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SYMPLECTIC 4-MANIFOLDS WITH KODAIRA DIMENSION ZERO

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Abstract

We discuss the notion of the Kodaira dimension for symplectic manifolds in dimension 4. In particular, we propose and partially verify Betti number bounds for symplectic 4-manifolds with Kodaira dimension zero.

1. Introduction

Ever since Thurston [**42**] discovered that any orientable T^2-bundle over T^2 with $b_1 = 3$ admits symplectic structures but not Kähler structures, many constructions of closed non-Kähler symplectic 4-manifolds have appeared. For instance, Gompf [**16**] used the normal connected sum construction to build, for any finitely presented group G, a closed non-Kähler symplectic 4-manifold M_G with $\pi_1(M_G) = G$. As a result, symplectic 4-manifolds are much harder to classify. Nevertheless one could still search for a coarse classification scheme. In this regard, the notion of Kodaira dimension is a perfect place to start.

The Kodaira dimension for a Kähler surface is a measure of how positive the canonical bundle is in terms of the growth of plurigenera. The first extension of this notion to closed symplectic 4-manifolds manifold (M, ω) appeared in [**30**]. It measures the positivity of the *symplectic* canonical class K_ω as an element in integral cohomology. More specifically, for a minimal symplectic 4-manifold, it is defined in terms of the positivity of $K_\omega \cdot [\omega]$ and $K_\omega \cdot K_\omega$. Like the case of a Kähler surface, it also takes four values: $-\infty, 0, 1$ and 2. To extend it to general symplectic 4-manifolds, one needs to use results on the (uniqueness of) minimal models. In this paper we slightly modify their definition and show that results from Seiberg-Witten theory imply that the new extension is well-defined and actually an invariant of the *oriented diffeomorphism* type.

When examined under the lens of the Kodaira dimension, except certain orientable T^2-bundles over T^2, all the known non-Kähler symplectic 4-manifolds have positive values. For instance, the Kodaira dimension

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of the manifolds M_G is equal to one. In fact it follows from [**28**] that symplectic 4-manifolds with Kodaira dimension $-\infty$ are all Kähler surfaces. The goal of this paper is to provide evidence that the world of symplectic 4-manifolds with Kodaira dimension zero is not really much bigger than its Kähler counterpart.

It turns out that *minimal* symplectic 4-manifolds with Kodaira dimension zero are exactly those with torsion symplectic canonical class. In addition a closed symplectic 4-manifold with $b^+ > 1$ and torsion canonical class actually has trivial canonical class, and hence is a spin manifold. Known examples of closed minimal symplectic 4-manifolds with torsion canonical class are either Kähler surfaces with Kodaira dimension zero or orientable T^2-bundles over T^2. They all have small Betti numbers: $b^+ \leq 3$, $b^- \leq 19$ and $b_1 \leq 4$; and the signatures are between -16 and 0. As a first step towards the classification of closed symplectic 4-manifolds with Kodaira dimension zero, we make the following conjecture [1] .

Conjecture 1.1 (Betti Number Conjecture)**.** A closed minimal symplectic 4-manifold with Kodaira dimension zero has $b^+ \leq 3, b^- \leq 19$ and $b_1 \leq 4$.

In this paper we can achieve the bounds of b^+ and b^- if we assume the *desired* bound of b_1.

Theorem 1.2. *Let M be a closed minimal symplectic 4-manifold with Kodaira dimension zero. If $b_1 \leq 4$, then $b^+ \leq 3$ and $b^- \leq 19$. Consequently, the signature of M is between -16 and 0.*

In fact we can show that M shares more topological properties, like the real homology group, the intersection form, and the spin type with either a Kähler surface with Kodaira dimension zero or an orientable T^2-bundle over T^2.

The proof is divided into two cases: case 1. $b^+ \leq b_1 + 1$; case 2. $b^+ \geq b_1 + 2$. The first case is the easier one. From the definition of the Kodaira dimension, M satisfies $2\chi + 3\sigma = 0$, where χ and σ are the Euler number and the signature respectively. This relation on Betti numbers, together with the fact that M is spin if $b^+ > 1$ and hence satisfies Rokhlin's congruence on signature, readily gives the theorem in this case.

Our approach to the second case is to show that, on a closed smooth oriented 4-manifold with $2\chi + 3\sigma = 0$, $b_1 \leq 4$ and $b^+ > \max\{3, b_1 + 1\}$, the Mod 2 Seiberg-Witten invariant of any reducible Spin^c structure vanishes. In this paper we will call a Spin^c structure reducible if it admits a reduction to a spin structure. We remark that such a vanishing result was first proved in [**32**] with the assumption that $b_1 = 0$ and

[1] This conjecture has been recently confirmed by the author in [**24**].

$b^+ > 3$. To obtain the vanishing result we use the Bauer-Furuta stable cohomotopy refinement of the Seiberg-Witten invariant in [**2**] (see also [**1**], [**7**], [**8**]). Under the assumption $b^+ \geq b_1 + 2$, one can pass from the refined Seiberg-Witten invariant to the ordinary Seiberg-Witten invariant. In particular, the Mod 2 Seiberg-Witten invariant can be shown to be trivial using Pin (2)-equivariance and techniques in [**9**]. On the other hand, the fundamental result in [**38**] implies that, on a closed symplectic 4-manifold with trivial canonical class, a certain canonical reducible Spin^c structure has Seiberg-Witten invariant one. The theorem in this case then follows by comparing the vanishing result against Taubes' non-vanishing result.

The organization of the paper is as follows. In §2 we give a complete treatment of the notion of the Kodaira dimension for a symplectic 4-manifold. It has appeared in several places (see e.g., [**30**], [**23**], [**17**]), but none is complete. In §3 we list all the known closed symplectic 4-manifolds with torsion canonical class. In §4.1 we introduce quaternionic vector bundles over a space with an involution and discuss the associated Pin (2)-symmetry. In §4.2, we set up the correspondence between certain Fredholm maps and elements in certain stable cohomotopy groups. After writing down the Pin (2)-equivariant Seiberg-Witten equations in §5.1, we introduce the Bauer-Furuta refinement of the Seiberg-Witten invariant as a stable cohomotopy class and describe how to derive the Mod 2 Seiberg-Witten invariant from it. For our purpose, we restrict ourselves to reducible Spin^c structures. In §6 we prove Theorem 1.2 and derive some interesting consequences.

In the body of this paper all the 4-manifolds are assumed to be closed, smooth and oriented.

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2. The Kodaira dimension

Let (M, ω) be a symplectic 4-manifold. Its symplectic canonical class K_ω is the first Chern class of the cotangent bundle with any ω-compatible almost complex structure. Such ω-compatible almost complex structures are nonempty and homotopic, so K_ω is well-defined.

As we mentioned in the introduction, we will first define the Kodaira dimension of (M, ω) when it is minimal, so we need to recall the notion of minimality.

Let $\mathcal{E}_M$ be the set of homology classes which have square -1 and are represented by smoothly embedded spheres. We say that M is smoothly minimal if $\mathcal{E}_M$ is empty. Let $\mathcal{E}_{M,\omega}$ be the subset of $\mathcal{E}_M$ which are represented by embedded ω-symplectic spheres. We say that (M, ω) is symplectically minimal if $\mathcal{E}_{M,\omega}$ is empty. When (M, ω) is non-minimal, one can blow down some of the symplectic -1 spheres to obtain a minimal symplectic 4-manifold (N, μ), which is called a (symplectic) minimal model of (M, ω) ([Mc]). We summarize the basic facts about the minimal models in the following proposition.

Proposition 2.1 ([**22**], [**26**], [**29**], [**40**])**.** *Let M be a closed oriented smooth 4-manifold and ω a symplectic form on M compatible with the orientation of M.*

1. *M is smoothly minimal if and only if (M, ω) is symplectically minimal. In particular the underlying smooth manifold of the (symplectic) minimal model of (M, ω) is smoothly minimal.*
2. *If (M, ω) is not rational nor ruled, then it has a unique (symplectic) minimal model. Furthermore, for any other symplectic form ω' on M compatible with the orientation of M, the (symplectic) minimal models of (M, ω) and (M, ω') are diffeomorphic as oriented manifolds.*
3. *If (M, ω) is rational or ruled, then its (symplectic) minimal models are diffeomorphic to CP^2 or an S^2-bundle over a Riemann surface.*

Here a rational symplectic 4-manifold is a symplectic 4-manifold whose underlying smooth manifold is $S^2 \times S^2$ or $CP^2 \# k\overline{CP^2}$ for some non-negative integer k. A ruled symplectic 4-manifold is a symplectic 4-manifold whose underlying smooth manifold is the connected sum of a number of (possibly zero) $\overline{CP^2}$ with an S^2-bundle over a Riemann surface.

Now we are ready to define the symplectic Kodaira dimension.

Definition 2.2. For a minimal symplectic 4-manifold M with symplectic form ω and symplectic canonical class K_ω, the Kodaira dimension of (M, ω) is defined in the following way:

$$\kappa(M,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0. \end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

Remark 2.3. In [**30**], [**23**] and [**17**] the Kodaira dimension of a minimal symplectic 4-manifold (M, ω) is defined to be $-\infty$ if $K_\omega \cdot [\omega] < 0$, and zero if $K_\omega \cdot [\omega] = 0$. Our modification is to take into account the sign of $K_\omega \cdot K_\omega$ as well in these two cases. Since, for any minimal ruled surface with negative $K_\omega \cdot K_\omega$, there are symplectic forms ω with $K_\omega \cdot [\omega]$ non-negative, this slight modification is necessary for the Kodaira dimension to be well-defined for all symplectic 4-manifolds. Moreover, the crucial Lemma 2.5 seems to have first appear here.

For a minimal symplectic 4-manifold, its Kodaira dimension has the following properties.

Theorem 2.4. *Let M be a closed oriented smooth 4-manifold and ω a symplectic form on M compatible with the orientation of M. If (M,ω) is symplectically minimal, then*

1. *The Kodaira dimension of (M,ω) is well-defined.*
2. *(M,ω) has Kodaira dimension $-\infty$ if and only if it is rational or ruled.*
3. *(M,ω) has Kodaira dimension 0 if and only if K_ω is a torsion class.*
4. *Furthermore, the Kodaira dimension of (M,ω) only depends on the oriented diffeomorphism type of M, i.e., if ω' is another symplectic form on M compatible with the orientation of M, then $\kappa(M,\omega) = \kappa(M,\omega')$.*

Proof. We first show that the Kodaira dimension is well-defined for minimal manifolds. It amounts to show that any minimal symplectic 4-manifold must satisfy one and only one of the four conditions above. We will need the following lemma.

Lemma 2.5. *If (M,ω) is minimal with $K_\omega \cdot [\omega] = 0$ and $K_\omega \cdot K_\omega \geq 0$, then K_ω is a torsion class and hence $K_\omega \cdot K_\omega = 0$.*

Proof. By [**40**] and [**25**], if M is not an S^2-bundle over a Riemann surface of genus at least 2 and K_ω is not a torsion class, then nK_ω is represented by a symplectic surface for some nonzero integer n and therefore $K_\omega \cdot \omega \neq 0$. Since an S^2-bundle over a Riemann surface of genus at least 2 has negative $K_\omega \cdot K_\omega$, our assumption leads to the conclusion that K_ω is a torsion class. The proof of the lemma is complete. q.e.d.

We continue with the proof of Theorem 2.4. The two intersection numbers $K_\omega \cdot [\omega]$ and $K_\omega \cdot K_\omega$ have two possibilities:

Case 1. $K_\omega \cdot [\omega] < 0$ or $K_\omega \cdot K_\omega < 0$,

Case 2. $K_\omega \cdot [\omega] \geq 0$ and $K_\omega \cdot K_\omega \geq 0$.

The first case corresponds to the condition of the Kodaira dimension being $-\infty$. In the second case, there are four subcases: the one with $K_\omega \cdot [\omega] = 0$ and $K_\omega \cdot K_\omega > 0$ never occurs due to Lemma 2.5, and the remaining three correspond to the conditions of the Kodaira dimension being $0, 1$ and 2 respectively. So part 1 is proved.

Now we deal with part 2. Taubes [**40**] proved that if $K_\omega \cdot [\omega] < 0$ or $K_\omega \cdot K_\omega < 0$, then M must have $b^+ = 1$. Furthermore, in [**28**] (see also [**33**]) it was proved that if M has $b^+ = 1$ and satisfies $K_\omega \cdot [\omega] < 0$ or $K_\omega \cdot K_\omega < 0$, then M is diffeomorphic to CP^2, or an S^2-bundle. Thus the only if part is established. For an S^2-bundle over a Riemann surface of genus at least 2, $K_\omega \cdot K_\omega$ is negative for any symplectic structure ω. For $M = CP^2$ or an S^2-bundle over a sphere or a torus, it was shown in [**25**] that, for any symplectic structure ω, $-K_\omega$ is represented by a symplectic surface and so $-K_\omega \cdot [\omega] > 0$ for any ω. Therefore, if (M, ω) is rational or ruled, it has Kodaira dimension $-\infty$. Thus we have finished the proof of part 2.

If $K_\omega \cdot [\omega] = 0$ and $K_\omega \cdot K_\omega = 0$, then, by Lemma 2.5, K_ω must be a torsion class. Conversely, if K_ω is a torsion class, it is obvious that $K_\omega \cdot [\omega] = 0$ and $K_\omega \cdot K_\omega = 0$. So any (M, ω) with Kodaira dimension zero must have torsion canonical classes, and vice versa. This finishes the proof of part 3.

Finally we show that (M, ω) is an oriented diffeomorphism invariant. Notice that, from part 2, if (M, ω) has Kodaira dimension $-\infty$, then so does (M, ω') for any symplectic form ω'.

Assume now that $\kappa(M, \omega) = 0$ and ω' is a different symplectic form on M. When $b^+ > 1$, Taubes' result in [**38**] implies that if $K_{\omega'} \neq K_\omega$, then $0 \leq K_{\omega'} \cdot [\omega] < K_\omega \cdot [\omega]$. Such a $K_{\omega'}$ cannot exist since $K_\omega \cdot [\omega] = 0$. Therefore $K_{\omega'} = K_\omega$ is a torsion class. When $b^+ = 1$, it was shown in [**25**] that $K_{-\omega} = \pm K_\omega$, so $K_{\omega'}$ is a torsion class as well. It follows from part 3 that if (M, ω) has Kodaira dimension 0, then so does (M, ω') for any symplectic form ω'.

What we have just proved implies that if $\kappa(M, \omega) \geq 1$ then $\kappa(M, \omega') \geq 1$ as well. Together with the oriented homotopy invariance of $K_\omega \cdot K_\omega$, we have that if (M, ω) has Kodaira dimension 1 or 2, then $\kappa(M, \omega') = \kappa(M, \omega)$ for any symplectic form ω'. Therefore the proof of part 4 and hence the theorem is complete. q.e.d.

For a general symplectic 4-manifold, it follows from Proposition 2.1 that its Kodaira dimension has similar properties.

Theorem 2.6. *$\kappa(M, \omega)$ is well-defined for any symplectic 4-manifold (M, ω) and only depends on the oriented diffeomorphism type of M.*

Proof. If (M, ω) is not rational or ruled, it has a unique (symplectic) minimal model by Proposition 2.1 (2), so $\kappa(M, \omega)$ is well defined by Theorem 2.4 (1). Furthermore, the (symplectic) minimal models for different symplectic forms are diffeomorphic as oriented manifolds by Proposition 2.1 (2), so $\kappa(M, \omega)$ only depends on the oriented diffeomorphism type of M by Theorem 2.4 (4).

If (M, ω) is rational or ruled, it has non-diffeomorphic (symplectic) minimal models. However, the different minimal models are still rational or ruled by Proposition 2.1 (3), so all have Kodaira dimension $-\infty$ by Theorem 2.4 (2). The proof is complete. q.e.d.

Since we have shown that $\kappa(M, \omega)$ only depends on the oriented diffeomorphism type of M, we will denote it simply by $\kappa(M)$.

If (M, J) is a complex surface, a holomorphic -1 curve in (M, J) is a smooth rational curve with square -1. (M, J) is called holomorphically minimal if it does not contain any holomorphic -1 curve. A basic fact is that every complex surface can be blown down to a holomorphically minimal surface. We refer to [**3**] and [**12**] for basic facts about complex surfaces. In particular it is known (see [**3**]) that properties 2.2 characterize the holomorphic Kodaira dimension of minimal Kähler surfaces. Since the holomorphic Kodaira dimension is birationally invariant, we conclude that $\kappa(M)$ is indeed an extension of the holomorphic Kodaira dimension of a Kähler surface.

Proposition 2.7. *The holomorphic Kodaira dimension of a Kähler surface coincides with the Kodaira dimension of the underlying symplectic* 4*-manifold.*

As an immediate consequence of Theorem 2.4 and Proposition 2.7 we get the known property of the holomorphic Kodaira dimension of a Kähler surface, that it is an invariant of the oriented diffeomorphism type (see e.g., [**11**]).

We would like to see whether it is possible to define $\kappa(M, \omega)$ for higher dimensional symplectic manifolds such that it is invariant under 'symplectic birational operations' (see [**34**]). Again we would first define it for 'minimal' manifolds with dimension $2n$ as follows: $\kappa(M, \omega)$ is defined to be $-\infty$ if $K_\omega^i \cdot [\omega]^{n-i}$ is negative for some i; and $\kappa(M, \omega) = i$ if $K_\omega^j \cdot [\omega]^{n-j} = 0$ for any $j \geq i+1$ and $K_\omega^j \cdot [\omega]^{n-j} > 0$ for any $j < i+1$. To show it is well-defined we need to prove the analogue of Lemma 2.4. Then we would extend it to general manifolds by requiring the birational invariance. Of course this is just a speculation as there are many issues to be settled.

3. The known manifolds with Kodaira dimension zero

In the previous section we have seen that symplectic 4-manifolds with $\kappa = -\infty$ are either rational or ruled; in particular, all of them admit Kähler structures. Any Kähler surface with holomorphic Kodaira dimension 0 has $\kappa = 0$ by Proposition 2.7. Such Kähler surfaces have been classified: the K3 surface, the Enriques surface and the hyperelliptic surfaces. It is not hard to find non-Kähler ones with $\kappa = 0$. In fact the first example of a non-Kähler symplectic manifold, the Kodaira-Thurston manifold, has $\kappa = 0$. The Kodaira-Thurston manifold is an example of a T^2-bundle over T^2. T^2-bundles over T^2 are classified in

[**36**]. It turns out that all orientable T^2-bundles over T^2 are symplectic and have Kodaira dimension zero.

Let M be an orientable T^2-bundle over T^2. We first write down the Betti numbers of M. Since χ is multiplicative for a fiber bundle and T^2 has zero χ, M has zero χ as well. Since the Betti numbers of M are bounded by those of the trivial bundle, which is the 4-torus, we have $b_1 \leq 4$ and $b_2 \leq 6$. The bound on b_2 implies that $|\sigma|$ is bounded. For any integer d, since T^2 admits a covering by itself of degree d, M admits a covering by another T^2-bundle over T^2 of degree d. The existence of such coverings, together with the multiplicativity of σ under finite coverings, implies that the bound of $|\sigma|$ and hence σ itself must be zero. Therefore the Betti numbers of M satisfy

$$b^+ = b^- = b_1 - 1, \qquad b^+ \leq 3. \tag{3.1}$$

It is shown in [**15**] that every orientable T^2-bundle over T^2 admits symplectic structures. The argument is based on the explicit representation as a geometric manifold $\Gamma\backslash\mathbf{R}^4$ in [**43**]. Alternatively, the Thurston construction gives rise to symplectic forms on any surface bundle over surface with homology essential fibers. So any orientable T^2-bundle over T^2 with homology essential fibers admits symplectic structures. For an orientable T^2-bundle over T^2 with homology inessential fibers, it is also observed in [**15**] that it is a principal S^1-bundle over a principal S^1-bundle over T^2, and thus admits symplectic structures by the construction in [**10**].

From the homotopy exact sequence of a fiber bundle, it is easy to see that a T^2-bundle over T^2 is aspherical. In particular it has trivial π_2 and therefore is minimal. Now we show that every orientable T^2-bundle over T^2 has $\kappa = 0$.

Proposition 3.1. *Let M be an orientable T^2-bundle over T^2. There exists a symplectic form ω on M such that K_ω is a torsion class.*

Proof. We will need the following lemma.

Lemma 3.2. *Let (M, ω) be a minimal symplectic 4-manifold.*

1. *If (M, ω) admits a smooth Lagrangian bundle structure over $B = T^2$, then $K_\omega = 0$.*
2. *If $K_\omega = K_{-\omega}$ and $K_\omega \cdot K_\omega = 0$, then $K_\omega \cdot [\omega] = 0$.*

Proof. The first statement follows from Lemma 3.1 in [**37**]. Let us prove the second statement. If $K_\omega \cdot [\omega] < 0$, then (M, ω) has Kodaira dimension $-\infty$. However,

$$K_{-\omega} \cdot (-[\omega]) = K_\omega \cdot (-[\omega]) > 0.$$

Therefore $(M, -\omega)$ has Kodaira dimension one. This is a contradiction since the Kodaira dimension only depends on the oriented diffeomorphism type. The case $K_\omega \cdot [\omega] > 0$ can be similarly ruled out. Thus $K_\omega \cdot [\omega]$ must be zero and the lemma is proved. q.e.d.

Now we are back to the proof of Proposition 3.1. In the case $b^+ = 3$, M is the four torus and the standard symplectic form has vanishing symplectic canonical class. Based on the explicit representation of M as a geometric manifold $\Gamma\backslash\mathbf{R}^4$ in [**43**], the following was proved in [**15**]: if M has $b^+ = 2$, or $b^+ = 1$ and the fiber is inessential, then there exists ω_0 on M such that (M, ω_0) admits a smooth Lagrangian bundle structure over T^2. Therefore $K_{\omega_0} = 0$ by part 1 of Lemma 3.2. The case $b^+ = 2$ can also be settled by part 2 of Lemma 3.2, as both ω_0 and $-\omega_0$ lie in an S^1 family of symplectic forms; in particular, K_{ω_0} is the same as $K_{-\omega_0}$.

Now assume that M has $b^+ = 1$ and the fibers are essential. Then $b_2 = b_1 = 2$ from equation (3.1) and M falls into cases (c), (e), (f), (g) and (h) in Table I in [**15**]. In the case (c), M is a hyperelliptic surface and thus has $\kappa = 0$. For the remaining cases, using the explicit representation $\Gamma\backslash\mathbf{R}^4$, we can follow [**30**] to find a basis for $H_2(M; \mathbf{R})$ represented by square zero symplectic tori and apply the adjunction formula for an embedded symplectic surface to conclude that K_ω is rationally trivial. Since the symplectic structures constructed in [**15**] are compatible with the fibrations in these cases, we can choose the fiber class to be one element of such a basis. In cases (g) and (h), we choose the other one to be the class of the torus $\{(0, 0, z, t)\}$; and in cases (e) and (f), we use the class of the torus $\{(0, y, 0, t)\}$. Since the symplectic forms in [**15**] are of the form $\alpha dx \wedge dy + \beta dz \wedge dt$ and $\alpha dy \wedge dt + \beta(dx \wedge dz - x dx \wedge dy)$ respectively for (g,h) and (e,f) with $\alpha\beta \neq 0$, these tori are symplectic. The proof is complete. q.e.d.

In fact it would follow from Proposition 6.3 that the symplectic canonical class is not just a torsion class but actually trivial.

Let us comment on the existence of complex or Kähler structure on a T^2-bundle over T^2, which was determined in [**15**]. We say a smooth manifold is Kähler (complex) if it admits a Kähler (complex) structure. Any orientable T^2-bundle over T^2 with $b^+ = 3$ is the 4-torus and hence Kähler. Those with $b^+ = 2$ are complex but not Kähler, because any Kähler surface has odd b^+ by the Hodge decomposition. For orientable T^2-bundles over T^2 with $b^+ = 1$, they are not complex (hence not Kähler), except when they are the hyperelliptic surfaces (which are Kähler).

Now we list minimal Kähler surfaces with $\kappa = 0$ and orientable T^2-bundles over T^2 in the following table according to their homology type.

Table 1

class	b^+	b_1	χ	σ	b^-	known as
a)	3	0	24	-16	19	$K3$
b)	3	4	0	0	3	4-torus
c)	2	3	0	0	2	primary Kodaira surface
d)	1	0	12	-8	9	Enriques surface
e)	1	2	0	0	1	hyperelliptic surface if complex

For the manifolds in the table, an important feature is that they are all fibred by tori. Those in classes b), c) and e) are orientable T^2-bundles over T^2, and those in classes a) and d) fiber over S^2. In fact it is not hard to see that, if (M, ω) has torsion canonical class and admits a genus g Lefschetz fibration structure, then the adjunction formula applied to the symplectic fiber class leads to the conclusion that $g = 1$. Now, as in [**37**], from the classification of genus one Lefschetz fibration of Moishezon and Matsumoto, we see easily that M is either K3 or a torus bundle over torus.

To the author's knowledge, no potentially new minimal symplectic 4-manifolds with $\kappa = 0$ have been constructed so far. For instance, Fintushel and Stern's knot surgery ([**13**]) on a fibred knot is a powerful technique to produce infinitely many families of homeomorphic but non-diffeomorphic symplectic 4-manifolds. In order to get one with torsion canonical class, one has to start with such a manifold which is known, e.g., the K3 surface, and apply this surgery to a fibred knot with trivial Alexander polynomial. Though there are many knots with trivial Alexander polynomial, the only fibred one is the trivial knot. Therefore the knot surgery produces nothing new in this context. It is natural to wonder whether all minimal symplectic 4-manifolds with $\kappa = 0$ are in Table 1. To identify the diffeomorphism types of such symplectic 4-manifolds, we think the parametrized SW theory in [**27**] could play a role, at least when M has a winding family. A winding family of symplectic forms is an S^{b^+-1} family of symplectic forms, which represents the generator of $\pi_{b^+-1}(\mathcal{P})$ where $\mathcal{P}$ is the cone of real classes with positive square. Every known manifold of Kodaira dimension zero carries such a family. We would like to see whether it is possible to use the parametrized Seiberg-Witten and Gromov-Taubes theories to construct a torus fibration assuming the existence of a winding family. If the answer is yes, following from Moishezon, such a manifold is either Kähler or an orientable T^2-bundle.

We end this section by speculating on the moduli space of symplectic structures. In the case $\kappa = -\infty$ it is known ([**40**] and [**26**]) that there is a unique symplectic structure up to deformation on a minimal rational or ruled manifold M, and the set of symplectomorphism types (i.e., the moduli space of symplectic structures) was shown to be $\mathcal{P}/D(M)$ in [**20**], where $D(M)$ is the image of $\mathrm{Diff}(M)$ in the automorphism group of the cohomology lattice (the non-minimal case was recently worked out in [**25**]). For a minimal manifold M with Kodaira dimension zero, Proposition 5.3 implies that there is a unique symplectic canonical class, and according to [**25**], for all the manifolds listed in Table 1, the cone of classes of symplectic forms is the positive cone $\mathcal{P}$. We speculate that there is a unique deformation class of symplectic forms and the moduli space of symplectic structures is similarly given by $\mathcal{P}/D(M)$.

4. Some equivariant topology

4.1. Quaternionic vector bundles and $\mathrm{Pin}\,(2)$**.** In this subsection we introduce quaternionic vector bundles over an involutive space and show that they are naturally acted upon by the group $\mathrm{Pin}\,(2)$. This subsection is a synopsis of Chapter 3 in [**9**].

Let J be a space with an involution ι_J and isolated fixed point set. The case we are interested in is the torus $T^m = \mathbf{R}^m/2\mathbf{Z}^m$ with ι_J given by $x \longrightarrow -x$ using the coordinates of $\mathbf{R}^m$. In this case we use O_J to denote the image of the origin in $\mathbf{R}^m$.

Recall that a bundle map between complex bundles is called anti-complex if it anti-commutes with the right multiplication by $i = \sqrt{-1}$. A complex bundle Q on J with an anti-complex lift ι_Q of ι_J is called a *quaternionic* vector bundle if $\iota_Q \circ \iota_Q = -1$. We should warn the readers that a *quaternionic* bundle here is not a bundle over $\mathbf{H}$. A better name might be 'a complex bundle with quaternionic structure ι_Q' suggested by the referee. In particular, the rank of a *quaternionic* bundle is its rank as a complex bundle. The Grothendick group of the *quaternionic* vector bundles is denoted by $KQ(J)$ (first appeared in Dupont's work [**5**]).

Let $\underline{\mathbf{H}}$ be the rank two quaternionic vector bundle $J\times\mathbf{H}$ with the anti-complex map $\iota_{\underline{\mathbf{H}}} : (x, q) \longrightarrow (\iota_J x, qj)$. A quaternionic vector bundle is called trivial if it is isomorphic to a direct sum of $\underline{\mathbf{H}}$.

To understand quaternionic bundles, we will also need to consider certain complex bundles with involutive anti-complex lifts, which we call I-bundles in this paper (as pointed out by the referee, such a bundle is called a 'real vector bundle' by Atiyah, a 'vector bundle with real structure' by Adams, or a 'Real vector bundle' by Lawson-Michelson). Rank one I-bundles arise naturally as the complex determinant bundle of quaternionic bundles. In particular let $\underline{\mathbf{C}}$ be the I-bundle $J \times \mathbf{C}$ with the involutive anti-complex lift $\iota_{\underline{\mathbf{C}}} : (x, z) \longrightarrow (\iota_J x, \bar{z})$. $\underline{\mathbf{C}}$ is called the trivial rank one I-bundle. An important fact proved in [**9**] is that any rank one I-bundle over J is isomorphic to $\underline{\mathbf{C}}$ when J is a torus T^m with the involution specified above.

Let F be the fixed point set of J. Let $GL_I(\underline{\mathbf{C}})$ be the complex automorphism of $\underline{\mathbf{C}}$ as an I-bundle. By restricting to F, $GL_I(\underline{\mathbf{C}})$ maps to $GL_I(\underline{\mathbf{C}}|_F) = \mathrm{Map}(F, \mathbf{R}^*)$, where $\mathbf{R}^*$ is the set of non-zero real numbers. Via the non-trivial homomorphism $\mathbf{R}^* \longrightarrow \mathbf{Z}_2$, we obtain a homomorphism $GL_I(\underline{\mathbf{C}})$ to $\mathrm{Map}(F, \mathbf{Z}_2)$. Let $A(F, \mathbf{Z}_2)$ be the image subgroup.

Given any (complex) rank $2k$ quaternionic bundle Q, a trivialization $t_F : Q|_F \cong \underline{\mathbf{H}}^k|_F$ induces a trivialization of the determinant bundle $\det(t_F) : \det Q|_F \cong \underline{\mathbf{C}}|_F$. On the other hand there is a global trivialization $t_J : \det_Q \cong \underline{\mathbf{C}}$ as we just mentioned. The composition $(t_J|_F)(\det\, t_F)^{-1}$ is an element of $GL_I(\underline{\mathbf{C}}|_F) \cong \mathrm{Map}(F, \mathbf{R}^*)$, which maps to an element in $\mathrm{Map}(F, \mathbf{Z}_2)$. Define $c_F(Q)$ to be the equivalence class of $(t_J|_F)(\det\, t_F)^{-1}$ in $\mathrm{Map}(F, \mathbf{Z}_2)/A(F, \mathbf{Z}_2)$. Using c_F one can further define a $\mathbf{Z}_2$-valued homomorphism on $KQ(J)$ if $\dim J \geq 2$:

$$\epsilon(Q) = \sum_{x \in F} c_F(Q)(x),$$

which is well defined since it can be shown that $\sum_{x \in F} f(x) = 0$ for $f \in A(F, \mathbf{Z}_2)$.

Quaternionic vector bundles over 4-dimensional tori were classified in [**9**]; in particular, we have the following:

Proposition 4.1. *Let J be the torus T^4 with the involution $\iota : x \longrightarrow -x$. Then*

1. *Any quaternionic vector bundle Q over J is the direct sum of a trivial quaternionic vector bundle and a rank 2 quaternionic vector bundle. Furthermore Q satisfies $c_2(Q) \equiv \epsilon(Q) \pmod 2$.*
2. *Two rank 2 quaternionic bundles Q and Q' on J are isomorphic if they have the same c_2 and c_F.*
3. *Moreover, for any two quaternionic rank 2 bundles over J, there exists a $\mathbf{C}$ linear homomorphism $\phi : Q \longrightarrow Q'$ satisfying the following properties:*
   - 3a. *ϕ is transverse to the zero section and $\iota_{Q'} \circ \phi = \phi \circ \iota_Q$.*
   - 3b. *For each point x where ϕ_x is not an isomorphism, there exist a diffeomorphism from an open neighborhood U_x of x to an open neighborhood U of 0 in $\mathbf{H}$, and trivializations $Q|_U \cong U \times \mathbf{H}$, $Q'|_U \cong U \times \mathbf{H}$ as complex vector bundles, such that ϕ is described as $\phi|_v(q) = vqi$ or $\phi|_v(q) = \bar{v}qi$ in terms of this local coordinate and these trivializations. And if n^+ (respectively n^-) is the number of x of the former type (respectively the latter type), then $c_2(Q') - c_2(Q) = n^+ - n^-$.*

Remark 4.2. In fact it was remarked in [**9**], that if Q and Q' are two quaternionic bundles over a torus of dimension at most 3, then they are isomorphic if they have the same rank and the same c_F. If $J = T^m$, then $A(F, \mathbf{Z}_2)$ was shown in [**9**] to be isomorphic to $H^1(J; \mathbf{Z}_2) \oplus \mathbf{Z}_2$, in particular it is of order 2^{m+1}. Since $\mathrm{Map}(F, \mathbf{Z}_2)$ is of order 2^{2^m}, it follows that any quaternionic bundles over T^1 is trivial, and there is a unique non-trivial rank 2 quaternionic bundle over T^2.

Just as complex vector bundles are acted upon by $U(1)$ via the complex multiplication, quaternionic vector bundles are naturally acted upon by the group $\mathrm{Pin}\,(2)$, which is generated by $U(1)$ and the symbol ι with the relations

$$\iota^2 = -1, \qquad \iota z \iota^{-1} = z^{-1} \quad \text{for} \quad z \in U(1).$$

Clearly $\mathrm{Pin}\,(2)$ fits into the short exact sequence

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Pin}\,(2) \longrightarrow \{\pm 1\} \longrightarrow 1.$$

Notice that $\mathrm{Pin}\,(2)$ is isomorphic to the subgroup of $\mathbf{H}$ generated by $U(1) = \{\cos\theta + i \sin\theta\}$ and j.

We first specify the $\mathrm{Pin}\,(2)$ action on J: it is simply defined via the surjection of $\mathrm{Pin}\,(2)$ to the order 2 group $\{id, \iota_J\}$. For a quaternionic vector bundle V over J, since ι_V is anti-complex, $\mathrm{Pin}\,(2)$- acts on V via the complex multiplication and ι_V. We will also need the simple fact that, for a real vector space W, the trivial real vector bundle $J \times W$ is also $\mathrm{Pin}\,(2)$-equivariant via the involution $\iota_W : (x, a) \longrightarrow (\iota_J x, -a)$ and the surjection $\mathrm{Pin}\,(2) \longrightarrow \{id, \iota_W\}$.

4.2. Fredholm maps and equivariant stable homotopy. In this section we closely follow §2 in [**1**] and [**2**].

Let $\mathcal{E}$ and $\mathcal{F}$ be infinite dimensional Hilbert space bundles over a compact space B. The structure group is the orthogonal group with its norm topology. Every such Hilbert bundle is trivial according to a theorem of Kuiper [**19**]. Let $\mathcal{U}$ be a fiber of $\mathcal{F}$ and $p : \mathcal{F} \cong B \times \mathcal{U} \longrightarrow \mathcal{U}$ be a trivialization.

Suppose $l : \mathcal{E} \longrightarrow \mathcal{F}$ is a continuous, fiberwise linear Fredholm map. Let $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$ be the space of fiberwise continuous maps $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ such that $\phi - l$ is fiberwise compact, and preimages of bounded sets are bounded. Equip $\mathcal{P}_l(\mathcal{E}, \mathcal{F})$ with the topology induced by the metric

$$d(\phi, \psi) = \sup_{e \in \mathcal{E}} |j\phi(e) - j\psi(e)|,$$

where $j : \mathcal{F} \longrightarrow \mathbf{R} \times \mathcal{F}$ denote the embedding $f \longrightarrow (1 + |f|^2)^{-1}(1 - |f|^2, 2f)$ into the unit sphere bundle in $\mathbf{R} \times \mathcal{F}$ over B.

Let $U \subset \mathcal{U}$ be a finite dimensional linear subspace such that the index of l is represented by the difference $E - \underline{U}$ of finite dimensional vector bundles on B. Here $\underline{U}$ denotes the trivial vector bundle $p^{-1}(U)$ and $E = l^{-1}(\underline{U})$. The one-point compactification TE of E is called the Thom space of E.

We now construct a stable cohomotopy group with twisting $\mathrm{ind} l$. This relies on the notion of the Spanier-Whitehead category indexed by a universe. Objects and morphisms in this category are defined through colimit constructions. The relevant universe here is the Hilbert space $\mathcal{U}$, and the index category consists of the finite dimensional linear subspaces W of $\mathcal{U}$ with $U \subset W$.

Any pointed space A canonically defines an object $\Sigma^{-U}A$ in this category. Here $\Sigma^{-U}A$ is the spectrum associating to W the space $A_W = S^V \wedge A$, where V is the orthogonal complement to U in W and S^V is the one point compactification of V with the point at infinity as the base point. The suspension morphism for $W \subset W'$ with $Z = W^{\perp} \cap W'$ is the suspension map $\sigma_{W,W'} : S^Z \wedge A_W \to A_{W'}$. The morphism set between two objects in this category is the colimit

$$\{\Sigma^{-U}A, \Sigma^{-U}C\}_{\mathcal{U}} = \mathrm{colimit}_{W\in\mathcal{U}}[A_W, C_W],$$

over the maps

$$[A_W, C_W] \longrightarrow [S^Z \wedge A_W, S^Z \wedge C_W] \longrightarrow [A_{W'}, C_{W'}].$$

With the preceding understood, the stable cohomotopy with twisting $\mathrm{ind}\, l$ is defined by

$$\pi^0_{\mathcal{U}}(B; \mathrm{ind}\, l) = \{\Sigma^{-U}TE, \Sigma^{-U}U\}_{\mathcal{U}}.$$

Notice that $\Sigma^{-U}U$ is just the ordinary suspension spectrum associated to S^0. And if we denote $\Sigma^{-U}TE$ by $T(\mathrm{ind}\, l)$, then $\pi^0_{\mathcal{U}}(B; \mathrm{ind}\, l)$ can be better written as $\{T(\mathrm{ind}\, l), S^0\}_{\mathcal{U}}$.

The following beautiful result associates to any $\phi \in \mathcal{P}_l(\mathcal{E}, \mathcal{F})$ an element in this stable cohomotopy group.

Theorem 4.3 ([**1**]). *A projection $p : \mathcal{F} \cong B \times \mathcal{U} \longrightarrow \mathcal{U}$ induces a natural bijection*

$$\pi_0(\mathcal{P}_l(\mathcal{E}, \mathcal{F})) \cong \pi^0_{\mathcal{U}}(B; \mathrm{ind}\, l).$$

Now suppose a compact Lie group G acts on $\mathcal{E}$ and $\mathcal{F}$ as isometries and there exists an equivariant projection $p : \mathcal{F} \cong B\times\mathcal{U} \longrightarrow \mathcal{U}$. Suppose further that any irreducible G-representation is either not contained in $\mathcal{U}$ or has infinite multiplicity. Then the above concepts carry over to an equivariant setting. In particular there is still a natural bijection

$$\pi_0(\mathcal{P}_l(\mathcal{E}, \mathcal{F})^G) \cong \pi^0_{G,\mathcal{U}}(B; \mathrm{ind} l)$$

as before. Moreover $\pi^0_{G,\mathcal{U}}(B; \mathrm{ind} l)$ is a group if the G-universe contains trivial G-representations.

5. The Bauer-Furuta refinement of Seiberg-Witten invariants

5.1. The reducible Spin^c structures and parametrized Dirac operators. In this subsection M is a closed oriented smooth 4-manifold and c is a Spin^c structure. We first describe the construction of the family of the Dirac operators over the torus $J = T^{b_1(M)}$. When c is a reducible Spin^c structure and a reduction of c to a spin structure is fixed, following [**9**] we will show that this family of Dirac operators gives rise to an element in $KQ(J)$.

Recall that there is a lifting of $SO(4)$ to $\mathrm{Spin}^c(4)$; that is to say, the quotient of $\mathrm{Spin}^c(4)$ by its center is isomorphic to $SO(4)$. This is easily seen from the isomorphisms

$$\begin{aligned} \mathrm{Spin}^c(4) &\cong U(1) \times SU(2) \times SU(2)/\{\pm(1,1,1)\}, \\ SO(4) &\cong SU(2) \times SU(2)/\{\pm(1,1)\}. \end{aligned}$$

Fix a Riemannian metric on M and let $P_{SO(4)}$ be the oriented orthogonal frame bundle of TM. The Spin^c structure c on M is a principal $\mathrm{Spin}^c(4)$-bundle $\tilde{P}_c$ which is a lift of $P_{SO(4)}$. If identifying $SU(2)$ with the unit quaternions in $\mathbf{H}$, we can write any element in $\mathrm{Spin}^c(4)$ as a triple (z, q_0, q_1) with $z \in U(1)$ and q_i a unit quaternion for $i = 0, 1$. In this notation the Riemannian cotangent bundle TM^* is isomorphic to the fiber product $\tilde{P}_c \times_{\mathrm{Spin}^c(4)} {}_1\mathbf{H}_{\bar{0}}$, where ${}_1\mathbf{H}_{\bar{0}}$ is a copy of $\mathbf{H}$ with the representation of $\mathrm{Spin}^c(4)$ given by $(z, q_0, q_1)(s) = q_1 s \bar{q}_0$. Similarly the bundle of self-dual two forms $\Lambda^+ TM^*$ is isomorphic to the fiber product $\tilde{P}_c \times_{\mathrm{Spin}^c(4)} {}_0(\mathrm{Im}\mathbf{H})_{\bar{0}}$, where ${}_0\mathbf{H}_{\bar{0}}$ is a copy of the imaginary quaternions $\mathrm{Im}\mathbf{H}$ with the representation of $\mathrm{Spin}^c(4)$ given by $(z, q_0, q_1)(s) = q_0 s \bar{q}_0$.

Let S^0 and S^1 be the spinor bundles associated to c via the $\mathrm{Spin}^c(4)$ modules ${}_0\mathbf{H}_{\bar{z}}$ and ${}_1\mathbf{H}_{\bar{z}}$, where the actions are given by $(z, q_0, q_1)(s) = q_0 s \bar{z}$ and $(z, q_0, q_1)(s) = q_1 s \bar{z}$ respectively. Since the right multiplication by i commutes with the isometric $\mathrm{Spin}^c(4)$ actions, S^0 and S^1 are complex rank two Hermitian bundles. The Clifford multiplication $C : TM^* \otimes S^0 \longrightarrow S^1$ is then induced by the $\mathrm{Spin}^c(4)$-equivariant homomorphism ${}_1\mathbf{H}_{\bar{0}} \times {}_0\mathbf{H}_{\bar{z}} \longrightarrow {}_1\mathbf{H}_{\bar{z}}$. The determinant line bundles $\det_{\mathbf{C}} S^0$ and $\det_{\mathbf{C}} S^1$ are isomorphic to the Hermitian line bundle L associated to the homomorphism ρ from $\mathrm{Spin}^c(4)$ to $U(1)$ sending (z, q_0, q_1) to z^2. Let $P_{U(1)}$ be the principal $U(1)$ bundle of L; then $\tilde{P}_c$ is a finite covering of $P_{SO(4)} \times P_{U(1)}$. In particular, any Hermitian connection A on L, together with the Levi-Civita connection on TM, uniquely determines a connection on $\tilde{P}_c$, and hence a $\mathrm{Spin}^c(4)$-connection ∇_A on S^0. The Dirac operator associated to A is the linear differential operator $D_A = C \circ \nabla_A : \Gamma(S^0) \longrightarrow \Gamma(S^1)$.

Fix a Hermitian connection A_0 on L. Let $\mathcal{H}^1(M, \mathbf{R})$ be the space of harmonic 1-forms, and consider the affine space $\mathcal{A}_0$ of Hermitian connections on L of the form $A = A_0 + ai$ for $a \in \mathcal{H}^1(M, \mathbf{R})$ (here we use the identification of the Lie algebra of $U(1)$ and $\mathbf{R} \cdot i$). Since $\nabla_A s = \nabla_{A_0} + \frac{1}{2} ai \otimes s$, the Dirac operator D_A is given by

$$D_A s = D_{A_0} s + \frac{1}{2} C_a s i,$$

where $C_a : S^0 \longrightarrow S^1$ is the Clifford multiplication by a. Let $\mathcal{H}^0(M, U(1))$ be the group of harmonic maps from M to $U(1)$. Consider the gauge group action of $g \in \mathcal{H}^0(M, U(1))$ on $A \in \mathcal{A}_0$ specified by

$$\nabla_{g(A)} s = [\nabla_A(sg^{-1})]g = \nabla_A s + dg^{-1} \otimes sg = \nabla_A s - g^{-1} dg \otimes s.$$

Then $g(A) = A - 2(g^{-1}dg)$ still lies in $\mathcal{A}_0$. Fix a base point $x_0 \in M$ and let $\mathcal{H}^0_0(M, U(1))$ be the subgroup consisting of the harmonic maps sending x_0 to the identity. Then $\mathcal{H}^0(M, U(1))$ is the product of $U(1)$ and $\mathcal{H}^0_0(M, U(1))$, where $U(1)$ is the subgroup of constant maps. Clearly $U(1)$ acts trivially on $\mathcal{H}^1(M, \mathbf{R})$. Let $p_M : \mathcal{A}_0 \times M \longrightarrow M$ be the projection map. We lift the action of $\mathcal{H}^0(M, U(1))$ to the bundle $p_M^*(S^0 \oplus S^1)$ on $\mathcal{A}_0 \times M$ by

$$g : (A, x, q) \longrightarrow (g(A), x, qg).$$

The action of the subgroup $U(1)$ coincides with the action coming from the complex structure. For $g \in \mathcal{H}^0(M, U(1))$,

$$D_{g(A)}(sg) = C(\nabla_{g(A)}(sg)) = C(\nabla_A(sgg^{-1})g) = (D_A s)g.$$

Therefore this family of operators $\{D_A\}$ over $\mathcal{A}_0$ is $\mathcal{H}^0(M, U(1))$-equivariant. Since $\mathcal{H}^0_0(M, U(1))$ acts freely on $\mathcal{A}_0$, we can divide out everything by $\mathcal{H}^0_0(M, U(1))$. Thus, over J which is the quotient of $\mathcal{A}_0$ by $\mathcal{H}^0_0(M, U(1))$, we have two infinite dimensional complex vector bundles:

$$\begin{aligned} \tilde{V}^0 &= (\mathcal{A}_0 \times \Gamma(p_M^* S^0))/\mathcal{H}^0_0(M, U(1)), \\ \tilde{V}^1 &= (\mathcal{A}_0 \times \Gamma(p_M^* S^1))/\mathcal{H}^0_0(M, U(1)), \end{aligned}$$

and a smooth family of Dirac operators $\{D_a\}_J$ which is equivariant with respect to $\mathcal{H}^0(M, U(1))/\mathcal{H}^0_0(M, U(1)) = U(1)$. If we further notice that $\mathcal{H}^0_0(M, U(1))$ is isomorphic to $H^1(M; 2\mathbf{Z})$, and $\mathcal{H}^1(M, \mathbf{R})$ is isomorphic to $H^1(M; \mathbf{R})$, then J is seen to be simply the quotient of $H^1(M; \mathbf{R})$ by $H^1(M; 2\mathbf{Z})$, which is a torus of dimension b_1.

Now let us suppose c is a reducible Spin^c structure. Recall that the group Spin(4) is isomorphic to $SU(2) \times SU(2)$ and its quotient by its center $\{\pm(1,1)\}$ is $SO(4)$. A spin structure on M is a principal Spin(4) bundle which is a lift of $P_{SO(4)}$. Spin(4) embeds into Spin$^c(4)$ by sending (q_0, q_1) to $(1, q_0, q_1)$. A spin reduction of $\tilde{P}_c$ is an embedding of a principal Spin(4)-bundle P in $\tilde{P}_c$ such that $\tilde{P}_c = P \times_{\mathrm{Spin}(4)} \mathrm{Spin}^c(4)$. Under such a reduction S^0 and S^1 are 1 dimensional vector bundles over $\mathbf{H}$ (we consider the action of $\mathbf{H}$ as right multiplication). The Spin^c structure c is reduced to a spin structure if and only if L is trivial. Moreover a spin reduction corresponds to a trivialization of the square root of L, or equivalently, a gauge equivalence class of trivial connections on L, so each reducible Spin^c structure admits 2^{b_1} spin reductions.

Coming with a reduction of c to a spin structure are the involution on J and the enlarged Pin(2) symmetry. This can be seen as follows. Fix a trivial connection A_0 on L corresponding to the spin reduction, and let D_0 be the associated Dirac operator. Consider the involution

$\iota_{\mathcal{A}_0}$ on $\mathcal{A}_0$ by sending a to $-a$. We lift $\iota_{\mathcal{A}_0}$ to the bundle $p_M^*(S^0 \oplus S^1)$ by

$$\iota_S : (a, x, s) \longrightarrow (-a, x, sj),$$

where sj is the right multiplication of s by the quaternion number j. The identities $ij = -ji$, $j^2 = -1$, and $g^{-1} = \bar{g}$ yield $\iota_S \circ \iota_S = -1$ and $\iota_S^{-1} g \iota_S = g^{-1}$. Thus, $p_M^* S^0$ and $p_M^* S^1$ are quaternionic bundles. When dividing out by $\mathcal{H}_0^0(M, U(1))$, $\iota_{\mathcal{A}_0}$ induces the standard involution ι_J on the torus J (in fact the fixed point set F corresponds to the set of spin structures that c reduces to). Furthermore, ι_S induces the anti-complex lifts $\iota_{\tilde{V}^0}$ and $\iota_{\tilde{V}^1}$ of ι_J on the bundles $\tilde{V}^0$ and $\tilde{V}^1$, which make them quaternionic bundles, hence Pin (2)-equivariant. Since D_0 is a $\mathbf{H}$-linear operator, i.e., $D_0(sj) = (D_0 s)j$, and $ij = -ji$, we have $D_{-a}(sj) = (D_a s)j$. This is equivalent to say that D_a is Pin (2)-equivariant.

Let $\mathrm{Ind}\{D_a\}_J$ be the index bundle over J by taking the difference of the kernel and cokernel of D_a. $\mathrm{Ind}\{D_a\}_J$ lies in $KQ(J)$ due to the Pin (2)-equivariance of D_a. We end this subsection with the formula

$$\mathrm{rank}_{\mathbf{C}} \mathrm{Ind}\{D_a\}_J = -\sigma(M)/8, \tag{4.1}$$

which follows from the Atiyah-Singer index theorem.

5.2. The refined and Mod 2 Seiberg-Witten invariants. In this subsection c is still a reducible Spin^c structure. We first write down the Seiberg-Witten equations associated to a spin reduction of c on M. There are two important features: one is the Pin (2)-equivariance, and the other is that the equations are considered as parametrized by the b_1-dimensional torus J.

Consider two trivial infinite dimensional real vector bundles over J:

$$\begin{aligned} \tilde{W}^0 &= J \times d^*(\Gamma(\Lambda^2 TM^*)), \\ \tilde{W}^1 &= J \times \Gamma(\Lambda^+ TM^*), \end{aligned}$$

with trivial $U(1)$-action. Since $zi\bar{z} = i$ for any $z \in U(1)$, the map $s \longrightarrow si\bar{s}$ is compatible with the Spin$^c(4)$-actions on ${}_0\mathbf{H}_z$ and ${}_0\mathbf{H}_{\bar{0}}$ and it is $U(1)$-equivariant, and thus it defines a $U(1)$-equivariant map from $\tilde{V}^0$ to $\tilde{W}^1$.

Now we are in a position to write down the Seiberg-Witten equations. Let $\mathcal{C}$ be the set of Hermitian connections on L, which can be identified with $A_0 + \Gamma(TM^*)i$, where A_0 is a trivial connection on L corresponding to the spin reduction. The configuration space for the Seiberg-Witten equations is the space $\mathcal{C} \times \Gamma(S^0)$ and the gauge group is $\mathrm{Map}(M, U(1))$. For our purpose it is convenient to restrict to the slice

$$\{A_0 + [\mathcal{H}^1(M; \mathbf{R}) + d^*(\Gamma(\Lambda^2 TM^*))]i\} \times \Gamma(S^0).$$

With this restriction, the gauge group reduces to $\mathcal{H}^0(M, U(1))$. After dividing out the action of the based gauge group $\mathcal{H}_0^0(M, U(1))$, the

Seiberg-Witten equations are a $U(1)$-equivariant bundle map $\tilde{f}_{SW}$ between the infinite dimensional bundles $\mathcal{E} = \tilde{V}^0 \oplus \tilde{W}^0$ and $\mathcal{F} = \tilde{V}^1 \oplus \tilde{W}^1$, which, at a point $a \in J$, is of the form

$$\tilde{f}_{SW}|_a(s, b) = (D_a s + C(b)s(2\pi i), d^+ b + si\bar{s}).$$

By the discussion at the end of §4.1, the real trivial bundles $\tilde{W}^0$ and $\tilde{W}^1$ are Pin(2)-equivariant with ι sending a vector b over $a \in J$ to the vector $-b$ over $-a \in J$. Since D_a is Pin(2)-equivariant, and

$$d^+(-b) + (sj)i\overline{(sj)} = -(d^+ b + si\bar{s}), \quad C(-b)sj(2\pi i) = C(b)s(2\pi i)j,$$

the monopole map $\tilde{f}_{SW}$ is Pin(2)-equivariant.

Now we are in a position to put the monopole map in the framework of §4.2. For a fixed $k > 2$, consider the fiberwise L^2_k Sobolev completion $\mathcal{E}_k$ of $\mathcal{E}$ and the fiberwise L^2_{k-1} Sobolev completion $\mathcal{F}_{k-1}$ of $\mathcal{F}$. The monopole map $\tilde{f}_{SW}$ extends to a continuous map $\mathcal{E}_k \longrightarrow \mathcal{C}_{k-1}$ between Hilbert bundles over J, which will also be denoted by $\tilde{f}_{SW}$.

Fiberwise, the linear part of $\tilde{f}_{SW}$ is the Fredholm map $l = \{D_a\}_J + d^+$. Since a multiplication map $\mathcal{E}_k \times \mathcal{E}_k \longrightarrow \mathcal{F}_k$ is continuous for $k > 2$, and the restriction map $\mathcal{F}_k \longrightarrow \mathcal{F}_{k-1}$ is compact, the non-linear part c is a compact map. Furthermore, it is shown in [**2**] that $\tilde{f}_{SW}$ satisfies the boundedness property: preimages of bounded sets of $\tilde{f}_{SW}$ are bounded.

Thus $\tilde{f}_{SW}$ gives rise to an element in $\mathcal{P}_l(\mathcal{E}_k, \mathcal{F}_{k-1})^{\mathrm{Pin}(2)}$. Let $\mathcal{U}$ be the L^2_{k-1} completion of the sum $\Gamma(S^1) \oplus \Gamma(\Lambda^+ TM^*)$. It contains two types of $U(1)$-representations, the standard one and the trivial one, both with infinite multiplicities. Applying the $U(1)$-equivariant version of Theorem 4.3, we see that the monopole map $\tilde{f}_{SW}$ induces an element $[\tilde{f}_{SW}]$ in the stable cohomotopy group

$$\pi^0_{U(1),\mathcal{U}}(J; \mathrm{ind}\, l).$$

Notice that $\mathrm{ind}\, l = \mathrm{ind}\,\{D_a\}_J - \underline{H^+(M; \mathbf{R})}$.

The element $[\tilde{f}_{SW}]$ in $\pi^0_{U(1),\mathcal{U}}(J; \mathrm{ind}\, l)$ is called the Bauer-Furuta stable cohomotopy refinement of the Seiberg-Witten invariant. It can be related to the integer valued Seiberg-Witten invariant. (In fact, it is finer than the integer valued Seiberg-Witten invariant. See the discussion in section 8 in [**1**].) As a particular case of Proposition 3.3 in [**2**], we have

Proposition 5.1 ([**2**]). *Suppose M is a 4-manifold with $b^+ \geq b_1 + 2$ and c is a reducible* Spin^c *structure. Then a homology orientation of M (i.e., an orientation of the vector space $H^1(M; \mathbf{R}) \oplus H^+(M; \mathbf{R})$) determines a homomorphism $t : \pi^0_{U(1),\mathcal{U}}(J; \mathrm{ind}\, l) \longrightarrow \mathbf{Z}$, which maps $[\tilde{f}_{SW}]$ to the integer valued Seiberg-Witten invariant of c.*

For our purpose, we just need to describe geometrically the composition of t and the reduction $\mathbf{Z} \longrightarrow \mathbf{Z}_2$ in the case $2\chi(M) + 3\sigma(M) = 0$.

First of all, any element in $\pi^0_{U(1),\mathcal{U}}(J; \operatorname{ind} l)$ is represented by a $U(1)$-equivariant pointed map $\phi : TE \longrightarrow S^W$ from the Thom space of a bundle E over J, where $E - \underline{W} = \operatorname{ind} l$. Notice that, since $2\chi(M) + 3\sigma(M) = 0$ and c is reducible, by (4.1) and (6.2), the dimension of the total space of E is 1+dim$W$. When $b^+ \geq b_1 + 2$, the connecting homomorphism in equivariant cohomotopy for the pair of a disc bundle and a sphere bundle of E is an isomorphism, and hence we get an equivariant map f from the sphere bundle $S(E)$ to the sphere $S(W)$.

One can associate a mod 2 degree to two such maps $f_0, f_1 : S(E) \longrightarrow S(W)$ as follows. Let $F(f_0, f_1)$ be the set of $U(1)$-equivariant homotopies

$$\tilde{f} : S(E) \times [0,1] \longrightarrow W$$

connecting f_0 and f_1, and satisfying

1. The restriction of $\tilde{f}$ to the $U(1)$-fixed point set $S(E^{U(1)}) \times [0,1]$ does not vanish anywhere.
2. $\tilde{f}$ is transverse to the zero section.

To construct such a $\tilde{f}$, we start with a $U(1)$-equivariant homotopy u of f_0 and f_1 as maps to W (such a u exists since W is $U(1)$-equivariantly homotopic to a point). Notice that u maps $S(E^{U(1)}) \times [0,1]$ into $W^{U(1)}$ by the $U(1)$-equivariance. The condition

$$\dim \left(S(E^{U(1)}) \times [0,1]\right) - \operatorname{rank} W^{U(1)} = b_1 - b^+ \leq -2$$

enables us to perturb u near $S(E^{U(1)}) \times [0,1]$ to a $U(1)$-equivariant homotopy v such that v does not vanish on $S(E^{U(1)}) \times [0,1]$. Now the zero set of v is away from $S(E^{U(1)}) \times [0,1]$ and thus has a neighborhood where $U(1)$ acts freely. We can obtain a $\tilde{f} \in F(f_0, f_1)$ by perturbing v in the quotient of this neighborhood.

Given $\tilde{f} \in F(f_0, f_1)$, denote the zero set of $\tilde{f}^{-1}(0)$ by $\hat{\mathcal{M}}$. Let $\hat{B}$ be the complement of $S(E^{U(1)})$ in $S(E)$. Then $\hat{\mathcal{M}}$ is a 1-dimensional closed submanifold in $\hat{B} \times (0,1)$. Since $U(1)$ acts freely on $\hat{B}$, $B = \hat{B}/U(1)$ is smooth and $\mathcal{M} = \hat{\mathcal{M}}/U(1)$ is a 0-dimensional smooth closed submanifold of $B \times (0,1)$. The cardinality of $\mathcal{M}$ modulo 2 is defined to be the mod 2 degree of the pair f_0 and f_1, and is denoted by $\gamma'(f_0, f_1)$. It is straightforward to see that γ' is additive in the sense that $\gamma'(f_0, f_1) + \gamma'(f_1, f_2) = \gamma'(f_0, f_2)$. In particular γ' only depends on the $U(1)$-equivariant homotopy classes.

Now consider constant equivariant maps from $S(E)$ to $S(W)$. By the equivariance, they must land in $S(W^{U(1)})$. Since $b^+ - b_1 > 1$, all such maps are homotopic. Let $[f_0]$ be this unique homotopy class of constant maps. For any $[f] \in [S(E), S(W)]^{U(1)}$ we define the mod 2 degree of $[f]$ to be

$$\gamma([f]) = \gamma'([f_0], [f]).$$

In our case, c is a reducible Spin^c structure, and the monopole map $\tilde{f}_{SW}$ is actually Pin(2)-equivariant. Therefore the map $f : S(E) \longrightarrow S(W)$ representing $[\tilde{f}_{SW}]$ can also be chosen to be Pin(2)-equivariant. The following result, as suggested by the referee, should be regarded as a variant of an equivariant Hopf index theorem in [**4**].

Proposition 5.2. *Suppose $f_0, f_1 : S(E) \longrightarrow S(W)$ are* Pin(2)*-equivariant. Then they have the same* mod 2 *degree.*

Proof. First of all, we can construct a $\tilde{f} \in F(f_0, f_1)$ which is Pin(2)-equivariant. This is because, as $U(1)$, Pin(2) acts freely away from $S(E^{(U(1)})$ and $S(W^{U(1)})$. Observe now that Pin(2) acts freely on $\hat{B}$ and hence on $\hat{\mathcal{M}}$. Therefore $\mathcal{M}$ is acted upon freely by the group $\mathbf{Z}_2 = \{1, \iota\}$. This is equivalent to say that $\gamma'([f_1], [f_2]) = 0$. Together with the additivity of γ', this vanishing result implies the proposition. q.e.d.

6. Bounds of Betti numbers

In this section we will prove Theorem 1.2. Let (M, ω) be a symplectic 4-manifold. As we have mentioned, the symplectic form ω determines a unique homotopy class of ω-compatible almost complex structures, and hence the symplectic canonical class K_ω. Via the homomorphism

$$\begin{aligned} U(2) &= U(1) \times SU(2)/\{\pm(1,1)\} \\ &\longrightarrow \mathrm{Spin}^c(4) \cong U(1) \times SU(2) \times SU(2)/\{\pm(1,1,1)\} \end{aligned}$$

sending $((\det A)^{\frac{1}{2}}, A(\det A)^{-\frac{1}{2}})$ to

$$((\det A)^{\frac{1}{2}}, \mathrm{diag}((\det A)^{\frac{1}{2}}, -(\det A)^{\frac{1}{2}}), A(\det A)^{-\frac{1}{2}}),$$

any such almost complex structure J in turn induces the canonical symplectic Spin^c structure, $\mathcal{K}_\omega$, with $S^0 \cong \underline{\mathbf{C}} \oplus \det\,(TM^*, J)$ and $S^1 \cong (TM^*, J)$ as complex bundles, where $\underline{\mathbf{C}}$ is the trivial complex line bundle. Notice that K_ω is the c_1 of the determinant line bundle of both S^0 and S^1. Therefore, if K_ω is trivial, $\mathcal{K}_\omega$ is reducible, i.e., it admits a reduction to a spin structure. Finally there is a canonical orientation of $\det(M)$ determined by ω, which we call the symplectic orientation.

The Gromov–Taubes invariant $Gr_\omega(A)$ of a non-zero class $A \in H^2(M; \mathbf{Z})$ is a certain count of symplectic submanifolds representing the Poincáre dual to A (see [**39**]). For the zero class $Gr_\omega(0)$ is defined to be 1. In the case when A is a torsion class, its Gromov-Taubes invariant can be completely determined.

Lemma 6.1. *When A is a torsion class, $Gr_\omega(A)$ is equal to zero if A is non-zero, and $Gr_\omega(A)$ is equal to one if A is the zero class.*

Proof. If $A \neq 0$ and $Gr(A) \neq 0$, then A is represented by a symplectic submanifold. But the integral of ω over a symplectic submanifold is positive, while the pairing of a torsion class with the symplectic form is certainly zero. This proves the first half of the statement. The second half is just the definition of $Gr_\omega(0)$.

Now we state a fundamental non-vanishing result of Taubes on the ordinary Seiberg-Witten invariants and the Gromov-Taubes invariants of a symplectic 4-manifold. q.e.d.

Theorem 6.2 ([**38**], [**41**]). *Suppose (M, ω) is a symplectic 4-manifold with $b^+ > 1$ and symplectic canonical class K_ω. Then, with the symplectic orientation of the Seiberg-Witten moduli space, the values of the ordinary Seiberg-Witten invariant of the* Spin^c *structure $\mathcal{K}_\omega$ and the Gromov-Taubes invariant of K_ω are both equal to $(-1)^l$ with $l = \frac{b^+ - b_1 + 1}{2}$. In particular the ordinary Seiberg-Witten invariant of the* Spin^c *structure $\mathcal{K}_\omega$ is nonzero modulo 2.*

We next collect some general properties of symplectic 4-manifolds with torsion canonical class (which should be known to experts).

Proposition 6.3. *Let (M, ω) be a symplectic 4-manifold with torsion canonical class K_ω. Then*

1. $2\chi + 3\sigma = 0$.
2. *M has even intersection form.*
3. *K_ω is either trivial, or of order two which only occurs when M is an integral homology Enriques surface.*
4. *M is spin except when M is an integral homology Enriques surface.*

Proof. The first statement follows from $K_\omega \cdot K_\omega = 2\chi + 3\sigma$. The second statement follows from another property of K_ω: for any class $e \in H^2(M; \mathbf{Z})$, $e \cdot e = e \cdot K_\omega$ modulo 2.

Since M is spin if and only if the second Stieffel-Whitney class $w_2(M)$ is trivial and $w_2(M)$ is the mod 2 reduction of K_ω, the last statement follows from the third statement.

Now we prove the third statement. If K_ω is non-zero, then $Gr_\omega(K_\omega) = 0$ by Lemma 6.1. In the case $b^+ > 1$, this is impossible due to Theorem 6.2; therefore K_ω is the zero class.

The case $b^+ = 1$ was already understood in [**30**], based on the observation that, when A is a torsion class, there is the following formula (6.1)

$$|Gr_\omega(A) - Gr_\omega(K_\omega - A)| = \begin{array}{ll} 1 & \text{if } b_1 = 0, \\ 0 & \text{if } b_1 > 0. \end{array}$$

When $b_1 > 0$, if we choose A to be the zero class, then we find from (6.1) that $Gr_\omega(K_\omega) = 1$. So as in the case $b^+ > 1$, K_ω must be trivial.

When $b^+ = 1$ and $b_1 = 0$, then $b^- = 4 - 4b_1 + 5b^+ = 9$ by the first statement. Since M has even intersection form, it is a rational homology

Enriques surface. What remains to prove is that $H_1(M;\mathbf{Z}) \cong \mathbf{Z}_2$. If we choose A to be the zero class, then we find from (6.1) that $Gr_\omega(K_\omega) = 0$ or 2. By Lemma 6.1, $Gr_\omega(K_\omega) = 0$ and K_ω is non-zero. Now choose A to be a non-zero torsion class, then $Gr_\omega(A) = 0$ by Lemma 6.1. Therefore, according to (6.1), the Gromov-Taubes invariant of the torsion class $(K_\omega - A)$ is ± 1. By applying Lemma 6.1 again, we find the value of $Gr_\omega(K_\omega - A)$ is one and A is equal to K_ω. Thus, the torsion subgroup of $H^2(M;\mathbf{Z})$ is $\mathbf{Z}_2$ and K_ω is of order 2. Since $b_1 = 0$, $H_1(M;\mathbf{Z})$ is isomorphic to its torsion subgroup, which is the same as the torsion subgroup of $H^2(M;\mathbf{Z})$ by the Universal Coefficient Theorem. The proof is complete. q.e.d.

Remark 6.4. It is well-known that the canonical line bundles of hyperelliptic surfaces are non-trivial as holomorphic line bundles (see e.g., [**3**]). However, it follows from statement 3 as well as from inspection of the proof in [**3**] that the canonical classes of hyperelliptic surfaces are actually trivial as integral cohomology classes.

In view of Theorem 6.2 and Proposition 6.3, it is useful to compute the Mod 2 Seiberg-Witten invariant of spin manifolds with $2\chi + 3\sigma = 0$. We succeed to do so in the case $b^+ \geq b_1 + 2$ and $b_1 \leq 4$ via the refined Seiberg-Witten invariant, Proposition 4.1, and Proposition 5.4. We begin with a consequence of Proposition 4.1.

Lemma 6.5. *Suppose $J = T^{b_1}$, V_0, V_1 are quaternionic bundles over J with $rank_{\mathbf{C}} V_0$-$rank_{\mathbf{C}} V_1 = 2p \geq 0$ and with V_1 trivial, and W_0, W_1 are trivial real bundles with $rank_{\mathbf{R}} W_0$-$rank_{\mathbf{R}} W_1 = -(4p + b_1 - 1)$. Then $W_1 = W_0 \oplus \underline{\mathbf{R}}^{4p+b_1-1}$. If $b_1 \leq 1$, then we have*

$$(V_0, V_1) = (\underline{\mathbf{H}}^p \oplus V_1, V_1),$$

and if $2 \leq b_1 \leq 4$, then

$$(V_0, V_1) = (Q \oplus \underline{\mathbf{H}}^p \oplus V_0', \underline{\mathbf{H}} \oplus V_0'),$$

where V_0' is a trivial quaternionic bundle, and Q is some rank 2 quaternionic vector bundle.

Proof. The statement for W_0 and W_1 is obvious. By Remark 4.2, if $b_1 \leq 1$, V_i is trivial for $i = 0, 1$. Thus we can assume (V_0, V_1) is as claimed. If $b_1 = 4$, as quaternionic vector bundles, V_0 is isomorphic to $Q \oplus \underline{\mathbf{H}}^{r+p}$ for some rank 2 quaternionic vector bundle Q by Proposition 4.1, and V_1 is isomorphic to $\underline{\mathbf{H}} \oplus \underline{\mathbf{H}}^r$. We claim the same is true if $b_1 \leq 3$. Let p be the Pin(2)-equivariant projection from $J \times T^{4-b_1}$ to the first factor, and e the Pin(2)-equivariant embedding of J to $J \times O_{T^{4-b_1}}$ in $J \times T^{4-b_1}$. Then for $i = 0, 1$, as a quaternionic bundle over T^4, $p^* V_i$ splits as above. Therefore, as we claimed, $e^* p^* V_i = V_i$ splits as well. Thus we can assume that V_0 and V_1 are as stated when $2 \leq b_1 \leq 4$. The proof is finished. q.e.d.

Proposition 6.6. *Suppose $J = T^{b_1}$ with $b_1 \leq 4$, V_0, V_1 are quaternionic bundles and W_0, W_1 are trivial real bundles over J of the form*

$$(V_0 \oplus W_0, V_1 \oplus W_1) = (\underline{\mathbf{H}}^p, \underline{\mathbf{R}}^{4p+b_1-1}),$$

if $b_1 \leq 1$; and

$$(V_0 \oplus W_0, V_1 \oplus W_1) = (Q \oplus \underline{\mathbf{H}}^p, \underline{\mathbf{H}} \oplus \underline{\mathbf{R}}^{4p+b_1-1}),$$

if $2 \leq b_1 \leq 4$, where Q is a rank 2 quaternionic bundle. Suppose further $p \geq 2$ if $b_1 = 0$. Then the mod 2 *degree of any* Pin (2)*-equivariant map $f : S(V_0 \oplus W_0) \longrightarrow S(V_1 \oplus W_1)$ is zero.*

Proof. By Proposition 5.4, it suffices to construct f_0 and f_1 in $\mathrm{Map}\,(S(V_0 \oplus W_0), S(V_1 \oplus W_1))^{U(1)}$ with f_0 a constant map and f_1 Pin (2)-equivariant, such that $\tilde{f} = (1-t)f_0 + tf_1$ does not vanish on $S(V_0 \oplus W_0) \times [0,1]$.

We begin with the case $b_1 \leq 1$. Write $\mathbf{R}^{4p+b_1-1}$ as $(\mathrm{Im}\mathbf{H})^p \oplus \mathbf{R}^{p+b_1-1}$. Our assumption on p implies that $p + b_1 - 1 \geq 1$. Consider f_0 and f_1 in $\mathrm{Map}(S(\underline{\mathbf{H}}^p), S((\underline{\mathrm{Im}\mathbf{H}})^p \oplus \underline{\mathbf{R}}^{p+b_1-1}))$, defined for any $z \in J$ by

$$\begin{array}{lllllllll} f_0|_z(q_1, ..., q_p) & = (& -i, & ..., & -i, & -1, & ..., & -1 &), \\ f_1|_z(q_1, ..., q_p) & = (& q_1 i \bar{q}_1, & ..., & q_p i \bar{q}_p, & 0, & ..., & 0 &). \end{array}$$

Clearly f_0 is a non-vanishing $U(1)$-equivariant constant map. Since $qi\bar{q}$ is a quaternionic homomorphism from $\mathbf{H}$ to $\mathrm{Im}\,\mathbf{H}$, f_1 is Pin (2)-equivariant. Furthermore $q_i i \bar{q}_i = 0$ only if $q_i = 0$, so f_1 is also non-vanishing. Define $\tilde{f} : S(V_0 \oplus W_0) \times [0,1] \longrightarrow V_1 \oplus W_1$ by $\tilde{f} = (1-t)f_0 + tf_1$. Clearly the last $p + b_1 - 1$ coordinates of $(1-t)f_0 + tf_1$ vanish only if $t = 1$. Since $p + b_1 - 1 \geq 1$, and f_1 is nowhere vanishing, $\tilde{f}$ does not vanish.

Now we consider the case $b_1 = 4$. Let $\phi : Q \longrightarrow \underline{\mathbf{H}}$ be a quaternionic homomorphism defined in Proposition 4.1. Let D be the finite subset where ϕ vanishes. Then, for each $x \in D$, there is a neighborhood U_x of x and an identification of U_x with a neighborhood of U of 0 in $\mathbf{H}$, with the property that

1. $Q|_{U_x} \cong U_x \times \mathbf{H}$, as complex vector bundles.
2. Over U_x, ϕ is described as $\phi|_v(q) = vqi$ or $\phi|_v(q) = \bar{v}qi$ under the identification of U_x with $U \subset \mathbf{H}$ and the trivialization in 1.
3. U_x is disjoint from U_y if $x \neq y$, and $U \cup_{x \in D} U_x$ is ι_J invariant.

For each $x \in D$ let

$$\chi : Q|_{U_x} \longrightarrow U_x \times \mathrm{Im}\mathbf{H}$$

be the Pin (2)-equivariant map defined by

$$\chi|_v(q) = qi\bar{q}.$$

We extend χ to Q over J by an ι_J invariant cut off function. Write $\mathbf{R}^{4p+b_1-1} = \mathbf{R}^{4p+3}$ as $\mathrm{Im}\mathbf{H} \oplus (\mathrm{Im}\mathbf{H})^p \oplus \mathbf{R}^p$. Consider f_0 and f_1 in

$$\mathrm{Map}(S(Q \oplus \underline{\mathbf{H}}^p), S(\underline{\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}} \oplus (\underline{\mathrm{Im}\mathbf{H}})^p \oplus \underline{\mathbf{R}}^p))^{U(1)}$$

as defined for any $z \in J = T^4$ by

$$\begin{aligned} f_0|_z(q_0, q_1, ..., q_p) &= (0, \quad -i, \quad -i, \quad ..., \quad -i, \quad -1, \quad ..., \quad -1), \\ f_1|_z(q_0, q_1, ..., q_p) &= (\phi|_z(q_0),\ \chi|_z(q_0),\ q_1 i\bar{q}_1,\ ...,\ q_p i\bar{q}_p,\ 0, \quad ..., \quad 0\ ), \end{aligned}$$

where q_0 is in Q. f_1 is Pin (2)-equivariant since $\phi : Q \longrightarrow \underline{\mathbf{H}}$ is a quaternionic homomorphism. We claim that f_1 is also non-vanishing. Suppose $f_1|_z = 0$ at $(q_0, q_1, ..., q_p)$. Since $q_j i\bar{q}_j = 0$ only if $q_j = 0$, q_j is equal to zero for $j \geq 1$. Similarly $\chi|_z(q_0) = 0$ implies that either $q_0 = 0$ or z is away from D. Finally, since $vq_0 i = 0$ only if $v = 0$ or $q_0 = 0$, the first term $\phi|_z(q_0) = 0$ only when $q_0 = 0$ or $z \in D$. Hence there are no such points in the sphere bundle.

Define $\tilde{f} = (1-t)f_0 + tf_1$ as above. Since the last p coordinates of $(1-t)f_0 + tf_1$ vanish only if $t = 1$, the zero set $\mathcal{M} = \tilde{f}^{-1}(0)$ is empty if $p \geq 1$.

For the case $b_1 = 2$ or 3, by pulling back V_i via the projection p and the embedding e as in the previous lemma, we can get a Pin (2)-equivariant homomorphism $\phi : Q \longrightarrow \underline{\mathbf{H}}$ as well. Notice that we can also simply restrict χ to $J = T^{b_1}$. Write $\mathbf{R}^{4p+b_1-1}$ as $\mathrm{Im}\mathbf{H} \oplus (\mathrm{Im}\mathbf{H})^p \oplus \mathbf{R}^{p+b_1-4}$. As long as $p + b_1 - 4$ is positive, we can use the same f_0 and f_1 as in the case $b_1 = 4$, and conclude that $\tilde{f} = (1-t)f_0 + tf_1$ does not vanish on the sphere bundle.

When $b_1 = 3$, $p + b_1 - 4$ fails to be positive if $p = 1$. In this case

$$V_0 = Q \oplus \underline{\mathbf{H}}, \quad W_0 = 0, \quad V_1 = \underline{\mathbf{H}}, \quad W_1 = \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}}.$$

We need to stabilize $(V_0 \oplus W_0, V_1 \oplus W_1)$ by adding a copy of $\underline{\mathbf{R}}$. This is also achieved by considering the projection $p : J \times T^1 \longrightarrow J$ in Lemma 6.5 with $J = T^3$. Let B^1 be an invariant disc of T^1 around O_{T^1} and $\tau : B^1 \longrightarrow \mathbf{R}$ be an equivariant diffeomorphism. On $T^3 \times B^1$ denote by D' the set of points where ϕ fails to be an isomorphism. By shrinking B^1 if necessary we can assume that each point in D' is of the form $z \times 0$ with $z \in T^3$. Let $D'' = p(D') \subset T^3$. Define χ on $T^3 \times B^1$ as above, first around D', and then extend it to $T^3 \times B^1$ by an invariant cut off function.

Via τ we identify $p^*Q|_{T^3 \times B^1}$ and $p^*\underline{\mathbf{H}}|_{T^3 \times B^1}$ with the bundles $Q \oplus \underline{\mathbf{R}}$ and $\underline{\mathbf{H}} \oplus \underline{\mathbf{R}}$ over T^3 respectively. Notice that these identifications are Pin (2)-equivariant since τ is. Via these identifications the homomorphism $\phi : p^*Q \longrightarrow p^*\underline{\mathbf{H}}$ induces a Pin (2)-equivariant bundle map (not a homomorphism)

$$\phi' : Q \oplus \underline{\mathbf{R}} \longrightarrow \underline{\mathbf{H}}$$

by the formula

$$p^*\phi'|_z(q_0, s) = \phi|_{z \times \tau^{-1}(s)}(p^*q_0),$$

where $z \in T^3, q_0 \in Q$ and $s \in \underline{\mathbf{R}}$. Similarly define

$$\chi' : Q \oplus \underline{\mathbf{R}} \longrightarrow \underline{\mathrm{Im}\mathbf{H}}$$

by the formula

$$p^*\chi'|_z(q_0, s) = \chi|_{z\times\tau^{-1}(s)}(p^*q_0).$$

Consider f_0 and f_1 in

$$\mathrm{Map}(S(Q \oplus \underline{\mathbf{H}} \oplus \underline{\mathbf{R}}), S(\underline{\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathbf{R}}))^{U(1)}$$

as defined by

$$\begin{array}{ll} f_0|_z(q_0, q_1, s) = (& 0, \quad -i, \quad -i, \quad -1 \quad), \\ f_1|_z(q_0, q_1, s) = (& \phi'|_z(q_0, s), \quad \chi'|_z(q_0, s) + si, \quad q_1 i\bar{q}_1, \quad 0 \quad). \end{array}$$

f_1 is clearly Pin (2)-equivariant. Let us show that f_1 does not vanish. The first term $\phi'|_z(q_0, s)$ vanishes only if $\phi|_{z\times\tau^{-1}(s)}(p^*q_0) = 0$. This occurs either when $q_0 = 0$, or when $z \times \tau^{-1}(s) \in D'$.

When $q_0 = 0$, for any s, $\chi|_{z\times\tau^{-1}(s)}(p^*q_0) = 0$, so $\chi'|_z(q_0, s) = 0$. Now the second term $\chi'|_z(q_0, s) + si$ is just si and it vanishes only if $s = 0$.

By our assumption on D', $z \times \tau^{-1}(s) \in D'$ exactly when $z \in D''$ and $s = 0$. Now the second term $\chi'|_z(q_0, s) + si$ is equal to $q_0 i\bar{q}_0$ and it vanishes only if $q_0 = 0$.

We have just shown that the first two terms of f_1 vanish only if $q_0 = 0$ and $s = 0$. Since the third term $q_1 i\bar{q}_1$ vanishes only if $q_1 = 0$, we see that f_1 never vanishes. By considering the last coordinate, again we derive the conclusion that $\tilde{f} = (1-t)f_0 + tf_1$ does not vanish.

When $b_1 = 2$, $p + b_1 - 4$ fails to be positive if $p = 1$ or 2. The two cases are similar, so we only write down the maps for $p = 1$ in some detail. In this case

$$V_0 = Q \oplus \underline{\mathbf{H}}, \quad W_0 = 0, \quad V_1 = \underline{\mathbf{H}}, \quad W_1 = \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}}.$$

We use the same trick as in the case $b_1 = 3$ and $p = 1$ by adding two copies of $\underline{\mathbf{R}}$ via the projection $p : J \times T^2 \longrightarrow J$ with $J = T^2$. Via an equivariant identification of an invariant two disk with $\mathbf{R}^2$, we can similarly define

$$\begin{array}{ll} \phi' : & Q \oplus \underline{\mathbf{R}}^2 \longrightarrow \underline{\mathbf{H}}, \\ \chi' : & Q \oplus \underline{\mathbf{R}}^2 \longrightarrow \underline{\mathrm{Im}\mathbf{H}}. \end{array}$$

Consider f_0 and f_1 in

$$\mathrm{Map}(S(Q \oplus \underline{\mathbf{H}} \oplus \underline{\mathbf{R}}^2), S(\underline{\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathbf{R}}^2))^{U(1)}$$

as defined by

$$\begin{array}{l} f_0|_z(q_0, q_1, S) = (0, \quad -i, \quad -i, \quad -i, \quad -1, \quad -1), \\ f_1|_z(q_0, q_1, S) = (\phi'|_z(q_0, S), \chi'|_z(q_0, S) + \beta(S), q_1 i\bar{q}_1, q_2 i\bar{q}_2, \ 0, \quad 0\,), \end{array}$$

where $\beta(S) = s_1 i + s_2 j$ for $S = (s_1, s_2) \in \mathbf{R}^2$. Exactly the same arguments as in the case $b_1 = 3$ and $p = 1$ prove that $\tilde{f} = (1-t)f_0 + tf_1$ does not vanish.

For the last case $b_1 = 2, p = 2$,

$$V_0 = Q \oplus \underline{\mathbf{H}} \oplus \underline{\mathbf{H}}, \quad W_0 = 0, \quad V_1 = \underline{\mathbf{H}}, \quad W_1 = \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}} \oplus \underline{\mathrm{Im}\mathbf{H}}.$$

The only difference in defining f_0 and f_1 from the previous case is that q_1 is replaced by (q_1, q_2). The proof of Proposition 6.6 is complete. q.e.d.

The following is the vanishing result mentioned in the introduction.

Corollary 6.7. *Let M be a spin manifold with $2\chi + 3\sigma = 0$ and $b^+ \geq b_1 + 2$. Let c be a reducible* Spin^c *structure. We further assume $b_1 \leq 4$. If $b^+ > 3$, then the* Mod 2 *Seiberg-Witten invariant of c is trivial.*

Proof. When $2\chi + 3\sigma = 0$ the signature is given by

$$\sigma = (b^+ - b^-) = -4(1 - b_1 + b^+). \tag{6.2}$$

So the condition $b^+ \geq b_1+2$ is equivalent to the signature being negative. If we write $\sigma = -16p$, then $p \geq 1$ and the possible values of b^+ are

$$b^+ = 4p + b_1 - 1. \tag{6.3}$$

It follows from (6.3) that, in the case $b_1 \geq 1$, $b^+ > 3$ only if $p \geq 1$, and in the case $b_1 = 0$, $b^+ > 3$ only if $p \geq 2$. Thus, by Lemma 6.5, a representative for $[\tilde{f}_{SW}]$ gives rise to J, V_0, W_0, V_1, W_1 can be chosen exactly as in Proposition 6.6. Since the Mod 2 Seiberg-Witten invariant of c is given by the mod 2 degree, the result follows from Proposition 6.6. q.e.d.

Remark 6.8. When $b_1 \leq 1$, there is another way to show the vanishing of the Mod 2 Seiberg-Witten invariant. In this case the Dirac index bundle is trivial as a quaternionic bundle over J by Remark 4.2, and so according to Proposition 5.4, the Mod 2 Seiberg-Witten invariant only depends on b_1 and b^+ of the manifold. Now it suffices to construct a spin manifold with $2\chi+3\sigma = 0$, which has the same b_1 and b^+, and whose ordinary Seiberg-Witten invariants all vanish. For each $p \geq 1$ and $b_1 \leq 1$, consider the following family of spin 4-manifolds with $2\chi + 3\sigma = 0$,

$$N(p, b_1) = p(K3)\#b_1(S^1 \times S^3)\#(p + b_1 - 1)(S^2 \times S^2).$$

Since $N(p, b_1)$ is a connected sum of manifolds with $b^+ > 1$ if $p \geq 1$, the ordinary Seiberg-Witten invariants of $N(p, b_1)$ are trivial. Thus, $N(p, b_1)$ is the desired manifold.

Remark 6.9. Along the same line one can give another proof of the main result in [**32**] that the Mod 2 Seiberg-Witten invariant of the unique reducible Spin^c structure on a homotopy K3 is one. In this case $b_1 = 0$ and $b^+ = 3 \geq b_1 + 2$, so we can again apply Proposition 5.4. Again the fact that the Dirac index bundle is trivial implies that the Mod 2 Seiberg-Witten invariant has the same value for all homotopy K3. The non-vanishing then follows from the fact that the corresponding Seiberg-Witten invariant on the K3 is equal to one.

As for the remaining case $b^+ \leq b_1 + 1$, we can directly obtain the Betti number bounds.

Lemma 6.10. *Let M be a spin manifold with $2\chi + 3\sigma = 0$. If $b_1 \leq 4$ and $b^+ \leq b_1 + 1$, then $\sigma = 0$ and $b^+ = b^- \leq 3$.*

Proof. It is easy to see from equation (6.2) that, when $b_1 \leq 4$ and $b^+ \leq b_1 + 1$, σ is between -8 and 12. The Rokhlin congruence asserts that the signature of a spin manifold is divisible by 16. So σ must be equal to zero in this case. Since $b_1 \leq 4$, $\sigma = 0$ implies that $b^+ = b^- \leq 3$ by equation (6.2). q.e.d.

Now we are able to prove the following more refined version of Theorem 1.

Theorem 6.11. *Let M be a minimal symplectic 4-manifold with Kodaira dimension zero. If $b_1 \leq 4$, then $b^+ \leq 3$, $b^- \leq 19$, and $\sigma = 0, -8$ or -16. Moreover, the real homology group, the intersection form, and the spin type of M are the same as those of a Kähler surface with Kodaira dimension zero or an orientable T^2-bundle over T^2.*

Proof. If M is non-spin, then $b^+ = 1, b^- = 9$ and $\sigma = -8$ by Proposition 6.3. So we have the required bounds. Moreover, its real homology group agrees with the Enriques surface in Table I.

In the spin case it follows from Theorem 6.2, Proposition 6.3, Corollary 6.7 and Lemma 6.10 that $b^+ \leq 3$. In turn this bound on b^+, together with $b_1 \leq 4$, implies that $-16 \leq \sigma \leq 12$ by equation (6.2). The desired conclusion about σ then follows from the Rokhlin congruence. Since $b^- = b^+ - \sigma$, the bound on b^- is a direct consequence of the bounds on b^+ and σ. To show the real homology group of M coincides with that of certain manifold in Table I, we separate into two cases according to the two values of the signature. When $\sigma = 0$, we have $b^- = b^+ \leq 3$, and $b_1 = b^+ - 1$ by equation (6.2), so one of the manifolds in class b, c or e has the same real homology group as M. When $\sigma = -16$, we have $b^+ = 3 + b_1$ by equation (6.2). In particular, we have $b^+ \geq 3$. Therefore b^+ must be equal to 3, consequently, $b_1 = 0$ and $b^- = 19$, and the real homology group of M agrees with that of $K3$.

By Proposition 6.3, M always has even indefinite intersection form. Since indefinite even intersection forms are classified by the rank and the signature only, the claim about the intersection form follows from the claim about the real homology group. The proof of Theorem 6.11 is therefore complete. q.e.d.

Remark 6.12. A weaker version of the Betti number conjecture $b^+ \leq b_1 + 3$ would be a consequence of the conjectural generalized Noëther inequalities in [**14**] and [**31**]. A proof of the version in [**31**] for abundant manifolds with $b_1 = 0$ was announced in [**6**] (the abundance assumption is satisfied by all symplectic 4-manifolds with trivial canonical class).

Remark 6.13. By carefully examining the arguments in the proof of Theorem 6.11 we realize that we actually have shown that the Mod 2 Seiberg-Witten invariant of a reducible Spin^c structure on a spin manifold with $2\chi + 3\sigma = 0$ only depends on the real homology group as long as $b_1 \leq 4$ and $b^+ \geq b_1 + 2$. In [**35**] it was shown that, for a spin integral homology 4-torus M, the Mod 2 Seiberg-Witten invariant of a reducible Spin^c structure also depends on the product structure of $H^1(M; \mathbf{Z})$. In particular, it is non-trivial only if $\det H^1(M; \mathbf{Z})$ is odd. However, there is no contradiction between the two results, since a homology 4-torus does not satisfy the condition $b^+ \geq b_1 + 2$.

Remark 6.14. Together with the techniques in [**27**], it is actually possible to compute some of the Mod 2 Seiberg-Witten invariant of a reducible Spin^c structure without the restriction $b^+ \geq b_1 + 2$. In [**27**] we studied the family Seiberg-Witten invariants and showed that they are well-defined if the dimension of the family is at most $b^+ - 2$. When the dimension of the family exceeds $b^+ - 2$, we observed that there is the complication arising from their dependence on the chamber structure. When passing to a non-equivariant setting to define other versions of Seiberg-Witten invariants, the Bauer-Furuta Seiberg-Witten invariant behaves like a family Seiberg-Witten invariant parametrized by the b_1-dimensional torus, and thus the condition $b^+ \geq b_1 + 2$ naturally comes in. In [**27**] we also developed techniques to analyze the dependence of the family Seiberg-Witten invariant on the chamber structure. Thanks to the special feature of the Pin(2)-equivariance, we can show that the wall crossing number is zero as far as the Mod 2 Seiberg-Witten invariant of a reducible Spin^c structure is concerned. Furthermore, when $2 \leq b_1 \leq 4$, the Mod 2 Seiberg-Witten invariant of a reducible Spin^c structure on a 4-manifold with $2\chi + 3\sigma = 0$, e.g., a spin real homology torus, can be shown to be $\epsilon(\mathrm{Ind}\{D_a\}_J)$. Here ϵ is the $\mathbf{Z}_2$-valued homomorphism on $KQ(J)$ defined in §4.1. Such a result would be consistent with and imply the result in [**35**]. This is because, on one hand, $\det H^1(M; \mathbf{Z}) = c_2(\mathrm{Ind}\{D_a\}_J)$ up to sign by a family index computation as was done in [**35**]; on the other hand, $\epsilon(\mathrm{Ind}\{D_a\}_J) \equiv c_2(\mathrm{Ind}\{D_a\}_J) \pmod 2$ by Proposition 4.1. To end this long remark, we mention an amusing consequence: for an orientable T^2-bundle over T^2 with $b_1 = 2$, the Dirac index bundle of the canonical symplectic Spin^c structure is the unique non-trivial element in $KQ(T^2)$ (since the corresponding Mod 2 Seiberg-Witten invariant is nonzero by Theorem 6.2, and by Remark 4.2, $KQ(T^2)$ is isomorphic to $\mathbf{Z}_2$ via ϵ).

We end this section by providing more evidence that a minimal symplectic 4-manifold with Kodaira dimension zero resembles a manifold in Table 1.

Proposition 6.15. *Let M be a minimal symplectic 4-manifold with Kodaira dimension zero.*

1. [**32**]. *If $H_1(M;\mathbf{Z}) = 0$, then M is an integral homology K3 surface; moreover, if M is simply connected, then M is homeomorphic to the $K3$ surface.*
2. *If $\pi_1(M)$ is finite but non-trivial, then M is homeomorphic to the Enriques surface.*
3. *If $b_1 = 4$, then $H^*(M;\mathbf{R})$ is generated by $H^1(M;\mathbf{R})$, and hence isomorphic to $H^*(T^4;\mathbf{R})$ as a ring.*

Proof. In the first case, M has trivial b_1, so Theorem 6.11 applies. From Table I we find that either $b^+ = 3$ or $b^+ = 1$. By [**30**] (and the proof of Proposition 6.3), when $b^+ = 1$, H_1 is non-trivial. So $b^+(M)$ must be equal to 3 and M has the same intersection form as that of the K3 surface, which means that M is an integral homology K3 surface. If M is actually simply connected, then M is homeomorphic to the K3 surface due to Freedman's classification of simply connected topological 4-manifolds.

If $\pi_1(M)$ is finite, then the universal covering $\tilde{M}$ of M still has torsion canonical class. Therefore $\tilde{M}$ is homeomorphic to the K3 surface, and in particular it has signature -16. Since $\pi_1(M)$ is non-trivial, the signature of M cannot be -16. From the table we see the only possibility is that M has signature -8, which implies that $\pi_1(M) = \mathbf{Z}_2$. Again from Table I, M also has the same intersection form as that of the Enriques surface. By the extension of Freedman's result to the case $\pi_1 = \mathbf{Z}_2$ in [**18**], two smooth 4-manifolds with $\pi_1 = \mathbf{Z}_2$ and the same intersection form are homeomorphic if and only if they have the same w_2-type. For such a 4-manifold, there are three w_2-types: (I). $w_2(\tilde{M}) \neq 0$; (II). $w_2(M) = 0$; (III). $w_2(\tilde{M}) = 0$ but $w_2(M) \neq 0$. Clearly the Enriques surface has w_2-type (III). We claim this is the case for M as well. Since M has signature -8, M is not spin by Rokhlin's congruence Theorem. So $w_2(M)$ is non-zero. On the other hand $\tilde{M}$ has even intersection form and is simply connected, so it is a spin manifold and therefore has trivial w_2. The claim is verified and so M is homeomorphic to the Enriques surface.

By Theorem 6.11, $H^*(M;\mathbf{R})$ is isomorphic to $H^*(T^4;\mathbf{R})$ as a group. M is spin by Proposition 6.3, so M is a spin real homology 4-torus. In fact the argument in [**35**] can be applied in this situation to get the following result: choose an integral basis $\alpha_i, i = 1,2,3,4$ of $H^1(M;\mathbf{Z})/$ Torsion, then the product $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4$ modulo 2 is the same as the Mod 2 Seiberg-Witten invariant of a spin structure. This implies $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4$ is nonzero in our case. Therefore $\alpha_i \cdot \alpha_j$ with $i \neq j$ are linearly independent and span a six dimensional subspace of $H^2(M;\mathbf{R})$.

This subspace actually is the total space because $b_2(M) = 6$. Therefore $H^*(M; \mathbf{R})$, as a ring, is generated by α_i, $i = 1, 2, 3, 4$, and hence isomorphic to $H^*(T^4; \mathbf{R})$. q.e.d.

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