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SYMMETRY PROPERTIES OF POSITIVE SOLUTIONS TO SOME ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study symmetry properties of positive solutions to some semilinear elliptic problems with nonlinear Neumann boundary conditions. We give sufficient conditions to have symmetry around the $\mathbf{e}_n$-axis of positive solutions of problems on the half-space. The proofs are based on the moving plane method. Finally some symmetry results are given in the case when the domain is a ball.

0. Introduction and statement of the results. We are concerned with problems of the form

$$\begin{cases} -\Delta u = f(y,u) & y=(\bar{y},y_n)\in\mathbb{R}^n_+,\ n\geq 3 \\ -\frac{\partial u}{\partial y_n} = g(\bar{y},u) & \text{on } \ \mathbb{R}^{n-1}. \end{cases} \tag{P}$$

Our first goal is to prove symmetry around the $\mathbf{e}_n$-axis of positive solutions to (P), provided both f and g are symmetric around the $\mathbf{e}_n$-axis and are nonincreasing in $|\bar{y}|$ (see Theorem 1.1 and the Corollaries for the precise statements). Of course, whenever f and g do not depend on $\bar{y}$, symmetry is expected up to translations of the domain. Our arguments are based on the moving plane method. This technique was first introduced by Alexandroff and then used by several authors (we quote Serrin ([12]), Gidas, Ni and Nirenberg ([7], [8]), Berestycki and Nirenberg ([1], [2], Li [10], [11]); see also references therein) to prove symmetry and monotonicity of positive solutions to various nonlinear elliptic problems, both in bounded and unbounded domains. Roughly speaking, the moving plane technique consists in two main steps: first in reflecting the domain about a fixed hyperplane (say $y_1=\lambda$) and proving that the value of the solution at each reflected point is larger than the value at the point itself, and secondly in moving the hyperplane to a critical position; finally the solution results to be symmetric with respect to this limit hyperplane. When the domain is unbounded, a major difficulty consists in checking the first step: we shall show how some integrability conditions allow the application of the method. The advantage of this point of view is that the same argument can be applied in cases when f and g have subcritical, critical or supercritical growth. The following result is an easy application of Corollary 1.3:

Theorem 0.1. *Assume f and g satisfy*

$$\frac{f(s)-f(t)}{s-t}\leq \rho_1 t^{\mu_1},\quad \frac{g(s)-g(t)}{s-t}\leq \rho_2 t^{\mu_2}\qquad \forall\, 0<s\leq t\,,$$

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for some positive constants ρ_i and μ_i ($i = 1, 2$). Let u be a positive and smooth solution of

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^n_+ \\ -\frac{\partial u}{\partial y_n} = g(u) & \text{on } \mathbb{R}^{n-1} \end{cases}$$

and assume that $u^{\mu_1} \in L^{n/2}(\mathbb{R}^n_+)$ (and also $u \in L^\delta(\mathbb{R}^n_+)$ for some $\delta > 1$ if $\mu_1 \leq 2/n$), $u^{\mu_2} \in L^{n-1}(\mathbb{R}^{n-1})$. Then there is $\bar{y}_0 \in \mathbb{R}^{n-1}$ such that $u(y - \bar{y}_0)$ is symmetric around the $\mathbf{e}_n$-axis. Moreover $\nabla u(y - \bar{y}_0) \cdot (\bar{y} - \bar{y}_0) < 0$, for every $(\bar{y}, y_n) \in \mathbb{R}^n_+$.

Theorem 0.1 covers the case of powers, that is when $f(s) = \alpha s^{\theta_1}$ and $g(s) = \beta s^{\theta_2}$, provided $\theta_i > 1$. For this case, however, the existence of positive solutions may depend on the value of the exponents θ_1 and θ_2. In the subcritical case (i.e., $\theta_1 \leq (n+2)/(n-2)$ and $\theta_2 \leq n/(n-2)$) it can be shown, exploiting the same argument used by Hu in [9], that the solutions of the problem are functions of y_n only, without any decay assumption, provided at least one inequality strictly holds. Nonexistence of positive solutions can also be proved in the supercritical case, that is $\alpha(\theta_1 - (n+2)/(n-2)) \leq 0$ and $\beta(\theta_2 - n/(n-2)) \leq 0$ (with at least one strict inequality) under the further decay assumption that $u \in L^{\theta_1+1}(\mathbb{R}^n_+)$ and $u \in L^{\theta_2+1}(\mathbb{R}^{n-1})$. In order to apply our theorem, we have to assume that $u \in L^{(\theta_1-1)n/2}(\mathbb{R}^n_+)$ and $u \in L^{(\theta_2-1)(n-1)}(\mathbb{R}^{n-1})$ and it is not clear whether positive solutions satisfying those growth conditions exist. We wish to point out that Theorem 0.1 allows one to treat more general semilinear equations such as $-\Delta u + u = f(u)$ with boundary condition $-\partial u/\partial y_n + u = g(u)$.

The critical case, when $f(s) = \alpha s^{(n+2)/(n-2)}$ and $g(s) = \beta s^{n/(n-2)}$, is of special interest, because it is related to the Yamabe problem on manifolds with boundary. The following uniqueness result was first proved by Escobar in [6] and then generalized in [4]:

Theorem 0.2. *Let $u \in \mathcal{C}^1(\overline{\mathbb{R}^n_+})$ such that $u(y) = \mathcal{O}(|y|^{2-n})$ as $|y| \to \infty$ be a positive solution of*

$$\begin{cases} -\Delta u = \alpha u^{(n+2)/(n-2)} & \text{in } \mathbb{R}^n_+ \\ -\frac{\partial u}{\partial y_n} = \beta u^{n/(n-2)} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

(i) *If $\alpha = 0$ then u has the form*

$$u(y) = \frac{K}{|y - y_0|^{n-2}},$$

where K and y_0 satisfy the relation $K^{2/(n-2)} = -\frac{n-2}{\beta} y_0 \cdot \mathbf{e}_n$.

(ii) *If $\alpha > 0$ then u has the form*

$$u(y) = \frac{(n(n-2)/\alpha)^{(n-2)/4} \lambda^{(n-2)/2}}{(\lambda^2 + |y - y_0|^2)^{(n-2)/2}},$$

where λ and y_0 satisfy the relation $\frac{-y_0 \cdot \mathbf{e}_n}{\lambda} = \frac{\beta}{\alpha^{1/2}} \left(\frac{n}{n-2}\right)^{1/2}$.

We shall give an alternative proof of this result, using symmetry arguments and exploiting the conformal invariance of the problem: indeed, using Kelvin's inversion map, the half space $\mathbb{R}^n_+$ can be conformally mapped into the unit ball B. We shall prove the following results:

Theorem 0.3. *Let ℓ, α and β be given such that $\ell \leq (n-2)/2$, $\alpha(\theta_1-(n+2)/(n-2)) \leq 0$ and $\beta(\theta_2 - n/(n-2)) \leq 0$ and that at least one of these inequalities strictly holds. Then every positive solution $v \in \mathcal{C}^1(\overline{B})$ of*

$$\begin{cases} -\Delta v = \alpha v^{\theta_1} & \text{in } B \\ \frac{\partial v}{\partial \nu} + \ell v = \beta v^{\theta_2} & \text{on } \partial B \end{cases}$$

is radially symmetric.

Recalling that conformal maps in B are the composition of Kelvin's inversion map with rescaling and translations, the next result turns out to be equivalent to Theorem 0.2, in the case when $\alpha \geq 0$.

Theorem 0.4. *Let α, $\beta \in \mathbb{R}$ and $v \in \mathcal{C}^1(\overline{B})$ be a positive solution of*

$$\begin{cases} -\Delta v = \alpha v^{(n+2)/(n-2)} & \text{in } B \\ \frac{\partial v}{\partial \nu} + \frac{(n-2)}{2} v = \beta v^{n/(n-2)} & \text{on } \partial B. \end{cases}$$

Then there is a conformal map $R : B \to B$ such that the function w defined as $w(x) = |Jac\, R(x)|^{(n-2)/2n} v(R(x))$ is radially symmetric.

Preliminaries on symmetries. In the following we shall be concerned with symmetry properties of functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

Definition 0.5. Let $\mathbf{a}$ be a unit vector of $\mathbb{R}^n$ such that Ω is invariant under the reflection $y \to y - 2(y \cdot \mathbf{a})\mathbf{a}$. We say that u is $\pi_{\mathbf{a}}$-symmetric if

$$u(y - 2(y \cdot \mathbf{a})\mathbf{a}) = u(y) , \qquad \forall\, y \in \Omega ,$$

namely if u is symmetric with respect to the hyperplane orthogonal to $\mathbf{a}$.

Definition 0.6. We say that u is symmetric around the $\mathbf{b}$-axis if

$$u(y_1) = u(y_2) \quad \forall\, y_1, y_2 \in \Omega,\ |y_1| = |y_2|,\ y_1 \cdot \mathbf{b} = y_2 \cdot \mathbf{b}.$$

Definition 0.7. Let Ω be either a ball or the whole space. We say that u is radially symmetric if

$$u(y_1) = u(y_2) \quad \forall\, y_1, y_2 \in \Omega, |y_1| = |y_2|.$$

Remark 0.8. If $\Omega = \mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y \cdot \mathbf{e}_n > 0\}$, it is immediate to check that u is symmetric around the $\mathbf{e}_n$-axis if and only if it is $\pi_{\mathbf{a}}$-symmetric for every $\mathbf{a} \in \mathbb{R}^{n-1}$.

Remark 0.9. A function u is radially symmetric if and only if it is symmetric around any axis.

Proposition 0.10. *Let $\Omega = \mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y \cdot \mathbf{e}_n > 0\}$ and let $(\mathbf{e}_j)_{1 \leq j \leq n-1}$ be an orthornormal system of $\mathbb{R}^{n-1}$. Assume that u is $\pi_{\mathbf{e}_j}$–symmetric for every j and moreover that there are $\mathbf{a} \in \mathbb{R}^{n-1}$ and $\lambda_{\mathbf{a}} \in \mathbb{R}$ such that $u(y + \lambda_{\mathbf{a}}\mathbf{a})$ is $\pi_{\mathbf{a}}$-symmetric. Then u is $4\lambda_{\mathbf{a}}$-periodic in the $\mathbf{a}$-direction; that is,*

$$u(y - 4\lambda_{\mathbf{a}}\mathbf{a}) = u(y) , \quad \forall\, y \in \mathbb{R}^n_+.$$

Proof. Let us write $y = (\bar{y}, y_n)$; then we have $u(\bar{y} - 2(\bar{y} \cdot \mathbf{e}_j)\mathbf{e}_j, y_n) = u(\bar{y}, y_n)$, for every $j \in \{1, \dots, n-1\}$, in $\mathbb{R}^n_+$. In particular $u(A(\bar{y}, y_n)) = u(\bar{y}, y_n)$ in $\mathbb{R}^n_+$, where $A(\bar{y}, y_n) = (-\bar{y}, y_n)$. Saying that $u(y + \lambda_{\mathbf{a}}\mathbf{a})$ is $\pi_{\mathbf{a}}$-symmetric is equivalent to say that u is invariant with respect to $R(\bar{y}, y_n) = (\bar{y} - 2(\bar{y} \cdot \mathbf{a})\mathbf{a} + 2\lambda_{\mathbf{a}}\mathbf{a}, y_n)$. Taking the composition $ARAR(\bar{y}, y_n) = (\bar{y} - 4\lambda_{\mathbf{a}}\mathbf{a}, y_n)$, we obtain the thesis.

1. The moving plane method. We are concerned with positive solutions of boundary value problems of the type:

$$\begin{cases} -\Delta u = f(y, u) & \text{in } \mathbb{R}^n_+ \\ -\frac{\partial u}{\partial y_n} = g(\bar{y}, u) & \text{on } \mathbb{R}^{n-1}, \end{cases} \tag{1.1}$$

where, writing $y \in \mathbb{R}^n$ as $(\bar{y}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_n > 0\}$ and $\mathbb{R}^{n-1} = \{y \in \mathbb{R}^n : y_n = 0\}$. For $\lambda \in \mathbb{R}$ we consider the reflection

$$y = (y_1, \dots, y_n) \longrightarrow y^\lambda = (2\lambda - y_1, \dots, y_n),$$

where $y \in \Sigma^\lambda = \{y \in \mathbb{R}^n_+ \cup \mathbb{R}^{n-1} : y_1 < \lambda\}$, and we put

$$u^\lambda(y) = u(y^\lambda); \tag{1.2}$$

of course we have that

$$u^\lambda(y) = u(y), \qquad \text{for } \ y \in \Gamma^\lambda_0 = \{y \in \mathbb{R}^n_+ \cup \mathbb{R}^{n-1} : y_1 = \lambda\}. \tag{1.3}$$

We consider the following assumptions:
on f:

$$f \text{ is nondecreasing in the } y_1 \text{ direction for } y_1 < 0, \tag{$f1$}$$

$$\text{there are } \mu_1 > 0 \text{ and functions } \sigma_1 \in L^{n/2}(\mathbb{R}^n_+),\ \rho_1 \text{ such that} \tag{$f2$}$$
$$\frac{f(y,s) - f(y,t)}{s-t} \le \sigma_1(y) + \rho_1(y) t^{\mu_1}\ , \qquad \forall\, t \ge s \ge 0$$

on g:

$$g \text{ is nondecreasing in the } y_1 \text{ direction for } y_1 < 0, \tag{$g1$}$$

$$\text{there are } \mu_2 > 0 \text{ and functions } \sigma_2 \in L^{n-1}(\mathbb{R}^{n-1}),\ \rho_2 \text{ such that} \tag{$g2$}$$
$$\frac{g(\bar{y},s) - g(\bar{y},t)}{s-t} \le \sigma_2(\bar{y}) + \rho_2(\bar{y}) t^{\mu_2}\ , \qquad \forall t \ge s \ge 0$$

on u:

$$\rho_1 u^{\mu_1} \in L^{n/2}(\mathbb{R}^n_+), \tag{$u1$}$$
$$\rho_2 u^{\mu_2} \in L^{n-1}(\mathbb{R}^{n-1}). \tag{$u2$}$$

Theorem 1.1. *Let $u \in \mathcal{C}^1(\overline{\mathbb{R}^n_+}) \cap L^\delta(\mathbb{R}^n_+)$ $(\delta > 1)$ be a positive solution of* (1.1). *Under the above assumptions we have that either $u^\lambda > u$ on Σ^λ, for every $\lambda < 0$, or there exists $\lambda^* < 0$ such that $u^{\lambda^*} = u$ on Σ^{λ^*}. Moreover, we have $u_{y_1} > 0$ whenever $y_1 < 0$ in the former case and in the latter we have $u_{y_1} > 0$ whenever $y_1 < \lambda^*$.*

Corollary 1.2. *Under the assumptions of Theorem* 1.1, *assume moreover that, for any s, $f(\cdot, s)$ and $g(\cdot, s)$ are symmetric around the $\mathbf{e}_n$-axis, f and g are nonincreasing in $|\bar{y}|$ and that at least one among f and g is strictly decreasing in $|\bar{y}|$, for every s. Then u is symmetric around the $\mathbf{e}_n$-axis and $\nabla u(y) \cdot \bar{y} < 0$ for $\bar{y} \neq 0$.*

Corollary 1.3. *Under the assumptions of Theorem* 1.1, *let f and g be independent of y. Then there exists $\bar{y}_0 \in \mathbb{R}^{n-1}$ such that $u(y - \bar{y}_0)$ is symmetric around the $\mathbf{e}_n$-axis. Moreover, $\nabla u(y - \bar{y}_0) \cdot (\bar{y} - \bar{y}_0) < 0$ for $\bar{y} \neq \bar{y}_0$.*

Preliminaries. Let us write $v^\lambda = u^\lambda - u$; then v^λ satisfies a linear boundary problem of the type:

$$\begin{cases} -\Delta v^\lambda = a_\lambda(y) v^\lambda + h^\lambda(y) & \text{in } \Sigma^\lambda \cap \mathbb{R}^n_+ \\ -\frac{\partial v^\lambda}{\partial y_n} = b_\lambda(y) v^\lambda + k^\lambda(y) & \text{on } \Sigma^\lambda \cap \mathbb{R}^{n-1} \\ v^\lambda(y) = 0 \qquad \text{on } \Gamma_0^\lambda, \end{cases} \tag{1.4}$$

where

$$a_\lambda(y) = \frac{f(y, u^\lambda) - f(y, u)}{u^\lambda - u}, \qquad b_\lambda(\bar{y}) = \frac{g(\bar{y}, u^\lambda) - g(\bar{y}, u)}{u^\lambda - u},$$
$$h^\lambda(y) = f(y^\lambda, u^\lambda) - f(y, u^\lambda), \quad k^\lambda(\bar{y}) = g(\bar{y}^\lambda, u^\lambda) - g(\bar{y}, u^\lambda).$$

In order to study positiveness of solutions of (1.4) we prove the following result.

Lemma 1.4. *Let w satisfy*

$$\begin{cases} -\Delta w \geq a(y) w & \textit{in } \Sigma^\lambda \cap \mathbb{R}^n_+ \\ -\frac{\partial w}{\partial y_n} \geq b(y) w & \textit{on } \Sigma^\lambda \cap \mathbb{R}^{n-1} \\ w(y) = 0 & \textit{on } \Gamma_0^\lambda \end{cases} \tag{1.5}$$

and assume that $w \in L^\delta(\mathbb{R}^n_+) \cap W^{1,2}_{loc}(\mathbb{R}^n_+)$, for some $\delta > 1$. There are constants independent of w, $C_1 = C_1(n, \delta)$ and $C_2 = C_2(n, \delta)$, such that, putting $\Omega^- = \{y \in \Sigma^\lambda : w(y) < 0\}$, there holds

$$\textit{either } \mu(\Omega^-) = 0, \quad \textit{or} \quad C_1 \|a^+\|_{L^{n/2}(\Omega^-)} + C_2 \|b^+\|_{L^{n-1}(\Omega^- \cap \mathbb{R}^{n-1})} \geq 1\,. \tag{1.6}$$

Proof. Let $s = (\delta - 2)/2 > -1/2$. For $K > 1$, let us fix η such that $\eta(y) = 1$ for $|y| \leq K$, $\eta(y) = 0$ for $|y| \geq 2K$ and $|\nabla \eta| \leq 1$ and let $v = (w^-)^s \wedge K$. Define

$$\varphi = \eta^2 v^2 w^-, \qquad \psi = \eta v w^-.$$

We have, for any positive ε,

$$\int_{\Omega^-} |\nabla \psi|^2 = \int_{\Omega^-} \eta^2 |\nabla(v w^-)|^2 + 2\eta v w^- \nabla \eta \cdot \nabla(v w^-) + |\nabla \eta|^2 (v w^-)^2$$
$$\leq \int_{\Omega^-} (1 + \varepsilon) \eta^2 |\nabla(v w^-)|^2 + C_\varepsilon |\nabla \eta|^2 (v w^-)^2$$

and

$$\begin{aligned}\int_{\Omega^-} \nabla w^- \cdot \nabla\varphi &= \int_{\Omega^-} \nabla w^- \cdot \big(2\eta(v^2w^-)\nabla\eta + \eta^2\nabla(v^2w^-)\big) \\ &\geq \int_{\Omega^-} \eta^2\nabla w^- \cdot \nabla(v^2w^-) - \varepsilon\eta^2v^2|\nabla w^-|^2 - C_\varepsilon|\nabla\eta|^2(vw^-)^2 \\ &\geq \int_{\Omega^-} (1-\varepsilon)\eta^2\nabla w^- \cdot \nabla(v^2w^-) - C_\varepsilon|\nabla\eta|^2(vw^-)^2 \ .\end{aligned}$$

Since

$$\nabla w^- \cdot \nabla(v^2w^-) \geq \frac{1+2s}{(1+s)^2}|\nabla(vw^-)|^2 \ ,$$

we obtain that

$$\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{1+2s}{(1+s)^2}\Big)\int_{\Omega^-}|\nabla\psi|^2 \leq \int_{\Omega^-}\nabla w^- \cdot \nabla\varphi + C_\varepsilon|\nabla\eta|^2(vw^-)^2 \ . \tag{1.7}$$

On the other hand, testing (1.5) with φ and using Hölder's inequality we get

$$\begin{aligned}\int_{\Omega^-}\nabla w^- \cdot \nabla\varphi &\leq \int_{\Omega^-} a(y)\psi^2 + \int_{\Omega^-\cap\mathbb{R}^{n-1}} b(y)\psi^2 \\ &\leq \|a^+\|_{L^{n/2}(\Omega^-)}\|\psi\|^2_{L^{\frac{2n}{n-2}}(\Omega^-)} + \|b^+\|_{L^{n-1}(\Omega^-\cap\mathbb{R}^{n-1})}\|\psi\|^2_{L^{\frac{2n-2}{n-2}}(\Omega^-\cap\mathbb{R}^{n-1})} \ .\end{aligned}$$

Finally, using the Sobolev inequalities, we can conclude that

$$\begin{aligned}&\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{1+2s}{(1+s)^2}\Big)\int_{\Omega^-}|\nabla\psi|^2 \\ &\leq \big(S_1\|a^+\|_{L^{n/2}(\Omega^-)} + S_2\|b^+\|_{L^{n-1}(\Omega^-\cap\mathbb{R}^{n-1})}\big)\int_{\Omega^-}|\nabla\psi|^2 + \int_{\Omega^-} C_\varepsilon|\nabla\eta|^2(vw^-)^2 \ .\end{aligned}$$

Assuming that $\mu(\Omega^-)\neq 0$ and dividing the above inequality by $\|\nabla\psi\|_2^2$, we obtain the desired inequality for $K\to+\infty$; indeed, since $2+2s=\delta$ and $(w^-)^\delta$ is integrable, we have

$$\int_{\Omega^-} C_\varepsilon|\nabla\eta|^2(vw^-)^2 \leq \int_{\Omega^-\cap\{|y|\geq K\}} C'_\varepsilon(w^-)^{2+2s} \longrightarrow 0 \qquad \text{as } K\to+\infty.$$

Proof of Theorem 1.1. In our setting, by (f1) and (g1) we have

$$h^\lambda(y) \geq 0 \qquad \text{in } \ \mathbb{R}^n_+, \tag{1.8}$$

$$k^\lambda(\bar{y}) \geq 0 \qquad \text{in } \ \mathbb{R}^{n-1} \ . \tag{1.9}$$

Moreover, denoting $\Omega_\lambda^- = \{y\in\Sigma^\lambda : v^\lambda(y) < 0\}$, assumptions (f2) and (g2), together with (u1) and (u2) imply that

$$\begin{aligned}{a_\lambda}^+(y) &\leq \sigma_1(y) + \rho_1(y)u^{\mu_1}(y) \quad\ \forall\, y\in\Omega_\lambda^-, \\ {b_\lambda}^+(\bar{y}) &\leq \sigma_2(\bar{y}) + \rho_2(\bar{y})u^{\mu_2}(\bar{y}) \quad\ \forall\, \bar{y}\in\mathbb{R}^{n-1}\cap\Omega_\lambda^-;\end{aligned} \tag{1.10}$$

furthermore,

$$\sigma_1 + \rho_1 u^{\mu_1} \in L^{n/2}(\mathbb{R}^n_+), \qquad \sigma_2 + \rho_2 u^{\mu_2} \in L^{n-1}(\mathbb{R}^{n-1}). \tag{1.11}$$

Step 1. We claim that there is $\bar\lambda < 0$ such that $v^\lambda > 0$, for every $\lambda < \bar\lambda$. Indeed, let C_1 and C_2 as in Lemma 1.4. Thanks to (1.11) there exists $\bar\lambda$ such that, for every $\lambda \le \bar\lambda$,

$$C_1\|\sigma_1 + \rho_1 u^{\mu_1}\|_{L^{n/2}(\Sigma^\lambda)} + C_2\|\sigma_2 + \rho_2 u^{\mu_2}\|_{L^{n-1}(\Sigma^\lambda\cap\mathbb{R}^{n-1})} < 1\ .$$

Putting $\Omega^-_\lambda = \{y \in \Sigma^\lambda : v^\lambda(y) < 0\}$ we then obtain from (1.10), for $\lambda \le \bar\lambda$,

$$C_1\|{a_\lambda}^+\|_{L^{n/2}(\Omega^-_\lambda)} + C_2\|{b_\lambda}^+\|_{L^{n-1}(\Omega^-_\lambda\cap\mathbb{R}^{n-1})} < 1\ .$$

Therefore, we deduce by Lemma 1.4 that $v^\lambda \ge 0$, for $\lambda \le \bar\lambda$. By the strong maximum principle we then deduce that $v^\lambda > 0$ in Σ^λ, for $\lambda \le \bar\lambda$.

Step 2. Let us define $\lambda^* = \sup\{\bar\lambda < 0 : v^\lambda > 0\ , \forall\ \lambda < \bar\lambda\}$. If either $\lambda^* \ge 0$ or $v^{\lambda^*} \equiv 0$ in Σ^{λ^*} the proof is done. For the sake of contradiction, let us assume that $\lambda^* < 0$ and $v^{\lambda^*} \not\equiv 0$. Since we clearly have $v^{\lambda^*} \ge 0$, we deduce that $v^{\lambda^*} > 0$ in Σ^{λ^*} and also in $\Sigma^{\lambda^*} \cap \mathbb{R}^{n-1}$, by the strong maximum principle. From (1.11), for every $\varepsilon > 0$ there are $\delta_0 > 0$ and $R > 0$ such that, for every $\lambda \in (\lambda^* - \delta_0, \lambda^* + \delta_0)$ there holds

$$C_1\|\sigma_1 + \rho_1 u^{\mu_1}\|_{L^{n/2}(\Sigma^\lambda\setminus B_R)} + C_2\|\sigma_2 + \rho_2 u^{\mu_2}\|_{L^{n-1}(\Sigma^\lambda\cap(\mathbb{R}^{n-1}\setminus B_R))} < 1 - \varepsilon.$$

On the other hand, since v^{λ^*} is positive in $\Sigma^{\lambda^*} \cap B_R$ and also on $\Sigma^{\lambda^*} \cap (B_R \cap \mathbb{R}^{n-1})$, by convergence in measure of the v^λ's to v^{λ^*} we can find $0 < \delta_1 \le \delta_0$ such that

$$C_1\|\sigma_1 + \rho_1 u^{\mu_1}\|_{L^{n/2}(\Omega^-_\lambda\cap B_R)} + C_2\|\sigma_2 + \rho_2 u^{\mu_2}\|_{L^{n-1}(\Omega^-_\lambda\cap(\mathbb{R}^{n-1}\cap B_R))} < \varepsilon$$

for every $\lambda \in (\lambda^* - \delta_1, \lambda^* + \delta_1)$. In this way we have, using again (1.10),

$$C_1\|{a_\lambda}^+\|_{L^{n/2}(\Omega^-_\lambda)} + C_2\|{b_\lambda}^+\|_{L^{n-1}(\Omega^-_\lambda\cap\mathbb{R}^{n-1})} < 1, \qquad \forall \lambda \in (\lambda^* - \delta_1, \lambda^* + \delta_1)\ .$$

Then applying Lemma 1.4 we obtain $v^\lambda \ge 0$ and therefore $v^\lambda > 0$ for $\lambda \in [\lambda^*, \lambda^* + \delta_1)$, in contradiction with the definition of λ^*.

Let $u \in \mathcal{C}^1$; the v^λ's are positive solutions of linear problems in Σ^λ and they vanish on Γ^λ. Thus, by Hopf's lemma, we have that $\partial v^\lambda/\partial y_1 = -2\partial u/\partial y_1$ is negative on $\Gamma^\lambda_0 \cap \mathbb{R}^n_+$, for every $\lambda < \lambda^*$ and therefore that $\partial u/\partial y_1 > 0$ for $y_1 < \lambda^*$ and $y_n > 0$. If moreover f and g are differentiable we can differentiate the equation with respect to y_1 and, again by Hopf's lemma, we obtain that $\frac{\partial u}{\partial y_1} > 0$ also for $y_n = 0$.

Remark 1.5. It is clear that the above argument works more generally for $W^{1,2}_{loc}$ solutions. However, in our main applications, it can be shown using the general regularity theory (see [3]) that weak $W^{1,2}_{loc}$ solutions are actually in $\mathcal{C}^\infty$ up to the boundary.

Proof of Corollary 1.2. Following the proof of Theorem 1.1, we claim that $\lambda^* = 0$. If not $v^{\lambda^*} \equiv 0$ should solve an inhomogeneous linear boundary problem, (1.4), since at least one between h^{λ^*} and k^{λ^*} is not identically zero. Hence $u(-y_1, \ldots, y_n) \ge u(y_1, \ldots, y_n)$

for every $y_1 < 0$. Replacing $u(y_1, \ldots, y_n)$ with $u(-y_1, \ldots, y_n)$ we obtain that also the opposite inequality holds, so that $u(-y_1, \ldots, y_n) = u(y_1, \ldots, y_n)$ in $\mathbb{R}^n_+$. It is clear that we can repeat the same argument choosing as $\mathbf{e}_1$ any direction in $\mathbb{R}^{n-1}$. In this way we find that u is symmetric around the $\mathbf{e}_n$-axis.

Proof of Corollary 1.3. In this case the v^λ's always solve homogeneous linear problems so that λ^* is not necessarily zero. However, applying Theorem 1.1 to $u(y_1, \ldots, y_n)$ and to $u(-y_1, \ldots, y_n)$ we find the existence of $\lambda_1^* \in \mathbb{R}$ such that $u(2\lambda_1^* - y_1, \ldots, y_n) = u(y_1, \ldots, y_n)$ in $\mathbb{R}^n_+$. Of course the same argument works for any direction in $\mathbb{R}^{n-1}$. This means that for every unit vector $\mathbf{a} \in \mathbb{R}^{n-1}$ there is $\lambda_\mathbf{a} \in \mathbb{R}$ such that

$$u(y - 2(y \cdot \mathbf{a})\mathbf{a} + 2\lambda_a \mathbf{a}) = u(y) , \qquad \text{in } \mathbb{R}^n_+ .$$

Therefore, corresponding to a fixed orthonormal system $(\mathbf{e}_j)_{1 \le j \le n-1}$, we can find $n-1$ real numbers (λ_j) such that $u(y - 2(y \cdot \mathbf{e}_j)\mathbf{e}_j + 2\lambda_j \mathbf{e}_j) = u(y)$ in $\mathbb{R}^n_+$, for $j = 1, \ldots, n-1$. If $\bar{y}_0 = -\sum_j \lambda_j \mathbf{e}_j$, $w(y) = u(u - \bar{y}_0)$ still solves problem (1.1) and is $\pi_{\mathbf{e}_j}$-symmetric, for every $j = 1, \ldots, n-1$. Now let $\mathbf{a}$ be any direction in $\mathbb{R}^{n-1}$. As we pointed out before, there is $\lambda_\mathbf{a} \in \mathbb{R}$ such that $w(y + \lambda_\mathbf{a} \mathbf{a})$ is $\pi_\mathbf{a}$-symmetric. We claim that $\lambda_\mathbf{a} = 0$; if not, as proved in Proposition 0.10, w should be $4\lambda_\mathbf{a}$-periodic in the $\mathbf{a}$-direction, and the same holds for u. Since $u > 0$ is in L^δ for some δ, we obtain a contradiction. Since w is $\pi_\mathbf{a}$-symmetric for every $\mathbf{a} \in \mathbb{R}^{n-1}$, we obtain the thesis.

2. Inversion maps. In the following we shall consider the following maps:

$$\Phi : B \to \mathbb{R}^n_+ : x \longrightarrow \Phi(x) = \frac{x + \mathbf{e}_n}{|x + \mathbf{e}_n|^2} - (1/2)\mathbf{e}_n \tag{2.1}$$

and its inverse

$$\Phi^{-1} = \Psi : \mathbb{R}^n_+ \to B : y \longrightarrow \Psi(y) = \frac{y + (1/2)\mathbf{e}_n}{|y + (1/2)\mathbf{e}_n|^2} - \mathbf{e}_n \ . \tag{2.2}$$

These are both conformal maps, being the composition of translations with Kelvin's inversion map. Therefore they induce the following isometries Φ^* and $(\Phi^*)^{-1} = \Psi^*$:

$$\Phi^* : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \ \to \ H^1(B), \qquad \Psi^* : H^1(B) \ \to \ \mathcal{D}^{1,2}(\mathbb{R}^n_+)$$

defined as weighted pull-back of Φ and Ψ respectively:

$$\Phi^* u(x) = |\text{Jac } \Phi(x)|^{\frac{n-2}{2n}} u(\Phi(x)) = \frac{1}{|x + \mathbf{e}_n|^{n-2}} u(\Phi(x)), \tag{2.3}$$

$$\Psi^* v(y) = |\text{Jac } \Psi(y)|^{\frac{n-2}{2n}} u(\Psi(y)) = \frac{1}{|y + (1/2)\mathbf{e}_n|^{n-2}} u(\Psi(y)). \tag{2.4}$$

Indeed we have, for $v = \Phi^* u$,

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 = \int_B |\nabla v|^2 + \frac{n-2}{2} \int_{\partial B} v^2 d\sigma.$$

Moreover, to any weak solution $v \in H^1(B)$ of

$$\begin{cases} -\Delta v = \alpha v^{\theta_1} & \text{on } B \\ \frac{\partial v}{\partial \nu} + \ell v = \beta v^{\theta_2} & \text{on } \partial B \end{cases} \tag{2.5}$$

there corresponds a solution $u = \Psi^* v \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ of the problem

$$\begin{cases} -\Delta u = \alpha u^{\theta_1} |y + (1/2)\mathbf{e}_n|^{-n-2+\theta_1(n-2)} & \text{on } \mathbb{R}^n_+ \\ -\frac{\partial u}{\partial y_n} = \gamma(\ell) u |y + (1/2)\mathbf{e}_n|^{-2} + \beta u^{\theta_2} |y + (1/2)\mathbf{e}_n|^{-n+\theta_2(n-2)} & \text{on } \mathbb{R}^{n-1}, \end{cases} \tag{2.6}$$

where $\gamma(\ell) = \left(\frac{n-2}{2} - \ell\right)$. We shall study symmetry properties of positive solutions of problem (2.5). So let

$$f(y,s) = \alpha s^{\theta_1} |y + (1/2)\mathbf{e}_n|^{-n-2+\theta_1(n-2)}, \tag{2.7}$$

$$g(y,s) = \gamma(\ell) s |y + (1/2)\mathbf{e}_n|^{-2} + \beta s^{\theta_2} |y + (1/2)\mathbf{e}_n|^{-n+\theta_2(n-2)}. \tag{2.8}$$

Of course f and g are radially symmetric with respect to the $\mathbf{e}_n$-axis and they are nonincreasing in $|\bar{y}|$ if and only if $\ell \le (n-2)/2$, $\alpha(\theta_1 - (n+2)/(n-2)) \le 0$ and $\beta(\theta_2 - n/(n-2)) \le 0$. Moreover we have, for $0 \le s \le t$ and $y \in \mathbb{R}^n_+$,

$$\frac{f(y,s) - f(y,t)}{s-t} \le \alpha\theta_1 t^{\theta_1 - 1} |y + (1/2)\mathbf{e}_n|^{-n-2+\theta_1(n-2)} \qquad \text{if } \alpha > 0, \text{ and}$$

$$\frac{f(y,s) - f(y,t)}{s-t} \le 0 \qquad \text{if } \alpha \le 0$$

and, for $0 \le s \le t$ and $\bar{y} \in \mathbb{R}^{n-1}$,

$$\frac{g(\bar{y},s) - g(\bar{y},t)}{s-t} \le \gamma(\ell)|\bar{y} + (1/2)\mathbf{e}_n|^{-2} + \beta\theta_2 t^{\theta_2 - 1} |\bar{y} + (1/2)\mathbf{e}_n|^{-n+\theta_2(n-2)} \text{ if } \beta > 0,$$

$$\frac{g(\bar{y},s) - g(\bar{y},t)}{s-t} \le \gamma(\ell)|\bar{y} + (1/2)\mathbf{e}_n|^{-2} \qquad \text{if } \beta \le 0.$$

Thus (f2) and (g2) are fulfilled for

$$\sigma_1(y) = 0 \in L^{n/2}(\mathbb{R}^n_+), \quad \rho_1(y) = \alpha^+ \theta_1 |y + (1/2)\mathbf{e}_n|^{-n-2+\theta_1(n-2)}, \tag{2.9}$$

$$\sigma_2(\bar{y}) = \gamma(\ell)|\bar{y} + (1/2)\mathbf{e}_n|^{-2} \in L^{n-1}(\mathbb{R}^{n-1}), \quad \rho_2(\bar{y}) = \beta^+ \theta_2 |\bar{y} + (1/2)\mathbf{e}_n|^{-n+\theta_2(n-2)}.$$

3. Proof of Theorem 0.3. Let $v \in \mathcal{C}^1(\overline{B})$ be a positive solution of (0.1) and let $u = \Psi^* v$ be as in (2.4). Then $u \in \mathcal{D}^{1,2}(\mathbb{R}^n_+) \subset W^{1,2}_{loc}(\mathbb{R}^n_+) \cap L^{2n/(n-2)}(\mathbb{R}^n_+) \cap L^{(2n-2)/(n-2)}(\mathbb{R}^{n-1})$. Therefore, by Hölder's inequality,

$$\rho_1(y) u^{\theta_1 - 1} \in L^{n/2}(\mathbb{R}^n_+), \quad \rho_2(y) u^{\theta_2 - 1} \in L^{n-1}(\mathbb{R}^{n-1}).$$

If at least one of the inequalities in Theorem 0.3 strictly holds then at least one between f and g is strictly decreasing in $|\bar{y}|$. Thus Corollary 1.2 is applicable and we find that u is symmetric around the $\mathbf{e}_n$-axis. Since $v = \Phi^* u$, one immediately sees from (2.1) and (2.3) that also v is symmetric around the $\mathbf{e}_n$-axis. Repeating the same reasoning after any rotation of the ball we obtain the thesis.

4. Proof of Theorem 0.4. Let $v \in \mathcal{C}^1(\overline{B})$ be a positive solution of

$$\begin{cases} -\Delta v = \alpha v^{\frac{n+2}{n-2}} & \text{in } B \\ \frac{\partial v}{\partial \nu} + \frac{n-2}{2} v = \beta v^{\frac{n}{n-2}} & \text{in } \partial B. \end{cases} \tag{4.1}$$

Proposition 4.1. $\nabla_{tang}v(x_0) = 0$ *if and only if* v *is symmetric around the* x_0*-axis.*

Proof. By continuity of the gradient of v, if is clear that $\nabla_{tang}v(x_0) = 0$ whenever v is symmetric around the x_0-axis. To prove the converse, we assume that $\nabla_{tang}v(x_0) = 0$ and we prove symmetry around the x_0-axis. Let us choose a coordinate system over $\mathbb{R}^n$ such that $x_0 = \mathbf{e}_n$ and perform the inversion map $\Psi : \mathbb{R}^n_+ \to B$ as in (2.2). Again, if

$$u(y) = \frac{1}{|y + (1/2)\mathbf{e}_n|^{n-2}} v\Big(\frac{y + (1/2)\mathbf{e}_n}{|y + (1/2)\mathbf{e}_n|^2} - \mathbf{e}_n\Big),$$

then u solves

$$\begin{cases} -\Delta u = \alpha u^{\frac{n+2}{n-2}} & \text{on } \mathbb{R}^n_+ \\ -\frac{\partial u}{\partial y_n} = \beta u^{\frac{n}{n-2}} & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.2}$$

Then, as we already pointed out, Corollary 1.3 is applicable, because $u \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$. We deduce the existence of $\bar{y}_0 \in \mathbb{R}^{n-1}$ such that $u(y - \bar{y}_0)$ is symmetric around the $\mathbf{e}_n$-axis. Moreover, by the second part of Corollary 1.3, $\bar{y}_0$ is the only point of the boundary where the tangential derivative of u vanishes. A simple computation shows that $\nabla_{tang}v(\mathbf{e}_n) = 0$ implies that $\nabla_{tang}u(0) = 0$, hence we can conclude that $\bar{y}_0 = 0$ and therefore u is symmetric around the $\mathbf{e}_n$-axis. Finally also v possesses the same symmetry, since it is obtained by u through the inversion map Φ^* as

$$v(x) = \Phi^* u(x) = \frac{1}{|x + \mathbf{e}_n|^{n-2}} u\Big(\frac{x + \mathbf{e}_n}{|x + \mathbf{e}_n|^2} - (1/2)\mathbf{e}_n\Big)$$

and this operation clearly preserves symmetry around the $\mathbf{e}_n$-axis.

Remark 4.2. A direct consequence of the above result is that

$$\nabla_{tang}v(x_0) = 0 \iff \nabla_{tang}v(-x_0) = 0.$$

Proposition 4.3. *Let* $T_0 = \{x \in \partial B : \nabla_{tang}v(x) = 0\}$*. Then either* T_0 *consists of only two antipodal points or* $T_0 = \partial B$*.*

Proof. Since $v \in \mathcal{C}^1(\partial B)$, clearly $T_0 \neq \emptyset$. Moreover, by the above remark, T_0 consists of pairs of antipodal points. We also remark that if v is symmetric around an axis then also T_0 is symmetric around the same axis. That is, by Proposition 4.1, if $x, y \in T_0$ then, since v is symmetric around the x-axis, we have that $\Omega_x(y) = \{z \in \partial B : |z - x| = |y - x|\} \subset T_0$. On the other hand, since v is also symmetric around the y-axis we obtain that $\Omega_y(z) \subset T_0$, for every $z \in \Omega_x(y)$. Now it is easily seen that

$$\bigcup_{z \in \Omega_x(y)} \Omega_y(z) = \{w \in \partial B : |w - y| \leq 2\sqrt{1 - (x \cdot y)^2}\} .$$

Of course if $y \neq \pm x$ the above set contains a neighborhood of y in ∂B. Hence, assuming that T_0 contains at least two distinct pairs of antipodal points, we obtain that T_0 is open. Since, of course, T_0 is closed, we obtain the thesis.

Proposition 4.4. *Either v is constant on ∂B or v has exactly one minimum point, say x_0, and one maximum point $-x_0$ on ∂B.*

Proof. This is just a straightforward consequence of the above proposition.

Let us define the one-parameter family of conformal mappings

$$R_\mu = \Psi \circ S_\mu \circ \Phi \, : B \to B \tag{4.3}$$

obtained by composition of the inversions maps Φ and Ψ as in (2.1) and (2.3) with the scaling maps on $\mathbb{R}^n_+$:

$$S_\mu : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+ : y \longrightarrow \frac{y}{\mu}. \tag{4.4}$$

Let us remark that S_1 is the identity. We define

$$v_\mu(x) = |\text{Jac } R_\mu(x)|^{(n-2)/2n} v(R_\mu(x)).$$

Proposition 4.5. *Assume $\nabla_{tang} v(\mathbf{e}_n) = 0$; then there is a positive μ such that v_μ is constant on ∂B.*

Proof. First, the v_μ's still solve (4.1) in $H^1(B)$, by conformal invariance of this problem. Therefore, although the R_μ's are singular at $-\mathbf{e}_n$, it follows by the regularity result in [3] that the v_μ's are in $\mathcal{C}^\infty(\overline{B})$. Moreover, it is easy to check that all the v_μ's are symmetric around the $\mathbf{e}_n$-axis, by symmetry properties of the R_μ's. Simple computations show that

$$R_\mu(\mathbf{e}_n) = \mathbf{e}_n, \quad \lim_{x \to -\mathbf{e}_n} R_\mu(x) = -\mathbf{e}_n \qquad \forall \mu > 0\ .$$

Furthermore, since

$$|\text{Jac } R_\mu(x)|^{(n-2)/2n} = \frac{1}{\mu^{(n-2)/2} |x + \mathbf{e}_n|^{n-2}} \frac{1}{|\mu^{-1}(\frac{x+\mathbf{e}_n}{|x+\mathbf{e}_n|^2} - (1/2)\mathbf{e}_n) + (1/2)\mathbf{e}_n|^{n-2}}\ ,$$

we obtain that

$$|\text{Jac } R_\mu(\mathbf{e}_n)|^{(n-2)/2n} = \mu^{-(n-2)/2}, \quad \lim_{x \to -\mathbf{e}_n} |\text{Jac } R_\mu(x)|^{(n-2)/2n} = \mu^{(n-2)/2} \quad \forall \mu > 0$$

so that

$$v_\mu(\mathbf{e}_n) = \mu^{-(n-2)/2} v(\mathbf{e}_n), \quad v_\mu(-\mathbf{e}_n) = \mu^{(n-2)/2} v(-\mathbf{e}_n).$$

To conclude the proof, let us choose $\bar{\mu}^{n-2} = v(\mathbf{e}_n)/v(-\mathbf{e}_n)$, so that $v_{\bar{\mu}}(-\mathbf{e}_n) = v_{\bar{\mu}}(\mathbf{e}_n)$. Since $v_{\bar{\mu}}$ is radially symmetric around the $\mathbf{e}_n$-axis, by Proposition 4.4 we deduce that either $v_{\bar{\mu}}$ is constant on ∂B or $v_{\bar{\mu}}$ has only two critical points on ∂B: a minimum point and a maximum point. Since $\nabla_{tang} v(\pm \mathbf{e}_n) = 0$ and $v_{\bar{\mu}}(-\mathbf{e}_n) = v_{\bar{\mu}}(\mathbf{e}_n)$ we conclude that $\max_{\partial B} v_{\bar{\mu}} = \min_{\partial B} v_{\bar{\mu}}$; that is, $v_{\bar{\mu}}$ is constant on ∂B.

Proof of Theorem 0.4. If $v_{|\partial B}$ is constant, that is $\nabla_{tang} v \equiv 0$, we obtain from Proposition 4.1 that v is symmetric around any axis and hence it is radially symmetric. If not, we know from Proposition 4.5 that one of the v_μ's is constant on the boundary. Since all the v_μ's solve (4.1) we obtain the thesis.

Proof of Theorem 0.2. If $\alpha = 0$, the only radially symmetric solution of (4.1) is the constant function $v(x) \equiv ((n-2)/2\beta)^{(n-2)/2}$. From Theorem 0.4 and the fact that conformal maps in B are compositions of the inversion map with scaling and translations, it follows that every positive nonconstant solution of (4.1) has the form $v(x) = H|x - x_0|^{2-n}$, for suitable values of $H > 0$ and $x_0 \in \mathbb{R}^n \setminus \overline{B}$. If u is any positive $\mathcal{D}^{1,2}$ solution of (4.2), then $v = \Phi^* u$ is a weak $H^1(B)$ solution of (4.1). By the regularity result in [3], v is actually $\mathcal{C}^\infty$ up to the boundary and has the above expression. A simple computation shows that u has the form $u(y) = K|y - y_0|^{2-n}$.

If $\alpha > 0$, a similar argument applies, taking into account that radially symmetric solutions of (4.1) have the form $v(x) = H(\lambda^2 + |x|^2)^{(2-n)/2}$, for suitable values of $H > 0$ and λ. Indeed, for regular radially symmetric functions, the equation $-\Delta v = v^{(n+2)/(n-2)}$ is equivalent to a second-order ordinary differential equation in $r = |x|$, with zero first derivative at the origin. It can be shown that all positive solutions to such equations have the form written above. The constants H and λ are then determined by imposing the condition on the radial derivative at $r = 1$.

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