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SYMMETRY PROPERTIES OF DUAL TREE GRAPH N POINT AMPLITUDES

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A B S T R A C T

The dual tree graph N point functions are investigated from the point of view of symmetry properties. Using a cyclic symmetric contour integral for B_N, we deduce a set of simple relations between N-1 B_N functions and give rules for writing down more complicated relations. Secondly, we establish a new relationship between the cyclic symmetric integral and the conventional "multiperipheral" integral for B_N. Finally, both approaches are used to investigate the signature properties of B_N.

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1. INTRODUCTION

In this paper we investigate certain mathematical properties of the dual tree graph N point functions which are related to their symmetry properties. These B_N functions have been given by many authors [1], and several different representations are known.

For an investigation of the symmetry properties the cyclic symmetric integral representation of Koba and Nielsen [2),3)] is particularly suitable, and we work with this representation throughout. We write it as a many-dimensional contour integral, and by integrating over closed loops and using the symmetry properties of the integral, we establish a large set of linear relations for functions which are very closely related to B_N functions for different permutations of the external particles.

To indicate the character of these relations, let us briefly consider the (trivial) case $N = 4$. We extend the integration contour for the standard integral for the β function to a semicircle in the upper half plane. Using the Cauchy theorem and the changes of variable proposed by Fairlie and Jones [4)], this gives the relation

$$B(-\alpha_s,-\alpha_t) - e^{-i\pi\alpha_s} B(-\alpha_s,-\alpha_u+\delta) - e^{i\pi\alpha_t} B(-\alpha_t,-\alpha_u+\delta) = 0 , \tag{1.1}$$

where $B(-\alpha_s,-\alpha_t)$ is the four-point function, and

$$\alpha_s + \alpha_t + \alpha_u = -1-\delta . \tag{1.2}$$

Closing instead the contour in the lower half plane we get the same relation, but with opposite signs in the exponents. One of the functions, say, $B(-\alpha_t,-\alpha_u+\delta)$ may be eliminated, giving

$$\sin\pi(\alpha_s+\alpha_t) B(-\alpha_s,-\alpha_u+\delta) + \sin\pi\alpha_t B(-\alpha_s,-\alpha_t) = 0 .$$

This is of course a trivial consequence of Eq. (1.2) and the well-known properties of the β function, and only the simplest example of the relations we are going to establish.

For $N \geq 5$, however, we get non-trivial relations between the N-1 B_N functions having N-1 particles in a fixed ordering. Putting $\alpha_s = \alpha_{12}$, $\alpha_t = \alpha_{23}$, $\alpha_u = \alpha_{14}$, Eq. (1.1) can be seen to be also of this type. In obvious notation it is

$$\beta(1234) - e^{-i\pi\alpha_{12}}\beta(2134) - e^{i\pi\alpha_{23}}\beta(1324) = 0 . \tag{1.3}$$

It will be seen that this is just a particular example of the general relation (3.3).

The derivation of the linear relations made it necessary to look more carefully into the analyticity properties of the cyclic symmetric integral. We do this by investigating the conditions on the analytic structure coming from the requirement of cyclic symmetry of B_N. This determines the singularities and the branch cuts at all levels of integration, and based on this we establish simple rules for writing down a linear relation whenever at least one integration contour is closed.

Making simple linear transformations of the integration variables, we also find a new relationship between the cyclic symmetric contour integral and the "multiperipheral" Chan-type integral [5]. The integration variables in the latter are usually given as cross-ratios of the variables in the former [2),3)]. We show that they also may be given as simple ratios instead of cross-ratios, when a certain limit is taken.

As an application we consider the signature properties of B_N, and give two simple, different proofs. One uses a particular linear relation, the other is a result of the new method of going from the cyclic symmetric to the "multiperipheral" integral.

In Section 2 we define variables and functions used in the following Sections. The linear relations are established in Section 3, and in Section 4 we discuss some of their more important properties. The alternative interpretation of the cyclic symmetric integral is considered in Section 5. Finally, in Section 6, the signature properties of B_N are investigated.

2. THE CYCLIC SYMMETRIC CONTOUR INTEGRAL FOR B_N

Our work is based upon the manifestly cyclic symmetric integral for B_N given by Koba and Nielsen [2),3)]. The external particles are taken to be scalar mesons and we assume for simplicity that all trajectories are linear. We choose one (arbitrary) permutation as the standard permutation and label the mesons in this permutation from 1 to N in clockwise direction. The amplitude corresponding to the permutation $k_1, k_2 \ldots, k_N$ is denoted $B(k_1, k_2, \ldots, k_N)$ or simply B_N if we need not specify the ordering of the particles. We consider B as a function of the $\frac{1}{2}N(N-3)$ linearly independent trajectories *) $\alpha_{12}, \alpha_{123}, \ldots, \alpha_{23}, \alpha_{234}, \ldots,$ etc., defined by

$$\alpha_{i,i+1,\ldots j} = \alpha^0_{i,i+1,\ldots j} + \alpha'(p_i + p_{i+1} + \cdots + p_j)^2, \qquad \begin{array}{l} 1 \leq i \leq N-2, \\ i+1 \leq j \leq N-1 \text{ for } i>1, \\ 2 \leq j \leq N-2 \text{ for } i=1. \end{array} \tag{2.1}$$

All trajectories are assumed to have a common slope α'. We further define the quantities

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*) As is customary in the discussions of B_N, the non-linear constraints are only implicitly taken into account.

$$x_{ij} = \begin{cases} -\alpha_{ij}\,, & j = i+1\,, \\ -\alpha_{ikj} + \alpha_{ik} + \alpha_{kj}\,, & j = k+1 = i+2\,, \\ -\alpha_{ik_1\cdots k_r j} + \alpha_{ik_1\cdots k_r} + \alpha_{k_1 k_2\cdots k_r j} - \\ \qquad - \alpha_{k_1\cdots k_r}\,, & j = k_r+1 = \cdots = k_1 + r = i+r+1\,, \end{cases}$$

(2.2)

$$x_{i,i+1,\cdots j} = -\alpha_{i,i+1,\cdots j}\,. \tag{2.3}$$

The quantity x_{ij} corresponds, with a slight modification, to the quantity $g(A,B)$ defined by Koba and Nielsen [2].

From the definitions (2.2), (2.3) it follows a number of useful relations among the x variables. For later use we give the more important ones below. Others can be found by relabelling of the particles:

$$x_{12\cdots j} = x_{12} + x_{13} + \cdots + x_{1j} + x_{23\cdots j} - j + 2\,, \quad 2 \leq j \leq N-2\,. \tag{2.4}$$

For $j = N-2$ this implies

$$x_{12} + x_{13} + \cdots + x_{1N} = N-3\,, \tag{2.5}$$

or

$$x_{12} - 1 + x_{13} - 1 + \cdots + x_{1N} - 1 = -2\,. \tag{2.5'}$$

Repeated use of Eq. (2.4) implies

$$x_{12\cdots j} = \sum_{1\leq k<l\leq j} x_{kl} - \tfrac{1}{2}(j-1)(j-2) , \tag{2.6}$$

or

$$x_{12\cdots j} - 1 = \sum_{1\leq k<l\leq j} (x_{kl}-1) + j-2 , \tag{2.6'}$$

both valid for $2 \leq j \leq N-2$. Using Eq. (2.5) N times, we finally get

$$\sum_{1\leq k<l\leq N} x_{kl} = \tfrac{1}{2}N(N-3) , \tag{2.7}$$

or

$$\sum_{1\leq k<l\leq N} (x_{kl}-1) = -N . \tag{2.7'}$$

It is convenient to be able to extend the validity of Eqs. (2.6), (2.6') to $j = N-1$ and $j = N$. This is possible if we define

$$\left.\begin{aligned} x_{12\cdots N} &= -1, \\ x_{12\cdots N-1} &= 0. \end{aligned}\right\} \tag{2.8}$$

Eq. (2.6) may be considered as the solution of Eqs. (2.2), giving α variables in terms of x variables. We finally define an x variable for an arbitrary channel of any adjacent or non-adjacent mesons $j_1 j_2 \cdots j_r$ by

$$x_{j_1j_2\cdots j_r} = \sum_{1\leq k<\ell\leq r} x_{j_kj_\ell} - \tfrac{1}{2}(r-1)(r-2) . \tag{2.9}$$

It is easily checked that the x variables defined by Eq. (2.9) satisfy Eqs. (2.2) for arbitrary values of r, k_1, $k_2,\ldots,k_r$. For linear trajectories

$$x_{j_1j_2\cdots j_r} = -\alpha_{j_1j_2\cdots j_r} + c_{j_1j_2\cdots j_r} , \tag{2.10}$$

where $cj_1j_2,\ldots,j_r$ is a function of the masses of the particles and the intercepts of the trajectories involved.

Later we will need similar relations for the variables

$$E_{j_1j_2\cdots j_r} = e^{i\pi(x_{j_1j_2\cdots j_r} - 1)} . \tag{2.11}$$

They of course follow immediately from the x relations above:

$$\prod_{k\neq r} E_{kr} = 1 , \quad 1\leq r\leq N , \tag{2.12}$$

$$E_{j_1j_2\cdots j_r} = (-1)^{r-2} \prod_{1\leq k<\ell\leq r} E_{j_kj_\ell} , \tag{2.13}$$

$$\prod_{1\leq i<j\leq N} E_{ij} = (-1)^N . \tag{2.14}$$

With these definitions and relations, we are ready to write down the cyclic symmetric contour integral for B_N. It is

$$B(12\cdots N) = \int\cdots\int dG_3^{-1} \prod_{k=1}^{N} dz_k \prod_{1\leq i<j\leq N} (z_j - z_i)^{x_{ij}-1} , \tag{2.15}$$

where $z_j = \exp(i\theta_j)$ are N ordered points on the unit circle in the complex z plane such that

$$\theta_1 < \theta_2 < \cdots < \theta_N < \theta_1 + 2\pi . \tag{2.16}$$

The ordering of the points z_k is the same as the ordering of the particles, and z_k may be thought of as representing particle k. The differential dG_3 is defined by

$$dG_3 = \frac{dz_r\, dz_s\, dz_t}{(z_s - z_r)(z_t - z_s)(z_t - z_r)} , \tag{2.17}$$

where z_r, z_s, z_t are three arbitrary, but fixed points such that

$$\theta_r < \theta_s < \theta_t < \theta_r + 2\pi .$$

The integration in the remaining $N-3$ variables is over the parts of the circle consistent with the constraint (2.16). The "division" of differentials in Eq. (2.15) has been given a rigorous mathematical meaning by Koba and Nielsen [2),3)]. We emphasize that Eq. (2.15) is independent of the choice of r, s, and t and the values assigned to z_r, z_s, and z_t.

Expressions very similar to Eq. (2.15) have already been given [3),6)], but unfortunately the existing literature is somewhat confusing on the relationship between the integral (2.15) and, e.g., the conventional "multiperipheral" integral. We will therefore demonstrate this relationship in some detail.

This is done by introducing the angles θ_j as new variables. We insert

$$z_j - z_i = 2i\, e^{\frac{i}{2}(\theta_i+\theta_j)} \sin\tfrac{1}{2}(\theta_j-\theta_i)$$
$$= 2i\, e^{\frac{i}{2}(\theta_i+\theta_j)}\, \sigma_{ji}$$

into Eq. (2.15). Using Eqs. (2.5'), (2.7')

$$\prod_{1\le i<j\le N} (z_j - z_i)^{x_{ij}-1} = (2i)^{-N} \prod_{k=1}^{N} e^{-i\theta_k} \prod_{1\le i<j\le N} \sigma_{ji}^{\,x_{ij}-1} .$$

As $dz_k = iz_k d\theta_k$ and

$$dG_3 = \frac{1}{2^3}\, \frac{d\theta_r\, d\theta_s\, d\theta_t}{\sigma_{sr}\,\sigma_{ts}\,\sigma_{tr}} \equiv \frac{1}{2^3}\, dF_3 ,$$

we finally get the real integral

$$B(12\cdots N) = 2^{-N+3} \int\cdots\int dF_3^{-1} \prod_{k=1}^{N} d\theta_k \prod_{1\le i<j\le N} \sigma_{ji}^{\,x_{ij}-1} . \tag{2.18}$$

Introducing the "multiperipheral" variables u_k^j in the standard way [2], one can finally show that

$$B(12\cdots N) = \int dV^{(N-3)} \prod_{p} u_p^{\,x_p-1} ,$$

where the right-hand side is the integral over the product of "partitions" as defined by Chan and Tsou [7] with the invariant measure $dV^{(N-3)}$. Apart from a constant factor, Eq. (2.18) is equal to the formula given by Koba and Nielsen [8].

In writing down Eq. (2.15) we have chosen a particular ordering of the particles and also a particular one of the 2N permutations which, because of the cyclic symmetry and reflection symmetry of B_N, are equivalent to this. The other 2N-1 permutations are obtained by reversing the sign of some or all of the differences $z_j - z_i$. This introduces phase factors $E_{ij}^{\pm 1}$, and the branch cuts in the integrand must be chosen such that the phases cancel in this operation and really give the cyclic symmetry. The same applies of course to Eq. (2.18).

To simplify the notation in the following, for an arbitrary, but fixed ordering $k_1, k_2, \ldots, k_N$ of the particles such that

$$\theta_{k_1} < \theta_{k_2} < \cdots < \theta_{k_N} < \theta_{k_1} + 2\pi \ , \tag{2.19}$$

we define the function

$$\beta(k_1, k_2, \cdots, k_N) = 2^{-N+3} \int \cdots \int d|F_3|^{-1} \prod_{k=1}^{N} d\theta_k \prod_{1 \le i < j \le N} |\sigma_{ji}|^{x_{ij}-1} . \tag{2.20}$$

Here $d|F_3|$ is obtained by taking the modulus of the σ_{ji} in dF_3. The β_N functions are of course closely related to the B_N functions. Let us define

$$B(k_1, k_2, \cdots, k_N; -\alpha) \equiv B(k_1, k_2, \cdots, k_N) .$$

The relationship is then [for the linear trajectories (2.1)],

$$\beta(k_1, k_2, \cdots, k_N) = B(k_1, k_2, \cdots, k_N; -\alpha + c) , \tag{2.21}$$

where the constants c are defined by Eq. (2.10). In particular, since $\sigma_{ji} > 0$ for $\theta_i < \theta_j < \theta_i + 2\pi$,

$$\beta(12 \cdots N) = B(12 \cdots N) . \tag{2.22}$$

If and only if the trajectory functions α satisfy the "canonical constraints"

$$x_{j_1 j_2 \cdots j_r} = -\alpha_{j_1 j_2 \cdots j_r} , \tag{2.23}$$

for all the $\frac{1}{2}N(N-3)$ channels defining B_N for a particular permutation $k_1, k_2, \ldots, k_N$, one has

$$\beta(k_1, k_2, \cdots, k_N) = B(k_1, k_2, \cdots, k_N).$$

It is trivially true that the canonical constraints are sufficient. That they are also necessary follows for example by the requirement that the poles in β and B should coincide and have identical polynomial residues.

The attractive property of the β_N functions is their close relationship to the B_N functions, and that they are all defined by integrals differing only in the integration limits, such that the ordering of the particles is given by the ordering of the points z_i on the unit circle. Thus the integral (2.15) with the integration limits defined by the inequalities (2.19) is simply $\beta(k_1, k_2, \ldots, k_N)$ multiplied by a certain phase factor, as some of the differences $z_j - z_i$ will appear with the wrong sign. The determination of this phase factor is not trivial and will be considered in the following Section.

3. LINEAR RELATIONS FOR B_N

3.1 Simple linear relations

We are now prepared to derive the linear relations. To begin with we only consider the simplest type. More complicated relations are considered in Section 3.2.

Consider the first integration to be performed in the integral (2.15). Let it be the z_1 integration. The integral is

$$\int_{z_N}^{z_2} dz_1\,(z_2-z_1)^{x_{12}-1}(z_3-z_1)^{x_{13}-1}\cdots(z_N-z_1)^{x_{1N}-1}\,. \tag{3.1}$$

The integrand has branch points at $z_2, z_3, \ldots, z_N$. Assume that this integral corresponds to the permutation $B(123,\ldots,N)$. The z_1 integral for $B(23,\ldots,N,1)$ is then

$$\int_{z_N}^{z_2} dz_1\,(z_1-z_2)^{x_{12}-1}(z_1-z_3)^{x_{13}-1}\cdots(z_1-z_N)^{x_{1N}-1}\,,$$

with all the subsequent integrations unchanged. The only difference between the two integrals is factors $(-1)^{x_{1j}-1}$. If the branch cuts are chosen such that these factors have one and the same meaning for all points on the unit circle, it follows from Eq. (2.12) that the phase factors all cancel, and the two permutations are indeed equivalent. To obtain this the branch cuts must run along the tangent to the unit circle, all with the same orientation, e.g., as shown in Fig. 1. With this choice we have

$$(z_j-z_1)^{x_{1j}-1} = E_{1j}\,(z_1-z_j)^{x_{1j}-1}\,,\quad 2 \le j \le N\,, \tag{3.2}$$

which is valid also for all points inside the unit circle.

The integrand in (3.1) is then analytic inside the unit circle. We integrate z_1 around the whole circle and apply the Cauchy theorem. Consider the integral from z_k to z_{k+1}. The ordering of the points is then $2,3,\ldots,k,1,k+1,\ldots,N$. From the above discussion and the definition of the β_N functions, it gives $E_{12}E_{13}\cdots E_{1k}\,\beta(2,3,\ldots,k,1,k+1,\ldots,N)$. Thus, assuming the integrals converge, the relation is

$$\beta(12\cdots N) + E_{12}\beta(213\cdots N) + E_{12}E_{13}\beta(2314\cdots N) + \\ + \cdots + E_{12}E_{13}\cdots E_{1,N-1}\beta(23\cdots N-1,1,N) = 0, \tag{3.3}$$

and this is our main result in this Section. Taking the branch cuts in the opposite direction we get the "complex conjugate" relation

$$\beta(123\cdots N) + E_{12}^{-1}\beta(213\cdots N) + \cdots + E_{12}^{-1}E_{13}^{-1}\cdots E_{1N-1}^{-1}\beta(23\cdots N-1,1,N) = 0. \tag{3.3'}$$

By analytic continuation these relations are valid for all x. Later we will refer to Eq. (3.3) as

$$\sum{}^{+}(1;12\cdots N) = 0 \tag{3.4}$$

and Eq. (3.3') as

$$\sum{}^{-}(1;12\cdots N) = 0, \tag{3.5}$$

where it is indicated the initial permutation which has coefficient 1 and the particle that is "moved around".

Of course other relations of the same type may be obtained simply by relabelling the particles. Due to the relation (2.12) we get, loosely speaking, E_{kj} factors when particle k is moved to the right, and E_{kj}^{-1} factors when it is moved to the left, or vice versa.

3.2 Non-simple linear relations

It was shown in Section 3.1 how to choose the branch cuts in the first integration to ensure cyclic symmetry of the total integral. We will now use the same condition to determine the singularities and the branch cuts in the subsequent integrations. Knowing this we may close all these contours and obtain other, more complicated linear relations.

For this purpose it is convenient to write the integral (2.15) as

$$B_N = \int \cdots \int dG_3^{-1} \prod_{k=1}^{N} dz_k \prod_{k=1}^{N-1} (z_N - z_k)^{x_{kN}-1} \phi_k^{(N-1)}(z) \, ,$$

where

$$\phi_k^{(N-1)}(z) = \prod_{j=k+1}^{N-1} (z_j - z_k)^{x_{kj}-1} \quad .$$

It is implied by this that the integrations are performed in the order $z_1, z_2, \ldots, z_{N-3}$. Assume the branch cuts in the z_1 plane are chosen as before, Eq. (3.2). From the general discussion of singularities of functions defined by integrals [9], it follows that the only singularities of

$$\psi_2(z_2) = \int_{z_N}^{z_2} dz_1 (z_N - z_1)^{x_{1N}-1} \phi_1^{(N-1)}(z)$$

in the z_2 plane is an end-point singularity at $z_1 = z_2$. To exhibit this we make the change of variables $z_1 \to \mu_1$ defined by

$$z_1 = z_N + \mu_1 (z_2 - z_N) \, .$$

This implies

$$\left.\begin{aligned} z_N - z_1 &= \mu_1 (z_N - z_2)\,, \\ z_2 - z_1 &= (\mu_1 - 1)(z_N - z_2)\,. \end{aligned}\right\} \tag{3.6}$$

Using Eq. (2.4) we can write the z_2 integral as

$$- \int_{z_N}^{z_3} dz_2 (z_N - z_2)^{x_{N12}-1} \phi_2^{(N-1)}(z) \int_0^1 d\mu_1 \mu_1^{x_{N1}-1} (\mu_1 - 1)^{x_{12}-1} \prod_{k=3}^{N-1} \left[z_k - z_N - \mu_1 (z_2 - z_N) \right]^{x_{1k}-1}\,, \tag{3.7}$$

where the μ_1 integral is now an analytic function of z_2 (but not of the other z_j). As we have used, e.g.,

$$(z_N - z_1)^{x_{1N}-1} = \mu_1^{x_{1N}-1} (z_N - z_2)^{x_{1N}-1}\,,$$

the branch cuts in the z_2 plane are implicitly determined by this expression. Starting with Eq. (3.2), it follows that we must have

$$\left.\begin{aligned} (-\mu_1)^{x_{1N}-1} &= E_{1N}\, \mu_1^{x_{1N}-1}\,, \\ (z_N - z_2)^{x_{N12}-1} &= E_{N12} (z_2 - z_N)^{x_{N12}-1}\,, \end{aligned}\right\} \tag{3.8}$$

etc. If the branch cuts in $\phi_2^{(N-1)}(z)$ are chosen such that

$$(z_k - z_2)^{x_{2k}-1} = E_{2k} (z_2 - z_k)^{x_{2k}-1}\,, \quad 3 \le k \le N-1\,, \tag{3.9}$$

we can show that

$$B(12\cdots N) = B(34\cdots N12)\,.$$

We have already shown that $B(123\ldots N) = B(23\ldots N1)$, i.e., that the integral (3.7) is invariant under the reflections $z_j - z_1 \to \to z_1 - z_j$, $2 \leq j \leq N$. To show that $B(234\ldots N1) = B(34\ldots N12)$, make the reflections

$$z_j - z_2 \leftrightarrow z_2 - z_j \quad , \quad j = 1, 3, 4, \cdots N .$$

From Eq. (3.6) we see that the μ_1 integral is invariant except for the change

$$\mu_1^{x_{1N}-1} \to (-\mu_1)^{x_{1N}-1} ,$$

and using Eqs. (2.12), (2.13), (3.8), the change in the z_2 integral is

$$- E_{N12}^{-1} E_{23}^{-1} E_{24}^{-1} \cdots E_{2,N-1}^{-1} E_{1N} = 1 .$$

The minus sign comes from the change in $d\mu_1$. This proves our assertions and also shows that the determination of the branch cuts is consistent. From the point of view of obtaining linear relations it is of course Eq. (3.9), together with Eq. (3.2) which are the important results.

At this stage we are able to write down the relation obtained, e.g., by the integration limits

$$\oint dz_2 \int_{z_N}^{z_2} dz_1$$

with the remaining particles in fixed order. As an example, let us take $N = 5$. To write down the relation that follows, we proceed like this: the particles in fixed order are 345. Integrating z_2 around the closed loop gives the three orderings 2345, 3245, 3425. Finally, in each of the three cases, z_1 must be integrated from z_5 to z_2. Each final ordering gives the corresponding β_5 function multiplied by a phase factor which is determined by Eqs. (3.2), (3.9). We see the importance of the directions of the branch cuts being correlated. The relation obtained is

$$\beta(12345) + E_{23}\beta(13245) + E_{13}E_{23}\beta(31245) +$$
$$+ E_{23}E_{24}\beta(13425) + E_{13}E_{23}E_{24}\beta(31425) + E_{13}E_{14}E_{23}E_{24}\beta(34125) = 0.$$

In passing we note that as it can be written

$$\sum{}^{+}(2;12345) - E_{23}E_{24}E_{25}\sum{}^{+}(4;43125) = 0,$$

this relation is not independent of the simple ones.

Returning now to the general discussion, we can determine the branch cuts in the z_3 integral by introducing the new variable μ_2 defined by

$$z_2 = z_N + \mu_2(z_3 - z_N),$$

and proceed as above. We then get from Eqs. (3.2), (3.9) the conditions

$$(z_k - z_3)^{x_{3k}-1} = E_{3k}(z_3 - z_k)^{x_{3k}-1}, \quad 4 \leq k \leq N,$$

which can be shown to ensure $B(12345,\ldots,N) = B(45,\ldots,N123)$. Introducing consecutively μ_3, μ_4, etc., by

$$z_k = z_N + \mu_k(z_{k+1} - z_N), \quad 1 \leq k \leq N-3,$$

we finally end up with the set of conditions

$$(z_k - z_j)^{x_{jk}-1} = E_{jk}(z_j - z_k)^{x_{jk}-1}, \quad 1 \leq j < k \leq N, \tag{3.10}$$

which are necessary and sufficient to ensure the cyclic symmetry of B_N as given by Eq. (2.15).

This result may at first sight seem trivial, and we want to point out that it cannot be obtained by a simple inspection of the integral (2.15) and using Eq. (2.12).

The conditions (3.10) correlate the branch cuts in all the integrations, and it is no longer necessary, as in Section 3.1, to keep all particles except one in fixed order. We may now close the contour at any level of integration and use the Cauchy theorem. Given the limits of integration for all the N-3 variables integrated over, with at least one closed contour, the following rules determine the corresponding linear relation:

1) write down the particles that are kept in fixed order, say, $k_1, k_2, \ldots, k_j$, and define a positive direction of rotation;

2) consider the <u>last</u> of the remaining integrations, say, $\int dz_\ell$, and determine all permutations of $k_1, k_2, \ldots, k_j \ell$ consistent with its integration limits. Associate to each permutation ϕ a phase factor $\lambda_\ell(\phi)$ defined by

$$\lambda_\ell(\phi) = \epsilon \prod_i E_{\ell i}^{\epsilon} \tag{3.11}$$

where the product is over the set of particles with which ℓ has been permuted in ϕ, and $\epsilon = +1$ if the integration is in the positive direction and $\epsilon = -1$ if it is in the negative direction;

3) repeat this for the remaining integrations, starting with the last and ending with the first, always counting the permutations from the same, fixed permutation ϕ_0;

4) the linear relation is symbolically

$$\sum_\phi \prod_\ell \lambda_\ell(\phi) \beta(\phi) = 0, \tag{3.12}$$

where the sum is over all permutations $\emptyset$ consistent with the integration limits and the product is over all phase factors $\lambda_\ell(\emptyset)$ which have been associated to each final permutation by 2) and 3) above.

As an extreme case, consider the relation

$$\oint \cdots \oint dG_3^{-1} \prod_{k=1}^{N} dz_k \prod_{1\le i<j\le N} (z_j - z_i)^{x_{ij}-1} = 0 .$$

The left-hand side is the sum of all the $\frac{1}{2}(N-1)!$ different β_N functions, each multiplied by factors E_{ij} corresponding to the permutations in that particular function. On the other hand

$$S = \oint \cdots \oint d|G_3|^{-1} \prod_{k=1}^{N} d\theta_k \prod_{1\le i<j\le N} |z_j - z_i|^{x_{ij}-1}$$

is just the sum of all the β_N functions without coefficients. With the canonical constraints fulfilled, this is the result of Fairlie [6].

4. SOME PROPERTIES OF THE LINEAR RELATIONS

The most important consequence of the linear relations is of course that they provide information about β_N functions which are difficult to investigate by relating them to better known functions. In this way the relations may prove a useful tool for the investigation of, e.g., the multi-Regge limit of the full N point amplitude [*].

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*) See Ref. 10) for such a treatment of the case $N=5$.

Consider the reduction properties of the simple relations (3.3), (3.3') at non-positive integer values of the x variables. We introduce the notation

$$\left.\begin{aligned} r_J(12\cdots j|j+1\cdots N) &= \mathrm{Res}\left[\beta(12\cdots N)\right]_{x_{12\cdots j}=-J}, \\ R_J(12\cdots j|j+1\cdots N) &= \mathrm{Res}\left[B(12\cdots N)\right]_{x_{12\cdots j}=-J}. \end{aligned}\right\} \tag{4.1}$$

For $J=0$,

$$r_0(12\cdots j|j+1\cdots N) = \beta(12\cdots j\,j+1)\beta(j,j+1,\cdots N), \quad 3\leq j\leq N-2,$$
$$r_0(12\cdots N-2|N-1,N) = \beta(12\cdots N-1).$$

Only the two first terms in Eq. (3.3) have the 12-pole, so, for $x_{12}=-J$ it follows

$$r_J(12|3\cdots N) - (-1)^J r_J(21|3\cdots N) = 0,$$

which shows the signature properties of the β_N and B_N functions for two-particle trajectories. Next, consider $x_{12,\ldots,j}=0$, $3\leq j\leq N-2$. In this case $\Sigma^{\pm}(1;12,..,N)$ reduce to $\Sigma^{\pm}(1,12,\ldots,j+1)$, as the pole only occurs in the first j terms. For $x_{12,\ldots,j}=-J$, $J\neq 0$ one gets non-trivial relations between the residues of the β_N functions, but we have not investigated these.

One β function may be eliminated between Eqs. (3.3), (3.3'), giving a real relation between $N-2$ β_N functions. It seems likely that there are just $N-3$ linearly independent β_N functions (this is the case for $N=4$ and $N=5$), but we have not proved this. Even for $N=6$ the actual solution of the relations in terms of three β_6 functions seems unmanageable. However, there are more linear relations of the type (3.3) than different β_N functions. Thus they are not all independent and presumably also the non-simple relations can be derived from the simple ones.

Except for the trivial four-particle case, the only simple case is $N=5$. The B_5 function is simply related to a generalized hypergeometric ${}_3F_2(1)$ function [10),11)]. The ${}_3F_2$ functions obey a number of two-term and three-term relations which can be found in standard books on hypergeometric functions [12)]. A detailed investigation of the B_5 function from this point of view has recently been made [13)], and it has been shown that the 12 different hypergeometric series for given parameters just correspond to the 12 different β_5 functions. It can easily be seen from this that the present simple relations for β_5 are equivalent to the well-known three-term relations for ${}_3F_2$ functions and thus give more insight into the structure of the latter.

As an example consider the relation obtained by Białas and Pokorski [11)]. We can write it down immediately by eliminating $\beta(12453)$ from $\Sigma^+(3;12345)=0$ and $\Sigma^-(3;12345)=0$,

$$B(12345) = \frac{\sin\pi(x_{23}+x_{12}-x_{45})}{\sin\pi(x_{12}-x_{45})}\beta(13245) + \frac{\sin\pi(x_{34}+x_{45}-x_{12})}{\sin\pi(x_{45}-x_{12})}\beta(12435). \tag{4.2}$$

The interpretation of this formula should be clear from its diagrammatic representation in Fig. 2 and the discussion in Ref. 11). These authors also suggest to make Eq. (4.2) the starting point for a phenomenological analysis of five-particle reactions. However, it now reveals an ugly feature, the existence of a spurious pole at integer values of $x_{12}-x_{45}$ which vanishes only by a non-trivial cancellation of the two right-hand side terms at the pole. This is likely to make rather difficult any approximations to B_5 based on the above formula.

On the other side, as B_5 can be calculated fairly easily there is may be not much point in doing this. For $N \gtrsim 6$, however, this is not so, and it may be that one may find a corresponding formula in these cases where the spurious poles may be avoided.

Finally, let us consider the presence of satellite terms. In the integral (2.15), satellites appear as the presence of a function $f(z;x)$ in the integrand. The function $f(z;x)$ must be invariant under linear fractional transformations (Möbius transformations) of the points z_i. This property of the integrand in Eq. (2.15) is essential for the interpretation of the integral as a B_N function [2]. Also, only then is it possible to express $f(z;x)$ as a function of the conventional $N-3$ independent integration variables u_k^j for B_N.

If $f(z;x)$ is analytic such that it can be expanded in powers of (z_j-z_i), the modified integral can be written as a satellite series on and inside the unit circle. If, finally, $f(z;x)$ has the necessary symmetry properties, the linear relations are also valid in the presence of satellites.

5. ALTERNATIVE INTERPRETATION OF THE CYCLIC SYMMETRIC INTEGRAL FOR B_N

The standard "multiperipheral" integral representation for B_N corresponding to the configuration in Fig. 3 may be obtained from Eq. (2.15) by expressing its integration variables u_k^j as cross-ratios of the variables z_i [2),3)]. In this Section, we will show that the same integral representation can also be obtained by expressing the independent u_k^j as simple ratios of the differences z_j-z_i and going to a suitable limit after the integration has been carried out.

We write Eq. (2.15) as

$$B(12\cdots N) = (z_N-z_{N-1})(z_N-z_1)(z_{N-1}-z_1)\int_{z_1}^{z_{N-1}} dz_{N-2}\cdots\int_{z_1}^{z_4} dz_3 \int_{z_1}^{z_3} dz_2 \prod_{k=2}^{N}(z_k-z_1)^{x_{1k}-1}\,\phi_k(z)\ , \tag{5.1}$$

where

$$\phi_k(z) = \prod_{j=k+1}^{N} (z_j - z_k)^{x_{kj}-1} \quad ,$$

(5.2)

and perform the successive substitutions $z_k \to \mu_k$ defined by

$$\mu_k = \frac{z_k - z_1}{z_{k+1} - z_1} \,, \quad k = 2, 3, \cdots, N-2 \,.$$

(5.3)

After $j-2$ substitutions, the z_j integral is

$$\int_{z_1}^{z_{j+1}} dz_j \, (z_j - z_1)^{x_{12\cdots j}-1} \phi_j(z) \int_0^1 \cdots \int_0^1 \prod_{i=2}^{j-1} d\mu_i \, \mu_i^{x_{12\cdots i}-1}$$

$$\prod_{2 \le i \le k \le j-1} (1-\eta_{ik})^{x_{i,k+1}-1} \prod_{i=2}^{j-1} \prod_{k=j+1}^{N} \left[z_k - z_1 - \eta_{i,j-1}(z_j - z_1) \right]^{x_{ik}-1} ,$$

(5.4)

where

$$\eta_{ik} = \mu_i \mu_{i+1} \cdots \mu_k \,.$$

Now assume all the integrations have been performed, i.e., put $j = N-1$ in Eq. (5.4). As $x_{12,\ldots,N-1} = 0$ [Eq. (2.8)] we get

$$B(12\cdots N) = \int_0^1 \cdots \int_0^1 \prod_{i=2}^{N-2} d\mu_i \, \mu_i^{x_{12\cdots i}-1} \prod_{2 \le i \le k \le N-2} (1-\eta_{ik})^{x_{i,k+1}-1}$$

$$\left\{ (z_N - z_{N-1})^{x_{N-1,N}} (z_N - z_1)^{x_{1N}} \prod_{i=2}^{N-2} \left[z_N - z_1 - \eta_{i,N-2}(z_{N-1} - z_1) \right]^{x_{i,N-1}-1} \right\},$$

(5.5)

which, except for the curly bracket, is the "multiperipheral" integral. However, going to the limit

$$\left.\begin{array}{l} z_{N-1} - z_1 \rightarrow 0 , \\ z_N - z_{N-1} \neq 0 , \\ z_N - z_1 \neq 0 , \end{array}\right\} \tag{5.6}$$

and using Eq. (2.4), the curly bracket is simply 1, giving the desired result.

The necessity of the limit (5.6) to obtain the standard integral form may be understood in the following way: the transformations (5.3) are mappings of the unit circle described by z_k onto circles described by μ_k. The centres of the latter are at the points $\frac{1}{2}[1+i\cot(\theta_{k+1}-\theta_1)]$. For $0 < \theta_{k+1}-\theta_1 < \pi$ the situation is as indicated in Fig. 4. The integration contour for μ_k is along the lower part of the circle. In the limit (5.6), the centres of all the circles tend to $\frac{1}{2}+i\infty$, and the μ_k integration contours tend to the real axis from 0 to 1, reproducing the standard "multiperipheral" integral.

Next we will briefly show how, by essentially the same method, one may obtain an integral representation for the configuration in Fig. 5. The point is to perform the first j-2 integrations in opposite order. To be specific, we write

$$B(12\cdots N) = (z_N - z_{N-1})(z_N - z_1)(z_{N-1}-z_1)\int_{z_1}^{z_{N-1}} dz_{N-2} \cdots \int_{z_1}^{z_{j+1}} dz_j \int_{z_1}^{z_j} dz_2 \int_{z_2}^{z_j} dz_3 \cdots \int_{z_{j-2}}^{z_j} dz_{j-1} \prod_{k=2}^{N} (z_k - z_1)^{x_{1k}-1} \phi_k(z) . \tag{5.7}$$

As in the previous case we introduce new variables μ_k, but now by Eqs. (5.3) for $k \geq j$ and by

$$\mu_k = \frac{z_j - z_k}{z_j - z_{k-1}} \quad , 2 \leq k \leq j-1 \tag{5.8}$$

for the rest. After the first $j-2$ transformations, the z_j integral is now

$$\int_{z_1}^{z_{j+1}} dz_j (z_j - z_1)^{x_{12\cdots j}-1} \phi_j(z) \int_0^1 \cdots \int_0^1 \prod_{i=2}^{j-1} d\mu_i \, \mu_i^{x_{i,i+1,\cdots j}-1}$$

$$\prod_{2 \leq i \leq k \leq j-1} (1-\eta_{ik})^{x_{i-1,k}-1} \prod_{i=2}^{j-1} \prod_{k=j+1}^{N} \left[z_k - z_j + \eta_{2i}(z_j - z_1) \right]^{x_{ik}-1} .$$

(5.9)

This expression, which will be used in the final Section to investigate the signature, should be compared to Eq. (5.4). Performing the remaining substitutions according to Eqs. (5.8) and taking the limit (5.6), we finally get

$$B(12\cdots N) = \int_0^1 \cdots \int_0^1 \prod_{i=2}^{N-2} d\mu_i \prod_{i=2}^{j-1} \mu_i^{x_{i,i+1,\cdots j}-1} \prod_{k=j}^{N-2} \mu_k^{x_{12\cdots k}-1}$$

$$\prod_{2 \leq i \leq k \leq j-1} (1-\eta_{ik})^{x_{i-1,k}-1} \prod_{j \leq m \leq n \leq N-2} (1-\eta_{mn})^{x_{m,n+1}-1} \left[1 - \eta_{jn}(1-\eta_{2k}) \right]^{x_{k,n+1}-1} ,$$

(5.10)

which is the integral representation corresponding to the configuration in Fig. 4. This expression has also been given by Hopkinson and Chan [14]. The limit (5.6) can be given a similar interpretation in this case.

6. SIGNATURE PROPERTIES OF B_N

We will investigate the signature properties of B_N in two different ways: first by using a suitable non-simple linear relation, secondly by the formalism developed in Section 5. The problem has already been discussed in the literature [14)-16)]. In particular, Koba and Nielsen have obtained the same results as in the present paper, but by using a different method.

To investigate the channel $12,\ldots,j$, we need a linear relation where only the terms $\beta(12,\ldots,N)$ and $\beta(j,j-1, 2,1,j+1,\ldots \ldots,N)$ have the $12,\ldots,j$ pole. Such a relation is the following

$$\int \cdots \int dG_3^{-1} \prod_{k=j+1}^{N} dz_k \oint dz_j \int_{z_j}^{z_{j+2}} dz_{j-1} \int_{z_{j-1}}^{z_{j+2}} dz_{j-2} \cdots \int_{z_2}^{z_{j+2}} dz_1 \prod_{1\leq k<l\leq N} (z_l - z_k)^{x_{kl}-1} = 0, \tag{6.1}$$

where it is assumed that the particles $j+1, j+2,\ldots,N$ are kept in this fixed order. Using the rules established in Section 3, we can immediately write down this relation in terms of β_N functions. There is only one trick, in the term which eventually gives $\beta(12,\ldots,N)$, we integrate $z_{j-2}, z_{j-3},\ldots,z_1$ in the negative (anticlockwise) direction. Using Eq. (2.13), the resulting relation is

$$\beta(12\cdots N) + E_{12\cdots j}\,\beta(j,j-1,\cdots 2,1,j+1,\cdots N) + \Sigma = 0,$$

where Σ is the sum of all the terms in Eq. (6.1) not having the $12,\ldots,j$ pole. This implies

$$r_J(j,j-1,\cdots 2,1|j+1,\cdots N) = (-1)^J r_J(1,2,\cdots j|j+1,\cdots N). \tag{6.2}$$

We emphasize that r_J is the residue of the β_N function. Equation (6.2) then implies the decoupling of all the odd daughter trajectories in the $12,\ldots,j$ channel for the symmetric combination

$\beta(12,\ldots,N)+\beta(j,j-1,\ldots,2,1,j-1,\ldots,N)$. This is true for the same combination of B functions if and only if

$$R_J(j,j-1,\cdots 2,1|j+1,\cdots N) = r_J(j,j-1,\cdots 2,1|j+1,\cdots N) .$$

This happens if and only if the canonical constraints (2.23) are satisfied for all the channels in this B function which are dual to the $12,\ldots,j$ channel. The non-dual variables are of course the same in the two functions, and as the residues are polynomials of order J in the dual variables

$$R_J(1,2,\cdots j|j+1,\cdots N) + (-1)^J R_J(j,j-1,\cdots 2,1|j+1,\cdots N) = \mathcal{O}(x_{dual}^{J-1}),$$

such that in the symmetric/antisymmetric combination the odd/even leading trajectory decouples. This is the result of Hopkinson and Chan [14]. In Fig. 6 the poles present in the different cases are indicated.

For the second proof, which also gives a method for actually calculating the residues, let us turn back to the expressions (5.4) and (5.9). Repeating the derivation of (5.9) for the "twisted" permutation $j,j-1,\ldots,2,1,j+1,\ldots,N$ gives instead the expression

$$\int_{z_j}^{z_{j+1}} dz_1 (z_1-z_j)^{x_{12\cdots j}-1} \phi_j'(z) \int_0^1 \cdots \int_0^1 \prod_{i=2}^{j-1} d\mu_i \, \mu_i^{x_{12\cdots i}-1}$$

$$\prod_{2\le i\le k\le j-1} (1-\eta_{ik})^{x_{i,k+1}-1} \prod_{i=2}^{j-1} \prod_{k=j+1}^{N} \left[z_k - z_j + \eta_{i,j-1}(z_j-z_1)\right]^{x_{ik}-1},$$

where

$$\phi_j'(z) = \prod_{k=j+1}^{N} (z_k-z_1)^{x_{1k}-1} .$$

Comparing this to (5.4), one sees that the μ dependent parts are exactly identical, and in the remaining part of the integral z_1 and z_j are interchanged. Hence, the only difference of the two residues at $x_{12,\ldots,j} = -J$ is the factor $(-1)^J$, which is the above result. This is so because in the residue, $z_1 = z_j$. Strangely enough, the result is not easily seen from the expression (5.10) written for the "twisted" permutation.

For the results in this Section it is obviously a necessary condition that the trajectories are linear and that all trajectories dual to the $12,\ldots,j$ channel have a common slope. This implies narrow resonances (poles) but with the interpretation of the B_N function as a Born term of the physical N point amplitude, this is not in disagreement with experimental evidence. It has been shown that with a linear trajectory as input in the B_N function, the output trajectory in the iterated amplitude is no longer linear and have resonances with non-zero width associated with it [17),18)]. Koba and Nielsen [16),19)] have considered in some detail the question of the existence of realistic solutions of the canonical constraints (2.10) for particular reactions.

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FIGURE CAPTIONS

Figure 1 :

Branch cuts in the integrand of Eq. (3.1).

Figure 2: :

Diagram representing the relation of Bialas and Pokorski. The coefficients α_1 and α_2 are defined by Eq. (4.2).

Figure 3 :

The multiperipheral configuration.

Figure 4 :

The contour in the μ_k plane.

Figure 5 :

The "twisted" multiperipheral configuration.

Figure 6 :

Poles in the 12,...,j channel for the symmetric combinations

○ $B(12,\ldots,N)+B(j,j-1,\ldots,2,1,j+1,\ldots,N)$

□ $\beta(12,\ldots,N)+\beta(j,j-1,\ldots,2,1,j+1,\ldots,N)$.

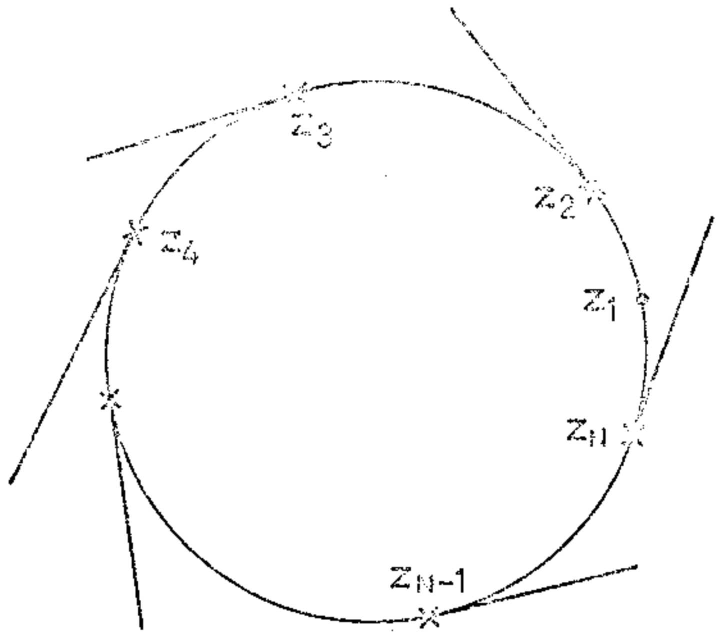

FIG. 1

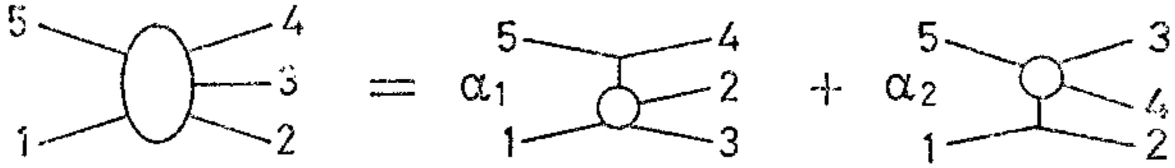

FIG. 2

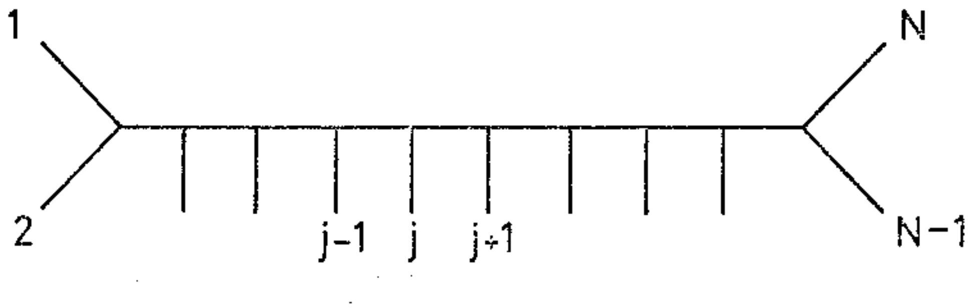

FIG. 3

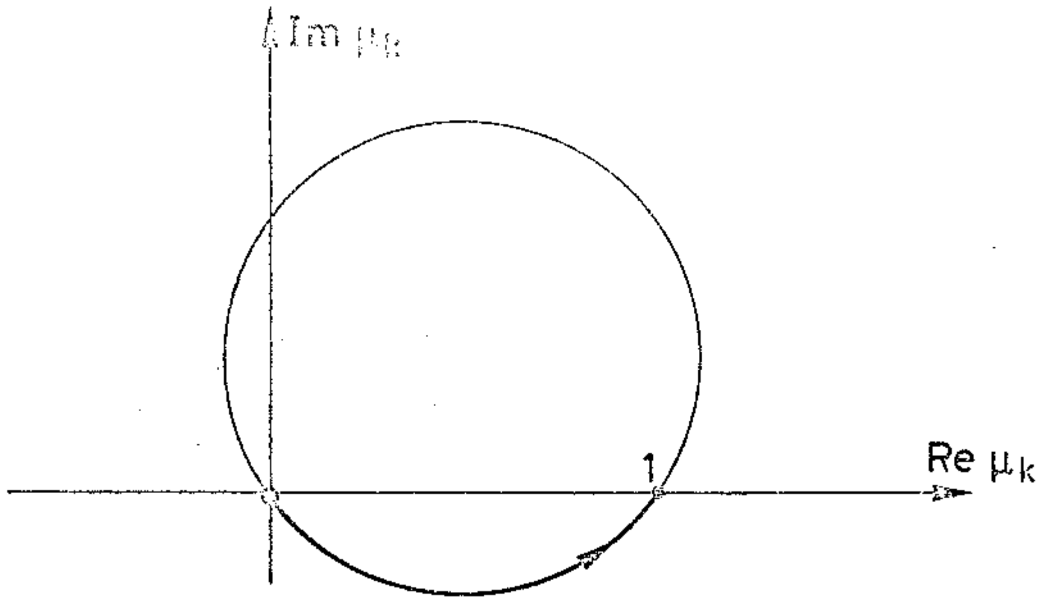

FIG. 4

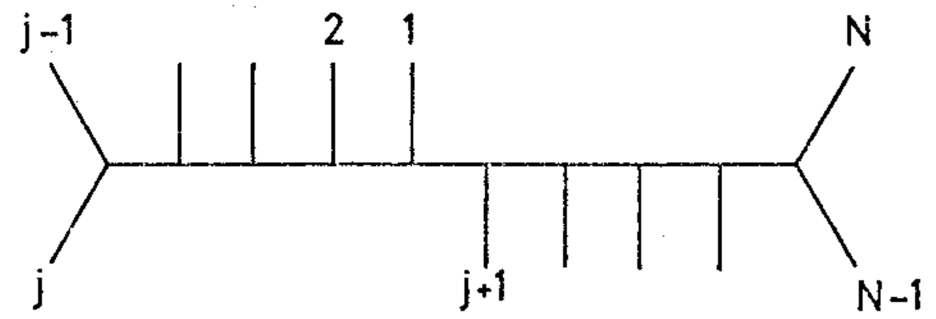

FIG. 5

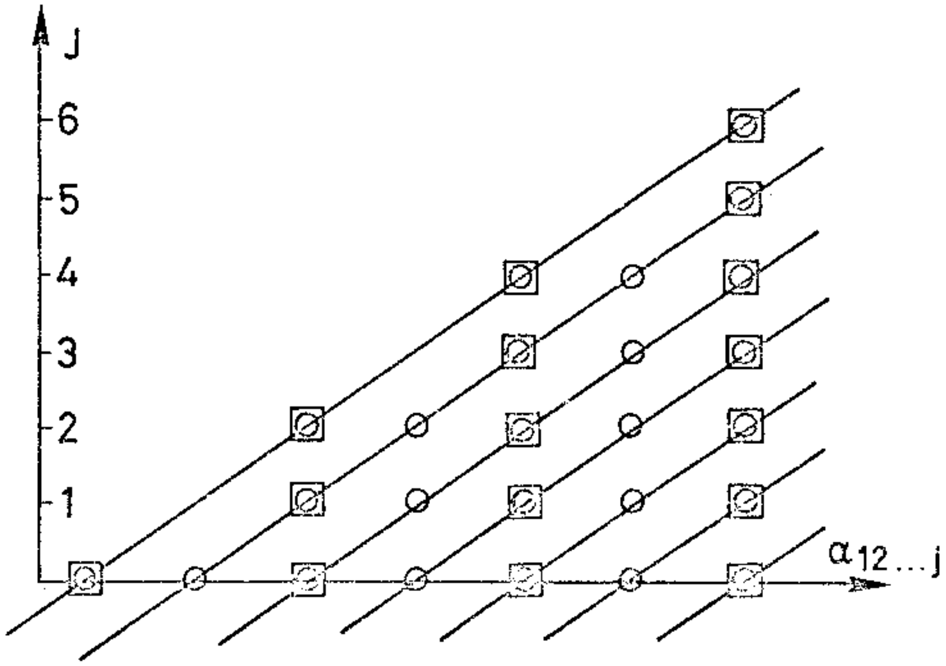

FIG. 6