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Symplectic Elasticity: Theory and Applications

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Many of the early works on symplectic elasticity were published in Chinese and as a result, the early works have been unavailable and unknown to researchers worldwide. It is the main objective of this paper to highlight the contributions of researchers from this part of the world and to disseminate the technical knowledge and innovation of the symplectic approach in analytic elasticity and applied engineering mechanics. This paper begins with the history and background of the symplectic approach in theoretical physics and classical mechanics and subsequently discusses the many numerical and analytical works and papers in symplectic elasticity. This paper ends with a brief introduction of the symplectic methodology. A total of more than 150 technical papers since the middle of 1980s have been collected and discussed according to various criteria. In general, the symplectic elasticity approach is a new concept and solution methodology in elasticity and applied mechanics based on the Hamiltonian principle with Legendre's transforma-tion. The superiority of this symplectic approach with respect to the classical approach is at least threefold: (i) it alters the classical practice and solution technique using the semi-inverse approach with trial functions such as those of Navier, Lévy, and Timosh-enko; (ii) it consolidates the many seemingly scattered and unrelated solutions of rigid body movement and elastic deformation by mapping with a series of zero and nonzero eigenvalues and their associated eigenvectors; and (iii) the Saint–Venant problems for plane elasticity and elastic cylinders can be described in a new system of equations and solved. A unique feature of this method is that bending of plate becomes an eigenvalue problem and vibration becomes a multiple eigenvalue problem.

Keywords: eigenvalue, eigenvector, energy, Hamiltonian, symplectic elasticity, symplecticity

1 Introduction

Symplecticity is a mathematical concept of geometry and it is an analog of "complex" in Greek first due to Weyl [1,2]. Symplectic geometry is a branch of differential geometry and differential topology. Hence, it is also called symplectic topology although, in reality, the latter is only an important subset of the former. Symplectic geometry was first developed due to its close relation to the Hamiltonian system in classical mechanics where the phase space can be constructed using symplectic manifold. The theory on symplectic geometry can be referred to Koszul and Zou [3]. The symplectic group is a mathematical group, previously known as the line complex group, and a symplectic group of dimension $2n$ is the group of $2n \times 2n$ matrices, which preserve a symplectic form where the latter can be constructed into a canonical form using symplectic basis [1,2]. Besides theoretical and classical mechanics, symplectic group has been applied in a number of fields in physics and mathematics for many years particularly in relativity and gravitation [4], quantum mechanics [5,6], and the Yang–Mills field theory [7].

2 Symplectic Numerical Analysis, Symplectic Elasticity, and Other Cross Disciplinary Areas

2.1 Symplectic Numerical Methods. Symplectic approach in applied mechanics was first developed due to the quest in accurate solutions for some nonlinear dynamical systems and stable numerical algorithm in computational mechanics constructed using Hamiltonian systems. The application of symplectic space and development of numerical methods for such systems was pioneered by Feng and his research group [8–34]. Feng first proposed symplectic algorithms based on symplectic geometry for Hamiltonian systems with finite and infinite dimensions, and on dynamical systems with Lie algebraic structures, such as contact systems and source free systems via the corresponding geometry and Lie group. These algorithms are superior to conventional algorithms in many practical applications, such as celestial mechanics and molecular dynamics. A collection of his works was published later in recognition of this contribution in applied mathematics and applied mechanics [32,33] in which Vol. II [33] are the papers related to symplectic algorithm and computation of Hamiltonian systems. The contribution of Feng in the development of symplectic numerical methods was particularly significant and important as highlighted in a memorial article dedicated to him by Lax [35].

2.2 Symplectic Elasticity in Applied Mechanics. In elasticity and applied mechanics, the development of symplectic space for analytical, engineering mechanics, and applied elasticity was attributed mainly to the research group led by Zhong [36–43]. A number of research texts have been published by this group of researchers and. In particular, Zhong et al. [36] published some early works on structural mechanics and control optimization; Zhong [37] introduced a new symplectic approach for systematically deriving analytical solution in engineering elasticity; Zhong [38–40] introduced the duality system and symplectic analytical solutions approach in applied mechanics; Zhong et al. [39,41] applied the symplectic methodology for control systems and problems; Yao et al. [42] systematically described the principle of symplectic elasticity and its application in applied engineering mechanics including Timoshenko beam, plate bending, and laminated plate analysis; and Zhong [43] recently published a very concise yet simple masterpiece of symplectic elasticity by putting forward an innovative way the seemingly mathematical and abstract concept of symplecticity and relating the common analytical engineering and/or mathematical tools such as state-space method

and matrix multiplication using very simple single-degree-of-freedom and multiple-degree-of-freedom spring-mass systems. Most of the research works of symplectic elasticity were originally pioneered by Zhong, Xu, Zhang, and Yao from Dalian University of Technology, PRC, and subsequently researched by Lim and Leung from City University of Hong Kong in collaboration with the former group of researchers.

In general, the symplectic elasticity methodology aims to develop a new analytical elasticity approach for deriving exact analytical and asymptotic solutions to some basic problems in solid mechanics and elasticity, as described in Yao et al. [42]. Some of these problems have long been bottlenecks in the development of elasticity, e.g., bending of plates with no opposite sides simply supported and bending of corner-supported plates. This symplectic elasticity approach is based on the use of the Hamiltonian principle with Legendre's transformation and subsequently analytical solutions could be obtained by expansion of eigenfunctions. It is a rational and systematic methodology with a clearly defined, step-by-step derivation procedure in which no trial functions are necessary. The advantage of this symplectic approach with respect to the classical approach, which relies on the semi-inverse method, is at least threefold. First, the symplectic approach alters the classical practice and concept of solution methodology and many basic problems previously unsolvable or too complicated to be solved can be resolved accordingly in a systematic and rational procedure. For instance, the conventional approach in plate and shell theories by Timoshenko [44] has been based on the semi-inverse method with trial 1D or 2D displacement functions, such as Navier's method [45] and Lévy's method [46] for plates. The trial functions, however, do not always exist except in some very special cases of boundary conditions such as plates with two opposite sides simply supported [47–50]. Using the symplectic approach, trial functions are not required. Second, it consolidates the many seemingly scattered and unrelated solutions of rigid body movement and elastic deformation by mapping with a series of zero and nonzero eigenvalues and their associated eigenvectors. Last but not least, the Saint–Venant problems for plane elasticity and elastic cylinders can be described in a new system of equations and solved. The difficulty of satisfying end boundary conditions in conventional problems, which could only be covered using the Saint–Venant principle can also be resolved.

2.3 Symplecticity for Elasticity. Many of the early works on symplectic elasticity were published in Chinese and as a result, the early works have been unavailable and unknown to researchers worldwide. The early works on analytic symplectic elasticity begins with Zhong and co-workers [51–57] and Yao [58] who presented the plane elasticity problems based on Hamiltonian systems [51,53], the reciprocal theorem, and adjoint symplectic orthogonality [48]; the Saint–Venant problems [54,55,58]; and symplectic eigensolutions for cylinders and body of revolution [56,57]. Recently, Zhao and Chen [59] reported some numerical studies for elasticity problems. The symplectic method with Hamiltonian state-space approach was also applied to solve analytical solutions for elastic circular solids including a circular elastic cylinder with self-weight [60], a transversely isotropic cylindrical body [61] and a circular cantilever [62].

2.4 Symplectic Methodology for Beams. Based on the foundations of symplectic methods for plane elasticity, duality in Hamiltonian systems, symplectic orthogonality, and the solutions to Saint–Venant problems, the approach is applied to applied mechanics problems for beams. The early works was reported by Leung et al. for nonlinear beam vibration and dynamics [63–65], Galerkin method [63], beam finite element [64] and frames [65], Zhong et al. [66] for curved beams, Xu et al. [67] and Ma et al. [68] for elastic structural vibration, Lü et al. [69] for bending of beams resting on elastic foundation, and for nonlinear thermal buckling of elastic beams [70,71].

2.5 Symplectic Elasticity for Plates. Due to the practical application and a quest for exact solutions, the symplectic approach for plate analysis in applied engineering mechanics has received very much attention [72–101]. Most of the research articles were based on the thin plate model [72–74,77,79–82,85–93,95–101] with a few others, which worked on thick plate problems [75,76,78,83,84,94]. Among the papers, many investigated rectangular plates [72–79,81–84,86–91,94,95,97–101], while only a few focused on circular or sector plates [80,85,92,93,96], and the majority analyzed isotropic plates [75,76,78–80,83–101], while a few worked on composite or laminated plates [72–74,77,81,82]. The papers can be further classified into a few categories according to the specific plate problems that were analyzed. Among the specific topics, plate bending [78–81,83–86,88,90–92,94,97–99,101] is the most popular subject of interest; others include vibration [56,76,87,89,95,100], buckling [93,96], wave propagation [87], and the Sain–Venant problem [74,77,80].

2.6 Symplectic Elasticity for Shells. Besides plate problems, the symplectic methods have also been applied for shell problems [102–108]. Due to the analytical complexity, only closed cylindrical shells have been considered because for such shells the circumferential deflection can be approximated with sinusoidal shape functions, while the other ends could be arbitrarily supported or free. Among these papers, the dynamic buckling problems for cylindrical shells subjected to axial impact were investigated by Xu and co-workers [102,103,105,106]. They also studied viscoelastic hollow circular cylinders [104], torsional buckling [107], and thermal buckling [108] of cylindrical shells.

2.7 Fracture Mechanics and Stress Singularity. The symplectic methodology has also been applied to solve other applied engineering mechanics problems and the most important issue has been in fracture mechanics for structural elements with cracks and/or stress singularity [109–123]. In these articles, the authors examined plane crack element [109], bimaterial crack singularity [110], wedge body problems [111–114,118,119,121], crack tip singularity [115], multimaterial crack singularity [116], application of the Dugdale model [117], and stress intensity factors [122]. Mode III crack and stress singularity problems were also investigated [120,123].

2.8 Perturbation and Symplectic Element. The development of numerical perturbation method using the symplectic approach was also reported mainly by Zhong and co-workers [124–129]. In particular, Zhong and Sun examined three different methods based on symplectic perturbation [124] and further developed symplectic conserved perturbation method [125], finite element models [126], time-space harmony element [127], and numerical integration techniques [128]. They also researched into dual variable symplectic principle [129].

2.9 Viscoelasticity. Structures with viscoelasticity have received relatively less attention. There are only three papers identified [104,130,131] where one focuses viscoelastic hollow circular cylinders [104] and the other two considered two-dimensional viscoelasticity without [130] and with thermoviscoelasticity [131].

2.10 Fluid Mechanics. Besides solid mechanics, the symplectic methodology could also be applied to fluid mechanics problems, due mainly to Xu and co-workers [132–136] particularly for Stokes flow. In specific, the eigenvalue problem and eigensolutions for Stokes flow were modeled [132], while the analytical and numerical solutions in two-dimensional domain were reported [133]. The influence of inlet radius on Stokes flow in a circular tube [134] and viscous fluid in lid-driven cavities [135] were further investigated. In nonlinear water waves driven by moving plates in shallow single- or double-layer fluid layers, dual variables associated with velocity potential and the wave elevation in a Hamiltonian system were introduced and the traveling waves solutions in elliptic cosine functions were presented [136].

2.11 Control. It is common knowledge for researcher in applied mechanics that control models bear many similarities with respect to dynamical systems in terms of governing equation, solution methodology, and numerical techniques. The very close relations and analogy with respect to symplecticity were first established by Zhong and co-workers [36,39,41] who discussed the duality systems for optimal control problems [39] and state-space control systems [41]. Recently, the time-varying optimal control problem [137] and optimal control for nonlinear dynamical systems [138] were analyzed.

2.12 Thermal Effects. The thermal effects on structures could also be solved using the symplectic method. One early work was the investigation of thermal stress for laminated plates [72]. The subjected did not receive much attention subsequently. Recently, the thermal effects were again reported for thermal buckling of beams [70,71], circular plates [93], cylindrical shells [108], and for analysis of thermal stress intensity factors [139,140].

2.13 Functionally Graded Effects. Materials that exhibit functional graded properties could also be solved using the symplectic method [141–144]. The topics of interest include statics and dynamics of plates [141], plane elasticity problems without [142] and with piezoelectric effects [143], and coupling of functionally graded and magneto-electro-elastic effects [141,144].

2.14 Piezoelectricity. The symplectic elasticity approach has been also extended to some cross-disciplinary research areas. One example is research in symplectic piezoelectricity [143,145–154] where the Hamiltonian systems are constructed to model the coupling of pressure and electricity for actuation and sensing purposes where driving force and/or electric power could be generated and vice versa. One early attempt was presented to develop the algorithms for analyzing piezoelectric materials [145]. Subsequent research works include analysis of piezoelectric two-dimensional transversely isotropic piezoelectric structures [146,147,151], cantilever beams [148,149], three-dimensional piezoelectric media [150,152], boundary layer phenomena [151], wedge body and crack singularity [121,123], composite structures such as laminated and cantilever plate [153,154], and statics and dynamics of functionally graded piezoelectric structures [143,152].

2.15 Electromagnetism and Waveguide. Electromagnetism also bears many similarities with Hamiltonian systems and hence it can be solved using the symplectic methodology. This cross-disciplinary subject was initiated by Zhong and co-workers who applied the approach and developed semi-analytical solutions to electromagnetic wave guides [155–158]. Zhong and Sun [159] later attempted to develop finite elements for electromagnetic resonant cavity.

2.16 Magneto-Electro-Elasticity. The coupling of magnetism, electricity, and elasticity was also reported [141,144,160,161]. Yao established symplectic solutions with the Saint–Venant principle for antiplane [160] and plane [161] magneto-electro-elastic solids. Lately, the statics and dynamics of functionally graded and layered magneto-electro-elastic plate/pipe [141] and plane problems for materials having coupled functionally graded and magneto-electro-elastic effects [144] were also reported.

3 Symplectic Space, Its Properties, and Correlation With Euclidean Space

All conservative real physical processes can be described by a suitable Hamiltonian system whose common mathematical fundamentals are the symplectic spaces. A symplectic space is different from a Euclidean space, which studies the metric properties such as length. It focuses on the study of area or the study of work and this is a mathematical structure present throughout the application of symplectic elasticity. The following presents a brief account of the fundamentals of symplectic space and subsequently discusses some of the special symplectic characteristics. Some proofs and more details could be referred to "Symplectic Elasticity" by Yao et al. [42].

For an n-dimensional linear space V defined in a real number field R, and a corresponding n-dimensional dual linear space V', a $2n$-dimensional phase space W in a real number field R can be defined as

$$W = V \times V' = \left\{ \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} \middle| \boldsymbol{q} \in V, \boldsymbol{p} \in V' \right\} \tag{1}$$

which is constructed by two n-dimensional vectors, $\mathbf{q}$ and $\mathbf{p}$, defined in V and V', respectively. Although there is no direct relation between these vectors and they could have absolutely different dimensions in actual problems, the product of their corresponding elements always has a specific physical meaning. For instance, one vector could represent displacement while the other stress and the product become work.

Let $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ denote the symplectic inner product for any two component vectors $\boldsymbol{x} = \{x_1, x_2, \ldots, x_{2n}\}^{\mathrm{T}}$ and $\boldsymbol{y} = \{y_1, y_2, \ldots, y_{2n}\}^{\mathrm{T}}$, then the symplectic space requires that the following four rules be satisfied absolutely:

(1) $\quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\langle \boldsymbol{y}, \boldsymbol{x} \rangle \tag{2a}$

(2) $\quad \langle k\boldsymbol{x}, \boldsymbol{y} \rangle = k\langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad k$ is an arbitrary real number $\tag{2b}$

(3) $\quad \langle \boldsymbol{x} + \boldsymbol{\gamma}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{\gamma}, \boldsymbol{y} \rangle \quad \boldsymbol{\gamma}$ is an arbitrary vector in W
$$\tag{2c}$$

(4) $\quad \boldsymbol{x} = \boldsymbol{0} \quad$ if $\quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for every vector $\boldsymbol{y}$ in $W \tag{2d}$

Any phase space that satisfies the four rules above for symplectic inner product constitutes a symplectic phase space or simply a symplectic space.

Consider a kind of symplectic inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = (\boldsymbol{x}, \boldsymbol{J}_{2n}\boldsymbol{y}) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{J}_{2n} \boldsymbol{y} \tag{3}$$

where

$$\boldsymbol{J}_{2n} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \boldsymbol{0} \end{bmatrix} \tag{4}$$

is the unit symplectic matrix of dimension $2n$, denoted briefly as $\boldsymbol{J}$, and $\boldsymbol{I}_n$ is the common unit matrix of dimension n. It is obvious that a unit symplectic matrix has the following properties:

$$|\boldsymbol{J}| = 1 \tag{5a}$$

$$\boldsymbol{J}^2 = -\boldsymbol{I} \tag{5b}$$

$$\boldsymbol{J}^{\mathrm{T}} = \boldsymbol{J}^{-1} = -\boldsymbol{J} \tag{5c}$$

where $|\boldsymbol{J}|$ indicates the determinant of $\boldsymbol{J}$. It is also obvious that the symplectic inner self-product of every vector must vanish, i.e., for every vector $\boldsymbol{x}$,

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \tag{6}$$

It is obvious that Eq. (3) satisfies the four rules of symplectic inner product defined in Eqs. (2a)–(2d), and therefore it forms a $2n$-dimensional symplectic space. For example, a two-dimensional real linear space with vectors, the symplectic inner product defined in Eq. (3) can be expressed as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_2 - x_2 y_1 \qquad (7)$$

Equation (7) forms a two-dimensional symplectic space and the symplectic inner product represents the area of a parallelogram constructed by $\boldsymbol{x}$, $\boldsymbol{y}$ as the adjacent sides.

It should be emphasized that the rules for a symplectic space defined in Eqs. (2a)–(2d) are global. There are various different definitions of symplectic inner product for a phase space and therefore there exist different symplectic spaces. The symplectic inner product defined in Eq. (3) is called the normal symplectic inner product in a $2n$-dimensional real vector space R^{2n} and this specific, normal symplectic space is always referred to throughout for all works in symplectic elasticity.

Any two component vectors $\boldsymbol{x}$, $\boldsymbol{y}$ defined in a symplectic space are symplectic orthogonal if their symplectic inner product vanishes as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0 \qquad (8)$$

Otherwise, they are symplectic adjoint. Hence, from Eq. (2d), there exists a symplectic adjoint nonzero vector for every nonzero vector, or $\boldsymbol{x}$ and $\boldsymbol{Jx}$ must be symplectic adjoint if $\boldsymbol{x} \neq \boldsymbol{0}$.

Another important property is the definition of a symplectic matrix $\boldsymbol{S}$, a $2n \times 2n$ matrix $\boldsymbol{S}$, which satisfies

$$\boldsymbol{S}^{\mathrm{T}} \boldsymbol{J} \boldsymbol{S} = \boldsymbol{J} \qquad (9)$$

A symplectic matrix has the following properties: (i) The inverse matrix of a symplectic matrix is a symplectic matrix, (ii) the transpose matrix of a symplectic matrix is a symplectic matrix, (iii) the determinant of a symplectic matrix is equal to either 1 or -1, and (iv) the product of two symplectic matrices is a symplectic matrix.

If a $2n \times 2n$ matrix $\boldsymbol{H}$ acting on arbitrary $2n$-dimensional vectors $\boldsymbol{x}, \boldsymbol{y}$ satisfies

$$\langle \boldsymbol{x}, \boldsymbol{H}\boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{H}\boldsymbol{x} \rangle \qquad (10)$$

then matrix $\boldsymbol{H}$ is a Hamiltonian matrix. It is obvious that a Hamiltonian matrix satisfies

$$(\boldsymbol{JH})^{\mathrm{T}} = \boldsymbol{JH} \qquad (11a)$$

$$\boldsymbol{JHJ} = \boldsymbol{H}^{\mathrm{T}} \qquad (11b)$$

There are many other important properties, definitions, and theorems in the symplectic space, such as the adjoint symplectic orthonormal vector set and normal adjoint symplectic orthonormal vector set. Some theorems without proofs and definitions include the facts that (i) an adjoint symplectic orthonormal vector set is a linearly independent vector set. (ii) Every adjoint symplectic orthonormal vector set in a $2n$-dimensional symplectic space can be extended to an adjoint symplectic orthonormal basis. (iii) Every normal adjoint symplectic orthonormal vector set in a $2n$-dimensional symplectic space can be extended to a normal adjoint symplectic orthonormal basis. (iv) The transformation matrix for normal adjoint symplectic orthonormal bases is a symplectic matrix. (v) If μ is an eigenvalue of a Hamiltonian matrix with multiplicity m, then $-\mu$ is also an eigenvalue with multiplicity m. If zero is an eigenvalue of a Hamiltonian matrix $\boldsymbol{H}$, then the multiplicity number is even. More details are available in Ref. [42].

One useful conclusion from the theorems and properties above is as follows. There exists an adjoint symplectic orthonormal basis composed of the basic eigenvectors and Jordan form eigenvectors of the Hamiltonian matrix $\boldsymbol{H}$ in a $2n$-dimensional symplectic space. Through normalization, a normal adjoint symplectic orthonormal basis can be formed. The matrix formed by the column vectors is indeed a symplectic matrix.

The readers are referred to Table 1.1 in Yao et al. [42] for a correlation between a Euclidean space and a symplectic space and it describes a clear summary of the relevant concepts and properties.

4 Some Applications of the Symplectic Methodology in Applied Mechanics

A few examples using the symplectic elasticity methodology is presented in this section to illustrate the application of the method for exact solutions to some common engineering mechanics problems. Contrary to the semi-inverse approach [44–50] where bending of beam is related to solving a nonhomogeneous set of equations while vibration of beam is an eigenvalue problem, from the following examples, it is obvious that bending of beam becomes an eigenvalue problem with the eigenvalue indicating the spatial wave frequency, while the vibration or wave propagation problem becomes a multiple eigenvalue problem with eigenvalues indicating the spatial and temporal frequencies.

4.1 Timoshenko Beam Theory. The Hamiltonian system for a Timoshenko beam is established first and subsequently the analytical solutions for bending and wave propagation are presented [42].

For bending of a Timoshenko beam of length L, let the x-axis be along the beam midaxis before deformation and the xz-plane be the deflection plane. The beam deflection at a point along the x-axis is denoted as $\widetilde{w}(x)$ and $\widetilde{\theta}(x)$ is the rotation of the cross section.

To construct an analytical system, first, the x-coordinate is modeled as the time coordinate of the Lagrangian system and the Hamiltonian system. Furthermore, denote

$$\boldsymbol{q} = \{w, \theta\}^{\mathrm{T}}, \quad \dot{\boldsymbol{q}} = \{\dot{w}, \dot{\theta}\}^{\mathrm{T}} \qquad (12)$$

where an overdot indicates differentiation with respect to the x-coordinate, i.e., $(\dot{\;}) = d/dx$. Application of the variational principle yields the Lagrange equation as

$$\frac{d}{dx}\left(\frac{\partial \mathrm{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathrm{L}}{\partial \boldsymbol{q}} = \boldsymbol{0} \qquad (13)$$

where $\mathrm{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrange density function and the temporal coordinate t is replaced by the spatial coordinate x.

A state vector is defined as

$$\boldsymbol{v} = \begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix} \qquad (14)$$

where the dual variable of $\boldsymbol{q}$ according to Legendre's transformation is

$$\boldsymbol{p} = \frac{\partial \mathrm{L}}{\partial \dot{\boldsymbol{q}}} \qquad (15)$$

Then the Hamiltonian density function can be obtained as

$$\mathrm{H}(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{p}^{\mathrm{T}} \dot{\boldsymbol{q}} - \mathrm{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \qquad (16)$$

The Hamiltonian dual system can be expressed as

$$\dot{\boldsymbol{q}} = \frac{\partial \mathrm{H}}{\partial \boldsymbol{p}} \qquad (17a)$$

$$\dot{\boldsymbol{p}} = \frac{\partial \mathrm{H}}{\partial \boldsymbol{q}} \qquad (17b)$$

which can also be expressed as

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} + \boldsymbol{h} \qquad (18)$$

where $\boldsymbol{H}$ is the Hamiltonian matrix and $\boldsymbol{h} = \{h_q, h_p\}^T$ is a $2n$-vector and $\boldsymbol{h}_q, \boldsymbol{h}_p$ are two n-vectors, which can be obtained from Eqs. (14)–(16), (17a), and (17b), and they are related to external loadings. The Hamiltonian density function, also known as the mixed energy density, can be expressed in terms of the Hamiltonian matrix [42].

The systems expressed in Eqs. (17a), (17b), and (18) are sys-

tems of first-order equations in which the spatial differential of the state vector is only present on the left-hand-side. Hence, the original system of n second-order differential equations (13) can be transformed into a system of $2n$ first-order differential equations (18) and the transformation from Lagrange system to Hamiltonian system is completed.

The physical meaning of the dual variable p can be interpreted as follows. For a Timoshenko beam, it is easily derived that

$$p = \{kGA(\dot{w} - \theta), \quad EI\dot{\theta}\}^{\mathrm{T}} = \{F_s, \quad M\}^{\mathrm{T}} \quad (19)$$

where k, E, G, A, I, F_s, and M are the shear correction factor, Young's modulus, shear modulus, cross-sectional area, second moment of area, shear force, and bending moment, respectively. Hence, the dual variable p can be interpreted as a the generalized internal force vector and the state vector becomes

$$v = \{w, \quad \theta; \quad F_s, \quad M\}^{\mathrm{T}} \quad (20)$$

4.1.1 Bending of Timoshenko Beam. For static bending of a Timoshenko beam [42], the Hamiltonian matrix is

$$H = \begin{bmatrix} 0 & 1 & (kGA)^{-1} & 0 \\ 0 & 0 & 0 & (EI)^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (21)$$

The eigenvalue μ and eigenvector ψ for H can be obtained from

$$H\psi = -\mu\psi \quad (22)$$

For nontrivial solution, the eigenvalue $\mu = 0$ has quadruple multiplicity. The corresponding eigenvectors are

$$v_0^{(0)} = \psi_0^{(0)} = \{1, \quad 0, \quad 0, \quad 0\}^{\mathrm{T}} \quad (23a)$$

which represents beam rigid body translation

$$v_0^{(1)} = \psi_0^{(1)} + x\psi_0^{(0)} = \{x, \quad 1, \quad 0, \quad 0\}^{\mathrm{T}} \quad (23b)$$

which represents beam rigid body rotation

$$v_0^{(2)} = \psi_0^{(2)} + x\psi_0^{(1)} + \tfrac{1}{2}x^2\psi_0^{(0)} = \left\{\tfrac{1}{2}x^2 + c, \quad x, \quad 0, \quad EI\right\}^{\mathrm{T}} \quad (23c)$$

which represents pure bending of beam, where c is an arbitrary constant, and

$$v_0^{(3)} = \psi_0^{(3)} + x\psi_0^{(2)} + \frac{1}{2}x^2\psi_0^{(1)} + \frac{1}{6}x^3\psi_0^{(0)}$$
$$= \left\{\frac{1}{6}x^3 + cx, \quad \frac{1}{2}x^2 + \frac{EI}{kGA} + c; \quad -EI, \quad EIx\right\}^{\mathrm{T}} \quad (23d)$$

which represents constant shear force bending of beam. The basic eigenvector pairs $\psi_0^{(0)}, \psi_0^{(1)}$ and $\psi_0^{(0)}, \psi_0^{(2)}$ are symplectic orthogonal, while $\psi_0^{(0)}, \psi_0^{(3)}$ are symplectic adjoint, i.e., $\langle\psi_0^{(0)}, \psi_0^{(1)}\rangle = 0$, $\langle\psi_0^{(2)}, \psi_0^{(0)}\rangle = 0$, and $\langle\psi_0^{(3)}, \psi_0^{(0)}\rangle = EI \neq 0$. It is also easy to prove that $\langle\psi_0^{(1)}, \psi_0^{(2)}\rangle = EI \neq 0$ and by assigning $c = -EI/2\kappa GA$, $\langle\psi_0^{(2)}, \psi_0^{(3)}\rangle = 0$. Hence, $\psi_0^{(0)}, \psi_0^{(1)}, \psi_0^{(2)}$, and $\psi_0^{(3)}$ form a set of adjoint symplectic orthonormal basis. Consequently, all basic eigenvectors and Jordan form eigenvectors for static bending of Timoshenko beam are established.

In the presence of external loading and/or external moment, $h \neq 0$ in Eq. (18) and a particular solution has to be determined. This particular solution depends cannot be explicitly derived without the type of loadings being known. This will be left as an exercise to the readers.

4.1.2 Wave Propagation for Timoshenko Beam. For wave propagation in a Timoshenko, the following solutions are stated without derivation. The readers are referred to Zhong [37] and Yao et al. [42] for more details.
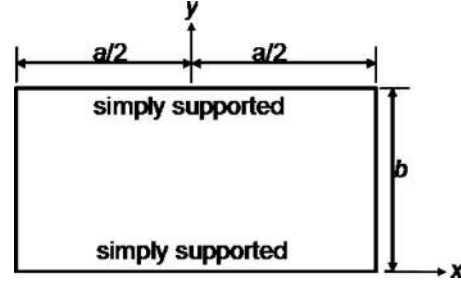
In this case, the Hamiltonian matrix H is given by

Fig. 1 Plan of a thin plate with support conditions

$$H = \begin{bmatrix} 0 & 1 & (kGA)^{-1} & 0 \\ 0 & 0 & 0 & (EI)^{-1} \\ -\rho\omega^2 A & 0 & 0 & 0 \\ 0 & -\rho\omega^2 I & -1 & 0 \end{bmatrix} \quad (24)$$

where ρ and ω are the material density and the wave frequency, respectively. The eigenvalue μ must satisfy that μ^2 be real. There are three different cases for μ^2, which can be classified according to a critical frequency as

$$\omega_{\mathrm{cr}}^2 = \frac{\kappa GA}{\rho I} \quad (25)$$

(i) For $\omega^2 > \omega_{\mathrm{cr}}^2$, there are two negative roots for μ^2. The solution in a Timoshenko beam indicates propagation of two pairs of waves with velocities ω/μ_1 and ω/μ_2, and traveling along $-x$ and $+x$, respectively.

(ii) For $\omega^2 < \omega_{\mathrm{cr}}^2$, there are a positive root and a negative root for μ^2. The negative root indicates propagation of a pair of waves along the positive and negative directions of x, similar to case (i) above The positive root indicates local vibration capable of creating resonance.

(iii) For $\omega^2 = \omega_{\mathrm{cr}}^2$, there are a negative root and a zero root for μ^2. The zero root indicates that there are two zero roots for μ and there will be Jordan form solutions. The wave propagation solutions for the roots are given in Ref. [42].

4.2 Bending of Rectangular Plates. Consider an isotropic rectangular Kirchhoff plate with uniform thickness h, length a, width b, Young's modulus E, and Poisson's ratio ν. Here only thin plates having at least a pair of opposite sides simply supported are considered. The plate, bounded within a domain $-a/2 \leq x \leq a/2$ and $0 \leq y \leq b$ where sides $y = 0, b$ are always simply supported, as shown in Fig. 1, is subjected to uniformly distributed load q. There are totally six cases with different boundary conditions. The analytical bending solutions are presented as follows and the details can be referred to Ref. [90]. Here C, S, and F denote the clamped support, the simple support, and the free support, respectively.

In this case, the Hamiltonian matrix is

$$H = \begin{bmatrix} 0 & \nu\dfrac{\partial}{\partial y} & D(1-\nu^2) & 0 \\ -\dfrac{\partial}{\partial y} & 0 & 0 & 2D(1-\nu) \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial y} \\ 0 & -\dfrac{1}{D}\dfrac{\partial^2}{\partial y^2} & \nu\dfrac{\partial}{\partial y} & 0 \end{bmatrix} \quad (26)$$

and the state vector is

$$\boldsymbol{v} = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}} \qquad (27)$$

where ν and D are the Poisson ratio and the flexural rigidity of

plate, and ϕ_x and ϕ_y are the bending moment functions, and κ_y and κ_{xy} are the curvature and twisting curvature, respectively.

For a SSSS plate, the xy-midplane deflection $w(x, y)$ solution can be expressed as [90]

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) + \frac{2q}{Db}\sum_{n=1}^{\infty}\frac{[\mu_n x \sinh(\mu_n x) - \cosh(\mu_n x)(2 + \alpha_n \tanh \alpha_n)]\sin(\mu_n y)}{\mu_n^5 \cosh \alpha_n} \qquad (28)$$

For a SFSF plate, simply supported at $y=0, b$ and free at $x = \pm a/2$, the deflection $w(x, y)$ solution is [90]

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) + \frac{4q}{(1-\nu)bD}\sum_{n=1}^{\infty}\frac{\{\nu \cosh(\mu_n x)[\alpha_n(1-\nu)\cosh \alpha_n - (1+\nu)\sinh \alpha_n] - \mu_n x(1-\nu)\nu \sinh \alpha_n \sinh(\mu_n x)\}\sin(\mu_n y)}{\mu_n^5[\alpha_n(1-\nu) - (3+\nu)\cosh \alpha_n \sinh \alpha_n]}$$

$$(29)$$

For a SCSC plate, simply supported at $y=0, b$, and clamped at $x = \pm a/2$, the deflection $w(x, y)$ solution is [90]

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) + \frac{4q}{bD}\sum_{n=1}^{\infty}\{\{e^{4\alpha_n}\sin(\mu_n y)[-2(3\alpha_n$$

$$+ \mu_n x)\cosh(\alpha_n - \mu_n x) + (\alpha_n + \mu_n x)\cosh(3\alpha_n - \mu_n x)$$

$$- 5\alpha_n \cosh(\alpha_n + \mu_n x) + 3\mu_n x \cosh(\alpha_n + \mu_n x) + 2\alpha_n \cosh(3\alpha_n$$

$$+ \mu_n x) - 2\mu_n x \cosh(3\alpha_n + \mu_n x)$$

$$+ 8 \cosh(2\alpha_n)\cosh(\mu_n x)\sinh \alpha_n - 4\alpha_n(\alpha_n - \mu_n x)\sinh(\alpha_n$$

$$+ \mu_n x)]\}/[\mu_n^5(1 + 8\alpha_n e^{4\alpha_n} - e^{8\alpha_n})]\} \qquad (30)$$

For a SFSC plate, simply supported at $y=0, b$, free at $x=-a/2$, and clamped at $x=a/2$, the deflection $w(x, y)$ solution is [90]

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) + \frac{8q}{bD}\sum_{n=1}^{\infty}\{\{e^{4\alpha_n}\sin(\mu_n y)\{[5 + 4\alpha_n^2(\nu$$

$$-1)^2 + \nu(2+\nu)]\cosh(\alpha_n - \mu_n x) + 3\cosh(3\alpha_n + \mu_n x)$$

$$+ \nu \cosh(\mu_n x)\{[1 + 4\alpha_n^2(\nu - 1) + \nu]\cosh \alpha_n + (-3 + \alpha_n$$

$$- 2\nu)\cosh(3\alpha_n) + \mu_n x[-5 \sinh \alpha_n - (\nu - 1)\sinh(3\alpha_n)]\} + \nu\{$$

$$- \mu_n x(1 + 2\nu)\cosh \alpha_n + \mu_n x(\nu - 1)\cosh(3\alpha_n) + [1 + 4\alpha_n^2(\nu$$

$$- 1) + \nu]\sinh \alpha_n + (\alpha_n - 1)\sinh(3\alpha_n)\}\sinh(\mu_n x)$$

$$- \alpha_n\{\cosh(\mu_n x)\{4\mu_n x(\nu - 1)\cosh \alpha_n - 3 \sinh \alpha_n$$

$$+ \nu\{\cosh(3\alpha_n) + [5 - 4\nu + 4\nu \cosh(2\alpha_n)]\sinh \alpha_n\}\} + [(3$$

$$- 7\nu)\cosh \alpha_n + 2\nu \cosh(3\alpha_n) + 4\mu_n x(\nu - 1)(2\nu - 1)\sinh \alpha_n$$

$$+ \nu \sinh(3\alpha_n)]\sinh(\mu_n x)\} + \mu_n x \sinh(\alpha_n - \mu_n x) - [\alpha_n(\nu - 3)$$

$$- \mu_n x(\nu - 1)(\nu + 3)]\sinh(3\alpha_n + \mu_n x)\}\}/\{\mu_n^5\{(\nu - 1)(\nu + 3)$$

$$+ e^{8\alpha_n}(\nu - 1)(\nu + 3) - 2e^{4\alpha_n}[5 + 8\alpha_n^2(\nu - 1)^2 + \nu(2+\nu)]\}\}\} \qquad (31)$$

For a SSSF plate, simply supported at $y=0, b$ and $x=-a/2$, and free at $x=a/2$, the deflection $w(x, y)$ solution is [90]

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) + \frac{2q}{bD}\sum_{n=1}^{\infty}\{\{e^{\alpha_n - \mu_n x}\{- 2e^{2(\alpha_n + \mu_n x)}[-1$$

$$+ 3\alpha_n(\nu - 1) + \mu_n x(\nu - 1) - \nu]\nu + e^{2\mu_n x}(-2 + \alpha_n + \mu_n x)(\nu - 1)$$

$$\times (3 + \nu) - 2e^{4\alpha_n}\nu[1 + 3\alpha_n(\nu - 1) + \mu_n x(\nu - 1) + \nu] + 2\nu[-1$$

$$- \alpha_n + \mu_n x + (-1 + \alpha_n - \mu_n x)\nu] + e^{2\alpha_n}[6 + 4\alpha_n^2(\nu - 1)^2$$

$$+ 3\mu_n x(\nu - 1)^2 - \alpha_n(5 + 4\mu_n x)(\nu - 1)^2 + 2\nu^2] + 2e^{6\alpha_n + 2\mu_n x}\nu[1$$

$$+ \alpha_n(\nu - 1) + \nu - \mu_n x(\nu - 1)] + e^{4\alpha_n + 2\mu_n x}[-4\alpha_n^2(\nu - 1)^2$$

$$+ 3\mu_n x(\nu - 1)^2 + \alpha_n(-5 + 4\mu_n x)(\nu - 1)^2 - 2(3$$

$$+ \nu)^2]\}\sin(\mu_n y)\}/\{\mu_n^5(\nu - 1)[8\alpha_n e^{4\alpha_n}(\nu - 1) + (1 - e^{8\alpha_n})(3$$

$$+ \nu)]\}\} \qquad (32)$$

For a SSSF plate, simply supported at $y=0, b$ and $x=-a/2$, and clamped at $x=a/2$, the deflection $w(x, y)$ solution is [90]

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) + \frac{4q}{bD}\sum_{n=1}^{\infty}\{\{e^{4\alpha_n}\sin(\mu_n y)\{- 2(3\alpha_n$$

$$+ \mu_n x)\cosh(\alpha_n - \mu_n x) + (\alpha_n + \mu_n x)\cosh(3\alpha_n - \mu_n x)$$

$$- 5\alpha_n \cosh(\alpha_n + \mu_n x) + 3\mu_n x \cosh(\alpha_n + \mu_n x) + 2\alpha_n \cosh(3\alpha_n$$

$$+ \mu_n x) - 2\mu_n x \cosh(3\alpha_n + \mu_n x)$$

$$+ 8 \cosh(2\alpha_n)\cosh(\mu_n x)\sinh(\alpha_n) - 4\alpha_n(\alpha_n - \mu_n x)\sinh(\alpha_n$$

$$+ \mu_n x)\}\}/[(1 + 8e^{4\alpha_n} - e^{8\alpha_n})\mu_n^5]\} \qquad (33)$$

4.3 Bending of Corner-Supported Rectangular Plates.
Consider an isotropic rectangular Kirchhoff plate with uniform thickness h, length $2a$, width $2b$, Young's modulus E, and Poisson's ratio ν. The Cartesian coordinate system is established with the origin at the center of plate such that $-a \leq x \leq a$ and $-b \leq y \leq b$, as illustrated in Fig. 2. The analytical solution is presented for bending of such a plate supported only at its four corners and subjected to uniformly distributed load q. The details can be referred to Lim et al. [91] and Lim and Yao [99]. It should be
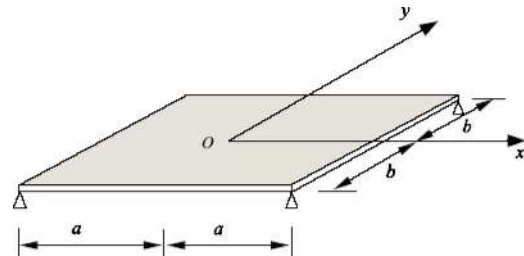
Fig. 2 Geometry and corner support conditions of a rectangular plate

highlighted here that all geometric and natural boundary conditions are satisfied and the analytical solutions can be considered exact to a very large extent except some stages toward the end of the analysis where numerical solutions have to be sought. In addition, the twisting moment at the corners derived from static bending equilibrium are also satisfied.

For such a corner-supported plate, the deflection $w(x,y)$ solution can be expressed as [91]

$$
\begin{aligned}
w = &\frac{q}{24D(-1+\nu)(1+\nu)^2}\{-6a^2(1+\nu)(-x^2+\nu y^2)+2b^2\nu[-(1\\
&+5\nu)x^2-(-3+\nu)(2+\nu)y^2]+(1+\nu)[-x^4+6\nu x^2 y^2+(-2\\
&+\nu)\nu y^4]\}+c_1\\
&+\sum_{n=1}^{\infty}\left\{f_n e^{\mu_n x}\left(\frac{[1+\nu-(3+\nu)\cos^2(\mu_n b)]\cos(\mu_n y)}{\mu_n D(1-\nu)^2}\right.\right.\\
&\left.-\frac{y\sin(\mu_n y)}{D(1-\nu)}\right)+\bar{f}_n e^{\bar{\mu}_n x}\left(\frac{[1+\nu-(3+\nu)\cos^2(\bar{\mu}_n b)]\cos(\bar{\mu}_n y)}{\bar{\mu}_n D(1-\nu)^2}\right.\\
&\left.-\frac{y\sin(\bar{\mu}_n y)}{D(1-\nu)}\right)\\
&+f_{-n}e^{\mu_{-n}x}\left(\frac{[1+\nu-(3+\nu)\cos^2(\mu_{-n}b)]\cos(\mu_{-n}y)}{\mu_{-n}D(1-\nu)^2}\right.\\
&\left.-\frac{y\sin(\mu_{-n}y)}{D(1-\nu)}\right)\\
&+\bar{f}_{-n}e^{\bar{\mu}_{-n}x}\left(\frac{[1+\nu-(3+\nu)\cos^2(\bar{\mu}_{-n}b)]\cos(\bar{\mu}_{-n}y)}{\bar{\mu}_{-n}D(1-\nu)^2}\right.\\
&\left.\left.-\frac{y\sin(\bar{\mu}_{-n}y)}{D(1-\nu)}\right)\right\}
\end{aligned}
\tag{34}
$$

4.4 Vibration of a Rectangular Plates.
For vibration of rectangular plate, the geometry, as illustrated in Fig. 2, is adopted except that the supports are not at the corners but they are along the boundaries with various conditions to be considered. At least one-pair of the sides at $x=\pm a$ are simply supported, while the other sides $y=\pm b$ could be free, simply supported, or clamped. Again only the analytical frequency solutions are presented and the details can be referred to Refs. [89,100] and Ref. [95].

From the property of symplectic adjoint orthogonality of eigenvectors and expansion of eigenvectors, the state vector for free vibration can be expanded as

$$
\begin{aligned}
\mathbf{v} = &\sum_{n=1}^{\infty}\left[f_1(n)e^{\mu_n x}\boldsymbol{\psi}_n(y)+f_2(n)e^{-\mu_n x}\boldsymbol{\psi}_{-n}(y)+f_3(n)e^{\mu'_n x}\boldsymbol{\psi}'_n(y)\right.\\
&\left.+f_4(n)e^{-\mu'_n x}\boldsymbol{\psi}'_{-n}(y)\right]
\end{aligned}
\tag{35}
$$

where $f_i(n), i=1,2,3,4$ are unknown functions depending on the boundary conditions at $x=\pm a$ and

$$
\begin{aligned}
\mu_n = &\pm\sqrt{\omega_n\sqrt{\frac{\rho h}{D}}+\frac{n^2\pi^2}{4b^2}},\quad\lambda_3\\
= &\sqrt{-2\omega_n\sqrt{\frac{\rho h}{D}}-\frac{n^2\pi^2}{4b^2}},\quad\left(\lambda_1=\pm\mathrm{i}\frac{n\pi}{2b}\right)
\end{aligned}
\tag{36a}
$$

$$
\begin{aligned}
\mu'_n = &\pm\sqrt{-\omega_n\sqrt{\frac{\rho h}{D}}+\frac{n^2\pi^2}{4b^2}},\quad\lambda_1\\
= &\sqrt{2\omega_n\sqrt{\frac{\rho h}{D}}-\frac{n^2\pi^2}{4b^2}},\quad\left(\lambda_3=\pm\mathrm{i}\frac{n\pi}{2b}\right)
\end{aligned}
\tag{36b}
$$

The circular frequencies ω_n are the only unknowns in the state vector in Eq. (35). Satisfaction of the boundary conditions at $x=\pm a$ will lead to the frequency equation.

For a SSSS plate, combining Eq. (35) with the boundary conditions yields a frequency relation

$$
\sinh\mu_n a\cosh\mu_n a\sinh\mu'_n a\cosh\mu'_n a=0
\tag{37}
$$

which can be subsequently solved for the free vibration frequency ω_n as

$$
\omega_n=\left[\frac{m^2\pi^2}{(2a)^2}+\frac{n^2\pi^2}{(2b)^2}\right]\sqrt{\frac{D}{\rho h}},\quad(m,n=1,2,\ldots)
\tag{38}
$$

which is identical to the well-known solution in any text on thin plate vibration [48,50].

For plates with other boundary conditions at $x=\pm a$, similar procedure as described above is followed. Combining Eqs. (35), (36a), and (36b) with the relevant boundary conditions yield the following frequency relations:

- CSCS

$$
\begin{aligned}
(\mu_n^2+&\mu_n'^2)\sinh(2\mu_n a)\sinh(2\mu'_n a)\\
&=2\mu_n\mu'_n[\cosh(2\mu_n a)\cosh(2\mu'_n a)-1]
\end{aligned}
\tag{39}
$$

- SSCS

$$
\mu_n\cosh(2\mu_n a)\sinh(2\mu'_n a)-\mu'_n\sinh(2\mu_n a)\cosh(2\mu'_n a)=0
\tag{40}
$$

- CSFS

$$
\begin{aligned}
2\mu_n\mu'_n[k^4-&(1-\nu)^2]+2\mu_n\mu'_n[k^4+(1\\
&-\nu)^2]\cosh(2\mu_n a)\cosh(2\mu'_n a)+(\mu_n^2+\mu_n'^2)[(1-2\nu)k^4\\
&-(1-\nu)^2]\sinh(2\mu_n a)\sinh(2\mu'_n a)=0
\end{aligned}
\tag{41}
$$

- SSFS

$$
\begin{aligned}
\mu_n[k^2-(1-\nu)]^2\cosh(2\mu_n a)\sinh(2\mu'_n a)=&\mu'_n[k^2+(1\\
&-\nu)]^2\sinh(2\mu_n a)\cosh(2\mu'_n a)
\end{aligned}
\tag{42}
$$

- FSFS

$$
\begin{aligned}
\{\mu_n^2[k^2-(1-\nu)]^4+&\mu_n'^2[k^2+(1\\
&-\nu)]^4\}\sinh(2\mu_n a)\sinh(2\mu'_n a)=2\mu_n\mu'_n[k^4-(1\\
&-\nu)^2]^2[\cosh(2\mu_n a)\cosh(2\mu'_n a)-1]
\end{aligned}
\tag{43}
$$

where $k^2=\alpha^2/l^2$, $\alpha^2=\omega_n\sqrt{\rho h/D}$, and $l^2=n^2\pi^2/4b^2$.

These frequency relations have to be solved numerical for the frequency solutions. Unlike the classical semi-inverse method [44–50], the frequency relations above are derived analytically and rigorously step-by-step in a rational manner without introducing any trial functions. Furthermore, unlike the semi-inverse trial functions, which only satisfy the geometric boundary conditions, these exact relations satisfy all natural and geometric boundary conditions at the outset. However, no closed form solutions for the transcendental frequency relation for ω_n in Eqs. (39)–(43) could be obtained. The numerical solutions are available in Ref. [95].

5 Conclusions

This paper presents a comprehensive review of the previous works on the theory and application of the symplectic methodol-

ogy in applied mechanics and engineering. It begins with a description of Feng's works on numerical symplectic algorithm in the middle 1980s, a discussion on Zhong's works on analytic symplectic elasticity since the beginning of the 1900s, and the many other works using the approach for closed form or nearly closed form solutions to elasticity problems for which exact or closed formed solutions have been impossible based on the semi-inverse method of Navier, Lévy, and Timoshenko [44–50]. A unique feature of this method is that bending of plate becomes an eigenvalue problem and vibration becomes a multiple eigenvalue problem. This paper also presents an introductory background and fundamentals of symplectic space and Hamiltonian dual system. This paper finally ends with a presentation of some analytical solutions to some beam and plate problems for bending and vibration for which only numerical solution are considered possible prior to symplectic methodology.

Acknowledgment

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