Physica 127A (1984) 599–612
North-Holland, Amsterdam

SYMMETRY TRANSFORMATIONS IN INDEFINITE METRIC SPACES

A GENERALIZATION OF WIGNER'S THEOREM

P.M. VAN DEN BROEK

*Department of Applied Mathematics, Twente University of Technology, P.O. Box 217, 7500 AE Enschede, the Netherlands**

Received 23 June 1983
Revised 20 February 1984

We formulate and prove a generalization to indefinite metric spaces of Uhlhorn's version of Wigner's theorem.

1. Introduction and results

Let V be a n-dimensional complex vector space, $n \geq 3$, with a scalar product denoted by $(\,,)$, which is linear in the second factor.

Let η be a Hermitian non-singular linear operator on V. The indefinite metric of V is given by

$$\langle \psi, \phi \rangle = (\psi, \eta\phi) \quad \forall \psi, \phi \in V. \tag{1.1}$$

Actually, this metric is a non-singular metric (which includes both the definite and indefinite cases).

If $\psi \in V$ and $\psi \neq 0$ then the ray $\boldsymbol{\psi}$ is defined by

$$\boldsymbol{\psi} = \{\lambda\psi \mid \lambda \in \mathbb{C}, \lambda \neq 0\}. \tag{1.2}$$

The rays $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ are said to be orthogonal (denoted by $\langle \boldsymbol{\psi} \cdot \boldsymbol{\phi} \rangle = 0$) if $\langle \psi, \phi \rangle = 0$ and non-orthogonal (denoted by $\langle \boldsymbol{\psi} \cdot \boldsymbol{\phi} \rangle \neq 0$) if $\langle \psi, \phi \rangle \neq 0$. This definition is independent of the choice of $\psi \in \boldsymbol{\psi}$ and $\phi \in \boldsymbol{\phi}$. The rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k$ are said to be independent if and only if $\psi_1, \ldots, \psi_k$ are linearly independent. This definition does not depend on the choice of $\psi_i \in \boldsymbol{\psi}_i$, $i = 1, \ldots, k$.

The rays of V form the projective space $\boldsymbol{V}$. A mapping $T: \boldsymbol{V} \to \boldsymbol{V}$ is called a symmetry transformation if

i) T is a bijection from $\boldsymbol{V}$ onto $\boldsymbol{V}$,
ii) $\langle \boldsymbol{\psi} \cdot \boldsymbol{\phi} \rangle = 0 \Leftrightarrow \langle T\boldsymbol{\psi} \cdot T\boldsymbol{\phi} \rangle = 0 \quad \forall \boldsymbol{\psi}, \boldsymbol{\phi} \in \boldsymbol{V}$. (1.3)

* Present address: Venderinklanden 59, 7542 MN Enschede, the Netherlands.

An operator $\mathcal{U}$ on V is called semilinear if

i) $\mathcal{U}$ is a bijection from V onto V,

$$\text{ii) } \mathcal{U}(\psi + \phi) = \mathcal{U}\psi + \mathcal{U}\phi \quad \forall \psi, \phi \in V, \tag{1.4}$$

$$\text{iii) } \mathcal{U}(\lambda\psi) = \zeta(\lambda)\mathcal{U}\psi \quad \forall \psi \in V, \forall \lambda \in \mathbb{C}, \tag{1.5}$$

where the mapping $\zeta : \mathbb{C} \to \mathbb{C}$ is determined by $\mathcal{U}$ and satisfies

$$\zeta(\lambda) = \lambda \quad \forall \lambda \in \mathbb{C}, \tag{1.6}$$

or

$$\zeta(\lambda) = \bar{\lambda} \quad \forall \lambda \in \mathbb{C}. \tag{1.7}$$

It is clear that to each semilinear operator $\mathcal{U}$ there corresponds a bijective mapping $T : \boldsymbol{V} \to \boldsymbol{V}$ defined by

$$T\boldsymbol{\psi} = \mathcal{U}\boldsymbol{\psi}. \tag{1.8}$$

$\mathcal{U}$ is said to induce T. We now have the following theorems:

Theorem 1. For each symmetry transformation T there exists a semilinear operator $\mathcal{U}$ which induces T.

Theorem 2. Two semilinear operators $\mathcal{U}$ and $\mathcal{U}'$ induce the same mapping $T : \boldsymbol{V} \to \boldsymbol{V}$ if and only if $\mathcal{U} = \lambda\mathcal{U}'$ for some $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.

Theorem 3. A semilinear operator $\mathcal{U}$ induces a symmetry transformation if and only if

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = C\zeta(\langle \psi, \phi \rangle) \quad \forall \psi, \phi \in V \tag{1.9}$$

for some real constant C.

So there is a 1–1 correspondence between the symmetry transformations T of $\boldsymbol{V}$ and those rays of semilinear transformations of V which contain a semilinear operator $\mathcal{U}$ such that either

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = \langle \psi, \phi \rangle \qquad \forall \psi, \phi \in V, \tag{1.10}$$

or

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = \langle \phi, \psi \rangle \quad \forall \psi, \phi \in V, \tag{1.11}$$

or

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = -\langle \psi, \phi \rangle \quad \forall \psi, \phi \in V, \tag{1.12}$$

or

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = -\langle \phi, \psi \rangle \quad \forall \psi, \phi \in V. \tag{1.13}$$

In Wigner's original theorem[1]), where η is the identity operator and n is allowed to take the value 2, condition (1.3) reads

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{\phi} \rangle = \langle T\boldsymbol{\psi} \cdot T\boldsymbol{\phi} \rangle \quad \forall \boldsymbol{\psi}, \boldsymbol{\phi} \in \boldsymbol{V}, \tag{1.14}$$

where $\langle \boldsymbol{\psi} \cdot \boldsymbol{\phi} \rangle$ is defined by

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{\phi} \rangle = \frac{|(\psi, \phi)|}{\|\psi\|\|\phi\|} \quad \forall \psi, \phi \in V, \tag{1.15}$$

which definition is independent of the choice of $\psi \in \boldsymbol{\psi}$ and $\phi \in \boldsymbol{\phi}$. This theorem has been generalized for indefinite metric spaces by Bracci, Morchio and Strocchi[2]). Uhlhorn[3]) proved a stronger version of Wigner's theorem: for $n \geq 3$ he replaced condition (1.14) by condition (1.3). So he actually showed that if $n \geq 3$ the conditions (1.14) and (1.3) are equivalent if η is the identity operator. This is the reason for our definition of symmetry transformation of V. Our theorems are the generalization of Uhlhorn's version of Wigner's theorem for indefinite metric spaces, so actually they combine the generalizations of Wigner's theorem of refs. 2 and 3.

The aim of this paper is to provide a complete proof of these theorems. This proof is a modification of the proof of Uhlhorn[3]). In order to make this paper self-contained also those parts of Uhlhorn's proof which need no modification are given in full detail. An application of our theorems is given in ref. 4.

2. Proof of theorem 1

Since the proof of theorem 1 is rather long we start with an overview of the proof, stating several lemmas which we will prove further on. In this section T denotes a fixed symmetry transformation.

2.1. Outline of the proof

A basic lemma is the following:

Lemma 1. The rays $\psi_1, \ldots, \psi_k$ are independent if and only if the rays $T\psi_1, \ldots, T\psi_k$ are independent.

Let S be a mapping from V into V with the property that $S0 = 0$ (0 denotes the null vector) and $S\psi$ is an element from the ray $T\psi$ for each non-zero $\psi \in V$. Thus S induces T. Let V' denote the set $V \backslash \{0\}$. Then we have the following lemma:

Lemma 2. There exists a complex function ω on $V' \times V'$ with the properties

$$\text{i) } \omega(\psi, \phi) \neq 0 \quad \forall \psi, \phi \in V', \tag{2.1}$$

$$\text{ii) } \omega(\psi, \phi)\omega(\phi, \xi) = \omega(\psi, \xi) \quad \forall \psi, \phi, \xi \in V', \tag{2.2}$$

$$\text{iii) } S(\psi + \phi) = \omega(\psi, \psi + \phi)S\psi + \omega(\phi, \psi + \phi)S\phi \tag{2.3}$$

for any pair of independent vectors $\psi, \phi \in V$.

Now define the mapping $\mathcal{U}$: $V \to V$ by $\mathcal{U}0 = 0$ and

$$\mathcal{U}\psi = \omega(\psi, \psi_0)S\psi \quad \forall \psi \in V' . \tag{2.4}$$

Here ψ_0 is some fixed vector from V'. It is clear that $\mathcal{U}$ induces T.

Lemma 3. $\mathcal{U}(\psi + \phi) = \mathcal{U}\psi + \mathcal{U}\phi \quad \forall \psi, \phi \in V.$ (2.5)

Lemma 4. $\mathcal{U}$ is a bijection from V onto V.

Lemma 5. $\mathcal{U}(\lambda\psi) = \xi(\lambda)\mathcal{U}\psi \quad \forall \lambda \in \mathbb{C}, \forall \psi \in V$ (2.6)
for some mapping ζ: $\mathbb{C} \to \mathbb{C}$.

Lemma 6. The mapping ζ from the preceding lemma satisfies

$$\zeta(\lambda) = \lambda \quad \forall \lambda \in \mathbb{C} , \tag{2.7}$$

or

$$\zeta(\lambda) = \bar{\lambda} \quad \forall \lambda \in \mathbb{C} . \tag{2.8}$$

The lemmas 3, 4, 5 and 6 imply that $\mathcal{U}$ is semilinear. Since $\mathcal{U}$ induces T, theorem 1 has been proved.

2.2. *Proof of lemma 1*

Let $\psi_1, \ldots, \psi_k$ be vectors from the rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k$ respectively. The rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k$ are independent if and only if the vectors $\psi_1, \ldots, \psi_k$ are linearly independent. The following proposition is obviously true:

The vectors $\psi_1, \ldots, \psi_k$ are linearly independent if and only if $\psi_1, \ldots, \psi_{k-1}$ are linearly independent and there exists a vector ϕ such that $(\psi_i, \phi) = 0$ for $i = 1, \ldots, k-1$ and $(\psi_k, \phi) \neq 0$.

Let $\phi' = \eta^{-1}\phi$. Then $\langle \psi_i, \phi' \rangle = (\psi_i, \eta\phi') = (\psi_i, \phi)$. So the following proposition is true: The rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k$ are independent if and only if the rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{k-1}$ are independent and there exists a ray $\boldsymbol{\phi}'$ such that $\langle \boldsymbol{\psi}_i, \boldsymbol{\phi}' \rangle = 0$ for $i = 1, \ldots, k-1$ and $\langle \boldsymbol{\psi}_k, \boldsymbol{\phi}' \rangle \neq 0$.

Lemma 1 is true for $k = 1$. Suppose that it is true for $k = p$. We will show that it is true for $k = p + 1$. The rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p+1}$ are independent if and only if the rays $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_p$ are independent and there exists a ray $\boldsymbol{\phi}'$ such that $\langle \boldsymbol{\psi}_i, \boldsymbol{\phi}' \rangle = 0$ for $i = 1, \ldots, p$ and $\langle \boldsymbol{\psi}_{p+1}, \boldsymbol{\phi}' \rangle \neq 0$. By our assumption and by eq. (1.3) this is true if and only if $T\boldsymbol{\psi}_1, \ldots, T\boldsymbol{\psi}_p$ are independent and there exists a ray $T\boldsymbol{\phi}'$ such that $\langle T\boldsymbol{\psi}_i, T\boldsymbol{\phi}' \rangle = 0$ for $i = 1, \ldots, p$ and $\langle T\boldsymbol{\psi}_{p+1}, T\boldsymbol{\phi}' \rangle \neq 0$. This is true if and only if $T\boldsymbol{\psi}_1, \ldots, T\boldsymbol{\psi}_{p+1}$ are independent which proves lemma 1.

2.3. *Proof of lemma 2*

Let ψ and ϕ be two linearly independent vectors. The three vectors ψ, ϕ and $\psi + \phi$ are linearly dependent but pairwise linearly independent. From lemma 1 it follows that $S\psi$, $S\phi$ and $S(\psi + \phi)$ are linearly dependent but pairwise linearly independent. Thus

$$S(\psi + \phi) = \lambda(\psi, \phi)S\psi + \mu(\psi, \phi)S\phi\,, \tag{2.9}$$

where the complex numbers $\lambda(\psi, \phi)$ and $\mu(\psi, \phi)$ are uniquely determined by ψ and ϕ and are non-zero. Since we also have

$$S(\phi + \psi) = \lambda(\phi, \psi)S\phi + \mu(\phi, \psi)S\psi\,, \tag{2.10}$$

it follows that

$$\mu(\psi, \phi) = \lambda(\phi, \psi)\,. \tag{2.11}$$

For any pair of linearly independent vectors ψ and ϕ we define

$$\omega(\psi, \phi) = \lambda(\psi, \phi - \psi)\,. \tag{2.12}$$

From the eqs. (2.9), (2.11) and (2.12) we immediately obtain eq. (2.3).

Let ψ, ϕ and ξ be three linearly independent vectors. (Remember that $\dim V \geqslant 3$). Then we can write

$$S(\psi+\phi+\xi)=\omega(\psi, \psi+\phi+\xi)S\psi+\omega(\phi+\xi, \psi+\phi+\xi)\omega(\phi, \phi+\xi)S\phi \\ +\omega(\phi+\xi, \psi+\phi+\xi)\omega(\xi, \phi+\xi)S\xi\,. \tag{2.13}$$

Interchanging ψ and ϕ in this expression gives

$$S(\phi+\psi+\xi)=\omega(\phi, \phi+\psi+\xi)S\phi+\omega(\psi+\xi, \phi+\psi+\xi)\omega(\psi, \psi+\xi)S\psi \\ +\omega(\psi+\xi, \phi+\psi+\xi)\omega(\xi, \psi+\xi)S\xi\,. \tag{2.14}$$

Since $S\psi$, $S\phi$ and $S\xi$ are linearly independent (lemma 1) the coefficients of $S\psi$ in eqs. (2.13) and (2.14) are equal:

$$\omega(\psi, \psi+\phi+\xi)=\omega(\psi+\xi, \phi+\psi+\xi)\omega(\psi, \psi+\xi)\,. \tag{2.15}$$

It follows that

$$\omega(\psi, \phi)\omega(\phi, \xi)=\omega(\psi, \xi) \tag{2.16}$$

for any three linearly independent vectors ψ, ϕ and ξ. Then

$$\omega(\psi, \phi)\omega(\phi, \psi)\omega(\psi, \xi)=\omega(\psi, \phi)\omega(\phi, \xi)=\omega(\psi, \xi) \tag{2.17}$$

and it follows, since $\omega(\psi, \xi)\neq 0$, that for any linearly independent ψ and ϕ

$$\omega(\psi, \phi)\omega(\phi, \psi)=1\,. \tag{2.18}$$

Now let ψ and ϕ be two linearly dependent vectors from V'. We define $\omega(\psi, \phi)$ by

$$\omega(\psi, \phi)=\omega(\psi, \xi)\psi(\xi, \phi)\,, \tag{2.19}$$

where ξ is any vector such that ψ and ξ are linearly independent.

We have to show that this definition is independent of the choice of ξ. Let χ be another vector such that ψ and χ are linearly independent. If ψ, ξ and χ are linearly independent then

$$\omega(\psi, \xi)\omega(\xi, \phi) = \omega(\psi, \xi)[\omega(\xi, \chi)\omega(\chi, \xi)]\omega(\xi, \phi)$$
$$= \omega(\psi, \chi)\omega(\chi, \phi) . \tag{2.20}$$

If ψ, ξ and χ are linearly dependent then choose a vector ζ such that ψ, ξ and ζ are linearly independent. Then also the vectors ψ, χ and ζ are linearly independent. Thus we have, as above,

$$\omega(\psi, \xi)\omega(\xi, \phi) = \omega(\psi, \zeta)\omega(\zeta, \phi) \tag{2.21}$$

and

$$\omega(\psi, \chi)\omega(\chi, \phi) = \omega(\psi, \zeta)\omega(\zeta, \phi) \tag{2.22}$$

and thus

$$\omega(\psi, \chi)\omega(\chi, \phi) = \omega(\psi, \xi)\omega(\xi, \phi) . \tag{2.23}$$

This shows that the definition of eq. (2.19) is independent of the choice of ξ. From this definition it follows that eq. (2.1) is true. We still have to prove eq. (2.2). Let ψ, ϕ and ξ be vectors from V'. Suppose first that ξ is not equal to a linear combination of ψ and ϕ. If ψ and ϕ are linearly independent then

$$\omega(\psi, \xi)\omega(\xi, \phi) = \omega(\psi, \phi) , \tag{2.24}$$

since ψ, ϕ and ξ are linearly independent (eq. (2.16)). If ψ and ϕ are linearly dependent, then eq. (2.24) is true by definition.

Now suppose that ξ is a linear combination of ψ and ϕ. Let χ be a vector from V' which is not a linear combination of ψ and ϕ. Then χ is neither a linear combination of ψ and ξ, nor a linear combination of ϕ and ξ. Thus, as in eq. (2.24),

$$\omega(\psi, \chi)\omega(\chi, \xi) = \omega(\psi, \xi) \tag{2.25}$$

and

$$\omega(\xi, \chi)\omega(\chi, \phi) = \omega(\xi, \phi) . \tag{2.26}$$

It follows that

$$\omega(\psi, \xi)\omega(\xi, \phi) = \omega(\psi, \chi)\omega(\chi, \xi)\omega(\xi, \chi)\omega(\chi, \phi)$$
$$= \omega(\psi, \chi)\omega(\chi, \phi) = \omega(\psi, \phi) , \tag{2.27}$$

where we used eq. (2.18). So also in this case eq. (2.24) is valid. This proves lemma 2.

2.4. *Proof of lemma 3*

The mapping $\mathcal{U}$, which is defined in section 2.1, satisfies

$$\mathcal{U}(\psi+\phi)=\mathcal{U}\psi+\mathcal{U}\phi \tag{2.28}$$

if at least one of the vectors ψ and ϕ is the null vector since $\mathcal{U}0=0$. Suppose that ψ and ϕ are linearly independent vectors from V'. Then we obtain, using eqs. (2.2), (2.3) and (2.4),

$$\begin{aligned}\mathcal{U}(\psi+\phi)&=\omega(\psi+\phi,\psi_0)S(\psi+\phi)\\&=\omega(\psi,\phi,\psi_0)\omega(\psi,\psi+\phi)S\psi+\omega(\psi+\phi,\psi_0)\omega(\phi,\psi+\phi)S\phi\\&=\omega(\psi,\psi_0)S\psi+\omega(\phi,\psi_0)S\phi=\mathcal{U}\psi+\mathcal{U}\phi\,.\end{aligned} \tag{2.29}$$

Suppose now that ψ and ϕ are linearly dependent vectors from V'. Let ξ be a vector such that ξ and ψ are linearly independent. Then

$$\mathcal{U}(\psi+\phi+\xi)=\mathcal{U}(\psi+\phi)+\mathcal{U}\xi \tag{2.30}$$

and

$$\mathcal{U}(\psi+\phi+\xi)=\mathcal{U}\psi+\mathcal{U}(\phi+\xi)=\mathcal{U}\psi+\mathcal{U}\phi+\mathcal{U}\xi\,. \tag{2.31}$$

It follows that

$$\mathcal{U}(\psi+\phi)=\mathcal{U}\psi+\mathcal{U}\phi\,, \tag{2.32}$$

which proves lemma 3.

2.5. *Proof of lemma 4*

From lemma 3 it follows that

$$\mathcal{U}(-\psi)=-\mathcal{U}\psi\quad\forall\psi\in V\,, \tag{2.33}$$

since $0=\mathcal{U}0=\mathcal{U}(\psi-\psi)=\mathcal{U}\psi+\mathcal{U}(-\psi)$.

Let ψ and ϕ be two vectors such that $\mathcal{U}\psi=\mathcal{U}\phi$. Thus

$$\mathcal{U}(\psi-\phi)=\mathcal{U}\psi-\mathcal{U}\phi=0\,. \tag{2.34}$$

From the definition of $\mathcal{U}$ it follows that

$$\psi - \phi \neq 0 \Rightarrow \mathcal{U}(\psi - \phi) \neq 0 . \tag{2.35}$$

From eqs. (2.34) and (2.35) it follows that $\psi = \phi$. So the mapping $\mathcal{U}$ is an injection.

To show that $\mathcal{U}$ is a surjection let ψ be some vector from V', let $\boldsymbol{\phi}$ be the ray such that $T\boldsymbol{\phi} = \boldsymbol{\psi}$ and let ξ be an element of V' such that $\boldsymbol{\xi}$ and $\boldsymbol{\phi}$ are independent. According to lemma 1 the rays $\mathcal{U}\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ are independent. Thus the rays $\mathcal{U}\boldsymbol{\xi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\psi} + \mathcal{U}\boldsymbol{\xi}$ are dependent, but pairwise independent. Let $\boldsymbol{\zeta}$ be the ray such that $T\boldsymbol{\zeta} = \boldsymbol{\psi} + \mathcal{U}\boldsymbol{\xi}$. From lemma 1 it follows that the rays $\boldsymbol{\xi}$, $\boldsymbol{\phi}$ and $\boldsymbol{\zeta}$ are dependent but pairwise independent. This implies that there exists $\phi \in \boldsymbol{\phi}$ and $\zeta \in \boldsymbol{\zeta}$ such that

$$\zeta = \phi + \xi . \tag{2.36}$$

Then

$$\mathcal{U}\zeta = \mathcal{U}(\phi + \xi) = \mathcal{U}\phi + \mathcal{U}\xi . \tag{2.37}$$

Since

$$\mathcal{U}\zeta = \lambda(\psi + \mathcal{U}\xi) \tag{2.38}$$

and

$$\mathcal{U}\phi = \mu\psi \tag{2.39}$$

for some $\lambda, \mu \in \mathbb{C}$, it follows that

$$\lambda(\psi + \mathcal{U}\xi) = \mu\psi + \mathcal{U}\xi . \tag{2.40}$$

Since ψ and $\mathcal{U}\xi$ are linearly independent it follows that $\lambda = \mu = 1$, and therefore $\mathcal{U}\phi = \psi$. Since $\mathcal{U}0 = 0$ it is proved that $\mathcal{U}$ is a surjection. This proves lemma 4.

2.6. Proof of lemma 5

For each $\psi \in V'$ and each $\lambda \in \mathbb{C}$ we define $\zeta(\lambda, \psi) \in \mathbb{C}$ by

$$\mathcal{U}(\lambda\psi) = \zeta(\lambda, \psi)\mathcal{U}\psi . \tag{2.41}$$

We will prove the lemma by showing that $\zeta(\lambda, \psi)$ is independent of ψ. Let ψ

and ϕ be two linearly independent vectors. Then

$$\mathcal{U}(\lambda(\psi+\phi)) = \zeta(\lambda, \psi+\phi)\mathcal{U}(\psi+\phi) = \zeta(\lambda, \psi+\phi)\mathcal{U}\psi + \zeta(\lambda, \psi+\phi)\mathcal{U}\phi \tag{2.42}$$

and

$$\mathcal{U}(\lambda\psi + \lambda\phi) = \mathcal{U}(\lambda\psi) + \mathcal{U}(\lambda\phi) = \zeta(\lambda, \psi)\mathcal{U}\psi + \zeta(\lambda, \phi)\mathcal{U}\phi\,. \tag{2.43}$$

Since $\mathcal{U}\psi$ and $\mathcal{U}\phi$ are linearly independent vectors, it follows from these equations that

$$\zeta(\lambda, \psi) = \zeta(\lambda, \phi)\,. \tag{2.44}$$

Now let ψ and ϕ be two linearly dependent vectors from V'. Let ξ be a vector such that ψ and ξ are linearly independent. Then

$$\zeta(\lambda, \psi) = \zeta(\lambda, \xi) = \zeta(\lambda, \phi)\,. \tag{2.45}$$

This proves lemma 5.

2.7. Proof of lemma 6

Let ψ be some vector from V'. Then for each $\lambda, \mu \in \mathbb{C}$ we have

$$\mathcal{U}((\lambda+\mu)\psi) = \zeta(\lambda+\mu)\mathcal{U}\psi \tag{2.46}$$

and

$$\mathcal{U}(\lambda\psi + \mu\psi) = \mathcal{U}(\lambda\psi) = \mathcal{U}(\mu\psi) = \zeta(\lambda)\mathcal{U}\psi + \zeta(\mu)\mathcal{U}\psi\,. \tag{2.47}$$

It follows that

$$\zeta(\lambda+\mu) = \zeta(\lambda) + \zeta(\mu) \quad \forall\lambda, \mu \in \mathbb{C}\,. \tag{2.48}$$

Furthermore, we have

$$\mathcal{U}(\lambda\mu\psi) = \zeta(\lambda\mu)\mathcal{U}\psi \tag{2.49}$$

and

$$\mathcal{U}(\lambda\mu\psi) = \zeta(\lambda)\mathcal{U}(\mu\psi) = \zeta(\lambda)\zeta(\mu)\mathcal{U}\psi\,, \tag{2.50}$$

from which it follows that

$$\zeta(\lambda\mu) = \zeta(\lambda)\zeta(\mu) \quad \forall\lambda, \mu \in \mathbb{C}\,. \tag{2.51}$$

Since the dimension of V is at least equal to three we may choose two eigenvectors ψ and ϕ of η with $(\psi, \phi) = 0$ whose eigenvalues have the same sign. With a suitable choice of (ψ, ψ) and (ϕ, ϕ) we then have

$$\langle \psi, \psi \rangle = \langle \phi, \phi \rangle \neq 0 \tag{2.52}$$

and

$$\langle \psi, \phi \rangle = 0 . \tag{2.53}$$

Let $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. Then

$$\left\langle \psi + \lambda\phi, \psi - \frac{1}{\bar{\lambda}} \phi \right\rangle = \langle \psi, \psi \rangle + \bar{\lambda} \langle \phi, \psi \rangle - \frac{1}{\lambda} \langle \psi, \phi \rangle - \langle \phi, \phi \rangle = 0 . \tag{2.54}$$

From eq. (1.3) it follows

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = 0 , \tag{2.55}$$

$$\langle \mathcal{U}\phi, \mathcal{U}\phi \rangle \neq 0 , \tag{2.56}$$

$$\left\langle \mathcal{U}(\psi + \lambda\phi), \mathcal{U}\left(\psi - \frac{1}{\bar{\lambda}} \phi\right) \right\rangle = 0 . \tag{2.57}$$

Thus

$$\left\langle \mathcal{U}(\psi + \lambda\phi), \mathcal{U}\left(\psi - \frac{1}{\bar{\lambda}} \phi\right) \right\rangle = \langle \mathcal{U}\psi, \mathcal{U}\psi \rangle + \overline{\zeta(\lambda)}\, \zeta\left(-\frac{1}{\bar{\lambda}}\right) \langle \mathcal{U}\phi, \mathcal{U}\phi \rangle = 0 , \tag{2.58}$$

and it follows that $\overline{\zeta(\lambda)}\, \zeta(-1/\bar{\lambda})$ is independent of λ.

From eqs. (2.48) and (2.51) it follows that

$$\zeta\left(-\frac{1}{\bar{\lambda}}\right) = -\zeta\left(\frac{1}{\bar{\lambda}}\right) = -[\zeta(\bar{\lambda})]^{-1} \tag{2.59}$$

and thus

$$\overline{\zeta(\lambda)}\zeta\left(-\frac{1}{\bar{\lambda}}\right) = -\overline{\zeta(\lambda)}[\zeta(\bar{\lambda})]^{-1} = -\overline{\zeta(1)}[\zeta(1)]^{-1} = -1 . \tag{2.60}$$

So we have

$$\zeta(\bar{\lambda}) = \overline{\zeta(\lambda)} , \tag{2.61}$$

and since this equation is also true for $\lambda = 0$, it is true for each $\lambda \in \mathbb{C}$. From eq. (2.61) it follows that $\zeta(\lambda) \in \mathbb{R}$ if $\lambda \in \mathbb{R}$. From eqs. (2.48) and (2.51) it follows that the function $\zeta(\lambda)$ for $\lambda \in \mathbb{R}$ is monotonically increasing, since, if μ is positive,

$$\zeta(\lambda + \mu) = \zeta(\lambda) + \zeta(\mu) = \zeta(\lambda) + \zeta(\sqrt{\mu})\zeta(\sqrt{\mu}) > \zeta(\lambda). \tag{2.62}$$

From eq. (2.48) it follows that if $r \in \mathbb{R}$ is a rational number then $\zeta(r) = r$. If $\lambda \in \mathbb{R}$ and r_1 and r_2 are rational numbers such that $r_1 \leq \lambda \leq r_2$ then $r_1 \leq \phi(\lambda) \leq r_2$. It follows that $\zeta(\lambda) = \lambda$ for each $\lambda \in \mathbb{C}$. From $\zeta(\mathrm{i})\zeta(\mathrm{i}) = -1$ it follows that either $\zeta(\mathrm{i}) = \mathrm{i}$ or $\zeta(\mathrm{i}) = -\mathrm{i}$. If $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ then

$$\zeta(\lambda) = \zeta(\lambda_1) + \zeta(\lambda_2)\zeta(\mathrm{i}) = \lambda_1 + \lambda_2\zeta(\mathrm{i}). \tag{2.63}$$

This proves lemma 6.

3. Proof of theorem 2

It is clear that if two semilinear operators $\mathcal{U}$ and $\mathcal{U}'$ satisfy $\mathcal{U} = \lambda\mathcal{U}'$ for some $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ they induce the same mapping $T: V \to V$. Let T be a symmetry transformation and let $\mathcal{U}$ and $\mathcal{U}'$ be the semilinear transformations which both induce T. Then for any $\psi \in V'$ there is a complex number $\lambda(\psi)$ such that

$$\mathcal{U}\psi = \lambda(\psi)\mathcal{U}'\psi \tag{3.1}$$

Let ψ and ϕ be two vectors from V'. Then

$$\mathcal{U}(\psi + \phi) = \lambda(\psi + \phi)\mathcal{U}'(\psi + \phi) = \lambda(\psi + \phi)\mathcal{U}'\psi + \lambda(\psi + \phi)\mathcal{U}'\phi \tag{3.2}$$

and

$$\mathcal{U}(\psi + \phi) = \mathcal{U}\psi + \mathcal{U}\phi = \lambda(\psi)\mathcal{U}'\psi + \lambda(\phi)\mathcal{U}'\phi. \tag{3.3}$$

If ψ and ϕ are linearly independent then $\mathcal{U}'\psi$ and $\mathcal{U}'\phi$ are linearly independent and it follows that $\lambda(\psi) = \lambda(\phi)$. If ψ and ϕ are linearly dependent then choose a vector ξ which is linearly independent of ψ. Then $\lambda(\psi) = \lambda(\xi) = \lambda(\phi)$. So $\lambda(\psi)$ is independent of ψ, and eq. (3.1) becomes

$$\mathcal{U}\psi = \lambda\mathcal{U}'\psi. \tag{3.4}$$

Since this equation is also true if ψ is the null vector it follows that $\mathcal{U} = \lambda\mathcal{U}'$, which proves theorem 2.

4. Proof of theorem 3

It is clear that a semilinear transformation $\mathcal{U}$ which satisfies eq. (1.9) induces a symmetry transformation.

Now suppose that $\mathcal{U}$ is a semilinear transformation which induces a symmetry transformation. Thus,

$$\langle \mathcal{U}\psi, \mathcal{U}\phi \rangle = 0 \Leftrightarrow \langle \psi, \phi \rangle = 0 \quad \forall \psi, \phi \in V . \tag{4.1}$$

Define the real number $C(\phi)$ by

$$C(\phi) = \frac{\langle \mathcal{U}\phi, \mathcal{U}\phi \rangle}{\zeta(\langle \phi, \phi \rangle)} \tag{4.2}$$

for each $\phi \in V$ with $\langle \phi, \phi \rangle \neq 0$. Let ψ and ϕ be two vectors with $\langle \psi, \psi \rangle \neq 0$ and $\langle \phi, \phi \rangle \neq 0$. Define

$$\xi = \langle \phi, \phi \rangle \psi - \langle \phi, \psi \rangle \phi . \tag{4.3}$$

Then $\langle \phi, \xi \rangle = 0$ and thus

$$0 = \langle \mathcal{U}\phi, \mathcal{U}\xi \rangle = \zeta(\langle \phi, \phi \rangle) \langle \mathcal{U}\phi, \mathcal{U}\psi \rangle - \zeta(\langle \phi, \psi \rangle) \langle \mathcal{U}\phi, \mathcal{U}\phi \rangle \tag{4.4}$$

which implies that

$$\langle \mathcal{U}\phi, \mathcal{U}\psi \rangle = C(\phi) \zeta(\langle \phi, \psi \rangle) . \tag{4.5}$$

Interchanging ϕ and ψ in this equation and taking the complex conjugate gives

$$\langle \mathcal{U}\phi, \mathcal{U}\psi \rangle = C(\psi) \zeta(\langle \phi, \psi \rangle) . \tag{4.6}$$

If $\langle \phi, \psi \rangle \neq 0$ it follows from eqs. (4.5) and (4.6) that $C(\phi) = C(\psi)$. If $\langle \phi, \psi \rangle = 0$ then there exists a $\lambda \in \mathbb{C}$ such that $\langle \phi, \psi + \lambda\phi \rangle \neq 0$, $\langle \psi, \psi + \lambda\phi \rangle \neq 0$ and $\langle \psi + \lambda\phi, \psi + \lambda\phi \rangle \neq 0$. Then $C(\phi) = C(\psi + \lambda\phi) = C(\psi)$. It follows that $C(\phi) \equiv C$ is independent of ϕ.

Thus we have

$$\langle \mathcal{U}\phi, \mathcal{U}\psi \rangle = C\zeta(\langle \phi, \psi \rangle) \tag{4.7}$$

for each pair $\phi, \psi \in V$ with $\langle \phi, \phi \rangle \neq 0$ and $\langle \psi, \psi \rangle \neq 0$.

Now suppose that ϕ and ψ are vectors with $\langle \phi, \phi \rangle \neq 0$ and $\langle \psi, \psi \rangle = 0$. Let χ

be a vector with $\langle\chi, \chi\rangle \neq 0$ and $\langle\psi+\chi, \psi+\chi\rangle \neq 0$. Then

$$\langle \mathscr{U}\phi, \mathscr{U}(\psi+\chi)\rangle = C\zeta(\langle\phi, \psi+\chi\rangle) = C\zeta(\langle\phi, \psi\rangle) + C\zeta(\langle\phi, \chi\rangle) \tag{4.8}$$

and

$$\langle \mathscr{U}\phi, \mathscr{U}\psi + \mathscr{U}\chi\rangle = \langle \mathscr{U}\phi, \mathscr{U}\psi\rangle + C\zeta(\langle\phi, \chi\rangle)\,. \tag{4.9}$$

It follows that eq. (4.7) also holds in this case. It therefore also holds in the case where $\langle\phi, \phi\rangle = 0$ and $\langle\psi, \psi\rangle \neq 0$.

Finally consider the case where ϕ and ψ are vectors with $\langle\phi, \phi\rangle = \langle\psi, \psi\rangle = 0$. Let χ be a vector with $\langle\chi, \chi\rangle \neq 0$ and $\langle\psi+\chi, \psi+\chi\rangle \neq 0$. Then

$$\langle \mathscr{U}(\phi+\chi), \mathscr{U}(\psi+\chi)\rangle = \langle \mathscr{U}\phi, \mathscr{U}\chi\rangle + C\{\zeta(\langle\phi, \chi\rangle) + \zeta(\langle\chi, \psi\rangle) + \zeta(\langle\chi, \chi\rangle)\} \tag{4.10}$$

and

$$\begin{aligned}\langle \mathscr{U}(\phi+\chi), \mathscr{U}(\psi+\chi)\rangle &= C\zeta(\langle\phi+\chi, \psi+\chi\rangle)\\ &= C\{\zeta(\langle\phi, \psi\rangle) + \zeta(\langle\phi, \chi\rangle) + \zeta(\langle\chi, \psi\rangle) + \zeta(\langle\chi, \chi\rangle)\}\,.\end{aligned} \tag{4.11}$$

It follows that eq. (4.7) also holds in this case. Thus eq. (4.7) holds for every pair $\phi, \psi \in V$. This proves theorem 3.

References

1) E.P. Wigner, Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra (Academic Press, New York, 1959).
2) L. Bracci, G. Morchio and F. Strocchi, Commun. Math. Phys. **41** (1975) 289.
3) U. Uhlhorn, Arkiv för Fysik **23** (1963) 307.
4) P.M. Van den Broek, Physica **122A** (1983) 587.