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Symmetry-protected topological orders of one-dimensional spin systems with $D_2 + T$ symmetry

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In Liu *et al.* [Phys. Rev. B **84**, 075135 (2011)] we studied eight gapped symmetric quantum phases in $S = 1$ spin chains/ladders which respect a discrete spin rotation $D_2 \subset SO(3)$ and time reversal T symmetries. In this paper, using a generalized approach, we study all the 16 possible gapped symmetric quantum phases of one-dimensional (1D) integer spin systems with $D_2 + T$ symmetry. Those phases are beyond Landau symmetry breaking theory and cannot be characterized by local order parameters since they do not break any symmetry. They correspond to 16 symmetry protected topological (SPT) orders. We show that all the 16 SPT orders can be fully characterized by the physical properties of the symmetry protected degenerate boundary states at the ends of a chain segment. So we can measure and distinguish all the 16 SPT orders experimentally. We also show that all these SPT orders can be realized in $S = 1$ spin chain or spin ladder models. The gapped symmetric phases protected by subgroups of $D_2 + T$ are also studied. Again, all these phases can be distinguished by physically measuring their end "spins."

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I. INTRODUCTION

In recent years, topological order[1,2] and symmetry protected topological order[3,4] for gapped quantum ground states has attracted much interest. Here "topological" means that this new kind of orders is different from the symmetry breaking orders.[5–7] The new orders include fractional quantum Hall states,[8,9] 1D Haldane phase,[10] chiral spin liquids,[11,12] Z_2 spin liquids,[13–15] non-Abelian fractional quantum Hall states,[16–19] quantum orders characterized by projective symmetry group (PSG),[3,20] topological insulators[21–26], etc.

Recent studies indicate that the patterns of entanglements provide a systematic and comprehensive point of view to understand topological orders and SPT orders.[27–30] The phases with long-ranged entanglement have intrinsic topological orders, while symmetric short-range entangled nontrivial phases are said to have SPT orders. With a definition of phase and phase transition via local unitary transformations, one can get a complete classification for all 1D gapped quantum phases,[31–33] and partial classifications for some gapped quantum phases in higher dimensions.[29,34–37]

In contradiction to the suggestion from the symmetry breaking theory, even when the ground states of two Hamiltonians have the same symmetry, sometimes they still cannot be smoothly connected by deforming the Hamiltonian, as long as the deformed Hamiltonians all respect the symmetry. So those two states with the same symmetry can belong to two different phases. Those kind of phases, if gapped, are called SPT phases. The Haldane phase of spin-1 chain[10] is the first example of SPT phase, which is known to be protected by the $D_2 = \{E, R_x = e^{i\pi S_x}, R_y = e^{i\pi S_y}, R_z = e^{i\pi S_z}\}$ symmetry.[38] Interestingly, when additional time reversal symmetry is present, more SPT phases emerges.[33,39]

Topological insulators[21–26] is another example of SPT phases which has attracted much interest in literature. Compared to the topological insulators formed by free electrons, most SPT phases (including the ones discussed in this paper) are strongly correlated. A particular kind of strongly correlated SPT phases protected by time reversal symmetry is called the fractionalized topological insulators by some people.[40,41] An interesting and important question is how to classify different 1D SPT phases even in presence of strong correlations/interactions. For the Haldane phase in spin chains, it was thought that the degenerate end states and nonlocal string order can be used to describe the hidden topological order. However, if we remove the spin rotation symmetry but keep the parity symmetry, the Haldane phase is still different from the $\otimes_i |z\rangle_i$ (here $|z\rangle$ is the eigenstate of S_z with $S_z|z\rangle = 0$) trivial phase despite that the degenerate end states and nonlocal string order are destroyed by the absence of spin rotation symmetry.[4,38,42]

Recently, it was argued in Ref. 43 that the entanglement spectrum degeneracy (ESD) can be considered as the criteria to tell whether a phase is topologically ordered or not. However, all 1D gapped states are short range entangled and have no intrinsic topological orders from entanglement point of view.[31,44] On the other hand, many gapped 1D phases have nontrivial ESD. So ESD cannot correspond to the intrinsic topological orders. Then, one may try to use it to characterize nontrivial SPT orders as suggested in Ref. 45. ESD does describe some nontranslation invariant SPT phases protected by on-site symmetry. In particular, it reveals an important connection to the projective representation of the on-site symmetry group.[45] However, if there are more than one nontrivial SPT phases, then it is possible that the ESD cannot distinguish all these SPT phases any longer.[39]

It turns out that a clear picture and a systematic classification of all 1D SPT phases can be obtained after realizing the deep connection between local unitary transformation and gapped (symmetry protected) topological phases.[31–33] In particular, for 1D systems, all gapped phase that do not break the symmetry are classified by the 1D representations and projective representations of the symmetry group G [by the group cohomology classes $H^1(G,U_T(1))$ and $H^2(G,U_T(1))$, see Appendix A].[31–33]

In our previous paper we have calculated the eight classes of unitary projective representations of the point group $D_{2h} =$

$D_2 + T$, based on which we predicted eight SPT phases in integer spin models that respect the D_{2h} symmetry. We realized four interesting SPT phases in $S = 1$ spin chains, and showed that these phases can be distinguished experimentally by their different responses of the end states to magnetic field. In the present paper we will show that the group $D_2 + T$ has totally 16 projective representations when the representation of T is anti-unitary. We then study the properties of the corresponding 16 SPT phases, such as the dimension of their degenerate end states and their response to perturbations. Interestingly, we find that all these SPT phases can be distinguished by their different responses of the end states to various physical perturbations. We also show that all these SPT phases can be realized in $S = 1$ spin chains or spin ladders.

This paper is organized as the following. In Sec. II we show that there are 16 SPT phases that respect $D_2 + T$ symmetry, and all these phases can be distinguished experimentally. The realization of the 16 SPT phases in $S = 1$ spin chains and spin ladders are given in Sec. III. In Sec. IV we discuss the projective representations and SPT phases of two subgroups of $D_2 + T$. Section V is the conclusion and discussion. Some details about the derivations, together with a brief introduction to projective representations, group cohomology, and classification theory of SPT phases are given in the Appendices.

II. DISTINGUISHING 16 SPT PHASES WITH $D_2 + T$ SYMMETRY

Our interest is focus on the the anti-unitary group $D_2 + T$. To begin with, we will give some information about its linear representations. Since all the group elements are commuting, all the linear representations are one dimensional. The number of linear representations of depends on the representation space. When acting on Hilbert space, the linear representations are classified by $H^1(D_2 + T, U_T(1)) = (Z_2)^2$, which contains four elements. When acting on Hermitian operators, the linear representations are classified by $H^1(D_2 + T, (Z_2)_T) = (Z_2)^3$, which contains eight elements. More details about linear representations and the first group cohomology are given in Appendix C. The eight linear representations (with Hermitian operators as the representation "bases") are shown in Table IV. These eight representations collapse into four if the representation space is a Hilbert space, because the bases $|1,x\rangle$ and $i|1,x\rangle$ (similarly $|1,y\rangle$ and $i|1,y\rangle$, $|1,z\rangle$ and $i|1,z\rangle$, $|0,0\rangle$ and $i|0,0\rangle$) are not independent. In later discussion we will assume the linear representation space is an Hermitian operator space if not mentioned otherwise.

The projective representations are classified by the group cohomology $H^2(D_2 + T, U_T(1))$. There are totally 16 different classes of projective representations for $D_2 + T$, as shown in Table I. More discussions about group cohomology and projective representation are given in Appendices A, B, D and E. The 16 classes of projective representations correspond to 16 SPT phases. Our result agrees with the classification in Ref. 33, and the correspondence is illustrated by the indices $\omega(D_2), \beta(T), \gamma(D_2)$.

In all these 16 SPT phases, the bulk is gapped and has no linear response to perturbations. The only way to distinguish these phases is from their different edge states, which are described by projective representations. We stress that all the physical properties of a SPT phase are determined by the edge states and can be detected experimentally. The basic idea of detecting these SPT phases is to add various perturbations to break the $D_2 + T$ symmetry, and to see how those perturbations split the degeneracy of the edge states.

Let us first consider the case that the effective end spin of a SPT phase is twofold degenerate. We have three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ to lift the end spin degeneracy. When acting on the ground state subspace, some of the perturbation operators of the system have the same matrix elements as the Pauli matrix σ_m $(m = x, y, z)$. These perturbations can lift the ground state degeneracy and will be called active operators of that SPT phase. To judge whether a perturbation is an active operator, one can study how it transforms under the symmetry group $D_2 + T$. If it varies in the same way as one of the three Pauli matrices, then it is an active operator. In different SPT phases, the end spins form different projective representations of the $D_2 + T$ group, and consequently the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ form different linear representations of $D_2 + T$. As a result, different SPT phases have different active operators.

Suppose O is a perturbation operator. Under the symmetry transformation g, O varies as

$$u(g)^\dagger O u(g) = \eta_g(O) O, \tag{1}$$

where $u(g)$ is the representation of symmetry transformation g on the physical spin Hilbert space, and $\eta_g(O)$ is a 1D linear representation of the symmetry group $D_2 + T$. On the other hand, the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ also form linear representations of $D_2 + T$. In the end spin space, the Pauli matrices σ_m $(m = x, y, z)$ transform as

$$M(g)^\dagger \sigma_m M(g) = \eta_g(\sigma_m) \sigma_m, \tag{2}$$

where $M(g)$ is the projective representation of g (see Table I) on the end spin Hilbert space. If the physical operator O and the end spin operator σ_m belong to the same linear representation of the symmetry group, namely, if $\eta_g(O) = \eta_g(\sigma_m)$, then they should have the same matrix elements (up to a constant factor) in the end spin subspace. In Table I, the sequence of operators (O_1, O_2, O_3) are the active operators corresponding to the end spin operators $(\sigma_x, \sigma_y, \sigma_z)$, respectively.

Similarly, in the case that the end spin is four dimensional, there will be 15 4×4 matrices that (partially) lift the degeneracy of the end states. These matrices are direct products of Pauli matrices and unit matrix, namely, $({}_x \otimes I, {}_y \otimes I, {}_z \otimes I, {}_x \otimes {}_x, {}_y \otimes {}_x, {}_z \otimes {}_x, I \otimes {}_x, {}_x \otimes {}_y, {}_y \otimes {}_y, {}_z \otimes {}_y, I \otimes {}_y, {}_x \otimes {}_z, {}_y \otimes {}_z, {}_z \otimes {}_z, I \otimes {}_z)$. The corresponding active operators for each SPT phase are given in Table I.

Since the active operators spilt the ground state degeneracy, from linear response theory, they correspond to measurable physical quantities. For example, if the spin operators S_x, S_y, S_z are active operators, they couples to a magnetic field through the interaction

$$H' = \sum_i (g_x \mu_B B_x S_{x,i} + g_y \mu_B B_y S_{y,i} + g_z \mu_B B_z S_{z,i}). \tag{3}$$

The end spins may be polarized by the above perturbation. In a real spin-chain materials, due to structural defects, there are considerable number of end spins. They behave as impurity

TABLE I. All the projective representations of group $D_{2h} = D_2 + T$. We only give the representation matrices for the three generators R_z, R_x, and T. K stands for the antilinear operator. The 16 projective representations corresponds to 16 different SPT phase. This result agrees with the classification of combined symmetry $D_2 + T$ given in Ref. 33. The indexes $(\omega,\beta,\gamma) \equiv (\omega(D_2),\beta(T),\gamma(D_2))$ show this correspondence. Five of these SPT phases can be realized in $S = 1$ spin chain models and others can be realized in $S = 1$ spin ladders or large-spin spin chains. The active operators are those physical perturbations which (partially) split the irreducible end states.

	R_z	R_x	T	ω,β,γ	Dimension	Active operators[a]	Spin models ($S = 1$)
E_0	1	1	K	$1, 1, A$	1		chain (trivial phase)
E_0'	I	I	$\sigma_y K$	$1, -1, A$	2	$(S_{xyz}, S_{xyz}, S_{xyz})$[b]	ladder
E_1	I	$i\sigma_z$	$\sigma_y K$	$1, -1, B_1$	2	(S_z, S_z, S_{xyz})	ladder
E_1'	I	$i\sigma_z$	$\sigma_x K$	$1, 1, B_1$	2	$(S_{xy}, S_{xy}, S_{xyz})$	ladder
E_3	σ_z	I	$i\sigma_y K$	$1, -1, B_3$	2	(S_x, S_x, S_{xyz})	ladder
E_3'	σ_z	I	$i\sigma_x K$	$1, 1, B_3$	2	$(S_{yz}, S_{yz}, S_{xyz})$	ladder
E_5	$i\sigma_z$	σ_x	IK	$-1, 1, A$	2	(S_{yz}, S_y, S_{xy})	chain (T_y phase)
E_5'	$I \otimes i\sigma_z$	$I \otimes \sigma_x$	$\sigma_y \otimes IK$	$-1, -1, A$	4	$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_{xz}, S^1_y, S^3_z, S^1_{xy})$[c]	ladder
E_7	σ_z	$i\sigma_z$	$i\sigma_x K$	$1, 1, B_2$	2	$(S_{xz}, S_{xz}, S_{xyz})$	ladder
E_7'	σ_z	$i\sigma_z$	$i\sigma_y K$	$1, -1, B_2$	2	(S_y, S_y, S_{xyz})	ladder
E_9	$i\sigma_z$	σ_x	$i\sigma_x K$	$-1, 1, B_3$	2	(S_{yz}, S_{xz}, S_z)	chain (T_z phase)
E_9'	$I \otimes i\sigma_z$	$I \otimes \sigma_x$	$\sigma_y \otimes i\sigma_x K$	$-1, -1, B_3$	4	$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_y, S^1_{xz}, S^3_{xy}, S^1_z)$[d]	ladder
E_{11}	$i\sigma_z$	$i\sigma_x$	$\sigma_z K$	$-1, 1, B_1$	2	(S_x, S_{xz}, S_{xy})	chain (T_x phase)
E_{11}'	$I \otimes i\sigma_z$	$I \otimes i\sigma_x$	$\sigma_y \otimes \sigma_z K$	$-1, -1, B_1$	4	$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_y, S^1_{xz}, S^3_z, S^1_{xy})$[e]	ladder
E_{13}	$i\sigma_z$	$i\sigma_x$	$i\sigma_y K$	$-1, -1, B_2$	2	(S_x, S_y, S_z)	chain (T_0 phase)
E_{13}'	$I \otimes i\sigma_z$	$I \otimes i\sigma_x$	$\sigma_y \otimes i\sigma_y K$	$-1, 1, B_2$	4	$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_{xz}, S^1_y, S^3_{xy}, S^1_z)$[f]	ladder

[a]In the ground states of SPT phases corresponding to the two-dimensional projective representations, the active operators behave as $(\sigma_x, \sigma_y, \sigma_z)$, and for the four-dimensional projective representations, the active operators behave as $({}_x \otimes I, {}_y \otimes I, {}_z \otimes I, {}_x \otimes {}_x, {}_y \otimes {}_x, {}_z \otimes {}_x, I \otimes {}_x, {}_x \otimes {}_y, {}_y \otimes {}_y, {}_z \otimes {}_y, I \otimes {}_y, {}_x \otimes {}_z, {}_y \otimes {}_z, {}_z \otimes {}_z, I \otimes {}_z)$.

[b]We notate $S_{mn} = S_m S_n + S_n S_m$, where $m,n = x,y,z$. For $S = 1$, S_{xyz} means a multispin operator, such as $S_{xy,i} S_{z,i+1}$.

[c]$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_{xz}, S^1_y, S^3_z, S^1_{xy}) = (S_{xyz}, S_{xyz}, S_{xyz}, S_x, S_x, S_x, S_{yz}, S_{xz}, S_{xz}, S_{xz}, S_y, S_z, S_z, S_z, S_{xy})$. Here S^3_x, for example, means that S_x appears for three times: $S^3_x \to S_x, S_x, S_x$. Also, these three S_x, S_x, S_x do not correspond to the same physical operator. They correspond to three different operators that transform in the same way as the S_x operator. For instance, they may correspond to S_x at three different sites near the end spin.

[d]$(S^3_{xyz}, S^3_x, S^1_{yz}, S^3_y, S^1_{xz}, S^3_{xy}, S^1_z) = (S_{xyz}, S_{xyz}, S_{xyz}, S_x, S_x, S_x, S_{yz}, S_y, S_y, S_y, S_{xz}, S_{xy}, S_{xy}, S_{xy}, S_z)$.

[e]$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_y, S^1_{xz}, S^3_z, S^1_{xy}) = (S_{xyz}, S_{xyz}, S_{xyz}, S_{yz}, S_{yz}, S_{yz}, S_x, S_y, S_y, S_y, S_{xz}, S_z, S_z, S_z, S_{xy})$.

[f]$(S^3_{xyz}, S^3_{yz}, S^1_x, S^3_{xz}, S^1_y, S^3_{xy}, S^1_z) = (S_{xyz}, S_{xyz}, S_{xyz}, S_{yz}, S_{yz}, S_{yz}, S_x, S_{xz}, S_{xz}, S_{xz}, S_y, S_{xy}, S_{xy}, S_{xy}, S_z)$.

spins (the gapped bulk can be seen as a paramagnetic material). Thus, the polarizing of the end spins can be observed by measuring the magnetic susceptibility,[39] which obeys the Curie law ($m = x,y,z$)

$$\chi_m(T) = \frac{N g_m^2 \mu_B}{3 k_B T},$$

where N is the number of end spins.

Notice that different projective representations have different active operators. Thus we can distinguish all of the 16 SPT phases experimentally. For instance, the active operators of the E_1 and E_1' phases are (S_z, S_z, S_{xyz}) and $(S_{xy}, S_{xy}, S_{xyz})$, respectively. Here $S_{mn} = S_m S_n + S_n S_m$ is a spin quadrupole operator, and S_{xyz} is a third order spin operator, such as $S_{xy,i} S_{zi+1}$ or $S_{x,i} S_{y,i+1} S_{z,i+2}$. We will show that the two SPT phases E_1 and E_1' can be distinguished by the perturbation (3). In E_1 phase, the active operators contain S_z, so it response to B_z. In consequence, the g factors g_z is finite, but $g_x, g_y = 0$ (because S_x, S_y are not active operators). However, in E_1' phase, none of S_x, S_y, S_z is active, so the end spins do not response to magnetic field at all. As a consequence, all components of the g factor approaches zero: $g_x, g_y, g_z = 0$. This difference distinguishes the two phases.

To completely separate all the 16 SPT phases, one need to add perturbations by the spin-quadrupole operators S_{xy}, S_{yz}, S_{xz} and the third-order spin operators such as $S_{xy,i} S_{z,i+1}$. Actually, these perturbations may be realized experimentally. For instance, the interaction between the spin-quadrupole and a nonuniform magnetic field is reasonable in principle:

$$H' = g_{xy} \left(\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x} \right) S_{xy} + \cdots.$$

One can measure the corresponding "quadrupole susceptibility" corresponding to the above perturbation. Similar to the spin susceptibility, different SPT phases have different coupling constants for the quadrupole susceptibility. Consequently, from the information of the spin dipole and quadrupole susceptibilities (and other information corresponding to the third-order spin operators), all the 16 SPT phases can be distinguished.

III. REALIZATION OF SPT PHASES IN $S = 1$ SPIN CHAINS AND LADDERS

In this section we will illustrate that all these 16 SPT phases can be realized in $S = 1$ spin chains or ladders.

A. Spin chains

1. SPT phases for nontrivial projective representations

In Ref. 39 we have studied four nontrivial SPT phases T_0,T_x,T_y,T_z in $S=1$ spin chains. The ground states of these phases are written as a matrix product state (MPS)

$$|\phi\rangle = \sum_{\{m_i\}} \mathrm{Tr}\big(A_1^{m_1}A_2^{m_2}\cdots A_N^{m_N}\big)|m_1m_2\cdots m_N\rangle,$$

where $m_i=x,y,z$. More information about MPS is given in Appendix B.

1. T_0 phase. The end spins of this phase belong to the projective representation E_{13}, and a typical MPS in this phase is

$$A^x = a\sigma_x,\quad A^y = b\sigma_y,\quad A^z = c\sigma_z, \tag{4}$$

where a,b,c are real numbers.[48] Table I shows that the active operators in this phase are S_x,S_y,S_z, so the end spins will response to the magnetic field along all the three directions.

2. T_x phase. The end spins of this phase belong to the projective representation E_{11}, and a typical MPS in this phase is

$$A^x = a\sigma_x,\quad A^y = ib\sigma_y,\quad A^z = ic\sigma_z, \tag{5}$$

where a,b,c are real numbers. Table I shows that there is only one active operator S_x in this phase, so the end spins will only response to the magnetic field along x direction.

3. T_y phase. The end spins of this phase belong to the projective representation E_5, and a typical MPS in this phase is

$$A^x = ia\sigma_x,\quad A^y = b\sigma_y,\quad A^x = ic\sigma_z, \tag{6}$$

where a,b,c are real numbers. Table I shows that there is only one active operator S_y in this phase, so the end spins will only response to the magnetic field along y direction.

4. T_z phase. The end spins of this phase belong to the projective representation E_9, and a typical MPS in this phase is

$$A^x = ia\sigma_x,\quad A^y = ib\sigma_y,\quad A^z = c\sigma_z, \tag{7}$$

where a,b,c are real numbers. Table I shows that there is only one active operator S_z in this phase, so the end spins will only response to the magnetic field along z direction.

2. SPT phases for trivial projective representations

Corresponding to the trivial projective IRs, we can also construct trivial phases. Here "trivial" means that the ground state is in some sense like a direct product state. In these phases the matrix A^m also vary as Eqs. (B3) and (D1), except that A^m is a 1D matrix, and $M(g)$ is a 1D representation of D_2+T. Since all the 1D representation belongs to the same class, there is only one trivial phase.

A simple example of the states in this phase is a direct product state

$$|\phi\rangle = |m\rangle_1|m\rangle_2\cdots|m\rangle_N.$$

This state can be realized by a strong (positive) on-site single-ion anisotropy term $(S_m)^2$, $m=x,y,z$. In this phase, there is no edge state, and no linear response to all perturbations.

B. Spin ladders

In last section we have realized 5 of the 16 different SPT phases (with only D_2+T symmetry) in $S=1$ spin chains. In this section we will show that all the other phases can be realized in $S=1$ ladders.

1. General discussion for spin ladders

For simplicity, we will consider the spin-ladder models without interchain interaction.[49] In that case, the ground state of the spin ladder is a direct product of the ground states of the independent chains. For example, for a two-leg ladder, the physical Hilbert space at each site is a direct product space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ spanned by bases $|m_1n_1\rangle = |m_1\rangle|n_1\rangle$, with $m_1,n_1=x,y,z$. If the ground state of the two chains are $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively,

$$\begin{aligned} |\phi_1\rangle &= \sum_{\{m\}} \mathrm{Tr}(A^{m_1}\cdots A^{m_N})|m_1\cdots m_N\rangle,\\ |\phi_2\rangle &= \sum_{\{n\}} \mathrm{Tr}(B^{n_1}\cdots B^{n_N})|n_1\cdots n_N\rangle, \end{aligned} \tag{8}$$

with

$$\begin{aligned} \sum_{m'} u(g)_{mm'}A^{m'} &= e^{i\alpha_1(g)}M(g)^\dagger A^m M(g),\\ \sum_{n'} v(g)_{nn'}B^{n'} &= e^{i\alpha_2(g)}N(g)^\dagger B^n N(g) \end{aligned} \tag{9}$$

for an unitary operator $\hat g$ and

$$\begin{aligned} \sum_{m'} u(T)_{mm'}(A^{m'})^* &= M(T)^\dagger A^m M(T),\\ \sum_{n'} v(T)_{nn'}(B^{n'})^* &= N(T)^\dagger B^n N(T) \end{aligned} \tag{10}$$

for the time reversal operator T. Then the ground state of the ladder is

$$\begin{aligned} |\phi\rangle &= |\phi_1\rangle\otimes|\phi_2\rangle\\ &= \sum_{\{m,n\}} \mathrm{Tr}(A^{m_1}\cdots A^{m_N})\mathrm{Tr}(B^{n_1}\cdots B^{n_N})|m_1n_1\cdots m_Nn_N\rangle\\ &= \sum_{\{m,n\}} \mathrm{Tr}[(A^{m_1}\otimes B^{n_1})\cdots(A^{m_N}\otimes B^{n_N})]\\ &\quad\times|m_1n_1\cdots m_Nn_N\rangle \end{aligned} \tag{11}$$

which satisfies

$$\begin{aligned} &\sum_{m,n,m',n'} [u(g)\otimes v(g)]_{mn,m'n'}(A^{m'}\otimes B^{n'})\\ &\quad= e^{i\alpha(g)}(M\otimes N)^\dagger(A^m\otimes B^n)(M\otimes N) \end{aligned} \tag{12}$$

for an unitary $\hat g$ [here $\alpha(g)=\alpha_1(g)+\alpha_2(g)$] and

$$\begin{aligned} &\sum_{m,n,m',n'} [u(T)\otimes v(T)]_{mn,m'n'}(A^{m'}\otimes B^{n'})^*\\ &\quad= (M\otimes N)^\dagger(A^m\otimes B^n)(M\otimes N) \end{aligned} \tag{13}$$

for the time reversal operator T. This shows that the ground state of the ladder is also a MPS which is represented by $A^m\otimes B^n$, and $M\otimes N$ is a projective representation of the symmetry group G.

TABLE II. Projective representations of group $\bar{D}_2 = \{E, R_zT, R_xT, R_y\}$. There are four classes of projective representations, meaning that the second group cohomology contains four elements.

Class	E	R_y	R_zT	R_xT	Dimension	Effective/active operators	Spin models ($S = 1$)
1	1	1	K	K	1		chain (trivial phase)
	I	σ_y	$i\sigma_z K$	$\sigma_x K$	2	$\sigma_x \sim S_z, S_{yz};\ \sigma_y \sim S_y;\ \sigma_z \sim S_x, S_{xy}$	chain
2	I	I	$\sigma_y K$	$\sigma_y K$	2	$\sigma_x, \sigma_y, \sigma_z \sim S_{xz}$	ladder
	$I \otimes I$	$I \otimes \sigma_y$	$\sigma_y \otimes i\sigma_z K$	$\sigma_y \otimes \sigma_x K$	4		ladder
3	I	$i\sigma_z$	$\sigma_y K$	$\sigma_x K$	2	$\sigma_x, \sigma_y \sim S_z, S_{yz};\ \sigma_z \sim S_{xz}$	chain
	I	σ_y	$i\sigma_y K$	$i I K$	2	$\sigma_x, \sigma_z \sim S_z, S_{yz};\ \sigma_y \sim S_{xz}$	chain
4	I	$i\sigma_z$	$\sigma_x K$	$\sigma_y K$	2	$\sigma_x, \sigma_y \sim S_x, S_{xy};\ \sigma_z \sim S_{xz}$	chain
	I	σ_y	$i I K$	$i\sigma_y K$	2	$\sigma_x, \sigma_z \sim S_x, S_{xy};\ \sigma_y \sim S_{xz}$	chain

Specially, if B^n is 1D and $N(g) = 1$ (representing a trivial phase), then we have

$$\sum_{m,n,m',n'} [u(g) \otimes v(g)]_{mn,m'n'} (A^{m'} \otimes B^{n'}) = e^{i\alpha(g)} M^\dagger (A^m \otimes B^n) M. \tag{14}$$

In general the projective representation $M(g) \otimes N(g)$ is reducible. This means that the end spin of the ladder is a direct sum space of several irreducible projective representations (IPRs). These IPRs are degenerate and belong to the same class. However, this degeneracy is accidental, because only irreducible representation protected by symmetry is robust. Notice that we did not consider the interchain interaction in the ladder. If certain interaction is considered, the degeneracy between the same classes of IPRs can be lifted, and only one IPR remains as the end spin in the ground state. This IPR (or more precisely the class it belongs to) determines which phase the spin ladder belongs to.

2. $S = 1$ *spin ladders in different SPT phases*

In Appendix E we show how to obtain all the other IPRs by reducing the direct product representations of E_{13}, E_{11}, E_5, E_9. We start with these four IPRs because the corresponding SPT phases T_0, T_x, T_y, T_z have been realized in spin chains. Actually, the reduction procedure provides a method to construct spin ladders from spin chains and to realize all the SPT phases.

By putting two different spin chains (belonging to the T_0, T_x, T_y, T_z phases) into a ladder, we obtain six new phases corresponding to $E_1, E_1', E_3, E_3', E_7, E_7'$, respectively. If we put one more spin chain into the ladder, then we obtain five more new phases corresponding to $E_0', E_5', E_9', E_{11}', E_{13}'$, respectively. Therefore, together with T_0, T_x, T_y, T_z and the trivial phase in spin chains, we have realized all the 16 SPT phases listed in Table I. Furthermore, if we have translational symmetry, then from Sec. III A1 and Eq. (14), we have totally $16 \times 4 = 64$ different SPT phases in spin ladders, in accordance with the result of Ref. 31.

IV. SPT PHASES FOR SUBGROUPS OF $D_2 + T$

From the projective representations of group $D_2 + T$, we can easily obtain the projective representations of its subgroups. According to Table I, the representation matrices for the subgroups also form a projective representation, but usually it is reducible. By reducing these matrices, we can obtain all the IPRs of the subgroup.

A. $\bar{D}_2 = \{E, R_zT, R_xT, R_y\}$

This group is also a D_2 group except that half of its elements are anti-unitary. Notice that T itself is not a group element. This group has four 1D linear representations. In Table V in Appendix C we list the representation matrix elements, representational bases of physical spin and spin operators (for $S = 1$) according to each linear representation.

The projective representations of the subgroup $\bar{D}_2$ are shown in Table II. By reducing the representation matrix of $D_2 + T$ we obtained eight projective representations. They are classified into four classes. This can be shown by calculating the corresponding 2-cocycles of these projective representations. Two projective representations belonging to the same class means that the corresponding 2-cocycle differ by a 2-coboundary (see Appendices A, B, and D).

As shown in Table II, the two-dimensional representation in class-1 is trivial (or linear), it belongs to the same class as the 1D representation. This means that the edge states in this phase is not protected by symmetry, the ground state degeneracy can be smoothly lifted without phase transition. The class-3 and class-4 nontrivial SPT phases can be realized in spin chains. These two phases can be distinguished by magnetic fields. The phase corresponding to the class-3 projective representation only response to the magnetic field along z direction, and the phase corresponding to class-4 projective representation only respond to the magnetic field along x direction. The remaining two nontrivial SPT phases of class 2 can be realized by spin ladders.

B. $Z_2 + T = \{E, R_z, T, R_zT\}$

This subgroup is also a direct product group. The linear representations and projective representations are given in Table VI (see Appendix C) and III, respectively. This group is isomorphic to $\bar{D}_2 = \{E, R_zT, R_xT, R_y\}$, so its projective representations and SPT phases are one to one corresponding to those in Table II. However, the corresponding SPT phases in Tables III and II are not the same, because they have different response to external perturbations.

Notice that this simple symmetry is very realistic for materials. For example, the quasi-1D anti-ferromagnets $CaRuO_3$[50] and $NaIrO_3$[51] respect this $Z_2 + T$ symmetry due to spin-orbital

TABLE III. Projective representations of group $Z_2 + T = \{E, R_z, T, R_zT\}$.

Class	E	R_z	T	R_zT	Dimension	Effective/active operators	Spin models ($S = 1$)
1	1	1	K	K	1		chain
	I	σ_y	$i\sigma_z K$	$\sigma_x K$	2	$\sigma_x \sim S_x, S_y$; $\sigma_y \sim S_{xy}$; $\sigma_z \sim S_{yz}, S_{xz}$	chain
2	I	I	$\sigma_y K$	$\sigma_y K$	2	$\sigma_x, \sigma_y, \sigma_z \sim S_z$	ladder
	$I \otimes I$	$I \otimes \sigma_y$	$\sigma_y \otimes i\sigma_z K$	$\sigma_y \otimes \sigma_x K$	4		ladder
3	I	$i\sigma_z$	$\sigma_y K$	$\sigma_x K$	2	$\sigma_x, \sigma_y \sim S_x, S_y$; $\sigma_z \sim S_z$	chain
	I	σ_y	$i\sigma_y K$	iIK	2	$\sigma_x, \sigma_z \sim S_x, S_y$; $\sigma_y \sim S_z$	chain
4	I	$i\sigma_z$	$\sigma_x K$	$\sigma_y K$	2	$\sigma_x, \sigma_y \sim S_{xz}, S_{yz}$; $\sigma_z \sim S_z$	chain
	I	σ_y	iIK	$i\sigma_y K$	2	$\sigma_x, \sigma_z \sim S_{xz}, S_{yz}$; $\sigma_y \sim S_z$	chain

coupling. Their ground state, if nonsymmetry breaking, should belong to one of the four SPT phases listed in Table III.

V. CONCLUSION AND DISCUSSION

In summary, through the projective representations, we studied all the 16 different SPT phases for integer spin systems that respect only $D_{2h} = D_2 + T$ on-site symmetry. We provided a method to measure all the SPT orders. We showed that in different SPT phase the end spins respond to perturbations differently. The perturbations include spin dipole (coupling to uniform magnetic fields) and quadrupole operators (coupling to nonuniform magnetic fields). We illustrated that the SPT orders in different SPT phases can be observed by experimental measurements, such as the temperature dependence of the magnetic susceptibility and asymmetric g factors. We illustrated that all the 16 SPT phases can be realized in $S = 1$ spin chains or ladders. Finally we studied the SPT phases for two subgroups of $D_2 + T$, one of the subgroup is the symmetry group of some interesting materials.[50,51] Certainly our method of studying SPT orders can be generalized to other symmetry groups.

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APPENDIX A: GROUP COHOMOLOGY

We consider a finite group $G = \{g_1, g_2, \ldots\}$ with its module space $U_T(1)$. The group elements of G are operators on the module space. A n-cochain $\omega_n(g_1, g_2, \ldots, g_n)$ is a function on the group space which maps $\otimes^n G \to U(1)$. The cochains can be classified with the *coboundary operator*.

Suppose the cochain $\omega_n(g_1, g_2, \ldots, g_n) \in U(1)$, then the coboundary operator is defined as

$$\begin{aligned}(d\omega_n)(g_1, g_2, \ldots, g_{n+1}) &= g_1 \cdot \omega_n(g_2, g_3, \ldots, g_{n+1}) \\ &\omega_n^{-1}(g_1 g_2, g_3, \ldots, g_{n+1})\, \omega_n(g_1, g_2 g_3, \ldots, g_{n+1}) \cdots \\ &\omega_n^{(-1)^i}(g_1, g_2, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \cdots \\ &\omega_n^{(-1)^n}(g_1, g_2, \ldots, g_n g_{n+1}) \omega_n^{(-1)^{n+1}}(g_1, g_2, \ldots, g_n)\end{aligned}$$

for $n \geqslant 1$, and

$$(d\omega_0)(g_1) = \frac{g_1 \cdot \omega_0}{\omega_0} \tag{A1}$$

for $n = 0$. Here $g \cdot \omega_n$ is a group action on the module space $U(1)$. If g is an unitary operator, it acts on $U(1)$ trivially $g \cdot \omega_n = \omega_n$. If g is anti-unitary (such as the time reversal operator T), then the action is given as $g \cdot \omega_n = \omega_n^* = \omega_n^{-1}$. We will use $U_T(1)$ to denote such a module space. We note that, if G contain no time reversal transformation, then $U_T(1) = U(1)$.

A cochain ω_n satisfying $d\omega_n = 1$ is called a n-cocycle. If ω_n satisfies $\omega_n = d\omega_{n-1}$, then it is called a n-coboundary. Since $d^2\omega = 1$, a coboundary is always a cocycle. The following are two examples of cocycle equations. 1-cocycle equation:

$$\frac{g_1 \cdot \omega_2(g_2)\omega(g_1)}{\omega_2(g_1 g_2)} = 1. \tag{A2}$$

2-cocyle equation:

$$\frac{g_1 \cdot \omega_2(g_2, g_3)\omega_2(g_1, g_2 g_3)}{\omega_2(g_1 g_2, g_3)\omega_2(g_1, g_2)} = 1. \tag{A3}$$

The *group cohomology* is defined as $H^n(G, U_T(1)) = Z^n/B^n$. Here Z^n is the set of n-cocycles and B^n is the set of n-coboundarys. If two n-cocycles ω_n and ω_n' differ by a n-coboundary $\tilde{\omega}_n$, namely, $\omega_n' = \omega_n \tilde{\omega}_n^{-1}$, then they are considered to be equivalent. The set of equivalent n-cocycles is called a equivalent class. Thus, the n-cocycles are classified with different equivalent classes, these classes form the (Abelian) cohomology group $H^n(G, U_T(1)) = Z^n/B^n$.

As an example, we see the cohomology of $Z_2 = \{E, \sigma\}$, where E is the identity element and $\sigma^2 = E$. Since this group Z_2 is unitary, it acts on the module space trivially and $U_T(1) = U(1)$: $g \cdot \omega_n = \omega_n$. From (A2) the first cohomology is the 1D representations

$$H^1(Z_2, U(1)) = Z_2.$$

The second cohomology classifies the projective representations (see Appendix B). It can be shown that all the solutions of (A3) are 2-coboundaries $\omega_2 = d\omega_1$. So all the 2-cocycles belong to the same class, consequently,

$$H^2(Z_2, U(1)) = 0.$$

Let us see another example, the time reversal group $Z_2^T = \{E, T\}$. Notice that the time reversal operator T is anti-unitary, it acts on $U_T(1)$ nontrivially: $T \cdot \omega_n = \omega_n^{-1}$. As a result, the cohomology of Z_2^T is different from that of Z_2:

$$H^1\big(Z_2^T, U_T(1)\big) = 0,$$
$$H^2\big(Z_2^T, U_T(1)\big) = Z_2.$$

The group Z_2^T have two orthogonal 1D representations (see Appendix C), but above result shows that these two 1D representations belongs to the same class. Furthermore, the nontrivial second group cohomology shows that Z_2^T has a nontrivial projective representation, which is well known: $M(E) = I, M(T) = i\sigma_y K$.

APPENDIX B: BRIEF REVIEW OF THE CLASSIFICATION OF 1D SPT ORDERS

A key trick to use local unitary transformation to study/classify 1D gapped SPT phases is the matrix product state (MPS) representation of the ground states. The simplest example is the $S = 1$ AKLT wave function[46] in the Haldane phase which can be written as a 2×2 MPS. Later it was shown that in 1D all gapped many-body spin wave functions (it was generalized to fermion systems) can be well approximated by a MPS as long as the dimension D of the matrix is large enough[47]

$$|\phi\rangle = \sum_{\{m_i\}} \mathrm{Tr}\big(A_1^{m_1} A_2^{m_2} \cdots A_N^{m_N}\big)|m_1 m_2 \cdots m_N\rangle. \quad \text{(B1)}$$

Here m is the index of the d-component physical spin and $A_i^{m_i}$ is a $D \times D$ matrix. Provided that the system is translationally invariant, then one set all the matrices A^m as the same over all sites.

In the MPS picture it is natural to understand that projective representations can be used as a label of different SPT phase. Suppose that a system has an on-site unitary symmetry group G which keep the ground state $|\phi\rangle$ invariant

$$\hat{g}|\phi\rangle = u(g) \otimes u(g) \otimes \cdots \otimes u(g)|\phi\rangle = (e^{i\alpha(g)})^N |\phi\rangle, \quad \text{(B2)}$$

where $\hat{g} \in G$ is a group element of G, $u(g)$ is its d-dimensional (maybe reducible) representation, and $e^{i\alpha(g)}$ is its 1D representation. We only consider the case that $u(g)$ is a linear presentation of G. The case that $u(g)$ forms a projective representation of G (such as half-integer spin chain) has been studied in Refs. 31,33. Equations (B1) and (B2) require that the matrix A^m should vary in the following way[31,45]:

$$\sum_{m'} u(g)_{mm'} A^{m'} = e^{i\alpha(g)} M(g)^\dagger A^m M(g), \quad \text{(B3)}$$

where $M(g)$ is an invertible matrix and is essential for the classification of different SPT phases. Notice that if $M(g)$ satisfies Eq. (B3), so does $M(g)e^{i\varphi(g)}$. Since $u(g_1 g_2) = u(g_1)u(g_2)$ and $e^{i\alpha(g_1 g_2)} = e^{i\alpha(g_1)}e^{i\alpha(g_2)}$, we obtain

$$M(g_1 g_2) = M(g_1)M(g_2)e^{i\theta(g_1,g_2)}. \quad \text{(B4)}$$

The above equation shows that up to a phase $e^{i\theta(g_1,g_2)}$, $M(g)$ satisfies the multiplication rule of the group. Furthermore, $M(g)$ satisfies the associativity condition $M(g_1 g_2 g_3) = M(g_1 g_2)M(g_3)e^{i\theta(g_1 g_2, g_3)} = M(g_1)M(g_2 g_3)e^{i\theta(g_1, g_2 g_3)}$, or equivalently

$$e^{i\theta(g_2,g_3)}e^{i\theta(g_1,g_2 g_3)} = e^{i\theta(g_1,g_2)}e^{i\theta(g_1 g_2,g_3)}.$$

The above equation coincide with the cocycle equation (A3) when G is unitary. The matrices $M(g)$ that satisfies above conditions are called *projective representation* of the symmetry group G. Above we also shows the relation between projective representations and 2-cocycle.

For a projective representation, the two-element function $e^{i\theta(g_1,g_2)}$ has redundant degrees of freedom. Suppose that we introduce a phase transformation $M(g_1)' = e^{i\varphi(g_1)}M(g_1)$, $M(g_2)' = e^{i\varphi(g_2)}M(g_2)$, and $M(g_1 g_2)' = e^{i\varphi(g_1 g_2)}M(g_1 g_2)$, then the function $e^{i\theta(g_1,g_2)}$ becomes

$$e^{i\theta(g_1,g_2)'} = \frac{e^{i\varphi(g_1 g_2)}}{e^{i\varphi(g_1)}e^{i\varphi(g_2)}} e^{i\theta(g_1,g_2)}. \quad \text{(B5)}$$

Notice that $e^{i\theta(g_1,g_2)'}$ and $e^{i\theta(g_1,g_2)}$ differs by a 2-coboundary, so they belong to the same class. Thus, the projective representations are classified by the second group cohomology $H^2(G, U_T(1))$. If $M(g)$ and $\tilde{M}(g)$ belong to different (classes of) projective representations, then they cannot be smoothly transformed into each other, therefore the corresponding quantum states A^m and $\tilde{A}^m$ fall in different phases. In other words, the projective representation $\omega_2 \in H^2(G, U_T(1))$ provides a label of a SPT phase. If the system is translationally invariant, then $e^{i\alpha(g)} \in H^1(G, U_T(1))$ is also a label of a SPT phase. In this case, the complete label of a SPT phase is (ω_1, α). If translational symmetry is absent, we can regroup the matrix A^m such that $e^{i\alpha(g)} = 1$, then each SPT phase is uniquely labeled by ω_2.

APPENDIX C: LINEAR REPRESENTATIONS AND ACTIVE OPERATORS

Generally, the 1D linear representations of a group G are classified by its first group cohomology $H^1(G)$. However, there is a subtlety to choose the coefficient of $H^1(G)$. We will show that if the representation space is a Hilbert space, the 1D representations are characterized by $H^1(G, U(1))$ [or $H^1(G, U_T(1))$ if G contains anti-unitary elements]; while if the representation space is a Hermitian operator space, then the 1D representations are characterized by $H^1(G, Z_2)$ [notice that $H^1(G, (Z_2)_T) = H^1(G, Z_2)$, there is no difference whether G contains anti-unitary elements or not].

Since the discusses for unitary group and anti-unitary group are very similar, we will only consider a group G which contains anti-unitary elements. First we consider the 1D linear representations on a Hilbert space $\mathcal{H}$. Suppose $\phi \in \mathcal{H}$ is a basis, and $g \in G$ is an anti-unitary element, then

$$\hat{g}|\phi\rangle = \zeta(g)K|\phi\rangle, \quad \text{(C1)}$$

where the number $\zeta(g)$ is the representation of g. Notice that g is antilinear, which may change the phase of $|\phi\rangle$. To see that we suppose $K|\phi\rangle = |\phi\rangle$ and introduce a phase transformation for the basis $|\phi\rangle$, namely, $|\phi'\rangle = |\phi\rangle e^{i\theta}$. Now we choose $|\phi'\rangle$ as the basis, then

$$\hat{g}|\phi'\rangle = \zeta(g)e^{i2\theta}K|\phi'\rangle, \quad \text{(C2)}$$

TABLE IV. Linear representations of $D_{2h} = D_2 + T$.

	E	R_x	R_y	R_z	T	R_xT	R_yT	R_zT	Bases	Operators
A_g	1	1	1	1	1	1	1	1	$\|0,0\rangle$	S_x^2,S_y^2,S_z^2
B_{1g}	1	-1	-1	1	1	-1	-1	1	$i\|1,z\rangle$	S_{xy}
B_{2g}	1	-1	1	-1	1	-1	1	-1	$i\|1,y\rangle$	S_{xz}
B_{3g}	1	1	-1	-1	1	1	-1	-1	$i\|1,x\rangle$	S_{yz}
A_u	1	1	1	1	-1	-1	-1	-1	$i\|0,0\rangle$	$(S_{x,i}S_{yz,i+1})$
B_{1u}	1	-1	-1	1	-1	1	1	-1	$\|1,z\rangle$	S_z
B_{2u}	1	-1	1	-1	-1	1	-1	1	$\|1,y\rangle$	S_y
B_{3u}	1	1	-1	-1	-1	-1	1	1	$\|1,x\rangle$	S_x

so the representation $\zeta(g)' = \zeta(g)e^{i2\theta}$ changes accordingly. This means that the 1D representation of the group G is $U(1)$ valued, and is characterized by the first cohomology group $H^1(G,U(1))$. In the case of $D_2 + T$ we have

$$H^1(D_2 + T, U_T(1)) = (Z_2)^2,$$

so $D_2 + T$ has four different 1D linear representations on Hilbert space, which can be labeled as A, B_1, B_2, B_3, respectively.

Now we consider the 1D representations on a Hermitian operator space. Suppose $O_1, O_2, \ldots, O_N$ are orthonormal Hermitian operators satisfying $\mathrm{Tr}(O_m O_n) = \delta_{mn}$, an anti-unitary element $g \in G$ act on these operators as

$$\hat{g}\, O_m = K M(g)^\dagger O_m M(g) K = \sum_n \eta(g)_{mn} O_n. \qquad \text{(C3)}$$

Here $M(g)K$ is either a linear or a projective representation of g, while $\eta(g)$ is always a linear representation. Since $[K M(g)^\dagger O_m M(g) K]^\dagger = K M(g)^\dagger O_m M(g) K$, we have $[\sum_n \eta(g)_{mn} O_n]^\dagger = \sum_n \eta(g)^*_{mn} O_n = \sum_n \eta(g)_{mn} O_n$, which gives

$$\eta(g)^* = \eta(g).$$

The same result can be obtained if G is unitary. So we conclude that *all the linear representations defined on Hermitian operator space are real.* Now we focus on 1D linear representations. Since g is either unitary or anti-unitary, we have $|\eta(g)| = 1$. On the other hand, $\eta(g)$ must be real, so $\eta(g) = \pm 1$. As a result, all the 1D linear representations on Hermitian operator space are Z_2 valued, which are characterized by the first group cohomology $H^1(G,(Z_2)_T)$. For the group $D_2 + T$,

$$H^1(D_2 + T, (Z_2)_T) = (Z_2)^3,$$

so there are eight different 1D linear representation, corresponding to eight classes of Hermitian operators as shown in Table IV. Since all the linear representations of $D_2 + T$ are 1D, these eight 1D representations are all of its linear representations.

TABLE V. Linear representations of $\bar{D}_2 = \{E, R_zT, R_xT, R_y\}$.

	E	R_zT	R_xT	R_y	Bases or operators	
A	1	1	1	1	$\|0,0\rangle,\|1,y\rangle$	S_y,S_x^2,S_y^2,S_z^2
B_1	1	1	-1	-1	$\|1,x\rangle,i\|1,z\rangle$	S_x,S_{xy}
B_2	1	-1	-1	1	$i\|0,0\rangle,i\|1,y\rangle$	S_{xz}
B_3	1	-1	1	-1	$\|1,z\rangle,i\|1,x\rangle$	S_z,S_{yz}

TABLE VI. Linear representations of $Z_2 + T = \{E, R_z, T, R_zT\}$.

	E	R_z	T	R_zT	Bases or operators	
A_g	1	1	1	1	$\|0,0\rangle,i\|1,z\rangle$	S_{xy},S_x^2,S_y^2,S_z^2
A_u	1	1	-1	-1	$i\|0,0\rangle,\|1,z\rangle$	S_z
B_g	1	-1	1	-1	$i\|1,x\rangle,i\|1,y\rangle$	S_{yz},S_{xz}
B_u	1	-1	-1	1	$\|1,x\rangle,\|1,y\rangle$	S_x,S_y

The above discussion is also valid for the subgroups of $D_2 + T$. In Tables V and VI we give the linear representations of its two subgroups (the number of 1D linear representations on Hilbert space is half of that on Hermitian operator space).

We have shown that for 1D linear representations defined on Hermitian operator space, there is no difference whether a group element is unitary or anti-unitary. This conclusion is also valid for higher dimensional linear representations (however, if the representation space is a Hilbert space, unitary or anti-unitary group elements will be quite different). The linear representations on Hermitian operator space are used to define the active operators.

For a general group G, each of its nontrivial projective representations correspond to a SPT phase. In a certain SPT phase, the active operators are defined in the following way: for a set of Hermitian operators $\{O_1^{\mathrm{ph}}, \ldots, O_n^{\mathrm{ph}}\}$ acting on the physical spin Hilbert space, if we can find a set of Hermitian operators $\{O_1^{\mathrm{in}}, \ldots, O_n^{\mathrm{in}}\}$ acting on the end-spin Hilbert space (or the projective representation space), such that $\{O^{\mathrm{ph}}\}$ and $\{O^{\mathrm{in}}\}$ form the same n-dimensional real linear representation of G, then the operators $\{O^{\mathrm{ph}}\}$ are called active operators. Different SPT phases have different set of active operators, so we can use these active operators to distinguish different SPT phases.

APPENDIX D: 16 PROJECTIVE REPRESENTATIONS OF $D_2 + T$ GROUP

We have shown in Appendices A and B that the projective representations are classified by the second group cohomology $H^2(G,U_T(1))$. However, usually it is not easy to calculate the group cohomology. So we choose to calculate the projective representations directly. In the following we give the method through which we obtain all the 16 projective representations of $D_2 + T$ in Table I.

The main trouble comes from the anti-unitarity of some symmetry operators, such as the time reversal operator T. Under anti-unitary operators (such as T), the matrix A^m varies as

$$\sum_{m'} u(T)_{mm'} (A^{m'})^* = M(T)^\dagger A^m M(T). \qquad \text{(D1)}$$

Notice that $e^{i\alpha(T)}$ is absent because we can always set it to be 1 by choosing proper phase of A^m. To see more difference between the unitary operator and anti-unitary operators we introduce an unitary transformation to the bases of the virtual

TABLE VII. Unitary projective representations of $D_{2h} = D_2 + T$, here we consider T as an unitary operator.

	R_z	R_x	T
A_g	1	1	1
B_{1g}	1	-1	1
B_{2g}	-1	-1	1
B_{3g}	-1	1	1
A_u	1	1	-1
B_{1u}	1	-1	-1
B_{2u}	-1	-1	-1
B_{3u}	-1	1	-1
E_1	I	$i\sigma_z$	σ_y
$E_2 = E_1 \otimes B_{3g}$	$-I$	$i\sigma_z$	σ_y
E_3	σ_z	I	$i\sigma_y$
$E_4 = E_3 \otimes B_{1g}$	σ_z	$-I$	$i\sigma_y$
E_5	$i\sigma_z$	σ_x	I
$E_6 = E_5 \otimes A_u$	$i\sigma_z$	σ_x	$-I$
E_7	σ_z	$i\sigma_z$	$i\sigma_x$
$E_8 = E_7 \otimes B_{1g}$	σ_z	$-i\sigma_z$	$i\sigma_x$
E_9	$i\sigma_z$	σ_x	$i\sigma_x$
$E_{10} = E_9 \otimes A_u$	$i\sigma_z$	σ_x	$-i\sigma_x$
E_{11}	$i\sigma_z$	$i\sigma_x$	σ_z
$E_{12} = E_{11} \otimes B_{3g}$	$i\sigma_z$	$i\sigma_x$	$-\sigma_z$
E_{13}	$i\sigma_z$	$i\sigma_x$	$i\sigma_y$
$E_{14} = E_{13} \otimes A_u$	$i\sigma_z$	$i\sigma_x$	$-i\sigma_y$

spin such that A^m becomes $\bar{A}^m = U^\dagger A^m U$. Then for an unitary symmetry operation g, Eq. (B3) becomes

$$\sum_{m'} u(g)_{mm'} \bar{A}^{m'} = e^{i\alpha(g)} \bar{M}(g)^\dagger \bar{A}^m \bar{M}(g),$$

where $\bar{M}(g) = U^\dagger M(g) U$. However, for the anti-unitary operator T, $\bar{A}^m$ varies as

$$\sum_{m'} u(T)_{mm'} (\bar{A}^{m'})^* = \tilde{M}(T)^\dagger \bar{A}^m \tilde{M}(T),$$

where $\tilde{M}(T) = U^\dagger M(T) U^* = U^\dagger [M(T)K] U$. Therefore, we can see that $M(T)K$ as a whole is the anti-unitary projective representation of T when acting on the virtual spin space.

The question is how to obtain the matrix $M(T)$. In Ref. 39 we first treated T as an unitary operator, and we got eight classes of unitary projective representations for the group D_{2h} (see Table VII). By replacing $M(T)$ by $M(T)K$, we obtained eight different classes of anti-unitary projective representations. However, not all the projective representations can be obtained this way. Notice that $[M(T)K]^2 = 1$ and $[M(T)K]^2 = -1$ belong to two different projective representations, the anti-unitary projective representations are twice as many as the unitary projective representations. Fortunately, all the remaining (anti-unitary) projective representations can be obtained from the known ones. Notice that the direct product of any two projective representations is still a projective representation of the group, which can be reduced to a direct sum of several projective representations. There may be new ones in the reduced representations that are different from the eight known classes. Repeating this procedure (until it closes), we finally obtain 16 different classes of projective representations (see Appendix E). Notice that the Clebsch-Gordan coefficients which reduce the product representation should be real, otherwise it does not commute with K and will not block diagonalize the product representation matrix of T (and other anti-unitary symmetry operators). Because of this restriction, we obtain four four-dimensional irreducible projective representations (IPRs) which are absent in the unitary projective representations.

E. REALIZATION OF SPT PHASES IN $S = 1$ SPIN LADDERS

From the knowledge of Sec. III A, together with Eqs. (12) and (14), we can construct different SPT phases with spin ladders. From the discussion in Sec. III B 1, the projective representation $M(g) \otimes N(g)$ is usually reducible. It can be reduced to several IPRs of the same class. This class of projective representation determines which phase the ladder belongs to. Thus, the decomposition of direct products of different projective representations is important. Since the SPT phases corresponding to E_{13}, E_{11}, E_5, E_9 (T_0, T_x, T_y, T_z, separately) have been already realized in spin chains, we will first study the decompositions of the direct product of two of them.

$E_5 \otimes E_9 = (\sigma_z, I, i\sigma_x) \oplus (\sigma_z, -I, i\sigma_x) = E_3' \oplus E_4'$;
$E_5 \otimes E_{11} = (I, i\sigma_z, \sigma_x) \oplus (-I, i\sigma_z, \sigma_x) = E_1' \oplus E_2'$;
$E_5 \otimes E_{13} = (\sigma_z, i\sigma_z, i\sigma_y) \oplus (\sigma_z, -i\sigma_z, i\sigma_y) = E_7' \oplus E_8'$;
$E_9 \otimes E_{11} = (\sigma_z, i\sigma_z, i\sigma_x) \oplus (\sigma_z, -i\sigma_z, i\sigma_x) = E_7 \oplus E_8$;
$E_9 \otimes E_{13} = (I, i\sigma_z, \sigma_y) \oplus (-I, i\sigma_z, \sigma_y) = E_1 \oplus E_2$;
$E_{11} \otimes E_{13} = (\sigma_z, I, i\sigma_y) \oplus (\sigma_z, -I, i\sigma_y) = E_3 \oplus E_4$.

In above decomposition, all the CG coefficients are real. The three matrices in each bracket are the representation matrices for the three generators R_z, R_x, T, separately. We omitted the anti-unitary operator K for the representation matrix of T. Furthermore, E_1 and E_2 (E_3 and E_4, so on and so forth) belong to the same class of projective representation, and differs only by a phase transformation. So with spin ladders, we realize six SPT phases corresponding to the projective representations $E_1, E_1', E_3, E_3', E_7, E_7'$.

Using these projective representations $E_1, E_1', E_3, E_3', E_7, E_7'$, together with E_{13}, E_{11}, E_5, E_9, we can repeat above procedure and obtain more projective representations and their corresponding SPT phases. The result is shown below:

$E_1 \otimes E_3 = (\sigma_z, -i\sigma_z, i\sigma_x) \oplus (\sigma_z, i\sigma_z, -i\sigma_x) = E_7 \oplus E_8$;
$E_1 \otimes E_5 = (-I \otimes i\sigma_z, I \otimes i\sigma_x, -\sigma_y \otimes \sigma_z) = E_{11}'$;
$E_1 \otimes E_7 = (\sigma_z, -I, i\sigma_y) \oplus (-\sigma_z, I, i\sigma_y) = E_3 \oplus E_4$;
$E_1 \otimes E_9 = (-i\sigma_z, -i\sigma_x, -i\sigma_y) \oplus (-i\sigma_z, i\sigma_x, -i\sigma_y) = E_{13} \oplus E_{14}$;
$E_1 \otimes E_{11} = (-I \otimes i\sigma_z, -I \otimes \sigma_x, \sigma_y \otimes I) = E_5'$;
$E_1 \otimes E_{13} = (-i\sigma_z, I, -i\sigma_x) \oplus (-i\sigma_z, -I, -i\sigma_x) = E_9 \oplus E_{10}$;
$E_1' \otimes E_3 = (-\sigma_z, -i\sigma_z, -i\sigma_y) \oplus (-\sigma_z, i\sigma_z, i\sigma_y) = E_7' \oplus E_8'$;
$E_1' \otimes E_5 = (-i\sigma_z, i\sigma_x, -\sigma_z) \oplus (-i\sigma_z, i\sigma_x, \sigma_z) = E_{11} \oplus E_{12}$;
$E_1' \otimes E_7 = (-\sigma_z, I, i\sigma_x) \oplus (-\sigma_z, -I, -i\sigma_x) = E_3' \oplus E_4'$;
$E_1' \otimes E_9 = (-I \otimes i\sigma_z, I \otimes i\sigma_x, -i\sigma_y \otimes \sigma_y) = E_{13}'$;
$E_1' \otimes E_{11} = (-i\sigma_z, -\sigma_x, I) \oplus (-i\sigma_z, -\sigma_x, -I) = E_5 \oplus E_6$;
$E_1' \otimes E_{13} = (-I \otimes i\sigma_z, -I \otimes \sigma_x, -i\sigma_y \otimes \sigma_x) = E_9'$;

$E_3 \otimes E_5 = (-I \otimes i\sigma_z, I \otimes \sigma_x, i\sigma_y \otimes \sigma_x) = E'_9$;
$E_3 \otimes E_7 = (-I, i\sigma_x, \sigma_y) \oplus (I, i\sigma_x, \sigma_y) = E_1 \oplus E_2$;
$E_3 \otimes E_9 = (-I \otimes i\sigma_z, I \otimes \sigma_x, -\sigma_y \otimes I) = E'_5$;
$E_3 \otimes E_{11} = (-i\sigma_z, i\sigma_x, i\sigma_y) \oplus (-i\sigma_z, i\sigma_x, -i\sigma_y) = E_{13} \oplus E_{14}$;
$E_3 \otimes E_{13} = (-i\sigma_z, i\sigma_x, -\sigma_z) \oplus (-i\sigma_z, -i\sigma_x, \sigma_z) = E_{11} \oplus E_{12}$;
$E'_3 \otimes E_5 = (-i\sigma_z, \sigma_x, -i\sigma_x) \oplus (-i\sigma_z, -\sigma_x, -i\sigma_x) = E_9 \oplus E_{10}$;
$E'_3 \otimes E_7 = (-I, i\sigma_x, -\sigma_z) \oplus (I, i\sigma_x, -\sigma_z) = E'_1 \oplus E'_2$;
$E'_3 \otimes E_9 = (-i\sigma_z, \sigma_x, -I) \oplus (-i\sigma_z, -\sigma_x, I) = E_5 \oplus E_6$;
$E'_3 \otimes E_{11} = (-I \otimes i\sigma_z, I \otimes i\sigma_x, i\sigma_y \otimes \sigma_y) = E'_{13}$;
$E'_3 \otimes E_{13} = (-I \otimes i\sigma_z, I \otimes i\sigma_x, \sigma_y \otimes \sigma_z) = E'_{11}$;
$E_7 \otimes E_5 = (-I \otimes i\sigma_z, I \otimes i\sigma_x, -\sigma_y \otimes \sigma_y) = E'_{13}$;
$E_7 \otimes E_9 = (-i\sigma_z, i\sigma_x, \sigma_z) \oplus (i\sigma_z, -i\sigma_x, \sigma_z) = E_{11} \oplus E_{12}$;
$E_7 \otimes E_{11} = (-i\sigma_z, \sigma_x, -i\sigma_x) \oplus (i\sigma_z, \sigma_x, i\sigma_x) = E_9 \oplus E_{10}$;
$E_7 \otimes E_{13} = (-I \otimes i\sigma_z, -I \otimes \sigma_x, -\sigma_y \otimes I) = E'_5$;
$E'_7 \otimes E_5 = (-i\sigma_z, i\sigma_x, i\sigma_y) \oplus (-i\sigma_z, i\sigma_x, -i\sigma_y) = E_{13} \oplus E_{14}$;
$E'_7 \otimes E_9 = (-I \otimes \sigma_z, I \otimes i\sigma_x, \sigma_y \otimes \sigma_z) = E'_{11}$;
$E'_7 \otimes E_{11} = (-I \otimes i\sigma_z, -I \otimes \sigma_x, -i\sigma_y \otimes \sigma_x) = E'_9$;
$E'_7 \otimes E_{13} = (-i\sigma_z, \sigma_x, -I) \oplus (-i\sigma_z, \sigma_x, I) = E_5 \oplus E_6$;
$E_1 \otimes E'_1 = (I, I, \sigma_y) \oplus (I, -I, -\sigma_y) = E'_0 \oplus E'_0$;
$E_3 \otimes E'_3 = (-I, I, -\sigma_y) \oplus (I, I, \sigma_y) = E'_0 \oplus E'_0$;
$E_7 \otimes E'_7 = (I, -I, \sigma_y) \oplus (-I, I, \sigma_y) = E'_0 \oplus E'_0$.

Above we get four SPT phases corresponding to $E'_5, E'_9, E'_{11}, E'_{13}$, all of them have four-dimensional end spins. We also get a SPT phase corresponding to E'_0, which has two-dimensional end spins.

Notice that the number of classes of unitary projective representations of D_{2h} is eight, but considering that T is anti-unitary such that T^2 can be either 1 or -1, we obtain 16 classes of projective representations for $D_2 + T$.

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[48] When $a = b = c = 1$ this state is invariant under $SO(3) + T$, where $SO(3)$ is generated by S_x, S_y, S_z and $T = e^{i\pi S_y} K$. From Ref. 14, systems with $SO(3) + T$ symmetry have four SPT phases. It seems strange that its subgroup $D_2 + T$ contains more SPT phases. Actually, there are four distinct $SO(3) + T$ groups which contain $D_2 + T$ as a subgroup. In these four groups, T is always defined as $T = e^{i\pi S_y} K$, but the $SO(3)$ parts are different. Except for the one mentioned above, we have three additional choices of generators: $-S_x, S_{xz}, S_{xy}$ or $S_{yz}, -S_y, S_{xy}$ or $S_{yz}, S_{xz}, -S_z$. Each

of the four groups contains four SPT phases, so their common subgroup $D_2 + T$ contains $4 \times 4 = 16$ SPT phases.

[49] Actually, provided that the the symmetry group of the Hamiltonian of the ladder is $D_2 + T$, interchain interactions must be considered [otherwise the symmetry group should be $(D_2 + T) \otimes (D_2 + T)$]. Here we take the limit that the strength of interchain interaction tends to zero.

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