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Research note

Symmetry of the wave equation and excitation of body waves

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Summary. The symmetry of the differential system for elastic waves, previously noted for plane geometry, is extended to any linear differential system and, in particular, the elastic-gravitational vibrations in a spherical earth. The result remains valid in a linearly viscoelastic medium. The symmetry allows the inverse of the propagator matrix to be obtained by simply 'transposing' the elements of the propagator. With this result, it is shown how the source excitation using a particular integral can be put in a more instructive form, comparable with the result for the excitation of normal modes.

1 Introduction

Symmetries of the wave equation and its solutions have been noted by several authors. Most fundamental of these is Betti's reciprocal theorem (Love 1944, pp. 173–174). In this research note we consider properties of the transformed wave equation in vertically inhomogeneous media.

Woodhouse (1974) first noted that symmetries of the differential system resulted in similar symmetries between the propagator and its inverse (we use the terminology introduced by Gilbert & Backus 1966). His results were for a plane, non-gravitating model and in this research note we extend these results to the spherical, gravitating system. In addition, we note that the symmetry of the propagator and its inverse allows a close analogy to be drawn between two different methods of calculating the source excitation. One technique includes the source in an inhomogeneous term in the differential system and finds the excitation from the particular integral (Gantmacher 1959, p. 131; Gilbert & Backus 1966; equation (21) below), and the other uses the complete set of orthogonal normal modes to represent the solution (Rayleigh 1877; Gilbert 1971; equation (22) below).

In the next section we give a general proof of the symmetry relationship previously proved only for the plane, non-gravitating system (Woodhouse 1974; Budden & Smith 1976; Kennett, Kerry & Woodhouse 1978). Earlier, Herrera (1964) and Alsop (1968) had investigated an orthogonality condition for the same case. Woodhouse (1980) has noted that the solution symmetry allows the sixth-order minor differential system to be reduced to a fifth-order system, a property first used by Watson (1970). The symmetry of the solution and its inverse in the spherical, non-gravitating system had previously been noted for homogeneous layers (Chapman 1969; Teng 1970) and extended to approximate solutions for inhomogeneous layers by Frazer (1977), but no exact, general proof has been given. In the final section of this note, we compare the two standard techniques for representing the seismic source. The particular integral (Gilbert & Backus 1966) and the normal mode representation (Gilbert 1971) are different. Using the solution symmetry, the equivalence can be illustrated. For plane stratified media the connection between the two methods of incorporating the source has recently been elaborated by Woodhouse (1980).

2 Symmetry of the wave equation

The transformed wave equations can be obtained using a variational principle. To obtain a linear, homogeneous system of equations the Lagrangian must be quadratic. Consider the most general quadratic Lagrangian:

$$L = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{X}\mathbf{q} + \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{Y}\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{W}\mathbf{q} \tag{1}$$

where $\mathbf{q}$ is an n-vector of 'coordinates', $\mathbf{X}$ and $\mathbf{Y}$ are, without loss of generality, symmetric matrices and $\mathbf{W}$ is an arbitrary matrix. The dot represents differentiation with respect to the vertical coordinate. Details for the plane system can be found in Kennett (1974) and Woodhouse (1974), and for the gravitating, spherical system in Pekeris & Jarosch (1958). From the Lagrangian (1), we obtain the 'conjugate momenta'

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \mathbf{Y}\dot{\mathbf{q}} + \mathbf{W}\mathbf{q} \tag{2}$$

and Lagrange's equations are

$$\dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{q}} = \mathbf{X}\mathbf{q} + \mathbf{W}^{\mathrm{T}}\dot{\mathbf{q}}. \tag{3}$$

These equations can be rewritten in the standard form (Gilbert & Backus 1966)

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} \tag{4}$$

where $\mathbf{y}$ is the $2n$-vector of 'coordinates' and 'momenta'

$$\mathbf{y} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}. \tag{5}$$

The $2n \times 2n$ coefficient matrix $\mathbf{A}$ has the structure

$$\mathbf{A} = \begin{pmatrix} -\mathbf{T} & \mathbf{C} \\ \mathbf{S} & \mathbf{T}^{\mathrm{T}} \end{pmatrix}, \tag{6}$$

where the $n \times n$ sub-matrices are

$$\begin{aligned} \mathbf{T} &= \mathbf{Y}^{-1}\mathbf{W} \\ \mathbf{C} &= \mathbf{Y}^{-1} \\ \mathbf{S} &= \mathbf{X} - \mathbf{W}^{\mathrm{T}}\mathbf{Y}^{-1}\mathbf{W}. \end{aligned} \tag{7}$$

As $\mathbf{C}$ and $\mathbf{S}$ are symmetric, it follows that

$$\mathbf{JA} + \mathbf{A}^{\mathrm{T}}\mathbf{J} = \mathbf{0} \tag{8}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \tag{9}$$

and $\mathbf{I}$ is the unit $n \times n$ matrix. Using $\mathbf{J}^{-1} = -\mathbf{J}$, and defining the 'transformation'

$$\tilde{\mathbf{A}} = -\mathbf{JA}^{\mathrm{T}}\mathbf{J} \tag{10}$$

we obtain the 'symmetry'

$$\mathbf{A} = -\tilde{\mathbf{A}}. \tag{11}$$

The 'symmetry' of $\mathbf{A}$ can be used to obtain 'orthogonality' conditions for the solutions of (4). If $\mathbf{y}$ and $\mathbf{y}'$ are any two solutions of (4), it follows that

$$(\mathbf{y}^{\mathrm{T}}\mathbf{J}\mathbf{y}')^{\cdot} = 0. \tag{12}$$

Applying (12) to the columns of the propagator matrix, $\mathbf{P}(\xi, \zeta)$, we obtain

$$\mathbf{P}^{\mathrm{T}}\mathbf{JP} = \mathbf{J} \tag{13}$$

where the constant of integration is obtained from $\mathbf{P}(\zeta, \zeta) = \mathbf{I}$. Expression (13) is equivalent to the 'orthonormality' condition

$$\tilde{\mathbf{P}}\mathbf{P} = \mathbf{I} \tag{14}$$

and we have the inverse propagator matrix

$$\mathbf{P}^{-1}(\xi, \zeta) = \mathbf{P}(\zeta, \xi) = \tilde{\mathbf{P}}(\xi, \zeta). \tag{15}$$

These symmetries, (11) and (14), are equivalent to Woodhouse's (1974) but are established here as general properties of linear differential systems, in particular the spherical, gravitating system. The explicit form of the matrix $\mathbf{A}$ for the plane case has already appeared in the literature (Woodhouse 1974) and we will not repeat it. For the spheroidal mode equations we follow the notation of Pekeris & Jarosch (1958) and use U, V and P to expand the displacement and gravitational potential. Many choices for the 'coordinates' $\mathbf{q}$ are possible, but the following is useful as it emphasizes the similarities with the plane differential system:

$$q_1 = rU, \qquad q_2 = \nu rV, \qquad q_3 = rP, \tag{16}$$

where $\nu = [n(n+1)]^{1/2}$ (the factor of r is introduced to remove the radial scaling of the solution, and ν is introduced as the spherical harmonics are not fully normalized). Using (2), we find the 'conjugate momenta' are

$$\begin{aligned} p_1 &= -(\lambda + 2\mu)r\dot{U} - \lambda(2U - \nu^2 V) \\ p_2 &= -\nu\mu(r\dot{V} - V + U) \\ p_3 &= -r[\dot{P} + (n+1)P/r - 4\pi G\rho U]/4\pi G. \end{aligned} \tag{17}$$

The non-zero elements of the matrices (7) are then

$$
\begin{aligned}
T_{11} &= \frac{\lambda - 2\mu}{\lambda + 2\mu}\,\frac{1}{r} \\
T_{12} &= \frac{\lambda}{\lambda + 2\mu}\,\frac{\nu}{r} \qquad\qquad T_{31} = 4\pi G\rho \\
T_{21} &= -\frac{\nu}{r} \qquad\qquad\qquad\quad T_{33} = -\frac{n}{r} \\
T_{22} &= \frac{2}{r}
\end{aligned}
\tag{18a}
$$

$$
\begin{aligned}
C_{11} &= -\frac{1}{\lambda + 2\mu} \\
C_{22} &= -\frac{1}{\mu} \\
C_{33} &= -4\pi G
\end{aligned}
\tag{18b}
$$

$$
\begin{aligned}
S_{11} &= \rho\omega^2 + \frac{4\rho g}{r} - \frac{2\gamma}{r^2} \\
S_{12} &= S_{21} = \nu\left(\frac{\rho g}{r} - \frac{\gamma}{r^2}\right) \\
S_{13} &= S_{31} = -\frac{\rho(n+1)}{r} \\
S_{23} &= S_{32} = \frac{\rho\nu}{r} \\
S_{22} &= \rho\omega^2 + \frac{2\mu}{r^2} - \frac{2\gamma\nu^2}{r^2}
\end{aligned}
\tag{18c}
$$

where $\gamma = 2\mu(3\lambda + 2\mu)/(\lambda + 2\mu)$. We omit details of the toroidal mode equations as they are very simple.

Because the variational principle (equations (1) to (3)) is real, the above results remain valid in anelastic media, i.e. if the elastic moduli, λ and μ, are complex, and if the frequency or wavenumber is complex. More usually, the variational principle and 'orthogonality' condition (12) include complex conjugation (Herrera 1964; Alsop 1968; Kennett 1974). In a perfectly elastic medium the matrix **A** is real, and result (12) can be replaced by

$$
(\mathbf{y}^{\dagger}\mathbf{J}\mathbf{y}')^{\cdot} = 0 \tag{19}
$$

(where † denotes the Hermitian conjugate). However, this form of the 'orthogonality' condition is only valid in perfectly elastic media (and is equivalent to the conservation of energy), while (12) remains valid in lossy media (and also if the frequency or wavenumber is complex).

3 Excitation of body waves

Two different techniques have been used to evaluate the source excitation. If the source terms are included in the equations of motion, an inhomogeneous term is included in the differential system (4), i.e.

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{w}. \tag{20}$$

The source vector, $\mathbf{w}$, is derived by transforming the body forces, $\mathbf{f}$, and the stress glut, $\mathbf{T}$. Details have been given by Kennett & Kerry (1979) and Ward (1980) for a point force and moment tensor source in plane and spherical geometries, respectively. The standard technique for solving the inhomogeneous equation (20) is to use the solutions of the homogeneous equation (4) and a particular integral (Gantmacher 1959, p. 131; Gilbert & Backus 1966). Thus

$$\mathbf{y}(\zeta) = \mathbf{P}(\zeta, \zeta_n)\mathbf{y}(\zeta_n) + \int_{\zeta_n}^{\zeta} \mathbf{P}(\zeta, \xi)\mathbf{w}(\xi)d\xi. \tag{21}$$

An alternative technique is to represent the solution in terms of the normal modes. Then the solution is (Gilbert 1971)

$$\mathbf{u}(\zeta) = \sum_n \mathbf{u}_n(\zeta) \frac{\int [\mathbf{u}_n^*(\xi)\cdot\mathbf{F}(\xi) + \boldsymbol{\epsilon}_n^*(\xi):\mathbf{T}(\xi)]d\xi}{(\omega_n^2 - \omega^2)}. \tag{22}$$

The orthonormalized normal mode, $\mathbf{u}_n(\zeta)$, has frequency ω_n, where n is an index used to enumerate the modes. The strain tensor, $\boldsymbol{\epsilon}_n$, is derived from the displacement, $\mathbf{u}_n$, in the usual fashion. Following Gilbert (1971), we shall assume that the modes are for a perfectly elastic body, so, without loss in generality, the conjugation in the orthonormality condition and equation (22) can be ignored provided the eigenfunctions are chosen to be real (i.e. real spherical harmonics are used). As the source vector, $\mathbf{w}$, in (21) depends on the body forces, $\mathbf{F}$, and the stress glut, $\mathbf{T}$, there is a superficial similarity between results (21) and (22). However, whereas in (22) each source component multiplies a different component of the mode, in (21) the different source components multiply different columns of the propagator which are the same components of different solutions, not different components of a solution (also, of course, the columns of $\mathbf{P}(\zeta, \xi)$ are solutions at ζ, not ξ). Using the reciprocal relation (15), result (21) can be rewritten

$$\mathbf{y}(\zeta) = \mathbf{P}(\zeta, \zeta_n)\mathbf{y}(\zeta_n) + \int_{\zeta_n}^{\zeta} \tilde{\mathbf{P}}(\xi, \zeta)\mathbf{w}(\xi)d\xi. \tag{23}$$

But $\tilde{\mathbf{P}}(\xi, \zeta)$ is obtained from $\mathbf{P}(\xi, \zeta)$ using the transformation rule (10). If we expand $\mathbf{P}(\xi, \zeta)$ into its sub-matrices

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{24}$$

we obtain

$$\tilde{\mathbf{P}} = \begin{pmatrix} \mathbf{P}_2^{\mathrm{T}} & -\mathbf{Q}_2^{\mathrm{T}} \\ -\mathbf{P}_1^{\mathrm{T}} & \mathbf{Q}_1^{\mathrm{T}} \end{pmatrix}. \tag{25}$$

It is now readily apparent that the columns of $\tilde{\mathbf{P}}$, which multiply each component of the source vector, $\mathbf{w}(\xi)$, in (23) correspond to components of the solution at ξ. By straightforward but very tedious algebra, it can be shown that each source component in expressions

(22) and (23) is completely analogous. For instance, the radial body-force term is contained in the fourth component of $\mathbf{w}(\xi)$. For a point source it contains the term $f_r\delta(\xi - r_0)/\xi$. In expression (23) this multiplies the fourth column of $\widetilde{\mathbf{P}}$ which corresponds to the first row of $\mathbf{P}$, i.e. $\xi U(\xi)$ from (16). Combining these terms we immediately obtain a term equivalent to the first element in the scalar product in (22). In a forthcoming paper, Ward (1981) has elaborated on the connection between expressions (22) and (23).

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