Symmetry Reductions and Exact Solutions of a class of Nonlinear Heat Equations

by

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August 10, 2002

Abstract

Classical and nonclassical symmetries of the nonlinear heat equation

$$u_t = u_{xx} + f(u), \tag{1}$$

are considered. The method of differential Gröbner bases is used both to find the conditions on $f(u)$ under which symmetries other than the trivial spatial and temporal translational symmetries exist, and to solve the determining equations for the infinitesimals. A catalogue of symmetry reductions is given including some new reductions for the linear heat equation and a catalogue of exact solutions of (1) for cubic $f(u)$ in terms of the roots of $f(u) = 0$.

1 Introduction

The nonlinear heat equation

$$u_t = u_{xx} + f(u), \tag{1.1}$$

where x and t are the independent variables, $f(u)$ is an arbitrary sufficiently differentiable function and subscripts denote partial derivatives, arises in several important physical applications including microwave heating (where $f(u)$ is the rate of absorption of microwave energy, cf. [80, 92]), in the theory of chemical reactions (where $f(u)$ is the temperature dependent reaction rate, cf. [2, 5, 31]) and in mathematical biology (where $f(u)$ represents the reaction kinetics in a diffusion process, cf. [65]).

The classical method for finding symmetry reductions of PDES is the Lie group method of infinitesimal transformations [3, 10, 11, 49, 73, 78, 86, 93]. To apply the classical method to the nonlinear heat equation (1.1) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{1.2}$$

where ε is the group parameter. Requiring that (1.1) is invariant under this transformation yields an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{1.3}$$

Classical symmetries of the nonlinear heat equation (1.1) have studied by Dorodnitsyn [24] who classified conditions on $f(u)$ for which symmetries exist, though he did not give the associated reductions (see also [32, 77]).

There have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [78] developed the method of partially invariant solutions. Bluman and Cole [9], in their study of symmetry reductions of the linear heat equation, proposed the so-called nonclassical method of group-invariant solutions (in the sequel referred to as the *nonclassical method*), which is also known as the "method of conditional symmetries" [33, 34, 36, 37, 59, 76, 81, 82, 103] and the "method of partial symmetries of the first type" [97]. In this method, the original PDE (1.1) is augmented with the invariant surface condition

$$\psi \equiv \xi(x,t,u)u_x + \tau(x,t,u)u_t - \phi(x,t,u) = 0, \tag{1.4}$$

which is associated with the vector field (1.3). By requiring that both (1.1) and (1.4) are invariant under the transformation (1.2) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$, which appear in both the transformation (1.2) and the supplementary condition (1.4). Since the number of determining equations arising in the nonclassical method is smaller than for the classical method and since all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, the solution set may be larger in the nonclassical case. For some equations, such as the Korteweg-de Vries equation, the infinitesimals arising from the classical and nonclassical methods coincide. It should be emphasized that the associated nonclassical vector fields do not form a vector space, still less a Lie algebra, since the invariant surface condition (1.4) depends upon the particular reduction. Subsequently, these methods were further generalized by Olver and Rosenau [75, 76] to include "weak symmetries" and, even more generally, "side conditions" or "differential constraints", and they concluded that "the unifying theme behind finding special solutions of PDES is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of PDES".

In this paper we consider both classical and nonclassical symmetries of the nonlinear heat equation (1.1), and find conditions on $f(u)$ under which symmetries, other than trivial translational ones, may exist. The method used to find the necessary conditions on $f(u)$, and from there to find the symmetries, is that of differential Gröbner bases (DGBS) [61, 63]. This method yields the "triangulation" of a nonlinear system of PDES from which solutions can be obtained in a systematic way.

The basic algorithm used to generate a DGB for a system of PDES is the Kolchin-Ritt algorithm, which forms an essential part of the StandardForm algorithm [83, 84] and the diffGbasis algorithm [63]. It appears to have been formally written down first by Carrà-Ferro [15]. The Kolchin-Ritt algorithm has been used for some time, intuitively if not formally, to yield integrability conditions for linear and orthonomic systems [57, 83]. With hindsight, one can see that the integrability conditions generated by the Janet-Riquier algorithm, which completes a system of orthonomic PDES to involutive form (cf. for example [52, 53, 89, 94, 95] and references therein), are produced by exactly the calculations performed in the Kolchin-Ritt algorithm. (It should be noted that several authors incorrectly assume that when the Kolchin-Ritt algorithm terminates, the output system is involutive, or passive. A counter-example can be found in [61].) The production of integrability conditions requires a total degree ordering on the derivative terms. On the other hand, the elimination ideals, or triangulation, of a system of PDES requires a purely lexicographical ordering.

Recently, the limits of the Kolchin-Ritt algorithm in calculating a DGB of a nonlinear system were found, and sufficient conditions for obtaining a DGB for a nonlinear system of PDES were formulated [61]. The Kolchin-Ritt algorithm generates a DGB only "up to" (in a sense to be defined) a certain set of differential coefficients, and these coefficients must not lie in the ideal. Proving that these coefficients do not lie in the ideal has no algorithmic solution as yet, and there are reasons as to why it may not have one at all. The condition is also one of the sufficient conditions in [63] for a DGB of a nonlinear system. Thus far, attempts to formulate algorithms that circumnavigate the difficulty, for example by calculating the algebraic Gröbner basis of successive prolongations of the system [15, 72, 79], lead to processes that do not terminate in finite time. Fortunately in many cases one can prove the condition directly, and there is a range of techniques that can be used when the condition fails. Despite the limitation, the Kolchin-Ritt algorithm is the fundamental tool in the theory, and improvements to efficiency and practicality of this algorithm is the most important problem to be faced from an applied point of view. A promising approach to this problem can be found in the thesis of Lisle [60], who rewrites the systems of PDES using first order noncommutative operators to effect an improvement in efficiency.

The next section contains an outline of the theory of differential Gröbner bases. After establishing the notation and formulae required, we show how to calculate the triangulation of a system of PDES using this theory. A variation of the Kolchin-Ritt algorithm is given that appears to be much faster than the one presented in [15, 63], at least for the examples calculated in this paper, and has been developed especially for use in solving the determining equations arising from the nonclassical method. Such systems of determining equations appear to have a natural ordering which can be exploited to improve the efficiency of the calcualtions. We also give a "skeletal" form of the Kolchin-Ritt algorithm which we found useful in extracting information to reduce the complexity of the calculations. We refer to this second algorithm as "Direct Search", which appears to be new.

In the third section of this paper, classical symmetries of the general nonlinear heat equation are catalogued for all those nonlinear $f(u)$ where a non-trivial symmetry is possible. Similarly, nonclassical symmetries are given in the fourth section, including some nonclassical symmetries for the linear heat equation. A catalogue of nonclassical symmetry reduction solutions, for cubic f, is given in the fifth section; the classification is in terms of the roots of $f(u)$. The paper concludes with some plots of several exact solutions of (1.1) derived from nonclassical symmetry reductions.

There is much current interest in the determination of symmetry reductions of PDEs which reduce the equations to ODEs. One frequently then checks if the resulting ODE is of *Painlevé type* (i.e., its solutions have no movable singularities other than poles). It appears to be the case that whenever the ODE is of Painlevé type, or can be transformed to one that is, then it can be solved explicitly, leading to exact solutions to the original equation. Conversely, if the resulting ODE is not of Painlevé type, then often one is unable to solve it explicitly.

Since solutions of PDEs asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, for reaction-diffusion equations such as (1.1), exact solutions arising from symmetry methods can often be effectively used to study properties such as "blow-up" [38, 39, 40]. Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; these solutions provide an important practical check on the accuracy and reliability of such integrators (cf. [4, 86, 91]).

Several exact solutions of equations of the form (1.1) have been derived in the literature. Ablowitz and Zeppetella [1] obtained an exact travelling wave solution of Fisher's equation [29]

$$u_t = u_{xx} + u(1-u) \tag{1.5}$$

by finding the special wave speed for which the resulting ODE is of Painlevé-type. Recently Guo and Chen [41] have used the Painlevé expansion method [67, 100, 101] to obtain some heteroclinic and homoclinic solutions of (1.5). Kaliappan [54] and Herrera, Minzoni and Ondarza [48] have derived travelling wave solutions of the generalized Fisher's equations

$$u_t = u_{xx} + u - u^k, \tag{1.6}$$
$$u_t = u_{xx} + u^p - u^{2p-1} \tag{1.7}$$

respectively. Cariello and Tabor [12, 13] found an exact solution of the real Newell-Whitehead equation [68] (or Kolmogoroff-Petrovsky-Piscounov equation [58]) equation

$$u_t = u_{xx} + u(1-u^2) \tag{1.8}$$

using a truncated Painlevé expansion and verified that it derives from a nonclassical symmetry reduction (see also [23]). Several authors have studied exact solutions of the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-a) \tag{1.9}$$

where a is an arbitrary parameter, which arises in population genetics [6, 7] and models the transmission of nerve impulses [30, 66]. Travelling wave solutions of the Fitzhugh-Nagumo equation (1.9) have been studied by several authors [6, 7, 28, 42, 96]. Exact solutions of (1.9) have been obtained using various techniques including Vorob'ev [97] (who calls the associated symmetry a "partial symmetry of the first type"), by Kawahara and Tanaka [55] using Hirota's bi-linear method [50], by Hereman [45] using the truncated Painlevé expansion method (see also [17]) and more recently by Nucci and Clarkson [71] using the nonclassical method (see also [21]). Exact solutions of the Huxley equation

$$u_t = u_{xx} + u^2(1-u) \tag{1.10}$$

have been obtained by Chen and Guo [17], using a truncated Painlevé expansion (see also [18, 27]), and by Clarkson and Mansfield [21] using the nonclassical method. Exact solutions of equation

$$u_t = u_{xx} - u^3 \tag{1.11}$$

have been obtained by Clarkson and Mansfield [21] using the nonclassical method.

A number of symbolic manipulation programmes have been developed, e.g. in MACSYMA [16], MAPLE [14, 83, 84, 85], MATHEMATICA [8, 47], MUMATH [44] and REDUCE [56, 69, 87, 88, 89, 90], that calculate the determining equations for classical Lie symmetries of PDES. A survey of the different packages presently available and a discussion of their strengths and applications is given in [46]. The determining equations for symmetries solved in this paper were calculated using the MACSYMA programme SYMMGRP.MAX [16]. The triangulations of these systems of PDES were calculated using the MAPLE package diffgrob2 [62]. The plots of some of the exact solutions of (1.1) derived here were drawn using MAPLE and MATHEMATICA.

2 Differential Gröbner Bases

In this section we discuss the triangulation of a system of PDES from which solutions may be obtained in a systematic way and give some algorithms using which a triangulation can be determined. First we introduce some required notation.

2.1 Some Notation

The determining equations for symmetries of PDES can be regarded as polynomials in the infinitesimals ξ, τ and ϕ, their derivatives, and the variables $\{x, t, u\}$, with complex coefficients. We establish our notation and formulae required for the statement of the algorithms in generic co-ordinates for ease of reference.

Let f be a differential polynomial; that is, a polynomial with complex coefficients in the independent variables $\{x_1, \ldots, x_n\}$, the functions $\{u_1, \ldots, u_m\}$ and the derivative terms $\{\mathrm{D}^\alpha u_j\}$ where

$$\mathrm{D}^\alpha u_j = \frac{\partial^{|\alpha|} u_j}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

The set of all such differential polynomials (d.p.'s) is denoted $R_{n,m}$ or
$_{\text{diff}}(x_1, \ldots, x_n; u_1, \ldots, u_m)$.

All the concepts depend upon an ordering on the derivative terms. There are many suitable orderings for different purposes. They need to satisfy $\mathrm{D}^\alpha u_j > \mathrm{D}^\beta u_k$ implies that $\mathrm{D}^\gamma \mathrm{D}^\alpha u_j > \mathrm{D}^\gamma \mathrm{D}^\beta u_k$ and $\mathrm{D}^\gamma u_j < \mathrm{D}^\gamma \mathrm{D}^\alpha u_j$. We assume that $u_j > x_k$ for all j and k. The standard lexicographic ordering based on $u_1 < u_2 < \ldots < u_m$ and $x_1 < x_2 < \ldots < x_n$ is given by

$\mathrm{D}^\alpha u_j > \mathrm{D}^\beta u_k$
if $u_j > u_k$,
else $j = k \alpha_n > \beta_n$,
else $\alpha_n = \beta_n, \ldots, \alpha_{n-j} = \beta_{n-j}, \alpha_{n-j-1} > \beta_{n-j-1}$ for some j such that $0 \le j \le n-2$.

The *highest derivative* term occurring in a d.p., f, is denoted $\mathrm{HDT}(f)$.
The *highest power* of the $\mathrm{HDT}(f)$ occurring in f is denoted $Hp(f)$.
The *highest coefficient*, $H\mathrm{coeff}(f)$, is defined to be $\mathrm{coeff}(f, \mathrm{HDT}(f)^{Hp(f)})$.

The *head* of f is $\mathrm{Head}(f) = H\mathrm{coeff}(f)\mathrm{HDT}(f)^{Hp(f)}$.

The *separant* of f, the highest coefficient of $\mathrm{D}^\alpha f$ for any non-zero multi-index α, is denoted $Sep(f)$.

A *pseudo-reduction* of a d.p. f by $G = \{g_1, g_2, \ldots, g_k\} \subset R_{n,m}$ effects elimination from f of any terms that are of the form $\mathrm{D}^\alpha\mathrm{HDT}(g_i)$, for some $i \in \{1, \ldots, k\}$ and $\alpha \in {}^n$.

Let a derivative term DT occur in f to some power p. Suppose there exists an $\alpha \in {}^n$ such that $\mathrm{D}^\alpha\mathrm{HDT}(g) = DT$ for some $g \in G$. If $\alpha = 0$ assume further that $p \geq Hp(g)$. A pseudo-reduction to f' of f by g is given by the formulae

$$f' = \begin{cases} \dfrac{H\mathrm{coeff}(\mathrm{D}^\alpha g)f - \mathrm{coeff}(f,\, DT^p)DT^{p-1}\mathrm{D}^\alpha g}{Z}, & \text{if } \alpha \neq 0, \\ \dfrac{H\mathrm{coeff}(g)f - \mathrm{coeff}(f,\, DT^p)\mathrm{HDT}(g)^{(p-Hp(g))}g}{Z}, & \text{if } \alpha = 0 \end{cases}$$

where Z is the greatest common divisor of $H\mathrm{coeff}(\mathrm{D}^\alpha g)$ and $\mathrm{coeff}(f,\, DT^p)$.

The *pseudo-normal* form of f with respect to a set G is obtained when no further pseudo-reduction with respect to any member of G is possible, and is denoted $\mathrm{normal}^p(f, G)$.

The *diffSpolynomial* of two d.p.'s is obtained first by cross-differentiating and then by cross-multiplying by the highest coefficients, and subtracting.

Let f_1 and f_2 be two d.p.'s with the same highest unknown. Let α_1 and α_2 be the smallest multi-indices possible such that $\mathrm{D}^{\alpha_1}\mathrm{HDT}(f_1) = \mathrm{D}^{\alpha_2}\mathrm{HDT}(f_2)$.

Let $Z = \gcd(H\mathrm{coeff}(\mathrm{D}^{\alpha_1} f_1),\, H\mathrm{coeff}(\mathrm{D}^{\alpha_2} f_2))$. If both $\alpha_1,\, \alpha_2 \neq 0$, define

$$\mathrm{diff}S\mathrm{poly}(f_1, f_2) = \frac{H\mathrm{coeff}(\mathrm{D}^{\alpha_1} f_1)\mathrm{D}^{\alpha_2} f_2 - H\mathrm{coeff}(\mathrm{D}^{\alpha_2} f_2)\mathrm{D}^{\alpha_1} f_1}{Z}.$$

If $\alpha_1 = 0$ and $\alpha_2 \neq 0$, then

$$\mathrm{diff}S\mathrm{poly}(f_1, f_2) = \frac{H\mathrm{coeff}(f_1)\mathrm{HDT}(f_1)^{(Hp(f_1)-1)}\mathrm{D}^{\alpha_2} f_2 - H\mathrm{coeff}(\mathrm{D}^{\alpha_2} f_2)f_1}{Z}$$

and similarly if $\alpha_1 \neq 0$ and $\alpha_2 = 0$.

If $\alpha_1 = \alpha_2 = 0$ so that $\mathrm{HDT}(f_1) = \mathrm{HDT}(f_2)$, or if f_1 and f_2 have different highest unknowns, then the differential S polynomial is defined to be

$$\mathrm{diff}S\mathrm{poly}(f_1,\, f_2) = \frac{\mathrm{Head}(f_2)f_1 - \mathrm{Head}(f_1)f_2}{\gcd(\mathrm{Head}(f_1),\, \mathrm{Head}(f_2))}.$$

For linear equations with different highest unknowns, we take their diffSpolynomial to be zero.

For $G \subset R_{n,m}$ let $S(G)$ be the multiplicative set in $R_{n,m}$ generated by the set of factors of all the highest coefficients and separants of the g in G. We assume that $\{1, -1\} \in S(G)$.

2.2 The Triangulation of a System of PDEs

Given a system of PDES, $\Sigma \subset R_{n,m}$, we define the ideal generated by Σ to be all those equations that can be obtained from the elements of Σ by differentiating and adding, and multiplying by arbitrary elements of $R_{n,m}$. In other words,

$$I(\Sigma) = \Big\{\sum_{\alpha,i} g_{\alpha,i}\mathrm{D}^\alpha f_i : f_i \in \Sigma,\, g_{\alpha,i} \in R_{n,m},\, \alpha \in {}^n\Big\}.$$

If the system is linear, we allow multiplication by polynomials in the variables $x_1, \ldots, x_n$ over only.

A *differential Gröbner basis* of $I(\Sigma)$ is defined to be a set of generators G of $I(\Sigma)$ such that every element of $I(\Sigma)$ pseudo-reduces to zero with respect to G.

Differential Gröbner Bases (DGBS) of ideals of differential polynomials have many useful properties and can be used to solve, in theory at least, a great many problems. These include being able to find all elimination ideals, integrability conditions and all compatibility conditions of a system of nonlinear PDES [63].

A DGB of a system of PDES depends on the ordering of derivative terms. Given a DGB for a system Σ of nonlinear PDES, we have the following theorem [63]:

THEOREM: **Elimination Ideals**. *If Σ is a* DGB *for $I(\Sigma)$ in a lexicographic ordering with $u_1 < u_2 < \ldots < u_m$ and $x_1 < x_2 < \ldots < x_n$, then $\Sigma_{p,q} := \Sigma \cap {}_{\text{diff}}(x_1, \ldots, x_p, u_1, \ldots, u_q)$ generates, up to $S(\Sigma)$ $I_{p,q} = I(\Sigma) \cap {}_{\text{diff}}(x_1, \ldots, x_p, u_1, \ldots, u_q)$. That is, for every $f \in I_{p,q}$ there exist an $s \in S(\Sigma)$ such that $sf \in \Sigma_{p,q}$ that is, sf is in the ideal generated by $\Sigma_{p,q}$.* □

In other words, every condition involving only the first q unknowns with derivatives with respect to the first p variables, obtainable from the original set of PDES by differentiating, adding, multiplying by like d.p.'s and so on, can be "read off" from a DGB of the system in a lexicographic order, up to $S(\Sigma)$, which represents the "margin of error" in the theory.

For systems Σ that are linear in their highest derivative terms, the set $S(\Sigma)$ is trivial. The difficulties posed by non-trivial $S(\Sigma)$ are caused by the use of pseudo-reduction in which one is allowed to multiply the expression by non-trivial coefficients. However replacing pseudo-reduction by reduction, in which only multiplication by constants or polynomials in the independent variables is allowed, leads to an algorithm that will not terminate on general nonlinear systems. Even if terminated artificially, algorithms using reduction have two additionally problems. First, the expression swell renders the method impractical even for small problems, while second, there is currently no method for deciding at which level of prolongation of the system to terminate.

The sets $I(\Sigma) \cap {}_{\text{diff}}(x_1, \ldots, x_p, u_1, \ldots, u_q)$ are called the elimination ideals of $I(\Sigma)$, while the DGB listed in decreasing order, with respect to the standard lexigraphic ordering described above, is called the "triangulation" of the system [83].

2.3 The Algorithms

The basic algorithm used to calculate a DGB is the Kolchin-Ritt algorithm [15, 61, 63, 83, 84]. In this algorithm, pairs of d.p.'s are cross-differentiated and cross-multiplied to make the leading derivative terms in each d.p. cancel. The result is then pseudo-reduced with respect to all other d.p.'s to obtain a new d.p. This is done systematically until the result of cross-differentiating all pairs of d.p.'s given or obtained pseudo-reduce to zero.

In practice, there are three difficulties in calculating a DGB. The first is the problem of the build-up of differential coefficients obtained by repeated cross-multiplication and pseudo-reduction. If the coefficient of a highest derivative term of any d.p. contains a derivative term, then Kolchin-Ritt no longer necessarily suffices to calculate a DGB. Sufficient conditions may be found in [63], but there remains the difficulty that one of the sufficient conditions given in [63] is not algorithmic. This condition is that none of the highest coefficients or separants of the equations comprising the system lie in the ideal they generate, that is, $S(\Sigma) \cap I(\Sigma) = \emptyset$ (cf., §2.2). In addition, if this condition fails then pseudo-reduction can lead to spurious zeroes in the calculation of the algorithms, resulting in loss of information. It is well known that the complexity of the Buchberger algorithm for computing the Gröbner basis of a polynomial ideal is doubly exponential [64]. Since the Kolchin-Ritt algorithm is modelled on the Buchberger algorithm, we have that the complexity of the Kolchin-Ritt algorithm is at least that of Buchberger's. However the main problem is that of "expression swell" in which the length of the expressions calculated exceed the memory limits of the available computer.

In spite of these problems, a DGB can often be obtained by a clever choice of the order in which pairs are cross-differentiated. We present here a version of the Kolchin-Ritt algorithm which for some systems is much faster than that given in [15, 63]. This version of the Kolchin-Ritt algorithm sorts the system after each new condition is found so that we are essentially finding all the lowest conditions (relative to the order) first. This seems to work well with those systems that are already partially sorted with respect to some lexicographical ordering on the dependent and independent variables, such as those in this paper.

The output statement of the algorithm shows the limits of the Kolchin-Ritt algorithm when one begins with, or obtains en route, an equation where the coefficient of the highest derivative contains differential terms. The proof of the output statement can be found in [63]. It shows that when the highest coefficient of a d.p. vanishes, for example when considering additional constraints or conditions, then one must re-do the calculation taking the additional conditions into account from the beginning. In particular, when an equation obtained factorises, and one wishes to consider a factor that does not include the highest derivative term for that equation, then one must re-start the algorithm including the desired factor at the outset.

We make some remarks about factors. It should be mentioned that if a product of functions is zero, say $f_1 f_2 = 0$, then in general one can only conclude that one of f_1, f_2 must be zero if the f_i are analytic. However, if one obtains, for example, a cubic in u_x, then the continuity of u_x ensures that u_x must be one of the roots of the polynomial.

Algorithm Kolchin-Ritt (sorting)

INPUT: Σ a system of d.p.'s, a termordering
OUTPUT: $\Sigma' \supset \Sigma$ such that
$S(\Sigma') \cap I(\Sigma') = \emptyset \Longrightarrow \forall f \in I(\Sigma')$
$\exists\ s \in S(\Sigma')$ such that sf pseudo-reduces to 0 with respect to Σ'.

```
pairsdone = {}
i = 1, j = 2
sort Σ into increasing order
Σ' = Σ
while i ≠ |Σ'| do
        let f_k be the k^th element of Σ' for 0 ≤ k ≤ |Σ'|
        if not {f_i, f_j} ∈ pairsdone then
            pairsdone=pairsdone ∪ {{f_i, f_j}}
            h =diffSpoly(f_i, f_j)
            h =normal^p(h, Σ')
            if h ≠ 0, then Σ' = Σ' ∪ {h}
                  sort Σ' into increasing order
                  i = 1, j = 2
            else if j = |Σ'| then j = i + 2, i = i + 1
            else j = j + 1
end
```

We give now a related algorithm, the "direct search" algorithm. The "direct search" algorithm recursively looks for conditions that are lower in the order than those given. Its output is a "skeleton" of that of Kolchin-Ritt, but often provides useful information that allows the triangulation of the system to proceed faster.

Algorithm Direct Search

INPUT: F a set of d.p.'s, a termordering
OUTPUT: $G \supset F$

```
G = F
sort F into increasing order
let f_k be the k^th element of F for 0 ≤ k ≤ |F|
i = |F| − 1
h = f_|F|
while i ≠ 0 do
        while h ≠ 0 do
            G = G ∪ {h}
            k = h
            h = normal^p(diffSpoly(h, f_i), G)
        h = min{k, f_i}
        i = i − 1
end
```

The direct search algorithm should be regarded as a "template" algorithm only. It is a natural generalization of the well-known process of setting coefficients to zero in the situation where one has an expression that is polynomial in $\{z_z, z_2, \ldots, z_m\}$ say, with coefficients that are independent of the z_i.

3 Classical Symmetries

The determining equations for the classical symmetries of the nonlinear heat equation (1.1) are

$$\begin{aligned} f_1 :&\quad \tau_u = 0, \\ f_2 :&\quad \tau_x = 0, \\ f_3 :&\quad \tau_t - 2\xi_x = 0, \\ f_4 :&\quad \xi_u = 0, \\ f_5 :&\quad \phi_{uu} = 0, \\ f_6 :&\quad 2\phi_{xu} + \xi_t - \xi_{xx} = 0, \\ f_7 :&\quad \phi_t - \phi_{xx} + \phi_u f - \phi f_u - 2f\xi_x = 0. \end{aligned}$$

These equations were calculated using a MACSYMA programme SYMMGRP.MAX [16]. The SPDE package in REDUCE [87] appears unable to handle equations such as (1.1) which contain arbitrary functions.

Assume the lexicographic ordering given by $\phi > \tau > \xi > f$ and $x > t > u$. From f_2 and f_3 we immediately obtain

$$f_8 : \quad \xi_{xx} = 0.$$

Calculating diff$S$poly(f_6, f_7) we obtain

$$f_9 : \quad -2\phi_{ut} + 2\phi f_{uu} + 4f_u\xi_x - \xi_{xt} = 0.$$

Performing the "direct search" algorithm on f_5 and f_9 and reducing the results with respect to $\{\xi_u, \xi_{xx}, \phi_{uu}\}$ yields

$$k_1 : \quad \phi_u f_{uu} + \phi f_{uuu} + 2f_{uu}\xi_x = 0,$$

$$\begin{aligned} k_2: &\quad \left\{-f_{uu}f_{uuuu}+2\left(f_{uuu}\right)^2\right\}\phi+2f_{uu}f_{uuu}\xi_x=0,\\ k_3: &\quad -\xi_x\left(f_{uu}\right)^2\left\{f_{uuuu}\left(f_{uuu}\right)^2-2f_{uu}\left(f_{uuuu}\right)^2+f_{uu}f_{uuu}f_{uuuuu}\right\}=0. \end{aligned}$$

Performing a "direct search" algorithm on f_6 and f_9 and reducing the results with respect to $\{\xi_u, \xi_{xx}, \phi_{uu}\}$ yields

$$\begin{aligned} j_1: &\quad \phi_{xu}f_{uu}+\phi_x f_{uuu}=0,\\ j_2: &\quad \xi_{tt}+2\phi_x f_{uu}=0,\\ j_3: &\quad f_{uuu}\xi_{tt}+\left(f_{uu}\right)^2\xi_t=0. \end{aligned}$$

Since $\xi_u=0$ we must have, differentiating j_3 with respect to u and cross-multiplying, that

$$j_4: \quad 2\xi_t\xi_{tt}\left\{\left(f_{uuu}\right)^2-f_{uu}f_{uuuu}\right\}f_{uu}=0.$$

We now consider the possibilities for $f(u)$; although it is possible that other "direct searches" will find other conditions, we can simplify the calculation of other conditions by putting $\xi_x=0$ for example.

We solve those factors in k_3 and j_4 involving derivatives of $f(u)$. Setting $H=f_{uu}$ in k_3 we obtain (assuming that $H\neq 0$ and $\xi_x\neq 0$)

$$H_{uu}(H_u)^2-2H(H_{uu})^2+HH_uH_{uuu}=0.$$

Dividing by HH_uH_{uu} we have

$$\frac{H_u}{H}-2\frac{H_{uu}}{H_u}+\frac{H_{uuu}}{H_{uu}}=0.$$

This integrates easily; the possibilities from k_3 are

(i) $f(u)=au^3+bu^2+cu+d$,
(ii) $f(u)=(au+b)^{n+2}+cu+d$, for $n\neq 0,1,2,3$,
(iii) $f(u)=a\mathrm{e}^{bu}+cu+d$,
(iv) $f(u)=[\ln(au+b)]/a+cu+d$,
(v) $f(u)=\left[(au+b)\ln(au+b)-(au+b)\right]/a^2+cu+d$, or,
(vi) $\xi_x=0$

where a, b, c and d are arbitrary constants.

Next we look at the equation j_4. In this case we obtain

(i) $f(u)=au^2+bu+c$,
(ii) $f(u)=\left[(au+b)\ln(au+b)-(au+b)\right]/a^2+cu+d$, or,
(iii) $\xi_t=0$.

We shall not consider $f(u)$ linear for the classical case as the symmetries in that case are already well known [9]. Hence for those solutions $f(u)$ not of the form $u\ln(u)+bu+c$ or au^2+bu+c, with b and c constants, we must have that $\xi_t=0$.

If $\xi_x=0$, from the conditions k_1 and k_2 one can conclude that if $f(u)$ is not of the form $u\ln(u)+bu+c$ or au^2+bu+c, then only the trivial solution, $\xi=\xi_0$, $\tau=\tau_0$, $\phi=0$ with ξ_0 and τ_0 constants, is possible. This solution exists for all choices of $f(u)$ and corresponds to the travelling wave reduction

$$u(x,t)=w(z), \qquad z=x-ct$$

where c is an arbitrary constant and $w(z)$ satisfies

$$w'' + cw' + f(w) = 0.$$

The cases where $f(u)$ is of the form $u\ln(u) + bu + c$ or $au^2 + bu + c$ require further attention since then the equation k_2 does not provide a condition for ϕ. For the other possibilities for $f(u)$, we must re-calculate the triangulation to obtain further conditions on ξ. For the remainder of this section, κ_1, κ_2, κ_3 and κ_4 are constants.

Case 3.1: $f(u) = u^2 + bu + c$. Here the triangulation of the system is (in addition to f_1, f_2, f_3, f_4):

$$4\phi + 8u\xi_x + 4b\xi_x + 3\xi_{xt} = 0, \qquad \xi_t = 0, \qquad \xi_{xx} = 0, \qquad \xi_x(b^2 - 4c) = 0.$$

Thus unless $f(u)$ is a perfect square, the only solution to the system is the trivial one. If $f(u) = u^2$ then

$$\xi = \kappa_1 x + \kappa_2, \qquad \tau = 2\kappa_1 t + \kappa_3, \qquad \phi = -2\kappa_1 u. \tag{3.1}$$

Hence for $\kappa_1 \neq 0$ (set $\kappa_1 = 1$) we obtain the symmetry reduction

$$u(x,t) = (t + \tfrac{1}{2}\kappa_3)^{-1} w(z), \qquad z = (x + \kappa_2)/(t + \kappa_3/2)^{1/2} \tag{3.2}$$

where $w(z)$ satisfies

$$w'' + \tfrac{1}{2}zw' + w + w^2 = 0. \tag{3.3}$$

It is easily shown that this equation is not of Painlevé-type.

Case 3.2: $f(u) = u\ln(u) + bu + c$. In the case $c = 0$, the triangulation of the system (in addition to f_1, f_2, f_3, f_4) is given by

$$u\phi_u - \phi = 0, \qquad \phi_t - \phi = 0, \qquad 2\phi_x + u\xi_t = 0, \qquad \xi_x = 0, \qquad \xi_{tt} - \xi_t = 0$$

which yields

$$\xi = \kappa_1 \mathrm{e}^t + \kappa_2, \qquad \tau = \kappa_3, \qquad \phi = u\left(-\tfrac{1}{2}\kappa_1 x + \kappa_4\right)\mathrm{e}^t. \tag{3.4}$$

Hence for $\kappa_3 \neq 0$ (set $\kappa_3 = 1$) we obtain the symmetry reduction

$$u(x,t) = w(z)\exp\left\{-\tfrac{1}{2}\kappa_1 x\mathrm{e}^t + \tfrac{1}{4}\kappa_1^2\mathrm{e}^{2t} + \left(\tfrac{1}{2}\kappa_1\kappa_2 + \kappa_4\right)\mathrm{e}^t\right\}, \qquad z = x - \kappa_1\mathrm{e}^t - \kappa_2 t \tag{3.5}$$

where $w(z)$ satisfies

$$w'' + \kappa_2 w' + bw + w\ln w = 0. \tag{3.6}$$

It is easily shown that this equation is not of Painlevé-type.

If $\kappa_3 = 0$ (set $\kappa_1 = 1$ and $\kappa_2 = 0$), we obtain the symmetry reduction

$$u(x,t) = v(s)\exp\left\{\frac{-\tfrac{1}{4}\kappa_1 x^2 + \kappa_4 x}{\kappa_1 + \kappa_2\mathrm{e}^{-t}}\right\} \qquad s = \mathrm{e}^t \tag{3.7}$$

where $v(s)$ satisfies

$$s\frac{\mathrm{d}v}{\mathrm{d}s} = v\ln(v) + \left(b + \kappa_4^2 - \tfrac{1}{2}\right)v \tag{3.8}$$

which has general solution

$$v(s) = \exp\left\{ks - b - \kappa_4^2 + \tfrac{1}{2}\right\} \tag{3.9}$$

with k an arbitrary constant. Hence we obtain the exact solution

$$u(x,t) = \exp\left\{k\mathrm{e}^t - \tfrac{1}{4}(x-2\kappa_4)^2 - b + \tfrac{1}{2}\right\}. \tag{3.10}$$

The solutions for the other cases of $f(u)$ are calculated the same way and are listed below.

Case 3.3: $f(u) = u^3 + au^2 + bu + c$. Unless $ab - 9c = 0$ and $a^2 - 3b = 0$ there is only the trivial solution. The two algebraic conditions together imply that $f(u)$ is a perfect cube. Setting $f(u) = u^3$ yields

$$\xi = \kappa_1 x + \kappa_2, \qquad \tau = 2\kappa_1 t + \kappa_3, \qquad \phi = -\kappa_1 u. \tag{3.11}$$

Hence for $\kappa_1 \neq 0$ (set $\kappa_1 = 1$) we obtain the symmetry reduction

$$u(x,t) = (t + \kappa_3/2)^{-1/2} w(z), \qquad z = (x+\kappa_2)/(t+\kappa_3/2)^{1/2} \tag{3.12}$$

where $w(z)$ satisfies

$$w'' + \tfrac{1}{2}zw' + \tfrac{1}{2}w + w^3 = 0. \tag{3.13}$$

It is easily shown that this equation is not of Painlevé-type.

Case 3.4: $f(u) = u^n + bu + c$, $n \neq 0, 1, 2, 3$. Unless $b = c = 0$ we obtain only the trivial solution. Otherwise for $f(u) = u^n$

$$\xi = \kappa_1 x + \kappa_2, \qquad \tau = 2\kappa_1 t + \kappa_3, \qquad \phi = -2\kappa_1 u/(n-1). \tag{3.14}$$

Hence for $\kappa_1 \neq 0$ (set $\kappa_1 = 1$) we obtain the symmetry reduction

$$u(x,t) = (t + \kappa_3/2)^{-1/(n-1)} w(z), \qquad z = (x+\kappa_2)/(t+\kappa_3/2)^{1/2} \tag{3.15}$$

where $w(z)$ satisfies

$$w'' + \tfrac{1}{2}zw' + \frac{w}{n-1} + w^n = 0. \tag{3.16}$$

It is easily shown that this equation is not of Painlevé-type.

Case 3.5: $f(u) = \mathrm{e}^{au} + bu + c$, $a \neq 0$. Unless $b = c = 0$ there is only the trivial solution. If $f(u) = \mathrm{e}^u$ then

$$\xi = \kappa_1 x + \kappa_2, \qquad \tau = 2\kappa_1 t + \kappa_3, \qquad \phi = -2\kappa_1. \tag{3.17}$$

Hence for $\kappa_1 \neq 0$ (set $\kappa_1 = 1$) we obtain the symmetry reduction

$$u(x,t) = w(z) - \ln(t + \kappa_3/2), \qquad z = (x+\kappa_2)/(t+\kappa_3/2)^{1/2} \tag{3.18}$$

where $w(z)$ satisfies

$$w'' + \tfrac{1}{2}zw' + 1 + \mathrm{e}^w = 0. \tag{3.19}$$

It is easily shown that this equation is not of Painlevé-type.

Case 3.6: $f(u) = \ln(u) + bu + c$. In this case we obtain only the trivial solution.

In Table 1 we summarise the infinitesimals obtained using the classical Lie method. For completeness, we have included the case when $f(u)$ is linear.

4 Nonclassical Symmetries

There are two sorts of nonclassical symmetries, those where the infinitesimal τ is non-zero, and those where it is zero. In the first case, we can assume without loss of generality that $\tau \equiv 1$, while in the second case we can assume that $\xi \equiv 1$.

In this section, we solve the determining equations for the nonclassical infinitesimals, and obtain the reductions. Solutions are catalogued in §5.

Table 1: Infinitesimals for equation (1.1) obtained using the classical Lie method

$f(u)$	ξ	τ	ϕ
0	$\alpha x+\beta t+\gamma xt+\delta$	$2\alpha t+\gamma t^2+\kappa$	$\left[\lambda-\frac{1}{2}\beta x-\left(\frac{1}{2}t+\frac{1}{4}x^2\right)\gamma\right]u+\phi_0(x,t)$
1	$\alpha x+\beta t+\gamma xt+\delta$	$2\alpha t+\gamma t^2+\kappa$	$\left[\lambda-\frac{1}{2}\beta x-\left(\frac{1}{2}t+\frac{1}{4}x^2\right)\gamma\right]u$ $+(2\alpha-\lambda)t+\frac{1}{2}\beta xt+\left(\frac{1}{4}x^2t+\frac{3}{2}t^2\right)\gamma$ $+\phi_0(x,t)$
u	$\alpha x+\beta t+\gamma xt+\delta$	$2\alpha t+\gamma t^2+\kappa$	$\left[\lambda+2\alpha t-\frac{1}{2}\beta x+\left(t^2-\frac{1}{2}t-\frac{1}{4}x^2\right)\gamma\right]u$ $+\psi_0(x,t)$
$u^n\ \ (n\neq 0,1)$	$\alpha x+\beta$	$2\alpha t+\gamma$	$-\dfrac{2\alpha u}{n-1}$
e^u	$\alpha x+\beta$	$2\alpha t+\gamma$	-2α
$u\ln(u)+bu$	$\alpha\mathrm{e}^t+\beta$	γ	$u\left(-\frac{1}{2}\alpha x+\delta\right)\mathrm{e}^t$
arbitrary	α	β	0

where ϕ_0 satisfies $\phi_{0,t}=\phi_{0,xx}$ and ψ_0 satisfies $\psi_{0,t}=\psi_{0,xx}+\psi_0$.

4.1 Nonclassical Symmetries, $\tau\equiv 1$

The determining equations for the nonclassical symmetries, given $\tau\equiv 1$, are

$$\begin{aligned}
f_1:&\qquad \xi_{uu}=0,\\
f_2:&\qquad \phi_{uu}-2\xi_{xu}+2\xi\xi_u=0,\\
f_3:&\qquad 2\phi_{xu}-2\phi\xi_u+3\xi_u f+\xi_t-\xi_{xx}+2\xi\xi_x=0,\\
f_4:&\qquad \phi_t-\phi_{xx}+\phi_u f-f_u\phi+2\phi\xi_x-2\xi_x f=0.
\end{aligned}$$

These were calculated using the MACSYMA programme SYMMGRP.MAX [16].

The result of the "direct search" algorithm on $\{f_1,f_2,f_3\}$ in the ordering $\phi>\xi>f$ and $x>t>u$ is the equation

$$\xi_u f_{uuuu}=0.$$

Hence, unless $\xi_u=0$, $f(u)$ must be cubic in u. In the case $\xi_u=0$, another "direct search" on the system led to the equation

$$-\xi_x\left(f_{uu}\right)^2\left\{-2f_{uu}\left(f_{uuuu}\right)^2+f_{uu}f_{uuu}f_{uuuuu}+f_{uuuu}\left(f_{uuu}\right)^2\right\}=0$$

which is the same condition k_3 for $f(u)$ obtained in the classical case. If $\xi_x=\xi_u=0$ then we obtain the condition

$$\phi\left\{2\left(f_{uuu}\right)^2-f_{uu}f_{uuuu}\right\}=0.$$

As in the previous Section, we consider the various possibilities for $f(u)$ separately.

4.1.1 $\tau\equiv 1,\ f(u)=u^3+bu^2+cu+d$

The direct search algorithm yields the equation $\xi_u(2\xi_u^2+9)=0$.

Case 4.1.1i $2\xi_u^2+9=0$. Here a triangulation of the system yields the infinitesimals

$$\xi=\tfrac{3}{2}\mathrm{i}\sqrt{2}\,(u+b/3),\qquad \phi=\tfrac{3}{2}(u^3+bu^2+cu+d). \tag{4.1}$$

and the associated invariant surface condition is

$$\tfrac{3}{2}\mathrm{i}\sqrt{2}\,(u+b/3)u_x+u_t-\tfrac{3}{2}(u^3+bu^2+cu+d)=0. \tag{4.2}$$

By solving this we obtain exact solutions (1.1) with $f(u)=u^3+bu^2+cu+d$ (see §5.1 below).
Setting $\xi_u=0$ we obtain the triangulation

$$\begin{aligned}
&\phi+\xi_x(u+b/3)=0,\\
&\xi_t-3\xi_{xx}+2\xi\xi_x=0,\\
&3\xi_{xxx}-3\xi\xi_{xx}-\xi_x\left(b^2-3c\right)=0,\\
&\xi_{xx}\left(-3\xi_{xx}+2\xi\xi_x\right)=0,\\
&\xi\xi_x^2\left(-b^2+3c\right)\left(3c-b^2-\xi\right)\left(9c-3b^2-2\xi^2+6\xi_x\right)=0,\\
&\xi_x\left(-27d-2b^3+9bc\right)=0.
\end{aligned}$$

Unless the algebraic condition $-27d-2b^3+9bc=0$ on the coefficients of $f(u)$ is satisfied, the only solution is the trivial one ($\xi=\xi_0$ a constant, $\phi=0$). So assume that $\xi_x\neq0$ and that $-27d-2b^3+9bc=0$. There are thus several cases to consider. Note that if both $b^2-3c=0$ and $-27d-2b^3+9bc=0$ then $f(u)=(u+b/3)^3$.
Case 4.1.1ii $\xi_u=0$, $\xi_{xx}=0$, $f(u)=(u+b/3)^3$. In this case we have from the triangulation equations that $\xi_t+2\xi\xi_x=0$ and we thus have that

$$\xi=(x+\kappa_1)/(2t+\kappa_2),\qquad \phi=-(u+b/3)/(2t+\kappa_2). \tag{4.3}$$

These are equivalent to (3.1) and thus also yield the (classical) scaling reduction (3.2).
Case 4.1.1iii $\xi_u=0$, $-3\xi_{xx}+2\xi\xi_x=0$, $f(u)=(u+b/3)^3$. In this case we have $\xi_t=0$ and the Kolchin-Ritt algorithm on the equations $\{-3\xi_{xx}+2\xi\xi_x=0,-3\xi_{xxx}+3\xi\xi_{xx}=0\}$ leads to the condition $2\xi_x(3\xi_x-\xi^2t)=0$. Thus the non-trivial solution in this case is

$$\xi=-\,3/(x+\kappa_1),\qquad \phi=-\,3(u+b/3)/(x+\kappa_1)^2. \tag{4.4}$$

Hence we obtain the nonclassical symmetry reduction

$$u(x,t)=(x+\kappa_1)w(z)-\tfrac{1}{3}b,\qquad z=\tfrac{1}{2}x^2+\kappa_1x+3t \tag{4.5}$$

where $w(z)$ satisfies

$$w''+w^3=0 \tag{4.6}$$

which is solvable in terms of elliptic functions (see §5.2 below).
Case 4.1.1iv $\xi_u=0$, $27d+2b^3-9bc=0$, $b^2-3c\neq0$. In this case we have

$$\xi_t=0,\qquad 9c-3b^2-2\xi^2+6\xi_x=0$$

leading to the solution

$$\xi=3\mu\tan(\mu x+\kappa_1)\qquad \phi=-\mu(3u+b)\sec^2(\mu x+\kappa_1) \tag{4.7}$$

where $\mu^2 = \frac{1}{6}(b^2 - 3c)$, and so we obtain the nonclassical symmetry reduction

$$u(x,t) = \mu \cot(\mu x + \kappa_1) w(z) - \tfrac{1}{3}b, \qquad z = \sin(\mu x + \kappa_1)\exp(-3\mu^2 t) \tag{4.8}$$

where $w(z)$ satisfies

$$z^2 w'' - 2zw' + 2w + w^3 = 0 \tag{4.9}$$

which is solvable in terms of elliptic functions (see §5.2 below).

We remark that the roots of the cubic $u^3 + bu^2 + cu + (9c - 2b^2)b/27 = 0$ are $-\frac{1}{3}b$, $-\frac{1}{3}b + \sqrt{2}\,\mu$ and $-\frac{1}{3}b - \sqrt{2}\,\mu$, where $\mu^2 = \frac{1}{6}(b^2 - 3c)$, i.e. they are collinear and the distances from the outer roots to the central one are equal. The condition that $27d + 2b^3 - 9bc = 0$ is the necessary and sufficient condition on $f(u) = u^3 + bu^2 + cu + d$ for the existence of nonclassical symmetry reductions of (1.1) to ODEs solvable in terms of elliptic functions. Furthermore, Nucci and Clarkson [71] show that elliptic functions solutions exist for the Fitzhugh-Nagumo equation (1.9) if either $a = -1$, $a = \frac{1}{2}$ or $a = 2$, which are precisely those the three cases when the distances from the outer roots to the central one are equal.

4.1.2 $\tau \equiv 1$, $f(u) = au + b$

A preliminary triangulation yields the condition $\xi_u^3 = 0$. Reducing the equations with respect to $\xi_u = 0$ we obtain $\phi_{uu} = 0$. Inserting $\phi = A(x,t)u + B(x,t)$ into the equations, we obtain the system [9, 43]

$$\begin{aligned}
g_1 : &\qquad 2A_x + \xi_t - \xi_{xx} + 2\xi\xi_x = 0,\\
g_2 : &\qquad A_t - A_{xx} + 2\,(A - a)\,\xi_x = 0,\\
g_3 : &\qquad B_t - B_{xx} + 2\,(B - b)\,\xi_x - aB + bA = 0.
\end{aligned}$$

Note that we no longer have an over-determined system. For an arbitrary function $v(x,t)$ let $\Phi(v;\xi) = v_t - v_{xx} + 2v\xi_x$, and set $\Phi(\xi;\xi) = \eta$. Let $A > \xi$. Then cross-differentiating g_1 and g_2 and reducing, we obtain

$$4\xi_{xx}(A - a) = \Phi(\eta;\xi).$$

Case 4.1.2i $\tau \equiv 1$, $\xi_{xx} \neq 0$, $f(u) = au + b$. Pseudo-reducing g_1, g_2 with respect to $\xi_{xx}(A - a) = \frac{1}{4}\Phi(\eta;\xi)$, we obtain two equations k_1 and k_2 in ξ whose highest derivative terms in a *total degree* ordering are ξ_{xxxxxx} and ξ_{xxxxx}. (A total degree ordering is determined first by the degree of the terms and then by a lexicographic ordering.) Reducing k_1 with respect to k_2 in this ordering yields a condition k_3 for ξ whose highest derivative term is ξ_{xxxxt}. Now $\{k_2, k_3\}$ forms a DGB for $I(\{k_1, k_2\})$ with $S(\{k_2, k_3\}) = \{\pm(\xi_{xx})^n : n \in \}$. (We have stipulated that $\xi_{xx} \neq 0$ in this case.) Our next step is to apply the Initial Data algorithm [83], from which we have that a formal solution to the equations for ξ depends on four arbitrary functions and one arbitrary constant. In fact we can assign $\xi(0,t)$, $\xi_x(0,t)$, $\xi_{xx}(0,t)$, $\xi_{xxx}(0,t)$ and $\xi_{xxxx}(0,0)$ and the other coefficients for a formal expansion for ξ will be determined.

Assuming that $\xi_{xx}(0,t)$ is a constant, and that the other arbitrary functions and constant are zero, one can derive a solution to the equations for ξ and A (i.e. g_1 and g_2):

$$\xi = \frac{3x^2}{3c - x^3}, \qquad A = a + \frac{3x}{3c - x^3} \tag{4.10}$$

where c is a constant, simply by summing the formal Taylor series obtained from k_2 and k_3. Solving g_3 for B with ξ and A as given in (4.10) yields

$$B = b + \frac{3bcx}{a(3 - cx^3)}.$$

Therefore the infinitesimals in this case are

$$\xi = \frac{3x^2}{3c - x^3}, \qquad \phi = (au + b) + \frac{3x(au + b)}{a(3c - x^3)} \tag{4.11}$$

and so we obtain the symmetry reduction

$$u(x,t) = xw(z)\,\mathrm{e}^{at} - b/a, \qquad z = \tfrac{1}{6}x^2 + t + c/x \tag{4.12}$$

where $w(z)$ satisfies

$$w'' = 0.$$

Hence we obtain the exact solution of (1.1) with $f(u) = au + b$ given by

$$u(x,t) = \left\{\alpha(\tfrac{1}{6}x^3 + xt + c) + \beta\right\} \mathrm{e}^{at} - b/a. \tag{4.13}$$

Case 4.1.2ii $\tau \equiv 1$, $\xi_{xx} = 0$, $f(u) = au + b$. In this case we obtain a classical reduction and so we omit details.

Nonclassical symmetry reductions for the linear equation

$$u_t = u_{xx} + au + b \tag{4.14}$$

where a and b are arbitrary constants, obtained using both Painlevé and symmetry analysis of the determining equations $\{g_1, g_2, g_3\}$ can be found in [98, 99].

4.1.3 Nonclassical, $\tau \equiv 1$, $f(u)$ neither cubic nor linear

In this case we have

$$-2f_{uu}\,(f_{uuuu})^2 + f_{uu}f_{uuu}f_{uuuuu} + f_{uuuu}\,(f_{uuu})^2 = 0$$

or

$$2\,(f_{uuu})^2 - f_{uu}f_{uuuu} = 0.$$

The various possibilities for $f(u)$ have already been listed in §3. For all these possibilities, we obtain the same symmetries as for the classical case, with the classical $\xi\,/\tau$ yielding the nonclassical ξ and the classical $\phi\,/\tau$ yielding the nonclassical ϕ. However, for some of the cases, this is harder to prove than for the classical determining equations.

We give the proof for the case $f = u^n + cu + d$ where $n \neq 0, 1, 2$ or 3, to show the ideas involved.

We have in this case that $\xi_u = 0$. The Direct Search algorithm yields that $d = 0$ else one has only the trivial solution, and that $\phi = -2u\xi_x/(n-1)$.

Reducing the determining equations with respect to ξ_u and $\phi + 2\,/(n-1)\;u\xi_x$ yields

$$\begin{aligned} h_1: &\quad 4\xi_{xxx} - 2(n-1)\xi\xi_{xx} + c(n-1)^2\xi_x = 0,\\ h_2: &\quad (n-1)\xi_t - (n+3)\xi_{xx} + 2(n-1)\xi\xi_x = 0. \end{aligned}$$

Take the ordering $t > x$. Then cross-differentiating h_1 and h_2, throwing away powers of c, ξ_x and $(n-1)$ that factor out, and iterating, we obtain successively equations with the following "vital statistics":

h_3: $\quad \mathrm{HDT}(h_3)^{\mathrm{Hp}(h_3)} = \xi_{xx}^2, \quad \mathrm{Sep}(h_3) = 24n\xi_{xx} + n^2(n-1)\xi\xi_x,$

h_4: $\quad \mathrm{HDT}(h_4)^{\mathrm{Hp}(h_4)} = \xi_{xx},$

$\qquad \mathrm{Sep}(h_4) = -12(n+1)\xi_x + (5n^2 - 2n^3 - 5)\xi^2 + 12c(n^2 - 120n + 108),$

h_5: $\quad \mathrm{HDT}(h_5)^{\mathrm{Hp}(h_5)} = \xi_x^3, \quad \mathrm{Sep}(h_5) = p(\xi_x, \xi)$

where p is a polynomial. Using the Implicit Function Theorem, we obtain for some function R, the ansatz $\xi_x = R(\xi)$. Note that if any of the separants are zero, leading to invalid reductions, we still obtain the same ansatz. Substituting the ansatz into h_1, h_2, one obtains

$$\begin{aligned}
k_1 : &\quad \xi_x - R(\xi) = 0,\\
k_2 : &\quad (n-1)\xi_t + 2(n-1)\xi R R_\xi - (n+3) R R_\xi = 0,\\
k_3 : &\quad (n-1)R^2 R_{\xi\xi} + 2(n-1)\xi R R_\xi - 4 R R_\xi^2 - 2(n-1)R^2 - n(n-2)cR = 0.
\end{aligned}$$

Cross-differentiating k_1 and k_2 ($\xi > R$) and reducing yields

$$R^2\left[(n+3)R_{\xi\xi} - 2(n-1)\right] = 0. \tag{4.15}$$

Solving for R using the second factor of (4.15), and substituting the result into k_3, yields a polynomial for ξ, that is, ξ is a constant.

Thus, since the only factors we threw away were c, ξ_x and $(n-1)$, it must be that for a nontrivial solution to exist, $c = 0$.

Setting $c = 0$ in h_1 and h_2 and performing the Kolchin-Ritt algorithm on these two equations yields the condition

$$(n-1)\xi_{xx}\left[(n^2-1)\xi\xi_x - 12\xi_{xx}\right]. \tag{4.16}$$

The system $\{\xi_{xx} = 0, \xi_t + 2\xi\xi_x = 0\}$ has for its solution

$$\xi = \frac{x+\kappa_2}{2t+\kappa_1}, \qquad \phi = -\frac{2u}{(n-1)(2t+\kappa_1)}$$

which is equivalent to the classical reduction of Case 3.3.

Putting the other factor of (4.16), namely $(n^2-1)\xi\xi_x - 12\xi_{xx}$, in addition to h_1 and h_2 (with $c = 0$) as input to the Kolchin-Ritt algorithm, we obtain

$$\begin{aligned}
&\xi_x(n-1)(n+1)\left[12\xi_x + (n-1)(n-5)\xi\right] = 0,\\
&\xi^4(n-1)^3(n-3)(n-5)^5 = 0.
\end{aligned}$$

Since $n \neq 1$ or 3 in this case, there are no other nontrivial solutions.

4.2 Nonclassical Symmetries, $\tau \equiv 0$, $\xi \equiv 1$

We come now to those nonclassical symmetries where $\tau \equiv 0$, and" without loss of generality we can take $\xi \equiv 1$. In this case we have the system consisting of the equation (1.1), the invariant surface condition (1.4) and one additional determining equation for ϕ:

$$\begin{aligned}
f_1 : &\qquad u_t - u_{xx} - f(u) = 0,\\
f_2 : &\qquad u_x - \phi(x,t,u(x,t)) = 0,\\
f_3 : &\qquad \phi_t - \phi_{xx} - \phi^2\phi_{uu} - 2\phi\phi_{xu} + f\phi_u - f_u\phi = 0.
\end{aligned}$$

The last equation was calculated using the MACSYMA package SYMMGRP.MAX [16]. Cross-differentiating f_1 and f_2 produces exactly f_3 so we obtain no new conditions. So we try three ansätze for ϕ, namely

Ansatz 4.2.1: $\phi = p_2u^2 + p_1u + p_0$ with $f(u) = -u^3 - bu^2 - cu - d$.

Ansatz 4.2.2: $\phi = q_1(x,t)u + q_0(x,t)$.

Ansatz 4.2.3: $\phi = r_1(t,u)x + r_0(t,u)$.

In the second and third ansätze, $f(u)$ is to be determined. We consider these three ansätze in turn. Note that in the equation f_3, u is to be regarded as an independent variable.

Ansatz 4.2.1. Reducing f_3 using the ansatz, and setting coefficients of powers of u to zero, we obtain from the coefficient of u^4 that $p_2(2p_2^2-1) = 0$. Setting $p_2 = \frac{1}{2}\sqrt{2}$ we have from the coefficients of u^2, u and 1 that:

$$\begin{aligned} g_1: &\qquad 2p_1^2 - \sqrt{2}\,bp_1 - \sqrt{2}\,p_0 + c = 0,\\ g_2: &\qquad 2p_0p_1 - \sqrt{2}\,bp_0 + d = 0,\\ g_3: &\qquad \sqrt{2}\,p_0^2 - cp_0 + dp_1 = 0. \end{aligned}$$

The coefficient of u^3 is identically zero. For generic b, c and d there are three possible solutions of $\{g_1, g_2, g_3\}$ given by

$$p_1 = -\tfrac{1}{2}\sqrt{2}\,(m_i + m_j), \qquad p_0 = \tfrac{1}{2}\sqrt{2}\,m_i m_j, \qquad 1 \le i < j \le 3, \tag{4.17}$$

where m_i and m_j are any two of the three distinct roots of the cubic

$$u^3 + bu^2 + cu + d = 0. \tag{4.18}$$

Hence there are three choices for ϕ

$$\phi = \tfrac{1}{2}\sqrt{2}\,(u - m_i)(u - m_j), \qquad 1 \le i < j \le 3. \tag{4.19}$$

Substituting this into f_2, solving for u and requiring that u also satisfies f_1 yields the solution

$$u_t = u_{xx} - u^3 - bu^2 - cu - d = u_{xx} - (u - m_1)(u - m_2)(u - m_3), \tag{4.20}$$

given by

$$u(x,t) = \frac{m_i\kappa_i\Theta_i(x,t) + m_j\kappa_j\Theta_j(x,t)}{\kappa_i\Theta_i(x,t) + \kappa_j\Theta_j(x,t)}, \qquad 1 \le i < j \le 3, \tag{4.21}$$

where

$$\Theta_k(x,t) = \exp\left\{\tfrac{1}{2}\sqrt{2}\,m_k x - m_k\left(m_1 + m_2 + m_3 - \tfrac{3}{2}m_k\right)t\right\}, \qquad 1 \le k \le 3.$$

We remark that the solution (4.21) is special case of the travelling wave solution and of the exponential solution (5.13) given in §5.1.1 below.

If $d = (9c - 2b^2)b/27$, then there is the solution of $\{g_1, g_2, g_3\}$ given by $p_1 = \frac{1}{3}\sqrt{2}\,b$ and $p_0 = \frac{1}{2}\sqrt{2}\,(c - \frac{2}{9}b^2)$. Thus, a similar procedure to the generic case above yields exact solutions of

$$u_t = u_{xx} - u^3 - bu^2 - cu - (9c - 2b^2)b/27,$$

given by

$$u(x,t) = \begin{cases} (c - \frac{1}{3}b^2)^{1/2}\tan\left(\frac{1}{2}\sqrt{2}\,(c - \frac{1}{3}b^2)^{1/2}x + \kappa_1\right) - \frac{1}{3}b, & \text{if } c \ne \frac{1}{3}b^2\\ \pm\dfrac{\sqrt{2}}{x + \kappa_2} - \frac{1}{3}b, & \text{if } c = \frac{1}{3}b^2 \end{cases}$$

where κ_1 and κ_2 are arbitrary constants. We remark that $d = (9c - 2b^2)b/27$ is the same relation as that we obtained for (1.1) with $f(u)$ cubic and $\tau \equiv 1$ which yielded special reductions in terms of elliptic solutions.

If $p_1 = 0$ then $p_0 = \frac{1}{2}\sqrt{2}\,c$ provided that $d = bc$. Hence we obtain the exact solution of

$$u_t = u_{xx} - u^3 - bu^2 - cu - bc,$$

given by

$$u(x,t) = \sqrt{c}\,\tan\left[\sqrt{c}\,(\tfrac{1}{2}\sqrt{2}\,x - bt)\right],$$

which is a special case of the travelling wave reduction.

If $p_2 = 0$, we then have that $p_1 = p_0 = 0$ implying that $\phi = 0$. Thus solving the equations f_1 and f_2 yields

$$u(x,t) = g(t), \qquad \text{where} \qquad \frac{\mathrm{d}g}{\mathrm{d}t} + ag^3 + bg^2 + cg + d = 0.$$

Ansatz 4.2.2. In this ansatz, we take

$$\phi = q_1(x,t)u + q_0(x,t) \tag{4.22}$$

Reducing f_3 using the ansatz, it can be shown that there are three cases to consider.

Case 4.2.2i $f = au^2 + bu + c$, $a \neq 0$. In this case we have $\phi = 0$ and solving the equations f_1 and f_2 yields

$$u(x,t) = g(t), \qquad \frac{\mathrm{d}g}{\mathrm{d}t} = ag^2 + bg + c.$$

Case 4.2.2ii $f = au + b$. In this case, it is easily shown that $\phi = (u + b/a)q_1$, where q_1 satisfies

$$q_{1,t} - q_{1,xx} - 2q_{1,x}q_1 = 0.$$

Combining this with the equations f_1 and f_2, we recover the well-known Cole-Hopf transformation [22, 51] relating a solution of Burgers' equation to a solution of the linear heat equation.

Case 4.2.2iii $f(u) = (u+k)\ln(u+k) - (u+k) + du + e$. In this case, it is easily shown that if $e = dk$ then $\phi = (u+k)q_1(x,t)$ where q_1 satisfies

$$q_{1,xx} + q_1 - q_{1,t} + 2q_1q_{1,x} = 0,$$

whilst if $e \neq dk$ then $\phi = 0$. The situation is analogous to that in Case 4.2.2ii above.

It appears that Bäcklund transformations arise in considering nonclassical symmetries in the $\tau = 0$ case. Further interesting examples have been obtained by Nucci [70].

Ansatz 4.2.3 $\phi = r_1(t,u)x + r_0(t,u)$. Reducing f_3 using the ansatz, it can be shown that there are three cases to consider: (i), $f(u) = au^2 + bu + c$, $a \neq 0$, (ii), $f(u) = au + b$, $a \neq 0$, or (iii), $f(u) = u\ln(u) - u + cu + d$.

Case 4.2.3i $f(u) = au^2 + bu + c$, $a \neq 0$. In this case, $\phi = 0$.

Case 4.2.3ii $f(u) = au + b$, $a \neq 0$. In this case we obtain the exact solution of

$$u_t = u_{xx} + au + b, \tag{4.23}$$

given by

$$u(x,t) = \frac{\kappa_4}{(t+\kappa_2)^{1/2}}\exp\left\{at - \frac{(x+\kappa_1)^2}{4(t+\kappa_2)}\right\} - \frac{b + \kappa_3\,\mathrm{e}^{at}}{a}, \tag{4.24}$$

where $\kappa_1, \kappa_2, \kappa_3$ and κ_4 are arbitrary constants.

Table 2: Infinitesimals for equation (1.1) obtained using the nonclassical method

$f(u)$	ξ	τ	ϕ
u^3+bu^2+cu+d	$\frac{3}{2}\mathrm{i}\sqrt{2}\,(u+b/3)$	1	$\frac{3}{2}(u^3+bu^2+cu+d)$
$-u^3-bu^2-cu-d$	$\frac{3}{2}\sqrt{2}\,(u+b/3)$	1	$-\frac{3}{2}(u^3+bu^2+cu+d)$
u^3	$-\dfrac{3}{x+\kappa_1}$	1	$-\dfrac{3(u+b/3)}{(x+\kappa_1)^2}$
$-u^3$	$-\dfrac{3}{x+\kappa_1}$	1	$\dfrac{3(u+b/3)}{(x+\kappa_1)^2}$
$u^3+bu^2+cu+b(9c-2b^2)/27$	$3\mu\tan(\mu x+\kappa_3)$	1	$-\mu(3u+b)\sec^2(\mu x+\kappa_3)$
$-u^3-bu^2-cu-b(9c-2b^2)/27$	$3\mu\tanh(\mu x+\kappa_4)$	1	$\mu(3u+b)\,\mathrm{sech}^2(\mu x+\kappa_4)$
$au+b$	$\dfrac{3x^2}{3\kappa_5-x^3}$	1	$au+b+\dfrac{3x(au+b)}{a(3\kappa_5-x^3)}$
$-(u-m_1)(u-m_2)(u-m_3)$	1	0	$\frac{1}{2}\sqrt{2}\,(u-m_1)(u-m_2)$
$-(u-m_1)(u-m_2)(u-m_3)$	1	0	$\frac{1}{2}\sqrt{2}\,(u-m_2)(u-m_3)$
$-(u-m_1)(u-m_2)(u-m_3)$	1	0	$\frac{1}{2}\sqrt{2}\,(u-m_3)(u-m_1)$
$-u^3-bu^2-cu-b(9c-2b^2)/27$	1	0	$\frac{1}{2}\sqrt{2}\left[(u+b/3)^2+(c-\frac{1}{3}b^2)\right]$
$-u^3-bu^2-cu-bc$	1	0	$\frac{1}{2}\sqrt{2}\,(u^2+c)$
$(u+k)\ln(u+k)+c(u+k)$	1	0	$-\frac{1}{2}(x+\kappa_6)(au+b)$
$au+b$	1	0	$-\dfrac{(x+\kappa_8)(au+b+\kappa_9\mathrm{e}^{at})}{2(t+\kappa_{10})}$

where $\kappa_1,\kappa_2,\ldots,\kappa_{10}$ are arbitrary constants.

Case 4.2.3iii $f(u)=(u+k)\ln(u+k)+c(u+k)$. In this case we obtain the exact solution of

$$u_t=u_{xx}+(u+k)\ln(u+k)+c(u+k), \tag{4.25}$$

given by

$$u(x,t)=\exp\left\{\kappa_2\mathrm{e}^t-\tfrac{1}{4}(x+\kappa_1)^2-c+\tfrac{1}{2}\right\}-k \tag{4.26}$$

where κ_1 and κ_2 are arbitrary constants.

5 Exact solutions of $u_t=u_{xx}+u^3+bu^2+cu+d$

In this section we obtain exact solutions of the PDEs

$$u_t=u_{xx}+u^3+bu^2+cu+d \tag{5.1}$$

and

$$u_t=u_{xx}-(u^3+bu^2+cu+d) \tag{5.2}$$

using the infinitesimals derived in §4.1.1 above. There are two types of solutions: those expressible in terms of exponentials arising from Case 4.1.1i (§5.1) and those expressible in terms of elliptic functions arising from Cases 4.1.1iii and 4.1.1iv (§5.2). It is interesting to note that the form of the exponential-type solution is closely related to the roots of the cubic

$$u^3 + bu^2 + cu + d = 0 \tag{5.3}$$

arising in the PDE we are considering. Furthermore the conditions when (5.1,5.2) possess solutions expressible in terms of elliptic functions are determined by the roots of (5.3). Recall that nonclassical symmetry reductions of (5.1), (5.2) are obtainable in the case when the roots of $f(u)$ are arbitrary (Case 4.1.1i), and when the roots of $f(u)$ are collinear and the distances from the outer roots to the middle root are equal (Case 4.1.1iv). These reductions lead to exponential and elliptic type solutions respectively. The second case is determined by the equivalent algebraic condition $27d + 2b^3 - 9bc = 0$ on the coefficients of $f(u) = u^3 + bu^2 + cu + d$. We note that a solution of (5.2) can be obtained from one of (5.1) by letting $x \to \pm \mathrm{i}x$ and $t \to -t$, and vice-versa.

5.1 Exponential-type solutions

In Case 4.1.1i above, we obtained the infinitesimals for (5.1) given by

$$\xi = \tfrac{3}{2}\mathrm{i}\sqrt{2}\,(u + b/3), \qquad \phi = \tfrac{3}{2}(u^3 + bu^2 + cu + d) \tag{5.4}$$

with associated invariant surface condition

$$\tfrac{3}{2}\mathrm{i}\sqrt{2}\,(u + b/3)u_x + u_t - \tfrac{3}{2}(u^3 + bu^2 + cu + d) = 0. \tag{5.5}$$

Following Nucci and Clarkson [71], this is solvable as follow. Using (5.5) to eliminate u_t in (5.1) yields

$$u_{xx} + \tfrac{3}{2}\mathrm{i}\sqrt{2}\,(u + b/3)u_x - \tfrac{1}{2}(u^3 + bu^2 + cu + d) = 0$$

which can be linearized by the transformation $u = -\sqrt{2}\,\mathrm{i}\,\Phi_x/\Phi$ yielding

$$2\sqrt{2}\,\Phi_{xxx} + 2\mathrm{i}b\Phi_{xx} - \sqrt{2}\,c\Phi_x - \mathrm{i}d\Phi = 0. \tag{5.6}$$

Suppose that the roots of the cubic (5.3) are m_1, m_2 and m_3, then the roots of

$$2\sqrt{2}\,p^3 + 2\mathrm{i}bp^2 - \sqrt{2}\,cp - \mathrm{i}d = 0 \tag{5.7}$$

are $\frac{1}{2}\sqrt{2}\,\mathrm{i}\,m_1$, $\frac{1}{2}\sqrt{2}\,\mathrm{i}\,m_2$ and $\frac{1}{2}\sqrt{2}\,\mathrm{i}\,m_3$.

The analogous infinitesimals for (5.2) are

$$\xi = \tfrac{3}{2}\sqrt{2}\,(u + b/3), \qquad \phi = -\tfrac{3}{2}(u^3 + bu^2 + cu + d) \tag{5.8}$$

with associated invariant surface condition

$$\tfrac{3}{2}\sqrt{2}\,(u + b/3)u_x + u_t + \tfrac{3}{2}(u^3 + bu^2 + cu + d) = 0. \tag{5.9}$$

Using this to eliminate u_t in (5.2) and making the linearizing transformation $u = \sqrt{2}\,\Psi_x/\Psi$ yields

$$2\sqrt{2}\,\Psi_{xxx} + 2b\Psi_{xx} + \sqrt{2}\,c\Psi_x + d\Psi = 0. \tag{5.10}$$

The roots of

$$2\sqrt{2}q^3 + 2bq^2 + \sqrt{2}\,cq + d = 0 \tag{5.11}$$

are $\frac{1}{2}\sqrt{2}\,m_1$, $\frac{1}{2}\sqrt{2}\,m_2$ and $\frac{1}{2}\sqrt{2}\,m_3$, where m_1, m_2 and m_3 are the roots of the cubic (5.3).

In the following four subsections we consider the cases when m_1, m_2 and m_3 are real and distinct, complex, two are equal and all three are equal, respectively. These exponential-type solutions of (5.2) are real for real x and t whereas those of (5.1) are complex for real x and t. Consequently we only give details for (5.2) and leave it for the reader to derive the analogous solutions for (5.1) using the transformation $x \to \pm\mathrm{i}x$, $t \to -t$.

5.1.1 Distinct Real Roots

Solving (5.10) in the case when m_1, m_2 and m_3 are distinct and real yields

$$\Psi(x,t) = \mu_1(t)\exp\left(\tfrac{1}{2}\sqrt{2}\,m_1x\right) + \mu_2(t)\exp\left(\tfrac{1}{2}\sqrt{2}\,m_2x\right) + \mu_3(t)\exp\left(\tfrac{1}{2}\sqrt{2}\,m_3x\right) \tag{5.12}$$

where $\mu_1(t)$, $\mu_2(t)$ and $\mu_3(t)$ are to be determined. By requiring that $u = \sqrt{2}\,\Phi_x/\Phi$ also satisfies (5.9), or equivalently (5.2), it is easily shown that

$$\mu_j(t) = \kappa_j \exp\left\{m_j\left(m_1+m_2+m_3-\tfrac{3}{2}m_j\right)t\right\}, \qquad j=1,2,3,$$

where κ_1, κ_2 and κ_3 are arbitrary constants (note that $m_1+m_2+m_3=-b$). Hence we have obtained the following exact solution of (5.2) given by

$$u(x,t) = \frac{\kappa_1 m_1\Psi_1(x,t)+\kappa_2 m_2\Psi_2(x,t)+\kappa_3 m_3\Psi_3(x,t)}{\kappa_1\Psi_1(x,t)+\kappa_2\Psi_2(x,t)+\kappa_3\Psi_3(x,t)} \tag{5.13}$$

where

$$\Psi_j(x,t) = \exp\left\{\tfrac{1}{2}\sqrt{2}\,m_jx - m_j\left(m_1+m_2+m_3-\tfrac{3}{2}m_j\right)t\right\}, \qquad j=1,2,3.$$

Setting $m_1 = a$, $m_2 = 1$ and $m_3 = 0$ in (5.13), which we may without loss of generality for the general cubic with real and distinct roots, yields the following exact solution of the Fitzhugh-Nagumo equation (1.9) for $a \neq 0$ and $a \neq 1$

$$u(x,t) = \frac{a\kappa_1\exp\left\{\tfrac{1}{2}\left(\pm\sqrt{2}\,ax+a^2t\right)\right\}+\kappa_2\exp\left\{\tfrac{1}{2}\left(\pm\sqrt{2}\,x+t\right)\right\}}{\kappa_1\exp\left\{\tfrac{1}{2}\left(\pm\sqrt{2}\,ax+a^2t\right)\right\}+\kappa_2\exp\left\{\tfrac{1}{2}\left(\pm\sqrt{2}\,x+t\right)\right\}+\kappa_3\exp\left(at\right)}. \tag{5.14}$$

This solution was obtained by Vorob'ev [97] (who calls the associated symmetry a "partial symmetry of the first type"), Kawahara and Tanaka [55] (using Hirota's bi-linear method [50]) and Hereman [45] (using the truncated Painlevé expansion method [67, 100, 101]). Plots of (5.14), with $\kappa_1 = \kappa_2 = \kappa_3 = 1$, for (i), $a=-0.5$, (ii), $a=0.4$, (iii), $a=0.7$, and (iv), $a=1.5$ are given in Figure 1; these plots were drawn using MAPLE.

5.1.2 Complex Roots

Suppose in this case that the roots of (5.3) are m (real) and $\alpha \pm \mathrm{i}\beta$. Then setting $m_1 = m$, $m_2 = \alpha+\mathrm{i}\beta$ and $m_3 = \alpha-\mathrm{i}\beta$ in (5.13) we obtain the exact solution of

$$u_t = u_{xx} - u^3 + (m+2\alpha)u^2 - (\alpha^2+\beta^2+2\alpha m)u + (\alpha^2+\beta^2)m \tag{5.15}$$

given by

$$u(x,t) = \frac{\kappa_1 m\exp\left\{\tfrac{1}{2}\sqrt{2}\,\mu x+\left(\tfrac{1}{2}\mu^2+\tfrac{3}{2}\beta^2\right)t\right\}+\kappa_2\left(\alpha^2+\beta^2\right)^{1/2}\cos\left(\tfrac{1}{2}\sqrt{2}\,\beta x-\mu\beta t+\theta_0+\delta_0\right)}{\kappa_1\exp\left\{\tfrac{1}{2}\sqrt{2}\,\mu x+\left(\tfrac{1}{2}\mu^2+\tfrac{3}{2}\beta^2\right)t\right\}+\kappa_2\cos\left(\tfrac{1}{2}\sqrt{2}\,\beta x-\mu\beta t+\delta_0\right)} \tag{5.16}$$

respectively, where κ_1, κ_2 and δ_0 are arbitrary constants, $\mu = m-\alpha$ and $\theta_0 = \tan^{-1}(\beta/\alpha)$.

Setting $m=\alpha=0$ and $\beta=1$ in (5.16) yields the exact solution of

$$u_t = u_{xx} - u(u^2+1) \tag{5.17}$$

given by

$$u(x,t) = \frac{\kappa_2 \sin\left(\frac{1}{2}\sqrt{2}\,x\right)}{\kappa_1 \exp\left(\frac{3}{2}t\right) + \kappa_2 \cos\left(\frac{1}{2}\sqrt{2}\,x\right)}. \tag{5.18}$$

Setting $m = 0$ and $\alpha = \beta = 1$ in (5.16) yields the exact solution of

$$u_t = u_{xx} - u(u^2 - 2u + 2) \tag{5.19}$$

given by

$$u(x,t) = \frac{\kappa_2 \left[\cos\left(\frac{1}{2}\sqrt{2}\,x - t\right) + \sin\left(\frac{1}{2}\sqrt{2}\,x - t\right)\right]}{\kappa_1 \exp\left(\frac{1}{2}\sqrt{2}\,x + 2t\right) + \kappa_2 \cos\left(\frac{1}{2}\sqrt{2}\,x - t\right)}. \tag{5.20}$$

Plots of (5.18), with $\kappa_1 = 2$ and $\kappa_2 = 1$, and (5.20), with $\kappa_1 = 1$ and $\kappa_2 = 1.1$, are given in Figure 2; these plots were drawn using MAPLE.

5.1.3 Two Equal Roots

Suppose that the cubic (5.3) has a single root m_1 and a double root m_2. Then the solution of (5.10) is given by

$$\Psi(x,t) = \mu_1(t) \exp\left(\frac{1}{2}\sqrt{2}\,m_1 x\right) + [\mu_2(t) + x\mu_3(t)] \exp\left(\frac{1}{2}\sqrt{2}\,m_2 x\right) \tag{5.21}$$

where $\mu_1(t)$, $\mu_2(t)$ and $\mu_3(t)$ are to be determined. A similar procedure to that used in the previous subsection yields the exact solution of

$$u_t = u_{xx} - (u - m_1)(u - m_2)^2 \tag{5.22}$$

given by

$$u(x,t) = -\frac{m_1\kappa_1 \exp\left\{\frac{1}{2}\left(\sqrt{2}\,\beta x + \beta^2 t\right)\right\} + \kappa_2\left[m_2\left(x - \sqrt{2}\,\beta t\right) + \sqrt{2}\right]}{\kappa_1 \exp\left\{\frac{1}{2}\left(\sqrt{2}\,\beta x + \beta^2 t\right)\right\} + \kappa_2\left(x - \sqrt{2}\,\beta t\right)} \tag{5.23}$$

where $\beta = m_2 - m_1$ and κ_1, κ_2 and κ_3 are arbitrary constants.

In particular, an exact solution of

$$u_t = u_{xx} - u^2(u + b) \tag{5.24}$$

is

$$u(x,t) = -\frac{b\kappa_1 \exp\left\{\frac{1}{2}\left(\sqrt{2}\,bx + b^2 t\right)\right\} + \sqrt{2}\,\kappa_2}{\kappa_1 \exp\left\{\frac{1}{2}\left(\sqrt{2}\,bx + b^2 t\right)\right\} + \kappa_2\left(x - \sqrt{2}\,bt\right)}. \tag{5.25}$$

Plots of this solution for $b = \pm 1$, $\kappa_1 = 1$ and $\kappa_2 = \frac{1}{2}$ are given in Figure 3; these plots were drawn using MAPLE.

5.1.4 Three Equal Roots

In the case when the cubic (5.3) has three equal roots, a similar procedure to that used in the previous three subsections yields the exact solution of

$$u_t = u_{xx} - u^3 \tag{5.26}$$

Table 3: Exponential-type solutions of equation (1.1)

$f(u)$	$u(x,t)$
$u(1-u)(u-a)$	$\dfrac{a\kappa_1 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,ax + a^2 t\right)\right\} + \kappa_2 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,x + t\right)\right\}}{\kappa_1 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,ax + a^2 t\right)\right\} + \kappa_2 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,x + t\right)\right\} + \kappa_3 \exp(at)}$
$-u(u^2+1)$	$\dfrac{\kappa_2 \sin\left(\frac{1}{2}\sqrt{2}\,x\right)}{\kappa_1 \exp\left(\frac{3}{2}t\right) + \kappa_2 \cos\left(\frac{1}{2}\sqrt{2}\,x\right)}$
$-u(u^2-2u+2)$	$\dfrac{\kappa_2 \left[\cos\left(\frac{1}{2}\sqrt{2}\,x - t\right) + \sin\left(\frac{1}{2}\sqrt{2}\,x - t\right)\right]}{\kappa_1 \exp\left(\frac{1}{2}\sqrt{2}\,x + 2t\right) + \kappa_2 \cos\left(\frac{1}{2}\sqrt{2}\,x - t\right)}$
$-u^2(u+b)$	$-\dfrac{b\kappa_1 \exp\left\{\frac{1}{2}\left(\sqrt{2}\,bx + b^2 t\right)\right\} + \sqrt{2}\,\kappa_2}{\kappa_1 \exp\left\{\frac{1}{2}\left(\sqrt{2}\,bx + b^2 t\right)\right\} + \kappa_2\left(x - \sqrt{2}\,bt\right)}$
$-u^3$	$\dfrac{\sqrt{2}\,(2x+\kappa_1)}{x^2+\kappa_1 x + 6t + \kappa_2}$

given by

$$u(x,t) = \frac{\sqrt{2}\,(2x+\kappa_1)}{x^2+\kappa_1 x + 6t + \kappa_2}. \tag{5.27}$$

A plot of this solution for $\kappa_1 = \kappa_2 = 0$ is given in Figure 4; this plot was drawn using MAPLE.

In Table 3, we list the nonclassical exponential-type solutions of equation (1.1) derived in this section for various canonical choices of $f(u)$.

5.2 Elliptic Function Solutions

In Case 4.1.1iii above we derived the nonclassical symmetry reduction of

$$u_t = u_{xx} + u^3 \tag{5.28}$$

given by

$$u(x,t) = (x+\kappa_1)w(z), \qquad z = \tfrac{1}{2}x^2 + \kappa_1 x + 3t \tag{5.29}$$

where $w(z)$ satisfies

$$w'' + w^3 = 0. \tag{5.30}$$

The solution of this equation is

$$w(z) = \tfrac{1}{2}\sqrt{2}\ \mathrm{sd}\left(z; \tfrac{1}{2}\sqrt{2}\right) \tag{5.31}$$

where $\mathrm{sd}(z;k)$ is the Jacobi elliptic function satisfying

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}z}\right)^2 = 1 + (2k^2-1)\eta^2 + k^2(k^2-1)\eta^4. \tag{5.32}$$

Hence we obtain the exact solution of (5.28) given by

$$u(x,t) = \tfrac{1}{2}\sqrt{2}\,(x+\kappa_1)\,\mathrm{sd}\left(\tfrac{1}{2}x^2 + \kappa_1 x + 3t; \tfrac{1}{2}\sqrt{2}\right). \tag{5.33}$$

A plot of this solution for $\kappa_1 = 0$ is given in Figure 5; this plot was drawn using MATHEMATICA.

The analogous nonclassical symmetry reduction of

$$u_t = u_{xx} - u^3 \tag{5.34}$$

is given by

$$u(x,t) = (x+\kappa_1)w(z), \qquad z = \tfrac{1}{2}x^2 + \kappa_1 x + 3t \tag{5.35}$$

where $w(z)$ satisfies

$$w'' - w^3 = 0. \tag{5.36}$$

The solution of this equation is

$$w(z) = \sqrt{2}\,\mathrm{ds}\left(z; \tfrac{1}{2}\sqrt{2}\right) \tag{5.37}$$

where $\mathrm{ds}(z;k)$ is the Jacobi elliptic function satisfying

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}z}\right)^2 = k^2(k^2-1) + (2k^2-1)\eta^2 + \eta^4. \tag{5.38}$$

Hence we obtain the exact solution of (5.34) given by

$$u(x,t) = \sqrt{2}\,(x+\kappa_1)\,\mathrm{ds}\left(\tfrac{1}{2}x^2 + \kappa_1 x + 3t; \tfrac{1}{2}\sqrt{2}\right). \tag{5.39}$$

In Case 4.1.1iv above we derived the nonclassical symmetry reduction of

$$u_t = u_{xx} + u^3 + bu^2 + cu + \frac{b(9c-2b^2)}{27} \tag{5.40}$$

given by

$$u(x,t) = \kappa_1\mu\sin(\mu x+\kappa_2)\exp\left(-3\mu^2 t\right)w(z) - \tfrac{1}{3}b, \quad z = \kappa_1\cos(\mu x+\kappa_2)\exp\left(-3\mu^2 t\right) \tag{5.41}$$

where $\mu^2 = \frac{1}{6}(b^2-3c)$, κ_1 and κ_2 are arbitrary constants and $w(z)$ satisfies (5.30). Hence for $b^2 > 3c$ we obtain the exact solution of (5.40) given by

$$u(x,t) = \tfrac{1}{2}\sqrt{2}\,\kappa_1\mu\sin(\mu x+\kappa_2)\exp\left(-3\mu^2 t\right)\mathrm{sd}\left[\kappa_1\cos(\mu x+\kappa_2)\exp\left(-3\mu^2 t\right); \tfrac{1}{2}\sqrt{2}\right] - \tfrac{1}{3}b \tag{5.42}$$

where $\mu^2 = \frac{1}{6}(b^2-3c)$. Two plots of this solution for $\kappa_1 = 3$, $\kappa_2 = 0$ and $\mu = \frac{1}{2}$ are given in Figure 6; these plots were drawn using MATHEMATICA. For $b^2 < 3c$ we obtain the exact solution of (5.40) given by

$$u(x,t) = \tfrac{1}{2}\sqrt{2}\,\kappa_1\lambda\sinh(\lambda x+\kappa_2)\exp\left(3\lambda^2 t\right)\mathrm{sd}\left[\kappa_1\cosh(\lambda x+\kappa_2)\exp\left(3\lambda^2 t\right); \tfrac{1}{2}\sqrt{2}\right] - \tfrac{1}{3}b \tag{5.43}$$

where $\lambda^2 = \frac{1}{6}(3c-b^2)$. A plot of this solution for $\kappa_1 = 3$, $\kappa_2 = 0$ and $\lambda = \frac{1}{2}$ is given in Figure 7; this plot was drawn using MATHEMATICA.

The analogous nonclassical symmetry reduction of

$$u_t = u_{xx} - u^3 - bu^2 - cu - \frac{b(9c-2b^2)}{27} \tag{5.44}$$

is given by

$$u(x,t) = \kappa_1\mu\sinh(\mu x+\kappa_2)\exp\left(3\mu^2 t\right)w(z) - \tfrac{1}{3}b, \qquad z = \cosh(\mu x+\kappa_2)\exp\left(3\mu^2 t\right) \tag{5.45}$$

Table 4: Elliptic Function solutions of equation (1.1)

$f(u)$	$u(x,t)$
u^3	$\frac{1}{2}\sqrt{2}\,(x+\kappa_1)\,\mathrm{sd}\left(\frac{1}{2}x^2+\kappa_1 x+3t;\frac{1}{2}\sqrt{2}\right)$
$-u^3$	$\sqrt{2}\,(x+\kappa_1)\,\mathrm{ds}\left(\frac{1}{2}x^2+\kappa_1 x+3t;\frac{1}{2}\sqrt{2}\right)$
$u(u^2-2)$	$\frac{1}{2}\sqrt{2}\,\kappa_1\sin(x+\kappa_2)\exp(-3t)\,\mathrm{sd}\left[\kappa_1\cos(x+\kappa_2)\exp(-3t)\,;\frac{1}{2}\sqrt{2}\right]$
$u(u^2+2)$	$\frac{1}{2}\sqrt{2}\,\kappa_1\sinh(x+\kappa_2)\exp(3t)\,\mathrm{sd}\left[\kappa_1\cosh(x+\kappa_2)\exp(3t)\,;\frac{1}{2}\sqrt{2}\right]$
$-u(u^2-2)$	$\sqrt{2}\,\kappa_1\sinh(x+\kappa_2)\exp(3t)\,\mathrm{ds}\left[\kappa_1\cosh(x+\kappa_2)\exp(3t)\,;\frac{1}{2}\sqrt{2}\right]$
$-u(u^2+2)$	$\sqrt{2}\,\kappa_1\sin(x+\kappa_2)\exp(-3t)\,\mathrm{ds}\left[\kappa_1\cos(x+\kappa_2)\exp(-3t)\,;\frac{1}{2}\sqrt{2}\right]$

where $\mu^2=\frac{1}{6}(b^2-3c)$, κ_1 and κ_2 are arbitrary constants and $w(z)$ satisfies (5.36). Hence for $b^2>3c$ we obtain the exact solution of (5.44) given by

$$u(x,t)=\sqrt{2}\,\kappa_1\mu\sinh(\mu x+\kappa_2)\exp\left(3\mu^2 t\right)\mathrm{ds}\left[\kappa_1\cosh(\mu x+\kappa_2)\exp\left(3\mu^2 t\right);\tfrac{1}{2}\sqrt{2}\right]-\tfrac{1}{3}b \qquad (5.46)$$

and for $b^2<3c$, obtain the exact solution

$$u(x,t)=\sqrt{2}\,\kappa_1\lambda\sin(\lambda x+\kappa_2)\exp\left(-3\lambda^2 t\right)\mathrm{ds}\left[\kappa_1\cos(\lambda x+\kappa_2)\exp\left(-3\lambda^2 t\right);\tfrac{1}{2}\sqrt{2}\right]-\tfrac{1}{3}b \qquad (5.47)$$

where $\lambda^2=\frac{1}{6}(3c-b^2)$.

In Table 4, we list the nonclassical elliptic function solutions of equation (1.1) derived in this section for various canonical choices of $f(u)$.

6 Discussion

In this paper we have demonstrated that the method of DGBS has enabled all symmetries of the nonlinear heat equation (1.1) to be found; the list it provides of possible analytic functions $f(u)$ for which symmetries may exist is definitive. The use of DGBS has made the analysis of overdetermined systems of PDES more tractable and whilst the diffgrob2 package needs to be used interactively at present, nevertheless it is effective in solving the overdetermined systems of determining equations for classical and nonclassical symmetries of (1.1).

It is not clear how the direct method, developed by Clarkson and Kruskal [20] for finding symmetry reductions of PDES may be applied to equations such as (1.1) which contain arbitrary functions. Nucci and Clarkson [71] (see also [21]) demonstrated that the symmetry reduction (5.14) of the Fitzhugh-Nagumo equation (1.9), is obtainable using the nonclassical method due to Bluman and Cole [9], though *not* using the direct method due to Clarkson and Kruskal [20]. Hence the nonclassical method is more general than the direct method, at least as it was originally formulated. Olver [74] has recently shown that the direct method is equivalent to the nonclassical method when the infinitesimals ξ and τ are independent of the dependent variable u (see also [19, 81, 104, 105]). In this case the associated vector field is of the form

$$\mathbf{w}=\xi(x,t)\partial_x+\tau(x,t)\partial_t+\phi(x,t,u)\partial_u,$$

and generates a group of "*fibre-preserving transformations*", meaning that the transformations in x and t do not depend upon u. A recent extension of the direct method by Estevez [26] does yield the symmetry reduction solution (5.14) of the Fitzhugh-Nagumo equation (1.9). These results pose the following important open question: "for which PDES does the nonclassical method yield more symmetry reductions than the direct method?" Furthermore, it remains an open question to determine *a priori* which PDES possess symmetry reductions that are not obtainable using the classical Lie group approach.

Fushchich and Serov [35] (see also [33]) claimed that

$$u_t + u_{xx} = f(u) \tag{6.1}$$

which is equivalent to (1.1), has a nonclassical symmetry, which they refer to as a conditional symmetry with respect to the vector field (1.3), if and only if (1.1) is equivalent to the special form

$$u_t + u_{xx} = au^3 + bu + c \tag{6.2}$$

where a, b and c are arbitrary constants, which is equivalent to (5.1) and (5.2) and that the associated infinitesimals are

$$\xi = \tfrac{2}{3}\sqrt{2a}\,u, \qquad \tau = 1, \qquad \phi = \tfrac{3}{2}(au^3 + bu + c). \tag{6.3}$$

Subsequently they consider the four special cases of (6.2) given by

$$\begin{aligned} u_t + u_{xx} &= au(u^2 - 1), && (6.4)\\ u_t + u_{xx} &= au(u^2 + 1), && (6.5)\\ u_t + u_{xx} &= a(u^3 - 3u + 2), && (6.6)\\ u_t + u_{xx} &= au^3 && (6.7) \end{aligned}$$

and whereas they integrate the invariant surface conditions, they do not write down the associated exact solutions of (6.1).

The results we have obtained in this paper show that there are more nonclassical symmetry of (6.1) other than those generated by the infinitesimals (6.3). In particular they do not obtain the nonclassical symmetry reductions corresponding to those obtained in Cases 4.1.1iii,iv and 4.2.3ii,iii above. Further even for the symmetry reductions associated with (6.3), the four "canonical" equations (6.4,6.5,6.6,6.7) do not include the case when the roots of the cubic $au^3 + bu + c$ are real, distinct and unequally spaced.

Acknowledgements

We thank Clara Nucci and Peter Olver for helpful discussions and the referees for constructive comments. We also thank the Program in Applied Mathematics, University of Colorado at Boulder, for their hospitality, where this work was completed. The support of SERC (grant GR/H39420) is gratefully acknowledged. PAC is also grateful for support through a Nuffield Foundation Science Fellowship and NATO grant CRG 910729.

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Figure Captions

Figure 1: Exponential solution (5.14) of $u_t = u_{xx} + u(1-u)(u-a)$ where (i) $a = -0.5$, (ii) $a = 0.4$, (iii) $a = 0.7$, (iv) $a = 1.5$.

Figure 2a: Exponential solution (5.18) of $u_t = u_{xx} - u(u^2+1)$ where $\kappa_1 = \kappa_2 = 1$.

Figure 2b: Exponential solution (5.20) of $u_t = u_{xx} - u(u^2-2u+2)$ where $\kappa_1 = \kappa_2 = 1$.

Figure 3: Exponential solution (5.25) of $u_t = u_{xx} - u(u^2+b)$ where (i) $b = -1$ and (ii) $b = 1$.

Figure 4: Rational solution (5.27) of $u_t = u_{xx} + u^3$ where $\kappa_1 = \kappa_2 = 0$.

Figure 5: Elliptic solution (5.33) of $u_t = u_{xx} + u^3$ where $\kappa_1 = 0$.

Figure 6: Elliptic solution (5.42) of $u_t = u_{xx} + u^3 + cu + b(9c - 2b^2)/27$ where $\mu^2 = \frac{1}{6}(b^2 - 3c) = \frac{1}{2}\sqrt{2}$, $\kappa_1 = 3$ and $\kappa_2 = 0$.

Figure 7: Elliptic solution (5.43) of $u_t = u_{xx} + u^3 + cu + b(9c - 2b^2)/27$ where $\lambda^2 = \frac{1}{6}(3c - b^2) = \frac{1}{2}\sqrt{2}$, $\kappa_1 = 3$ and $\kappa_2 = 0$.