Symmetry of steady deep-water waves with vorticity

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For a large class of vorticities we prove that a steady periodic deep-water wave must be symmetric if its profile is monotone between crests and troughs.

1 Introduction

Of all the various types of fluid wave motions that occur in Nature, surface water waves are the most easily observed. The importance of the influence of currents on waves has been known for centuries by navigators and a knowledge of the interaction of waves and currents is proving to be of increasing interest [13, 14, 20, 19, 21, 23]. Numerical calculations undertaken for a linearly sheared current (constant vorticity) in deep water [19, 23] confirm the existence of symmetric steady periodic two-dimensional waves (regular wave trains) on such currents. While there are many situations where the assumption of constant vorticity is valid (e.g. the majority of tidal flows have a non-constant vorticity approximately uniform with depth [20]), they are not universally applicable. Open ocean areas are dominated by deep water waves and the prime source of the ocean currents is long duration winds [13]. A current generated by wind is initially a pure surface process which gradually penetrates downward [13], and hence has a near-surface vorticity distribution [20]. We show that a steady periodic deep-water wave propagating against a wind-drift current must be symmetric if its profile is monotone between crests and troughs. This conclusion is consistent with previous results for uniform vorticity distributions (irrotational flows) [9, 16, 22] and extends to the deep water setting the recent results [3, 4, 5] valid for waves with vorticity in water of finite depth.

In §2 we present the governing equations for deep-water waves. §3 is devoted to some considerations about vorticity distributions for deep-water waves. In the last section, we formulate and prove the main result of this paper.

2 Formulation

In this section we recall the governing equations for the propagation of two-dimensional gravity deep water-waves and we give a reformulation suitable for our purposes.
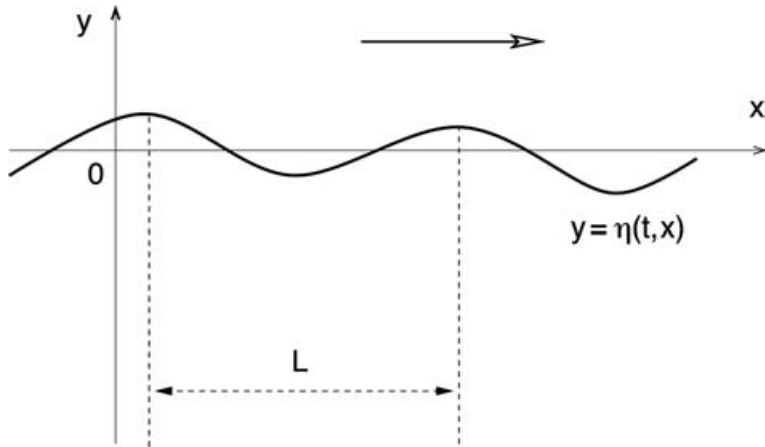
$$y = \eta(t,x)$$

FIGURE 1.

Since the motion is identical in any direction orthogonal to the direction of propagation of the wave, it suffices to analyze a cross-section of the flow, perpendicular to the crest line. We choose Cartesian coordinates (x,y) so that the horizontal x-axis is in the direction of wave propagation, the y-axis points vertically upwards and the origin lies in the mean water level. A suitable description of deep-water waves is obtained by assuming the water to be infinitely deep. The equation of the free surface is $y = \eta(t,x)$ with $\int_{\mathbb{R}} \eta(t,x)\,dx = 0$, and the fluid domain at time $t \geqslant 0$ is $D_\eta = \{(x,y) : x \in \mathbb{R},\ y < \eta(t,x)\}$. Let $(u(t,x,y), v(t,x,y))$ be the velocity field.

Homogeneity (constant density) is a good approximation for water [6], and it implies the equation of mass conservation

$$u_x + v_y = 0. \tag{2.1}$$

Neglecting viscosity, the equation of motion is Euler's equation

$$\begin{cases} u_t + u u_x + v u_y = -P_x, \\ v_t + u v_x + v v_y = -P_y - g, \end{cases} \tag{2.2}$$

where $P(t,x,y)$ denotes the pressure and g is the gravitational constant of acceleration. The boundary conditions for the water wave problem are the following. Ignoring the effects of surface tension, the dynamic boundary condition

$$P = P_0 \quad \text{on} \quad y = \eta(t,x), \tag{2.3}$$

P_0 being the constant atmospheric pressure, decouples the motion of the air from that of the water. The kinematic boundary condition

$$v = \eta_t + u\,\eta_x \quad \text{on} \quad y = \eta(t,x), \tag{2.4}$$

guarantees that the same fluid particles always form the free surface. At every instant $t \geqslant 0$, the boundary condition at the bottom

$$(u,v) \to (0,0) \quad \text{as} \quad y \to -\infty \quad \text{uniformly for } x \in \mathbb{R}, \tag{2.5}$$

expresses the fact that at great depths there is practically no motion. The deep-water

regime is characterized by the fact that the motion is confined to near-surface water layers [12, 15].

Given $c > 0$, we are considering periodic waves traveling at speed c, that is, the space-time dependence of the free surface, of the pressure, and of the velocity field has the form $(x - ct)$. Concerning regularity, we require that $\eta \in C^3(\mathbb{R})$ and $(P, u, v) \in C^1(\overline{D_\eta}) \times C^2(\overline{D_\eta}) \times C^2(\overline{D_\eta})$, where $\overline{D_\eta} = \{(x, y) \in \mathbb{R}^2 : -\infty < y \leqslant \eta(x)\}$ is the closure of the fluid domain. For our choice of coordinates the mean water level is $y = 0$ so that $\int_0^L \eta(x)\, dx = 0$, where $L > 0$ is the wavelength. We assume that $u < c$ throughout the fluid. This hypothesis is motivated by experimental evidence which indicates that for wave patterns that are not near the spilling or breaking state, the propagation speed of the surface wave is in general considerably larger than the speed of each individual water particle [1, 15]. Define a *stream function* $\psi(x, y)$ by

$$\psi_x = -v, \quad \psi_y = u - c, \tag{2.6}$$

and let

$$\omega = v_x - u_y$$

be the vorticity of the flow. Then $\omega \in C^1(\overline{D_\eta})$ and

$$\Delta \psi = -\omega \quad \text{for} \quad y < \eta(x). \tag{2.7}$$

Note that the stream function $\psi \in C^2(\overline{D_\eta})$, given by the explicit formula

$$\psi(x, y) = \psi_0 - \int_0^x v(\xi, -d)\, d\xi + \int_{-d}^y [u(x, \xi) - c]\, d\xi, \qquad y \leqslant \eta(x),$$

where $\psi_0 \in \mathbb{R}$ is a constant and $d > 0$ is chosen so that the horizontal line $y = -d$ lies entirely within the fluid domain, is periodic in the x-variable. Indeed, an explicit calculation shows that the expression $(\psi(x + L, y) - \psi(x, y)) = -\int_x^{x+L} v(\xi, -d)\, d\xi$ is a constant throughout the fluid. Thus (2.5) confirms the periodicity assertion.

The change of frame $(x - ct, y) \mapsto (x, y)$ eliminates time from the problem and transforms it into a problem in a fixed domain. In the new reference frame, in which the origin moves in the direction of propagation of the wave with the wave speed c, the wave is stationary and the flow is steady. In this moving reference frame the equations of motion (2.2) and the corresponding boundary conditions (2.3)–(2.4) are expressed as

$$\begin{cases} \psi_y \psi_{xy} - \psi_x \psi_{yy} = -P_x, \\ -\psi_y \psi_{xx} + \psi_x \psi_{xy} = -P_y - g, \end{cases} \quad \text{for} \quad y < \eta(x), \tag{2.8}$$

respectively

$$\begin{cases} \psi_x = -\psi_y \eta_x & \text{at} \quad y = \eta(x), \\ P = P_0 & \text{at} \quad y = \eta(x). \end{cases}$$

Here P, ψ, η are all required to have period L in the x-variable. The above form of the boundary conditions readily shows that ψ is constant on the free surface $y = \eta(x)$. We normalize ψ by choosing $\psi = 0$ on the free surface. The assumption $u < c$ ensures that $\psi_y < 0$ throughout the fluid. Moreover, $\psi_y \to -c$ as $y \to -\infty$ uniformly for $x \in \mathbb{R}$, in view of (2.5). This indicates that the coordinate transformation $(x, y) \mapsto (q, p)$ with $q = x$, $p = -\psi$, might be appropriate as it transforms the fluid domain $\overline{D_\eta}$ into the lower

half-plane $\{(q,p) \in \mathbb{R}^2 : q \in \mathbb{R},\, p \leqslant 0\}$. Since

$$\partial_x = \partial_q + v\,\partial_p, \quad \partial_y = (c-u)\,\partial_p,$$

we deduce that

$$\partial_q \omega = (\partial_x - v\,\partial_p)\,\omega = \left(\partial_x - \frac{v}{c-u}\,\partial_y\right)\omega.$$

On the other hand, taking the curl of the Euler equation (2.2), we obtain $(u-c)\,\omega_x + v\,\omega_y = 0$ in view of (2.7). Hence $\omega_q = 0$ so that ω is a function of p throughout the fluid. That is, $\omega = \gamma(\psi)$ with $\gamma \in C^1(\mathbb{R}_+, \mathbb{R})$. The vorticity function γ is a measure of the strength of the vorticity.

From (2.6) and (2.8) we obtain Bernoulli's Law, which states that

$$E := \frac{\psi_x^2 + \psi_y^2}{2} + gy + P - \int_\psi^0 \gamma(s)\,ds$$

is constant throughout the fluid. In view of Bernoulli's Law, the dynamic boundary condition (2.3) is equivalent to requiring that $\frac{\psi_x^2 + \psi_y^2}{2} + gy$ is constant on the free surface, that is,

$$|\nabla\psi|^2 + 2gy = C \quad \text{on} \quad y = \eta(x), \tag{2.9}$$

where $C := 2(E - P_0)$.

Summarizing the above considerations, from the governing equations for deep-water waves we obtain the free boundary value problem

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in} \quad -\infty < y < \eta(x), \\ |\nabla\psi|^2 + 2gy = C & \text{on} \quad y = \eta(x), \\ \psi = 0 & \text{on} \quad y = \eta(x), \\ \nabla\psi \to (0, -c) & \text{as} \quad y \to -\infty \quad \text{uniformly for } x \in \mathbb{R}, \end{cases} \tag{2.10}$$

to be satisfied for $\eta \in C^3(\mathbb{R})$ and $\psi \in C^2(\overline{D}_\eta)$, both L-periodic in the x-variable.

3 Vorticity of deep-water waves

In this section we present some considerations about the vorticity distribution for deep-water wave motions.

We are interested in the interaction of a regular irrotational wave train with an adverse steady current[1]. If the current is a laminar flow in the plane of the wave motion, i.e. its velocity components are $(u_c, 0)$, we say that the current is adverse if $u_c \leqslant 0$ throughout the fluid [20]. Being created by an external force that acted on the boundary of the flow (e.g. a wind stress), the current must be highly sheared with a non-uniform vorticity whose effect diminishes with depth. In other words ω_c (the vorticity of the current) depends monotonically on depth and vanishes deep down. The negative velocity u_c is also

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[1] The term 'current' is intended to indicate the presence of a water flow with a flat free surface. An adverse current is a current aligned opposite to the direction of wave propagation.

confined to a near-surface water layer. Accomodating these features, we conclude that an adverse current in deep water is negatively sheared ($\partial_y u_c \leqslant 0$) and with $\partial_y \omega_c \geqslant 0$. The last relation holds true in view of the monotone dependence of ω_c on depth since $\omega_c = 0$ at $y = -\infty$ and $\omega_c = -\partial_y u_c \geqslant 0$ at the surface. These considerations are much more complicated in the case of a wave-current interaction, as in this case the water flow is not laminar i.e. $v \not\equiv 0$. We would like to emphasize that experimental measurements [14, 20, 21] show that linear approximations yield a poor description with considerable errors in predictions so that a study taking fully into account the nonlinear character of the governing equations is necessary. Also, note that field observations and numerical calculations [13, 14, 20, 21] confirm the uniquity of symmetric wave trains propagating against currents. In this context, we prove the following result.

Proposition *Assume that (η, u, v) defines a symmetric non-trivial deep-water wave[2] with a monotone profile between crests and troughs. If the vorticity of the flow is non-decreasing with depth, i.e. $\partial_y \omega \geqslant 0$, and has bounded first-order partial derivatives, then it must be non-negative and vanishing in the limit $y \to -\infty$.*

Proof By assumption we have $0 \leqslant \partial_y \omega = \gamma'(\psi)\psi_y$ so that $\psi_y = u - c < 0$ yields that $\gamma'(\psi) \leqslant 0$ throughout the fluid. Without loss of generality, let us assume that the wave crest is located at $(0, \eta(0))$ and the wave trough at $(L/2, \eta(L/2))$, where $\eta(0) \geqslant \eta(L/2)$.

Since $\psi(x, \eta(x)) = 0$ for $x \in \mathbb{R}$, we deduce by differentiation that $\psi_x + \psi_y \eta_x = 0$ on the free surface $y = \eta(x)$. Since by assumption $\psi_y = u - c < 0$ throughout the fluid and $\eta_x \leqslant 0$ for $x \in [0, L/2]$, we infer that $v = -\psi_x \geqslant 0$ on the free surface from crest and trough. On the other hand, (2.7) implies that $\Delta \psi_x + \gamma'(\psi)\psi_x = 0$ for $y < \eta(x)$ since $\omega = \gamma(\psi)$. Thus

$$\Delta(-v) + \gamma'(\psi)(-v) = 0 \qquad (3.1)$$

in the fluid region $\{(x, y) \in \mathbb{R}^2 : 0 < x < L/2, \ y < \eta(x)\}$. By the symmetry assumption we know that $v(0, y) = 0$ for $y \leqslant \eta(0)$ and $v(L/2, y) = 0$ for $y \leqslant \eta(L/2)$. Since we proved that $-v(x, \eta(x)) \leqslant 0$ for $x \in [0, L/2]$, by the Phragmen–Lindelöf principle [18] we obtain

$$v(x, y) \geqslant 0, \quad 0 < x < L/2, \ y < \eta(x).$$

To show that above we actually have a strict inequality, assume that there is some (x_0, y_0) with $x_0 \in (0, L/2)$ and $y_0 < \eta(x_0)$, such that $v(x_0, y_0) = 0$. Choose $k_0 \in \mathbb{N}$ such that $-k_0 < y_0$. An application of the maximum principle [11] to (3.1) on the truncated domains

$$D_k = \{(x, y) \in \mathbb{R}^2 : 0 < x < L/2, \ -k < y < \eta(x)\}, \quad k \geqslant k_0,$$

implies that $v \equiv 0$ on D_k since we already know that v is non-negative. But then $v \equiv 0$, i.e. the flow is trivial. Therefore

$$v(x, y) > 0, \quad 0 < x < L/2, \ y < \eta(x). \qquad (3.2)$$

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[2] That is, in addition to the requirements of § 2, we assume that $\psi(x, y)$ is symmetric in the first variable throughout the fluid. This hypothesis is equivalent to asking for (η, u) to be symmetric and for v to be anti-symmetric in the first variable. We say that the wave is trivial if $v \equiv 0$.

We now claim that there is $a > 0$ sufficiently large that

$$v(x,y) - a \sin\left(\frac{2\pi x}{L}\right) e^{2\pi y/L} > 0 \quad \text{for} \quad x \in (0, L/2), \ y = \eta(L/2). \tag{3.3}$$

This is possible since the C^2-function v satisfies $v(0, \eta(L/2)) = v(L/2, \eta(L/2)) = 0$. Indeed, the mean-value ensures that for some $M > 0$ we have

$$0 < v(x, \eta(L/2)) \leqslant Mx, \quad x \in (0, L/2),$$

and

$$0 < v(x, \eta(L/2)) \leqslant M(L/2 - x), \quad x \in (0, L/2).$$

Since

$$\lim_{x \downarrow 0} \frac{\sin\left(\frac{2\pi x}{L}\right)}{x} = \lim_{x \uparrow L/2} \frac{\sin\left(\frac{2\pi x}{L}\right)}{L/2 - x} = \frac{2\pi}{L}$$

it is now plain that for $a > 0$ large enough (3.3) holds.

We now define the C^2-function

$$\theta(x,y) = v(x,y) - a \sin\left(\frac{2\pi x}{L}\right) e^{2\pi y/L} \quad \text{for} \quad (x,y) \in \overline{C}, \tag{3.4}$$

where $\overline{C}$ is the closure of the fluid region $C = \{(x,y) \in \mathbb{R}^2 : 0 < x < L/2, \ y < \eta(L/2)\}$. Note that $\theta(0, y) = \theta(L/2, y) = 0$ for $y \leqslant \eta(L/2)$, while (3.3) ensures that $\theta(x, \eta(L/2)) < 0$ for $x \in (0, L/2)$. Moreover, (2.5) shows that $\theta(x,y) \to 0$ as $y \to -\infty$ uniformly in $x \in [0, L/2]$. On the other hand, we have

$$\Delta\theta + \gamma'(\psi)v = 0 \quad \text{for} \quad (x,y) \in C,$$

if we take into account (3.1). Thus

$$\Delta\theta + \gamma'(\psi)\theta = -a\gamma'(\psi) \sin\left(\frac{2\pi x}{L}\right) e^{2\pi y/L} \geqslant 0 \quad \text{for} \quad (x,y) \in C.$$

Therefore, by the Phragmen–Lindelöf principle [18] we deduce that

$$\theta(x,y) \leqslant 0 \quad \text{for} \quad (x,y) \in C.$$

Taking into account (3.4), we obtain that

$$0 < v(x,y) \leqslant a \sin\left(\frac{2\pi x}{L}\right) e^{2\pi y/L} \quad \text{for} \quad (x,y) \in \overline{C}, \tag{3.5}$$

if we recall (3.2).

Because $\omega_y = \gamma'(\psi)\psi_y$ and $\psi_y = u - c < 0$ throughout the fluid, with $\lim_{y \to -\infty} \psi_y = -c$ uniformly for $x \in \mathbb{R}$, the boundedness of ω_y ensures that

$$\sup_{\psi \geqslant 0} |\gamma'(\psi)| < \infty. \tag{3.6}$$

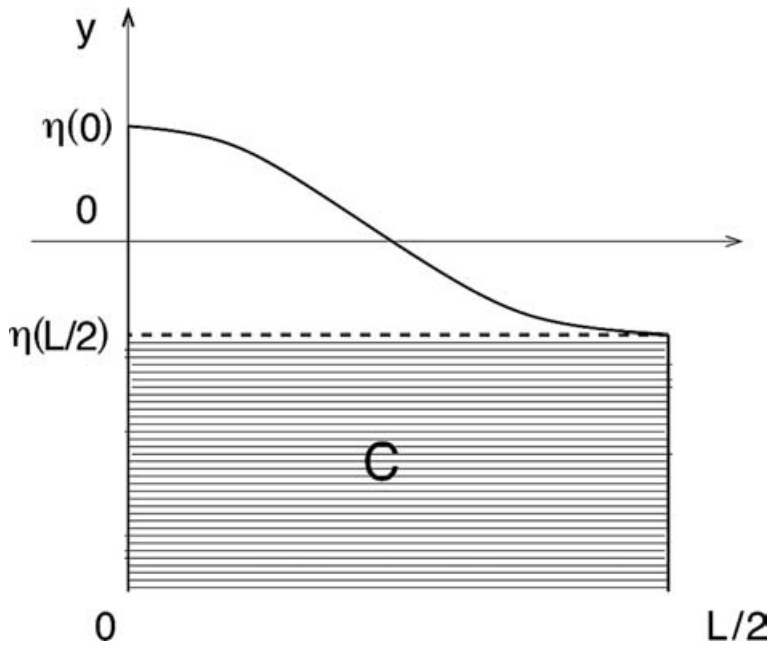Viewing now (3.1) as the Poisson equation $\Delta v = f(x,y)$ in C with right-hand side

FIGURE 2.

$f = -\gamma'(\psi)\, v$, classical gradient estimates [11, p. 37] yield

$$|(\nabla v)(L/4, y)| \leqslant \frac{L}{8} \sup_{(x,y)\in C} |f(x,y)| + \frac{8}{L} \sup_{(x,y)\in C} |v(x,y)|$$

$$\leqslant \left(\frac{L}{8} \sup_{\psi \geqslant 0} |\gamma'(\psi)| + \frac{8}{L} \right) \sup_{(x,y)\in C} |v(x,y)| \quad \text{for} \quad y < \eta(L/2) - L/4.$$

Combining the above estimate with (3.5)–(3.6), we deduce that

$$|\psi_{xx}(L/4, y)| = |v_x(L/4, y)| \leqslant K\, e^{2\pi y/L} \quad \text{for} \quad y \leqslant \eta(L/4) - L/4, \tag{3.7}$$

The statement of the Proposition follows at once if we prove that $\lim_{\psi\to\infty} \gamma(\psi) = 0$ since $\gamma'(\psi) \leqslant 0$ throughout the fluid. If this last assertion does not hold true, the monotonicity of the function γ forces $\lim_{\psi\to\infty} \gamma(\psi) = \alpha \neq 0$. But then (2.7) yields

$$|\psi_{yy}(L/4, y)| = |\gamma(\psi(L/4, y)) + \psi_{xx}(L/4, y)| \to |\alpha| \quad \text{as} \quad y \to -\infty,$$

if we take into account (3.7). By the mean-value theorem we would obtain that

$$\liminf_{n\to\infty} |\psi_y(L/4, -n-1) - \psi_y(L/4, -n)| \geqslant \frac{|\alpha|}{2} > 0.$$

However, since $\psi_y = u - c$, the previous relation contradicts (2.5). The proof is complete.

$\square$

Remark A simple consequence of the Proposition is that flows of constant non-zero vorticity do not describe deep-water waves. We refer elsewhere [19, 23] for numerical calculations of regular wave trains in water of constant vorticity and infinite depth.

4 Main result

In this section we will prove the following main result of the paper.

Theorem *A steady periodic deep-water wave with a monotone profile between crests and troughs, propagating against a current with a vorticity that is non-decreasing with depth and has bounded first-order partial derivatives, must be symmetric.*

The proof of the Theorem is based on the moving plane method and uses sharp maximum principles for elliptic partial differential equations, which we present now as a lemma.

Lemma *Let Ω be the open domain in the (x, y)-plane lying betweelow the graph $y = f(x)$ of a continuous function $f : [a, b] \to \mathbb{R}$. That is, $\Omega = \{(x, y) \in \mathbb{R}^2 : a < x < b, -\infty < y < f(x)\}$. For functions $b_1, b_2, c \in C(\overline{\Omega}, \mathbb{R})$ such that $c(x, y) \leqslant 0$ throughout $\overline{\Omega}$, define the elliptic operator*

$$\mathscr{L} = \partial_x^2 + \partial_y^2 + b_1(x, y)\, \partial_x + b_2(x, y)\, \partial_y + c(x, y).$$

(i) *If $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is such that $\mathscr{L}w \leqslant 0$ in Ω, $w \geqslant 0$ on the boundary $\partial\Omega$ of Ω, and $\lim_{y \to -\infty} w(x, y) = 0$ uniformly for $x \in [a, b]$, then $w > 0$ in Ω unless $w \equiv 0$ throughout $\overline{\Omega}$.*
(ii) *Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that $w \geqslant 0$ in $\overline{\Omega}$, $\mathscr{L}w \leqslant 0$ in Ω, and $w = 0$ at some point $Q \in \partial\Omega$. If Ω satisfies an interior sphere condition[3] at Q, then the outer normal derivative $\frac{\partial w}{\partial \nu}$ of w at Q, if it exists, satisfies the strict inequality $\frac{\partial w}{\partial \nu} < 0$, unless $w \equiv 0$ on $\overline{\Omega}$.*
(iii) *Assume that f is twice continuously differentiable and let T be the line containing the normal to $y = f(x)$ at some point $Q \in \partial\Omega$. Let Ω_0 then denote the portion of Ω lying on some particular side of T. Suppose that $w \in C^2(\overline{\Omega_0})$ satisfies $\mathscr{L}w \leqslant 0$ in Ω_0, while also $w \geqslant 0$ in Ω_0 and $w = 0$ at Q. Then either $\frac{\partial w}{\partial \mu} > 0$ or $\frac{\partial^2 w}{\partial \mu^2} > 0$ at Q unless $w \equiv 0$ on $\overline{\Omega_0}$, where μ is any direction at Q which enters Ω nontangentially.*

Assertion (i) follows from the Phragmen–Lindelöf principle and the Weak Maximum Principle [18]. Assertion (ii) is the Hopf Maximum Principle, whereas (iii) is a version of the Edge Point Lemma proved in Fraenkel [8].

Proof of the Theorem For simplicity we choose the crest of the wave at $x = 0$.
For $x_* \in (-L/2, 0]$ we define

$$D_* = \{(x, y) \in \mathbb{R}^2 : -\infty < y < \eta(x) \text{ for } -L/2 < x < x_*\}.$$

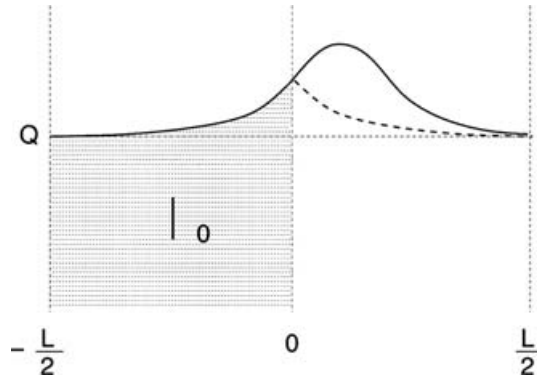[3] That is, there exists a small open ball contained in Ω with Q on its boundary.

FIGURE 3.

The map $(x, y) \mapsto (2x_* - x, y)$ reflects the domain D_* in the line $x = x_*$ into a domain D_*^R. Since $x = -L/2$ is the location of the wave trough, the monotonicity property of the free surface ensures the existence of some $\varepsilon > 0$ small enough such that the function $x \mapsto \eta(x)$ is nondecreasing on $(-L/2, -L/2 + \varepsilon)$. Therefore D_*^R is a subset of the fluid domain

$$D = \{(x, y) \in \mathbb{R}^2 : -\infty < y < \eta(x)\}$$

for all $x_* \in (-L/2, -L/2 + \varepsilon)$. As we increase x_* from $-L/2$ there is some maximal $x_0 \in (-L/2, 0]$ such that D_*^R is included in D for all $x_* \in (0, x_0)$. Note that D_0^R, corresponding to $x_* = x_0$, is still a subset of D. At $x = x_0$ one of the following three situations occurs:

(a) $x_0 = 0$;
(b) $x_0 < 0$ and the vertical line $x = x_0$ is normal to the free surface $y = \eta(x)$ at the crest point $(x_0, \eta(x_0))$;
(c) $x_0 < 0$ and D_0^R is internally tangent to the boundary $y = \eta(x)$ at some point.

Let us first assume that (a) occurs, like in Figure 3. Let $Q = (-L/2, \eta(-L/2))$ and define

$$w(x, y) = \psi(-x, y) - \psi(x, y), \qquad -L/2 \leqslant x \leqslant 0, \ -\infty < y \leqslant \eta(x),$$

where ψ is the stream function introduced in § 2. To obtain the statement of the theorem if (a) occurs, it suffices to show that $w \equiv 0$ in

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 : -L/2 < x < 0, \ -\infty < y < \eta(x)\}.$$

Indeed, then $\psi(-x, \eta(x)) = \psi(x, \eta(x))$ for all $x \in [-L/2, 0]$. Since the free surface $y = \eta(x)$ is given implicitly by $\psi = 0$, we infer that $\psi(-x, \eta(x)) = \psi(-x, \eta(-x)) = 0$ for all $x \in [-L/2, 0]$. The injectivity of the function $y \mapsto \psi(x, y)$ for every fixed x, ensured by $\psi_y = u - c < 0$, yields $\eta(x) = \eta(-x)$ for every $x \in [-L/2, 0]$. Therefore the wave is symmetric.

To prove that $w \equiv 0$ in $\overline{\Omega_0}$, we proceed as follows. Observe that $w \in C^2(\overline{\Omega_0})$. The periodicity property of ψ implies $w = 0$ on $x = {}^+_- L/2$. Moreover, by the mean-value

theorem we have

$$|w(x,y)| \leqslant 2|x| \sup_{-L/2 \leqslant x \leqslant L/2} |\psi_x(\xi,y)| \leqslant L \sup_{-L/2 \leqslant x \leqslant L/2} |v(\xi,y)|$$

so that (2.5) yields $w(x,y) \to 0$ as $y \to -\infty$, uniformly for $x \in [-L/2, L/2]$. Since $x_0 = 0$, we deduce that $(-x, \eta(x)) \in \overline{D}$ for all $x \in (-L/2, 0)$. Therefore $\psi(-x, \eta(x)) \geqslant 0$ for all $x \in (-L/2, 0)$, as $\psi \geqslant 0$ within the fluid. On the other hand, $\psi(x, \eta(x)) = 0$ for $x \in (-L/2, 0)$ in view of (2.10). Hence $w(x, \eta(x)) \geqslant 0$ for all $x \in (-L/2, 0)$. Thus $w \geqslant 0$ on the boundary $\partial \Omega_0$ of Ω_0. Since $\Delta \psi = -\gamma(\psi)$ throughout the fluid, we obtain that

$$\Delta w + \tilde{\gamma} = 0, \qquad -L/2 \leqslant x \leqslant 0, \ y \leqslant \eta(x),$$

where $\tilde{\gamma}(x,y) = \gamma(\psi(-x,y)) - \gamma(\psi(x,y))$. The mean-value theorem ensures the existence of some $s_0(x,y) \in \mathbb{R}$ such that $\tilde{\gamma}(x,y) = \gamma'(s_0) [\psi(-x,y) - \psi(x,y)]$. It follows that

$$\Delta w + \gamma'(s_0) w = 0, \qquad -L/2 \leqslant x \leqslant 0, \ y \leqslant \eta(x).$$

Since $w \geqslant 0$ on $\partial \Omega_0$, by the Lemma, part (i), we deduce that either $w > 0$ in Ω_0 or $w \equiv 0$ on $\overline{\Omega_0}$. Noticing that $w = 0$ at Q, part (iii) of the Lemma (with $T = \{x = -L/2\}$) yields $w \equiv 0$ in Ω_0 if at the point Q all partial derivatives of w of order less than or equal to two are equal to zero. We now show that this is the case. First of all, the way we defined the periodic function w guarantees that $w_y(Q) = w_{xx}(Q) = w_{yy}(Q) = 0$ since $w(Q) = 0$. Differentiating the relation $\psi(x, \eta(x)) = 0$, we obtain $\psi_x + \psi_y \eta' = 0$ on $y = \eta(x)$. But $\eta'(-L/2) = 0$ since Q is the wave trough, so that $\psi_x(Q) = 0$ and $w_x(Q) = -2\psi_x(Q) = 0$. It remains to show that $w_{xy}(Q) = 0$. Differentiating the nonlinear boundary condition on $y = \eta(x)$ from (2.10) with respect to x, we get

$$\psi_x(\psi_{xx} + \psi_{xy}\eta') + \psi_y(\psi_{xy} + \psi_{yy}\eta') + g\eta' = 0 \quad \text{on} \quad y = \eta(x).$$

Evaluating this at the wave trough Q, where $\eta' = \psi_x = 0$, we obtain $\psi_y(Q)\psi_{xy}(Q) = 0$. Since by assumption $\psi_y = u - c < 0$, we must have $\psi_{xy}(Q) = 0$. But $w_{xy}(Q) = -2\psi_{xy}(Q)$, and we conclude that $w_{xy}(Q) = 0$ since we already know that $w_x(Q) = 0$. Therefore the wave is symmetric if the case (a) occurs.

Let us now analyze alternative (b) (see Figure 4).

The defining property of $x_0 < 0$ ensures that the domain D_0^R, obtained by reflecting $D_0 = \{(x,y) \in \mathbb{R}^2 : -L/2 < x < x_0, \ y < \eta(x)\}$ in the line $x = x_0$ by means of the transformation $(x,y) \mapsto (2x_0 - x, y)$, is contained within the fluid domain D. Since $(x_0, \eta(x_0))$ is the wave crest, the wave profile $y = \eta(x)$ is decreasing on $[x_0, L/2]$. Therefore, letting $x_1 = 2x_0 + L/2$ and $x_2 = x_0 + L/2$, the reflection via the transformation $(x,y) \mapsto (2x_2 - x, y)$ of the domain

$$\{(x,y) \in \mathbb{R}^2 : x_2 < x < L/2, \ y < \eta(x)\}$$

in the line $x = x_2$, is also contained within D. Observe that this reflection maps the line $\{x = L/2\}$ into $\{x = x_1\}$. We now define

$$w(x,y) = \begin{cases} \psi(x,y) - \psi(2x_0 - x, y), & x_0 \leqslant x \leqslant x_1, \ y \leqslant \eta(2x_0 - x), \\ \psi(x,y) - \psi(2x_2 - x, y), & x_1 \leqslant x \leqslant x_2, \ y \leqslant \eta(2x_2 - x), \end{cases}$$
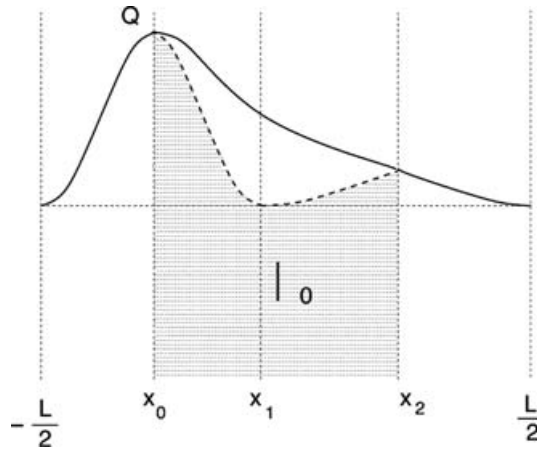
FIGURE 4.

and we claim that it suffices to show that $w \equiv 0$ on the closure of the domain

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 : x_0 < x < x_2, \ y < \tilde{\eta}(x)\}.$$

Here

$$\tilde{\eta}(x) = \begin{cases} \eta(2x_0 - x), & x_0 \leqslant x \leqslant x_1, \\ \eta(2x_2 - x), & x_1 \leqslant x \leqslant x_2. \end{cases}$$

Indeed, $w \equiv 0$ on $\overline{\Omega_0}$ implies that $\psi(2x_0 - x, \eta(2x_0 - x)) = \psi(x, \eta(2x_0 - x))$ for $x \in [x_0, x_1]$ and $\psi(2x_2 - x, \eta(2x_2 - x)) = \psi(x, \eta(2x_2 - x))$ for $x \in [x_1, x_2]$. Since $\psi_y = u - c < 0$ throughout $\overline{D}$ and the implicit equation of the free surface is $\psi(x, \eta(x)) = 0$, we deduce that $\eta(x) = \eta(2x_0 - x)$ for $x \in [-L/2, x_1]$ and $\eta(x) = \eta(2x_2 - x)$ for $x \in [x_1, L/2]$. That is, the wave profile $y = \eta(x)$ is symmetric with respect to $x = x_0$ on $[-L/2, x_1]$ and with respect to $x = x_2$ on $[x_1, L/2]$. But the profile is supposedly monotone between crest and trough, that is, on each of the intervals $[-L/2, x_0]$ and $[x_0, L/2]$. The obtained contradiction shows that the alternative (b) does not occur.

To verify that $w \equiv 0$ in $\overline{\Omega_0}$ we will apply part (iii) of the Lemma with $Q = (x_0, \eta(x_0))$ and $T = \{x = x_0\}$. First of all, note that $w \in C^2(\overline{\Omega_0})$ and the function $\tilde{\eta}$ is twice continuously differentiable on $[x_0, x_2]$ with $\tilde{\eta}'(x_1) = 0$. Similar to the case (a), we see that $w \geqslant 0$ on the top boundary of Ω_0, while $w = 0$ on the lateral boundaries of Ω_0 and $w(x, y) \to 0$ as $y \to -\infty$ uniformly for $x \in [x_0, x_2]$. Also, just like in the case (a), we see that

$$\Delta w + c(x, y) \, w = 0, \qquad (x, y) \in \Omega_0,$$

for some $c \in C(\overline{\Omega_0})$ with $c(x, y) \leqslant 0$ throughout $\overline{\Omega_0}$. Therefore, we may apply part (i) of the Lemma to infer that either $w > 0$ in Ω_0 or $w \equiv 0$ on $\overline{\Omega_0}$. Since $(x_0, \eta(x_0))$ is the crest of the wave, we have $\eta'(x_0) = 0$. An argumentation analogous to that pursued in the case of (a) confirms that at the point Q all partial derivatives of w of order less than or equal to two are equal to zero. But $w = 0$ at Q, so that by the Lemma, part (iii), we conclude that $w \equiv 0$ in $\overline{\Omega_0}$. As argues above, this leads to a contradiction.
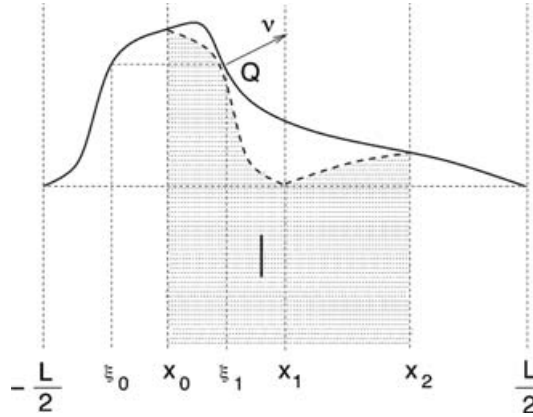
FIGURE 5.

It remains to investigate the last alternative (c), corresponding to Figure 5. Again, let $x_1 = 2x_0 + L/2$ and $x_2 = x_0 + L/2$. Since the contact point $Q = (\xi_1, \eta(\xi_1))$ of the upper boundaries of D_0^R and D has to be located on the decreasing part of the wave profile, the reflection of the domain

$$\{(x, y) \in \mathbb{R}^2 : x_2 < x < L/2, \ y < \eta(x)\}$$

in the line $x = x_2$, achieved through the transformation $(x, y) \mapsto (2x_2 - x, y)$, is contained in $\overline{D}$. This reflection maps the line $\{x = L/2\}$ into $\{x = x_1\}$.

Just like in the case of the alternative (b), it suffices to show that the function

$$w(x, y) = \begin{cases} \psi(x, y) - \psi(2x_0 - x, y), & x_0 \leqslant x \leqslant x_1, \ y \leqslant \eta(2x_0 - x), \\ \psi(x, y) - \psi(2x_2 - x, y), & x_1 \leqslant x \leqslant x_2, \ y \leqslant \eta(2x_2 - x), \end{cases}$$

is identically zero on the closure of the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x_0 < x < x_2, \ y < \tilde{\eta}(x)\},$$

where, as before,

$$\tilde{\eta}(x) = \begin{cases} \eta(2x_0 - x), & x_0 \leqslant x \leqslant x_1, \\ \eta(2x_2 - x), & x_1 \leqslant x \leqslant x_2. \end{cases}$$

Observe that $w \in C^2(\Omega)$ and $\tilde{\eta}$ is twice continuously differentiable on $[x_0, x_2]$.

Let us prove that $w \equiv 0$ on $\overline{\Omega}$. Since $\psi \geqslant 0$ below the free surface $y = \eta(x)$ and $\psi = 0$ on the free surface, we have that $w \geqslant 0$ on $y = \tilde{\eta}(x)$. The definition of w and the periodicity property of ψ ensure that $w = 0$ on $\{x = x_0\}$ and on $\{x = x_2\}$. Also, $w(x, y) \to 0$ as $y \to -\infty$ uniformly for $x \in [x_0, x_2]$ follows in view of (2.5) and the mean-value theorem, just like in the analysis made for the alternative (a). Similarly to the case (a), we have

$$\Delta w + c(x, y) w = 0, \quad (x, y) \in \Omega,$$

for some $c \in C(\overline{\Omega})$ with $c(x, y) \leqslant 0$ throughout $\overline{\Omega}$. Therefore, by part (i) of the Lemma, $w > 0$ in Ω unless $w \equiv 0$ on $\overline{\Omega}$. We now claim that $\frac{\partial w}{\partial v} = 0$ at Q, where v is the outer

normal to Ω at Q, implies $w \equiv 0$ on $\overline{\Omega}$. Indeed, the tangency property at Q ensures that Ω satisfies an interior sphere condition at Q. Moreover, note that $\eta(\xi_1) = \eta(2x_0 - \xi_1)$ yields $\psi(\xi_1, \eta(2x_0 - \xi_1)) = \psi(\xi_1, \eta(\xi_1)) = \psi(2x_0 - \xi_1, \eta(2x_0 - \xi_1)) = 0$ as $\psi = 0$ on the free surface. Therefore $w = 0$ at Q, and $\frac{\partial w}{\partial v}(Q) = 0$ implies $w \equiv 0$ on $\overline{\Omega}$ in view of part (ii) of the Lemma. To check that $\frac{\partial w}{\partial v}(Q) = 0$, let $\xi_0 = 2x_0 - \xi_1$. The tangency property at Q yields

$$\eta(\xi_0) = \eta(\xi_1) \quad \text{and} \quad \eta'(\xi_0) = -\eta'(\xi_1). \tag{4.1}$$

On the other hand, differentiating the relation $\psi(x, \eta(x)) = 0$ with respect to x, we obtain $\psi_x + \psi_y \eta' = 0$ on $y = \eta(x)$. Combining this with (4.1), we obtain that

$$\frac{\psi_x}{\psi_y}(\xi_0, \eta(\xi_0)) = -\frac{\psi_x}{\psi_y}(\xi_1, \eta(\xi_1)), \tag{4.2}$$

since $\psi_y = u - c < 0$ by assumption. Note also that (4.1) and the nonlinear boundary condition on $y = \eta(x)$ from (2.10) yield

$$|\nabla \psi|^2(\xi_0, \eta(\xi_0)) = |\nabla \psi|^2(\xi_1, \eta(\xi_1)). \tag{4.3}$$

Since $\psi_y = u - c < 0$ thoroughout $\overline{D}$, we deduce from (4.2)–(4.3) that

$$\psi_x(\xi_0, \eta(\xi_0)) = -\psi_x(\xi_1, \eta(\xi_1)) \quad \text{and} \quad \psi_y(\xi_0, \eta(\xi_0)) = \psi_y(\xi_1, \eta(\xi_1)).$$

This forces $\frac{\partial w}{\partial v} = 0$ at Q, if we take into account the definitions of ψ, α, ξ_1, and ξ_0, and note that $\eta(\xi_0) = \eta(\xi_1)$. The proof is complete. $\square$

Remark In view of the proposition, we see that the conclusion of the theorem is not only that the surface wave is symmetric, but also that the vorticity has to be non-negative and vanishing in the limit $y \to -\infty$.

5 Conclusion

We have shown that steady periodic deep-water waves which propagate against a current with a vorticity that is non-decreasing with depth and has a bounded gradient are symmetric if their profile is monotonic between crests and troughs. This symmetry property particularly holds true for irrotational waves. We also show that within the class of vorticity distributions described above the vorticity is non-negative and has to vanish at infinite depth. In particular, this means that, except of irrotational flows, there are no other deep-water waves of constant (non-zero) vorticity. Both results are based on sharp maximum principles, gradient estimates, and Phragmen-Lindelöf principles for elliptic boundary value problems.

Acknowledgement

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