we present the proof of Theorem 1.2.

Proof. Define $\Sigma_\kappa^i = \{x \in \Sigma_\kappa, u_i(x^\kappa) > u_i(x)\}$, $i = 1, 2, \ldots, m$, $u_i^\kappa = u_i(x^\kappa)$ and let Σ_κ^C be the complement of Σ_κ.

Step 1. We prove that there exists a real number $R < 0$, such that for $\kappa \leq R$, we have $u_i(x) \geq u_i(x^\kappa)$, $i = 1, 2, \ldots, m$ a.e.

As a result of Lemma 3.1, if we compare $u_i(x)$ and $u_i^\kappa(x)$ on Σ_κ with $\kappa < 0$, which implies $|x^\kappa| \geq |x|$ and $|x^\kappa - y| \geq |x - y|$, then we can obtain:

$$\begin{aligned} u_i(x^\kappa) - u_i(x) &\leq \sum_{j=1}^m \int_{\Sigma_\kappa^j} [l_i(|x|, |x-y|) - l_i(|x^\kappa|, |x^\kappa - y|)]\, h_i(|y^\kappa|) K_{i,j}(y,\kappa) dy \\ &\leq \sum_{j=1}^m \int_{\Sigma_\kappa^j} l_i(|x|, |x-y|) h_i(|y^\kappa|) K_{i,j}(y,\kappa) dy \end{aligned} \tag{4.1}$$

where $K_{i,j}(y,\kappa) = f_i(u_1, u_2, \ldots, u_j^\kappa, u_{j+1}^\kappa, \ldots, u_m^\kappa) - f_i(u_1, u_2, \ldots, u_j, u_{j+1}^\kappa, \ldots, u_m^\kappa)$.

By the assumptions, $0 \leq K_{i,j}(y,\kappa) \leq (u_j^\kappa)^{\beta_j}(u_j^\kappa - u_j)$, combining with those estimates about l_i, h_i, f_i together, we obtain:

$$u_i(x^\kappa) - u_i(x) \leq \sum_{j=1}^m \int_{\Sigma_\kappa^j} \frac{C}{|x-y|^{n-\alpha_i}} h_i(|y^\kappa|)(u_j^\kappa)^{\beta_j}(u_j^\kappa - u_j) dy. \tag{4.2}$$

We apply the Hardy-Littlewood-Sobolev inequality (2.2) to each mode. For $q_i > \frac{n}{n-\alpha_i}$, $i = 1, \ldots, m$:

$$\begin{aligned} \|u_i(x^\kappa) - u_i(x)\|_{L^{q_i}(\Sigma_\kappa{}^i)} &\leq \sum_{j=1}^m \left\| \int_{\Sigma_\kappa^j} \frac{C_i}{|x-y|^{n-\alpha_i}} h_i(|y^\kappa|)(u_j^\kappa)^{\beta_j}(u_j^\kappa - u_j) dy \right\|_{L^{q_i}(\Sigma_\kappa^i)} \\ &\leq \sum_{j=1}^m C \left\| h_i(|y^\kappa|)(u_j^\kappa)^{\beta_j}(u_j^\kappa - u_j) \right\|_{L^{\frac{nq_i}{q_i\alpha_i+n}}(\Sigma_\kappa^j)}. \end{aligned}$$

Then, by the Hölder inequality,

$$\begin{aligned} &\left\| h_i(|y^\kappa|)(u_j^\kappa)^{\beta_j}(u_j^\kappa - u_j) \right\|_{L^{\frac{nq_i}{q_i\alpha_i+n}}(\Sigma_\kappa^j)} \\ &\leq \left\| h_i(|y^\kappa|)(u_j^\kappa)^{\beta_j} \right\|_{L^{p_{i,j}}(\Sigma_\kappa^j)} \left\| u_j^\kappa - u_j \right\|_{L^{q_j}(\Sigma_\kappa^j)} \\ &\leq \left\| h_i(|y|)(u_j)^{\beta_j} \right\|_{L^{p_{i,j}}(\Sigma_\kappa^C)} \left\| u_j^\kappa - u_j \right\|_{L^{q_j}(\Sigma_\kappa^j)} \end{aligned} \tag{4.3}$$

where $p_{i,j} = \frac{nq_iq_j}{q_jq_i\alpha_i + n(q_j - q_i)}$, $q_i > \frac{n}{n-\alpha_i}$, $q_j > \frac{n}{n-\alpha_j}$, $i, j = 1, \ldots, m$.

Thus,

$$\|u_i^\kappa - u_i\|_{L^{q_i}(\Sigma_\kappa^i)} \le \sum_{j=1}^m C \left\| h_i(|y|)(u_j)^{\beta_j} \right\|_{L^{p_{i,j}}(\Sigma_\kappa^C)} \left\| u_j^\kappa - u_j \right\|_{L^{q_j}(\Sigma_\kappa^j)}. \tag{4.4}$$

By corresponding integrability assumptions, we can choose a real number $R < 0$ with $|R|$ large enough, so that for $\kappa \le R$ we have

$$C \left\| h_i(|y|)(u_j)^{\beta_j} \right\|_{L^{p_{i,j}}(\Sigma_\kappa^C)} \le \frac{1}{2m}, \quad \text{for all } i, j.$$

Then by (4.4),

$$\sum_{i=1}^m \|u_i^\kappa - u_i\|_{L^{q_i}(\Sigma_\kappa^i)} \le \sum_{i=1}^m \left[\frac{1}{2m} \sum_{j=1}^m \|u_j^\kappa - u_j\|_{L^{(q_j)}(\Sigma_\kappa^j)} \right] = \frac{1}{2} \sum_{j=1}^m \|u_j^\kappa - u_j\|_{L^{q_j}(\Sigma_\kappa^j)}.$$

This implies

$$\|u_i^\kappa - u_i\|_{L^{q_i}}(\Sigma_\kappa^i) = 0, \quad i = 1, 2, \ldots, m.$$

Therefore, the measure of Σ_κ^i must be zero for all i. This completes Step 1.

Step 2. Continuously moving the plane $x_1 = \kappa \le R$ to the right.

From step 1, we know that

$$u_i(x) \ge u_i(x^{\kappa_o}), \quad i = 1, 2, \ldots, m, \quad \kappa \le R, \quad \text{for some } R \text{ negatively large.}$$

In this step, we increase the value of R to the extreme and show the symmetry of the solutions about this extreme value of R. In fact, let

$$R_o = \text{Sup}\{R \mid \mu(\Sigma_\kappa^i) = 0,\ \kappa \le R \le 0 \ for\ i = 1, \ldots, m\} < \infty,$$

where μ is the notation of measure. We show that u_i, $i = 1, \ldots, m$, are symmetric about $x_1 = R_o$. Note if $R_o = 0$, we just choose to move the plane from positive infinity to the origin. For this reason, we make $R \le 0$.

Now we assume, without loss of generality, that $\mathbf{U}$ is not symmetric about R_o. Then we can derive a contradiction.

Letting $E_j = \{x \in \Sigma_{R_o} | u_j(x) = u_j(x^{R_o})\}$, $j = 1, \ldots, m$, and $E = \bigcap_{j=1}^m E_j = \{x \in \Sigma_{R_o} | \mathbf{U}(x) = \mathbf{U}(x^{R_o})\}$, from the assumption, we know $\mu(\Sigma_{R_o} \backslash E) \ne 0$. Now, we claim $\mu(E_j) = 0$, $j = 1, \ldots, m$.

Proof of the claim. Assume that $\mu(E_{i_l}) > 0$, $l = 1, \ldots, l_o$, $1 \le l_o \le m$, and $\mu(E_{i_k}) = 0$, $i_k \in S = \{1, 2, \ldots, m\} \backslash \{i_1, i_2, \ldots, i_{l_o}\}$.

For $x \in E_{i_l}$, $l = 1, \ldots, l_o$,

$$\begin{aligned} 0 &\equiv u_{i_l}(x^{R_o}) - u_{i_l}(x) \\ &\le \int_{\Sigma_{R_o}} \left[l_{i_l}(|x|, |x-y|) - l_{i_l}(|x^{R_o}|, |x^{R_o} - y|) \right] h_{i_l}(|y^{R_o}|) \left(f_{i_l}(U_{R_o}) - f_{i_l}(U) \right) dy \\ &\le 0 \end{aligned}$$

where $U_{R_o} = \{u_1(y^{R_o}), u_2(y^{R_o}), \ldots, u_m(y^{R_o})\}$.

In the interior of Σ_{R_o}, $|x| < |x^{R_o}|$ and $|x - y| < |x^{R_o} - y|$, so $l_{i_l}(|x|, |x-y|) - l_{i_l}(|x^{R_o}|, |x^{R_o} - y|) < 0$. h_{i_l} is positive, thus,

$$f_{i_l}(U_{R_o}) \equiv f_{i_l}(U) \text{ on } \Sigma_{R_o}, \text{ for } l = 1, \ldots, l_o.$$

If $l_o = m$, notice that $\mu(\Sigma_{R_o} \backslash E) \ne 0$, so $\sum_{i=1}^m f_i(\mathbf{U}(x)) < \sum_{i=1}^m f_i(\mathbf{U}(x^{R_o}))$. This is contradictory to $f_i(\mathbf{U}) = f_i(\mathbf{U}_{R_o})$ for $i = 1, \ldots, m$.

If $l_o \neq m$, by the definition of "essentially related" (1.6), we obtain the contradiction:

$$0 \equiv \sum_{l=1}^{l_o} (f_{i_l}(U_{R_o}) - f_{i_l}(U)) < 0, \tag{4.5}$$

in which $u_j(x) > u_j(x^{R_o})$ for $j \in S$.

Thus, we proved our claim $\mu(E_j) = 0,\ j = 1, \ldots, m$. □

In the following, we show that $\mathbf{U}_\kappa \leq \mathbf{U}$ on Σ_κ for $\kappa \leq R_o + \varepsilon$, which is a contradiction to the definition of R_o. More precisely, there exists an $\epsilon > 0$ depending on n, β_j and the solution $u_i(x)$ itself such that

$$u_i(x) \geq u_i(x^\kappa) \text{ on } \Sigma_\kappa \text{ for all } \kappa \text{ in } [R_o, R_o + \epsilon). \tag{4.6}$$

Letting $\Phi^i_{R_o} = \{x \in \Sigma_{R_o} \,|\, u_i(x) \leq u_i(x^{R_o})\}$, by the fact that $u_i(x) > u_i(x^\kappa)$ for all i in the interior of Σ_{R_o}, we know $\Phi^i_{R_o}$ has measure zero, and $\limsup_{\kappa \to R_o} \Sigma^i_\kappa \subset \Phi^i_{R_o}$. Let ${\Sigma^i_{R_o}}^*$ be the reflection of the set $\Sigma^i_{R_o}$ about the plane $x_1 = R_o$. From (4.3) and (4.4), we deduce

$$\|u_i^\kappa - u_i\|_{L^{q_i}(\Sigma^i_\lambda)} \leq \sum_{j=1}^m C \left\| h_i(|y|)(u_j)^{\beta_j} \right\|_{L^{p_{i,j}}({\Sigma^j_\lambda}^*)} \left\| u_j^\kappa - u_j \right\|_{L^{q_j}(\Sigma^j_\lambda)}. \tag{4.7}$$

The integrability conditions ensure that one can choose ϵ sufficiently small, so that for all κ in $[\kappa_o, \kappa_o + \epsilon)$,

$$C \left\| h_i(|y|)(u_j)^{\beta_j} \right\|_{L^{p_{i,j}}({\Sigma^j_\lambda}^*)} \leq \frac{1}{2m}, \quad \text{for all } i, j.$$

Now by (4.7), we have

$$\sum_{i=1}^m \|u_i^\kappa - u_i\|_{L^{q_i}(\Sigma^i_\lambda)} \leq \sum_{i=1}^m \left[\frac{1}{2m} \sum_{j=1}^m \|u_j^\kappa - u_j\|_{L^{q_j}(\Sigma^j_\lambda)} \right] = \frac{1}{2} \sum_{j=1}^m \|u_j^\kappa - u_j\|_{L^{q_j}(\Sigma^j_\lambda)}.$$

This implies $\|u_i^\kappa - u_i\|_{L^{q_i}}(\Sigma^i_\lambda) = 0$ for $i = 1, 2, \ldots, m$. Therefore, the measure of Σ^i_κ must be zero for all i, i.e. (4.6) holds.

This completes our proof for Theorem 1.2. □

Another application of Theorem 1.2 is the classification of the system (1.4), which has been discussed thoroughly in [9]. In this system, the integrability conditions are $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$. To apply Theorem 1.2 to this system, we simply let

$$u_1 = u,\ u_2 = v,\ L(|x|, |x-y|) = |x-y|^{\alpha - n},\ H(y) = 1,$$
$$\beta_1 = p - 1,\ \beta_2 = q - 1,\ q_1 = p + 1,\ q_2 = q + 1,\ f_1(u, v) = v^q \text{ and } f_2(u, v) = u^p.$$

When the system of integral equations (1.5) has only one equation, i.e. (1.7), it is the case stated in Theorem 1.6. As a special case of Theorem 1.2, the proof of Theorem 1.6 is easy. Here, we skip it.

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