SYMPLECTIC EMBEDDINGS OF ELLIPSOIDS

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ABSTRACT. We study the rigidity and flexibility of symplectic embeddings in the model case in which the domain is a symplectic ellipsoid. It is first proved that under the condition $r_n^2 \leq 2r_1^2$ the symplectic ellipsoid $E(r_1, \dots, r_n)$ with radii $r_1 \leq \dots \leq r_n$ does not symplectically embed into a ball of radius strictly smaller than r_n. We then use symplectic folding to see that this condition is sharp. We finally sketch a proof of the fact that any connected symplectic 4-manifold of finite volume can be asymptotically filled with skinny ellipoids.

INTRODUCTION AND RESULTS

Consider a connected smooth n-dimensional manifold M. A volume form on M is a smooth nowhere vanishing n-form Ω. It follows that M is orientable. We orient M such that $\int_M \Omega$ is positive, and we write $\text{Vol}(M, \Omega) = \int_M \Omega$. We endow each open (not necessarily connected) subset U of $\mathbb{R}^n$ with the Euclidean volume form

$$\Omega_0 = dx_1 \wedge \dots \wedge dx_n.$$

A smooth embedding $\varphi \colon U \hookrightarrow M$ is called volume preserving if

$$\varphi^* \Omega = \Omega_0.$$

Then $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$. The following proposition shows that this obvious condition for the existence of a volume preserving embedding is the only one.

Proposition 1 *The set U embeds into M by a smooth volume preserving embedding if and only if* $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$.

A proof of this result can be found in Appendix A of [11].

A symplectic manifold (M, ω) is a smooth manifold M endowed with a smooth non-degenerate closed 2-form ω. The non-degeneracy of ω implies that $\frac{1}{n!}\omega^n$ is a volume form, and that M is even dimensional,

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Date: May 22, 2003.
2000 *Mathematics Subject Classification.* Primary 53D35, Secondary 57R40.

$\dim M = 2n$. We endow each open subset U of $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

A smooth embedding $\varphi \colon U \hookrightarrow M$ is called symplectic if

$$\varphi^* \omega = \omega_0.$$

In particular, every symplectic embedding preserves the volume forms $\Omega_0 = \frac{1}{n!}\omega_0^n$ and $\frac{1}{n!}\omega^n$ induced by the symplectic forms. Given an open subset U of $\mathbb{R}^{2n}$ and $\lambda > 0$ we set $\lambda U = \{\lambda z \in \mathbb{R}^{2n} \mid z \in U\}$. In the symplectic world, the question behind Proposition 1 becomes

Problem 1 What is the largest number λ such that $(\lambda U, \omega_0)$ symplectically embeds into (M, ω)?

In dimension 2, an embedding is volume preserving if and only if it is symplectic, and so Problem 1 is completely solved by Proposition 1. In higher dimensions, however, strong symplectic rigidity phenomena appear. Denote the open $2n$-dimensional ball of radius r by $B^{2n}(\pi r^2)$ and the open $2n$-dimensional symplectic cylinder $B^2(a) \times \mathbb{R}^{2n-2}$ by

$$Z^{2n}(a) = B^2(a) \times \mathbb{R}^{2n-2}.$$

Examples 1. (Gromov's Nonsqueezing Theorem [4]) For $n \geq 2$, the ball $B^{2n}(a)$ symplectically embeds into the cylinder $Z^{2n}(A)$ only if $A \geq a$.

2. [6] For $n \geq 2$, there exist bounded starshaped domains $U \subset \mathbb{R}^{2n}$ which have arbitrarily small volume but do not symplectically embed into $B^{2n}(\pi)$.

On the other hand, the following two results suggest that the situation in Problem 1 becomes less rigid if U is "thin". We denote by

$$E(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^n \frac{\pi |z_i|^2}{a_i} < 1 \right\}$$

the open symplectic ellipsoid in $\mathbb{R}^{2n}$ with radii $\sqrt{a_i/\pi}$.

Examples 3. ([5, p. 335] and [3, p. 579]) Consider a $2n$-dimensional symplectic manifold (M, ω). For any $a > 0$ there exists a (possibly very small!) $\epsilon > 0$ such that the ellipsoid $E^{2n}(\epsilon, \dots, \epsilon, a)$ symplectically embeds into M.

4. (Traynor, [12, Theorem 6.4]) For every integer $k \geq 1$ and every $\epsilon > 0$ there exists a symplectic embedding

$$E\left(\frac{\pi}{k+1}, k\pi\right) \hookrightarrow B^4(\pi+\epsilon).$$

Examples 2 and 4 show that already the following special case of Problem 1 is interesting:

Problem 2 What is the smallest ball $B^{2n}(A)$ into which U symplectically embeds?

In this work we investigate the zone of transition between rigidity and flexibility in Problems 1 and 2. The main tool of detecting embedding obstructions will be special symplectic invariants, the so called symplectic capacities (see [7] and Section 1). Unfortunately, symplectic capacities can be computed only for very special sets. Therefore, we look at a model situation in which the set U is a symplectic ellipsoid $E(a_1, \dots, a_n)$. Since a permutation of the symplectic coordinate planes is a (linear) symplectic map, we may assume $a_1 \leq a_2 \leq \dots \leq a_n$. We first discuss our answers to Problem 2. Of course, the inclusion symplectically embeds $E(a_1, \dots, a_n)$ into $B^{2n}(A)$ if $A \geq a_n$. The following rigidity result shows that one cannot do better if $a_n \leq 2a_1$.

Theorem 1 *Assume $a_n \leq 2a_1$. Then there does not exist a smooth symplectic embedding of the ellipsoid $E(a_1, \dots, a_n)$ into the ball $B^{2n}(A)$ if $A < a_n$.*

Our proof uses the n'th Ekeland–Hofer capacity. In the special case $n = 2$, Theorem 1 was proved in [3] as an application of symplectic homology. The argument given here is much simpler and works in all dimensions.

Our first embedding result shows that Theorem 1 is sharp.

Theorem 2 *Assume $a_n > 2a_1$. Then there exists a smooth symplectic embedding of the ellipsoid $E(a_1, \dots, a_1, a_n)$ into the ball $B^{2n}(a_n - \delta)$ for every $\delta \in \left]0, \frac{a_n}{2} - a_1\right[$.*

The reader might ask why we assume $a_{n-1} = a_1$ in Theorem 2. This is because a much better result cannot be expected. Indeed, we will show using Ekeland–Hofer capacities that for $n \geq 3$ the ellipsoid $E^{2n}(a, 3a, \dots, 3a)$ does not symplectically embed into the ball $B^{2n}(A)$ if $A < 3a$. Ekeland–Hofer capacities also imply that $E^6(a, 2a, 3a)$ does not symplectically embed into $B^6(A)$ if $A < 2a$.

Question 1 *Does the ellipsoid $E^6(a,2a,3a)$ symplectically embed into $B^6(A)$ for some $A<3a$?*

In the special case $n=2$, Lalonde and Mc Duff observed in [8] that Theorem 2 can be proved by their technique of symplectic folding. A refinement of their method will prove Theorem 2 in all dimensions.

Theorem 2 can be substantially improved by folding more than once. For the sake of clarity we restrict ourselves to dimension 4. We can assume that $a_1=\pi$. The optimal function for the embedding problem $E(\pi,a)\hookrightarrow B^4(A)$ is the function $f(a)$ on $[\pi,\infty[$ defined by

$$f(a) \;=\; \inf\left\{A \,|\, E(\pi,a) \text{ symplectically embeds into } B^4(A)\right\}.$$

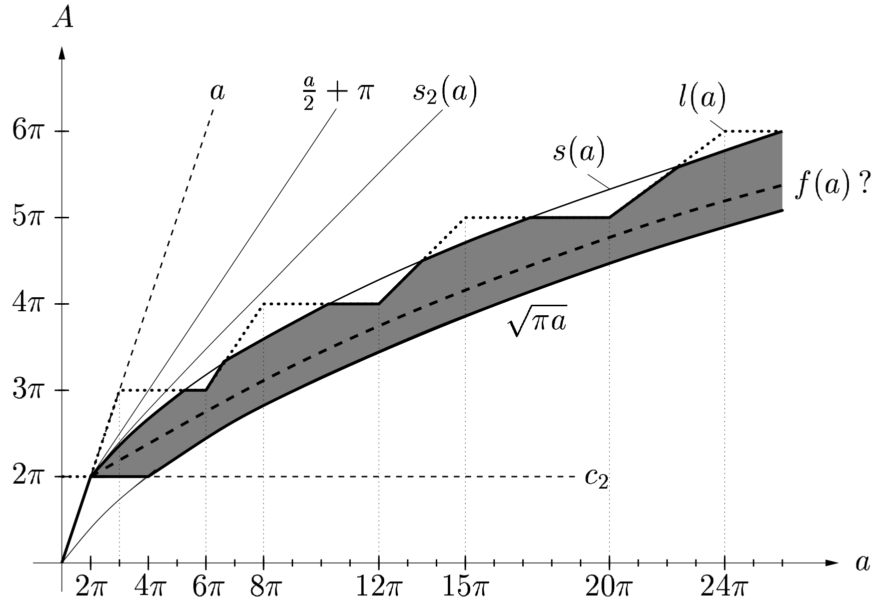

FIGURE 1. What is known about $f(a)$.

We illustrate the present knowledge of $f(a)$ with the help of Figure 1. In view of Theorem 1 we have $f(a)=a$ for $a\in[\pi,2\pi]$. For $a>2\pi$, the second Ekeland–Hofer capacity c_2 still implies that $f(a)\geq 2\pi$. This information is vacuous if $a\geq 4\pi$, since the volume condition $\operatorname{Vol}(E(\pi,a))\leq \operatorname{Vol}(B^4(f(a)))$ translates to $f(a)\geq\sqrt{\pi a}$. The estimate $f(a)\leq a/2+\pi$ stated in Theorem 2 is obtained by folding once. In view of Theorem 1 and Theorem 2 we are particularly interested in the behaviour of $f(a)$ as $a\to 2\pi^+$. Define the function $s_2(a)$ on $]2\pi,\infty[$ by

$$s_2(a) \;=\; 2\pi+(a-2\pi)\frac{a+\pi}{3a+\pi}.$$

We shall show by folding twice that for each $a > 2\pi$ and each $\epsilon > 0$ the ellipsoid $E(\pi, a)$ symplectically embeds into the ball $B^4\left(s_2(a) + \epsilon\right)$. Since $\frac{d}{da}s_2(2\pi) = \frac{3}{7}$, we conclude

Theorem 3 *We have*

$$\limsup_{\epsilon \to 0^+} \frac{f(2\pi + \epsilon) - 2\pi}{\epsilon} \;\leq\; \frac{3}{7}.$$

It will be clear from the 2-fold folding procedure described in the proof of Theorem 3 how one can associate to each $m \geq 3$ and to $a > 2\pi$ and $\epsilon > 0$ an m-fold folding procedure which symplectically embeds $E(\pi, a)$ into $B^4\left(s_m(a) + \epsilon\right)$. We shall compute $s_3(a)$ at the end of Section 3. In general, s_m is a strictly increasing rational function on $]2\pi, \infty[$, and $\frac{d}{da}s_m(2\pi) = \frac{3}{7}$ for all $m \geq 3$.

Question 2 *Is the estimate in Theorem 3 sharp?*

The family s_m is strictly decreasing. We denote the limit by

$$s(a) \;=\; \lim_{m \to \infty} s_m(a), \quad a > 2\pi.$$

The upper bound $s(a)$ of $f(a)$ is obtained from folding "infinitely many times". The graph of the function s is computed by a computer program, which in particular yields

$$f(3\pi) \leq s(3\pi) = 2.3801\ldots\pi \quad \text{and} \quad f(4\pi) \leq s(4\pi) = 2.6916\ldots\pi.$$

Finally, the piecewise linear upper bound $l(a)$ of $f(a)$ is a consequence of Traynor's theorem stated in Example 4, which she obtained from a Lagrangian folding method. We refer to Chapters 3 and 6 of [11] for a thorough analysis of the functions s_m, $m \geq 4$, and s as well as for a comparison of the functions s and l. There it is also shown that both differences $s(a) - \sqrt{\pi a}$ and $l(a) - \sqrt{\pi a}$ are bounded. It follows that the difference $f(a) - \sqrt{\pi a}$ is bounded. We in particular have

$$\lim_{a \to \infty} \frac{\operatorname{Vol}\left(E(\pi, a)\right)}{\operatorname{Vol}\left(B^4\left(f(a)\right)\right)} \;=\; 1,$$

i.e., a four dimensional ball can be asymptotically symplectically filled by thin ellipsoids. Symplectic folding can be used to prove such a result for any connected symplectic manifold (M^{2n}, ω) of finite volume $\operatorname{Vol}(M, \omega) = \frac{1}{n!}\int_M \omega^n$. For $a \geq \pi$ we define

$$e_a(M, \omega) \;=\; \sup_{\lambda} \frac{\operatorname{Vol}(\lambda E(\pi, \ldots, \pi, a))}{\operatorname{Vol}(M, \omega)}$$

where the supremum is taken over all those λ for which $\lambda E^{2n}(\pi, \dots, \pi, a)$ symplectically embeds into (M, ω).

Theorem 4 *Assume that (M, ω) is a connected symplectic manifold of finite volume. Then*

$$\lim_{a \to \infty} e_a(M, \omega) \;=\; 1.$$

This is our answer to Problem 1.

The paper is organized as follows: In Section i we prove Theorem i, i = 1,2,3. In Section 4 we outline a proof of Theorem 4 in dimension 4. A complete proof of Theorem 4 can be found in [11].

We write $|x|$ for the Euclidean norm of a point $x \in \mathbb{R}^n$ and $|U|$ for the Lebesgue measure of an open set $U \subset \mathbb{R}^n$. We work in the C^∞-category, i.e., all manifolds and diffeomorphisms are assumed to be C^∞-smooth, and so are all symplectic forms and maps.

Acknowledgements. This paper is part of my PhD thesis written at ETH Zürich. I would like to express my gratitude to my advisor Edi Zehnder for his support, his patience, and his continuous interest in my work. I am also indebted to Dusa Mc Duff who explained to me symplectic folding, a technique basic for this work.

1. Proof of Theorem 1

The main ingredient in the proof are Ekeland–Hofer capacities [2]. We recall the

Definition 1.1. An *extrinsic symplectic capacity on* $(\mathbb{R}^{2n}, \omega_0)$ is a map c associating with each subset S of $\mathbb{R}^{2n}$ a number $c(S) \in [0, \infty]$ in such a way that the following axioms are satisfied.

- A1. **Monotonicity:** $c(S) \le c(T)$ if there exists a symplectomorphism φ of $\mathbb{R}^{2n}$ such that $\varphi(S) \subset T$.
- A2. **Conformality:** $c(\lambda S) = \lambda^2 c(S)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.
- A3. **Nontriviality:** $0 < c(B^{2n}(\pi))$ and $c(Z^{2n}(\pi)) < \infty$.

The Ekeland–Hofer capacities form a countable family $\{c_i\}$, $i \ge 1$, of extrinsic symplectic capacities on $\mathbb{R}^{2n}$. For a symplectic ellipsoid $E = E(a_1, \dots, a_n)$ these invariants are given by

$$\{\, c_1(E) \le c_2(E) \le \dots \} \;=\; \{\, k a_i \mid k \in \mathbb{N},\, 1 \le i \le n \,\}, \tag{1}$$

see [2, Proposition 4].

Assume now that $a_n \leq 2a_1$ and that $E(a_1, \ldots, a_n)$ symplectically embeds into $B^{2n}(A)$. We need to show that $A \geq a_n$. By the Extension after Restriction Principle, for which we refer to [1] or [11, Appendix C], we find for each $\delta \in]0,1[$ a symplectomorphism φ_δ of $\mathbb{R}^{2n}$ such that $\varphi_\delta\left(E(\delta a_1, \ldots, \delta a_n)\right) \subset B^{2n}(A)$. By assumption, $\delta a_n \leq 2\delta a_1$, and so (1) implies that

$$c_n\left(E(\delta a_1, \ldots, \delta a_n)\right) = \delta a_n.$$

The monotonicity of the capacity c_n and $c_n\left(B^{2n}(A)\right) = A$ now yield $\delta a_n \leq A$. Since $\delta \in]0,1[$ was arbitrary we conclude that $A \geq a_n$, as claimed. □

We conclude this section by observing that the third Ekeland–Hofer capacity c_3 implies for $n \geq 3$ that the ellipsoid $E^{2n}(a, 3a, \ldots, 3a)$ does not symplectically embed into $B^{2n}(A)$ if $A < 3a$.

We refer to [11, Section 1.2] for a refinement of Theorem 1.

2. Proof of Theorem 2

2.1. **Reformulation of Theorem 2.** Recall from the introduction that the ellipsoid $E(a_1, \ldots, a_n)$ is defined by

$$E(a_1, \ldots, a_n) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} < 1 \right\}. \tag{2}$$

Theorem 2 of the introduction clearly can be reformulated as follows:

Theorem 2.1. *Assume $a > 2\pi$. Then $E^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into $B^{2n}\left(\frac{a}{2} + \pi + \epsilon\right)$ for every $\epsilon > 0$.*

The symplectic folding construction of Lalonde and Mc Duff considers a 4-ellipsoid as a fibration of discs of varying size over a disc and applies the flexibility of volume preserving maps to both the base and the fibers. It is therefore purely four dimensional in nature. We will refine the method in such a way that it allows us to prove Theorem 2.1 for every $n \geq 2$.

We shall conclude Theorem 2.1 from the following proposition in dimension 4.

Proposition 2.2. *Assume $a > 2\pi$. Given $\epsilon > 0$ there exists a symplectic embedding*

$$\Phi \colon E(a, \pi) \hookrightarrow B^4\left(\frac{a}{2} + \pi + \epsilon\right)$$

satisfying

$$\pi|\Phi(z_1,z_2)|^2 < \frac{a}{2}+\epsilon+\frac{\pi^2|z_1|^2}{a}+\pi|z_2|^2 \quad \textit{for all } (z_1,z_2)\in E(a,\pi).$$

We recall that $|\cdot|$ denotes the Euclidean norm. Postponing the proof, we first show that Proposition 2.2 implies Theorem 2.1.

Corollary 2.3. *Assume that Φ is as in Proposition 2.2. Then the composition of the permutation $E^{2n}(\pi,\dots,\pi,a) \to E^{2n}(a,\pi,\dots,\pi)$ with the restriction of $\Phi \times id_{2n-4}$ to $E^{2n}(a,\pi,\dots,\pi)$ embeds $E^{2n}(\pi,\dots,\pi,a)$ into $B^{2n}\left(\frac{a}{2}+\pi+\epsilon\right)$.*

Proof. Let $z=(z_1,\dots,z_n)\in E^{2n}(a,\pi,\dots,\pi)$. By Proposition 2.2 and the definition (2) of the ellipsoid,

$$\begin{aligned}
\pi\,|\Phi\times id_{2n-4}(z)|^2 &= \pi\left(|\Phi(z_1,z_2)|^2+\sum_{i=3}^n |z_i|^2\right)\\
&< \frac{a}{2}+\epsilon+\frac{\pi^2|z_1|^2}{a}+\pi\sum_{i=2}^n|z_i|^2\\
&= \frac{a}{2}+\epsilon+\pi\left(\frac{\pi|z_1|^2}{a}+\sum_{i=2}^n\frac{\pi|z_i|^2}{\pi}\right)\\
&< \frac{a}{2}+\epsilon+\pi,
\end{aligned}$$

as claimed. □

It remains to prove Proposition 2.2. In order to do so, we start with some preparations.

The flexibility of 2-dimensional area preserving maps is crucial for the construction of the map Φ. We now make sure that we may describe such a map by prescribing it on an exhausting and nested family of embedded loops. Recall that $D(a)$ denotes the open disc of area a centered at the origin.

Definition 2.4. A family $\mathcal{L}$ of loops in a simply connected domain $U\subset\mathbb{R}^2$ is called *admissible* if there is a diffeomorphism $\beta\colon D(|U|)\setminus\{0\}\to U\setminus\{p\}$ for some point $p\in U$ such that

(i) concentric circles are mapped to elements of $\mathcal{L}$,
(ii) in a neighbourhood of the origin β is a translation.

Lemma 2.5. *Let U and V be bounded and simply connected domains in $\mathbb{R}^2$ of equal area and let $\mathcal{L}_U$ and $\mathcal{L}_V$ be admissible families of loops in U and V, respectively. Then there is a symplectomorphism between U and V mapping loops to loops.*

Remark 2.6. The regularity condition (ii) imposed on the families taken into consideration can be weakened. Some condition, however, is necessary. Indeed, if $\mathcal{L}_U$ is a family of concentric circles and $\mathcal{L}_V$ is a family of rectangles with smooth corners and width larger than a positive constant, then no bijection from U to V mapping loops to loops is continuous at the origin. $\diamond$

Proof of Lemma 2.5. Denote the concentric circle of radius r by $C(r)$. We may assume that $\mathcal{L}_U = \{C(r)\}$, $0 < r < R$. Let β be the diffeomorphism parameterizing $(V \setminus \{p\}, \mathcal{L}_V)$. After reparametrizing the r-variable by a diffeomorphism of $]0, R[$ which is the identity near 0 we may assume that β maps the loop $C(r)$ of radius r to the loop $L(r)$ in $\mathcal{L}_V$ which encloses the domain $V(r)$ of area πr^2. We denote the Jacobian of β at $re^{i\varphi}$ by $\beta'(re^{i\varphi})$. Since β is a translation near the origin and U is connected, $\det \beta'(re^{i\varphi}) > 0$. By our choice of β,

$$\pi r^2 \;=\; |V(r)| \;=\; \int_{D(\pi r^2)} \det \beta' \;=\; \int_0^r \rho \, d\rho \int_0^{2\pi} \det \beta'(\rho e^{i\varphi}) \, d\varphi.$$

Differentiating in r we obtain

$$2\pi \;=\; \int_0^{2\pi} \det \beta'(re^{i\varphi}) \, d\varphi. \tag{3}$$

Define the smooth function $h \colon]0, R[\times \mathbb{R} \to \mathbb{R}$ as the unique solution of the initial value problem

$$\left.\begin{array}{rcll} \frac{d}{dt} h(r,t) & = & 1/\det \beta'(re^{ih(r,t)}), & t \in \mathbb{R} \\ h(r,t) & = & 0, & t = 0 \end{array}\right\} \tag{4}$$

depending on the parameter r. We claim that

$$h(r, t + 2\pi) \;=\; h(r,t) + 2\pi. \tag{5}$$

It then follows, since the function h is strictly increasing in the variable t, that for every r fixed the map $h(r, \cdot) \colon \mathbb{R} \to \mathbb{R}$ induces a diffeomorphism of the circle $\mathbb{R}/2\pi\mathbb{Z}$. In order to prove the claim (5) we denote by $t_0(r) > 0$ the unique solution of $h(r, t_0(r)) = 2\pi$. Substituting $\varphi = h(r,t)$ into formula (3) we obtain, using $\det \beta'(re^{ih(r,t)}) \cdot \frac{d}{dt} h(r,t) = 1$, that

$$2\pi \;=\; \int_0^{t_0(r)} dt \;=\; t_0(r).$$

Hence $h(r, 2\pi) = 2\pi$. Therefore, the two functions in t, $h(r, t+2\pi) - 2\pi$ and $h(r,t)$, solve the same initial value problem (4), and so the claim (5) follows. The desired diffeomorphism is now defined by

$$\alpha \colon U \setminus \{0\} \;\to\; V \setminus \{p\}, \quad re^{i\varphi} \;\mapsto\; \beta(re^{ih(r,\varphi)}).$$

It is area preserving. Indeed, representing α as the composition

$$re^{i\varphi} \mapsto (r,\varphi) \mapsto (r,h(r,\varphi)) \mapsto re^{ih(r,\varphi)} \mapsto \beta(re^{ih(r,\varphi)})$$

we obtain for the determinant of the Jacobian

$$\frac{1}{r}\cdot\frac{\partial h}{\partial\varphi}(r,\varphi)\cdot r\cdot \det\beta'(re^{ih(r,\varphi)}) \;=\; 1,$$

where we again have used (4). Finally, α is a translation in a punctured neighbourhood of the origin and thus smoothly extends to the origin. This finishes the proof of Lemma 2.5. $\square$

Consider a bounded domain $U \subset \mathbb{C}$ and a continuous function $f\colon U \to \mathbb{R}_{>0}$. The set $\mathcal{F}(U,f)$ in $\mathbb{C}^2$ defined by

$$\mathcal{F}(U,f) \;=\; \left\{(z_1,z_2)\in\mathbb{C}^2 \;\middle|\; z_1\in U,\ \pi\,|z_2|^2 < f(z_1)\right\}$$

is the trivial fibration over U having as fiber over z_1 the disc of capacity $f(z_1)$. Given two such fibrations $\mathcal{F}(U,f)$ and $\mathcal{F}(V,g)$, a symplectic embedding $\varphi\colon U\hookrightarrow V$ defines a symplectic embedding $\varphi\times id\colon \mathcal{F}(U,f)\hookrightarrow \mathcal{F}(V,g)$ if and only if $f(z_1)\le g(\varphi(z_1))$ for all $z_1\in U$.

Examples 2.7.

1. The ellipsoid $E(a,b)$ can be represented as

$$E(a,b) \;=\; \mathcal{F}\left(D(a), f(z_1) = b\left(1-\frac{\pi|z_1|^2}{a}\right)\right).$$

2. Define the open trapezoid $T(a,b)$ by $T(a,b)=\mathcal{F}(R(a),g)$, where

$$R(a) \;=\; \{\, z_1=(u,v) \,|\, 0<u<a,\ 0<v<1 \,\}$$

is a rectangle and $g(z_1)=g(u)=b(1-u/a)$. We set $T^4(a)=T(a,a)$. The example is inspired by [9, p. 54]. It will be very useful to think of $T(a,b)$ as depicted in Figure 2. $\diamond$

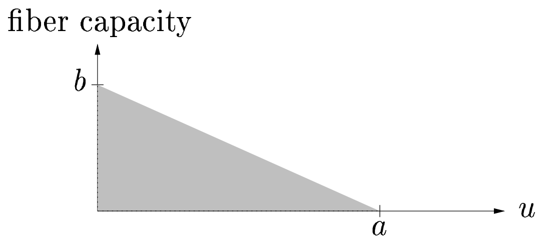

FIGURE 2. The trapezoid $T(a,b)$.

In order to reformulate Proposition 2.2 we shall prove the following lemma which later on allows us to work with more convenient "shapes".

Lemma 2.8. *Assume $\epsilon > 0$. Then*

(i) *$E(a,b)$ symplectically embeds into $T(a+\epsilon, b+\epsilon)$,*
(ii) *$T^4(a)$ symplectically embeds into $B^4(a+\epsilon)$.*

Proof. Set $\epsilon' = a\epsilon^2/(ab+a\epsilon+b\epsilon)$. We are going to use Lemma 2.5 to construct an area preserving diffeomorphism $\alpha\colon D(a) \to R(a)$ such that for the first coordinate in the image $R(a)$,

$$u(\alpha(z_1)) \le \pi|z_1|^2 + \epsilon' \quad \text{for all } z_1 \in D(a), \tag{6}$$

see Figures 3 and 4.

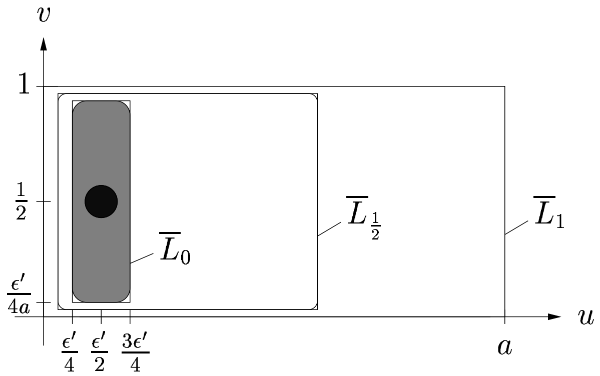

FIGURE 3. Constructing the embedding α.

In an "optimal world" we would choose the loops $\hat{L}_u$, $0 < u < a$, in the image $R(a)$ as the boundaries of the rectangles with corners $(0,0)$, $(0,1)$, $(u,0)$, $(u,1)$. If the family $\hat{\mathfrak{L}} = \{\hat{L}_u\}$ induced a map $\hat{\alpha}$, we would then have $u(\hat{\alpha}(z_1)) \le \pi|z_1|^2$ for all $(z_1,z_2) \in R(a)$. The non admissible family $\hat{\mathfrak{L}}$ can be perturbed to an admissible family $\mathfrak{L}$ in such a way that the induced map α satisfies the estimate (6). Indeed, choose the translation disc appearing in the proof of Lemma 2.5 as the disc of radius $\epsilon'/8$ centered at $(u_0,v_0) = \left(\frac{\epsilon'}{2}, \frac{1}{2}\right)$. For $r < \epsilon'/8$ the loops $L(r)$ are therefore the circles centered at (u_0,v_0). In the following, all rectangles considered have edges parallel to the coordinate axes. We may thus describe a rectangle by specifying its lower left and upper right corner. Let $\overline{L}_0$ be the boundary of the rectangle with corners $\left(\frac{\epsilon'}{4}, \frac{\epsilon'}{4a}\right)$ and $\left(\frac{3\epsilon'}{4}, 1-\frac{\epsilon'}{4a}\right)$, and let $\overline{L}_1$ be the boundary of $R(a)$. We define a family of loops $\overline{L}_s$ by linearly interpolating between $\overline{L}_0$ and $\overline{L}_1$, i.e., $\overline{L}_s$ is the boundary of the rectangle with corners

$$\left((1-s)\frac{\epsilon'}{4}, (1-s)\frac{\epsilon'}{4a}\right) \quad \text{and} \quad \left(u_s, 1-\frac{\epsilon'}{4a}+\frac{\epsilon'}{4a}s\right), \quad s \in [0,1],$$

where $u_s = \frac{3\epsilon'}{4} + s\left(a - \frac{3\epsilon'}{4}\right)$. Since $u_s < a$, the area enclosed by $\overline{L}_s$ is estimated from below by

$$\left(u_s - \frac{\epsilon'}{4}\right)\left(1 - 2\frac{\epsilon'}{4a}\right) > u_s - \frac{3\epsilon'}{4}. \tag{7}$$

Let $\{L_s\}$, $s \in [0,1[$, be the smooth family of smooth loops obtained from $\left\{\overline{L}_s\right\}$ by smoothing the corners as indicated in Figure 3. By choosing the smooth corners of L_s more and more rectangular as $s \to 1$, we can arrange that the set $\coprod_{0<s<1} L_s$ is the domain bounded by L_0 and $\overline{L}_1$. Moreover, by choosing all smooth corners rectangular enough, we can arrange that the area enclosed by L_s and $\overline{L}_s$ is less than $\epsilon'/4$. In view of (7), the area enclosed by L_s is then at least $u_s - \epsilon'$. Complete the families $\{L(r)\}$ and $\{L_s\}$ to an admissible family $\mathfrak{L}$ of loops in $R(a)$ and let $\alpha\colon D(a) \to R(a)$ be the map defined by $\mathfrak{L}$. Fix $(z_1, z_2) \in D(a)$. If $\alpha(z_1)$ lies on a loop in $\mathfrak{L} \setminus \{L_s\}_{0<s<1}$, then $u\left(\alpha(z_1)\right) < \frac{3\epsilon'}{4} \le \pi\left|z_1\right|^2 + \epsilon'$, and so the required estimate (6) is satisfied. If $\alpha(z_1) \in L_s$ for some $s \in\ ]0,1[$, then the area enclosed by L_s is $\pi\left|z_1\right|^2$, and so $\pi\left|z_1\right|^2 + \epsilon' > u_s \ge u\left(\alpha(z_1)\right)$, whence (6) is again satisfied. This completes the construction of a symplectomorphism $\alpha\colon D(a) \to R(a)$ satisfying (6). In the sequel, we will illustrate a map like α by a picture like in Figure 4.

To continue the proof of (i) we shall show that $(\alpha(z_1), z_2) \in T(a+\epsilon, b+\epsilon)$ for every $(z_1, z_2) \in E(a,b)$, so that the symplectic map $\alpha \times id$ embeds $E(a,b)$ into $T(a+\epsilon, b+\epsilon)$. Take $(z_1, z_2) \in E(a,b)$. Then, using the definition (2) of $E(a,b)$, the estimate (6) and the definition of ϵ' we find

$$\begin{aligned}
\pi|z_2|^2 < b\left(1 - \frac{\pi|z_1|^2}{a}\right) &\le b\left(1 - \frac{u\left(\alpha(z_1)\right)}{a} + \frac{\epsilon'}{a}\right) \\
&< b\left(1 - \frac{u\left(\alpha(z_1)\right)}{a+\epsilon}\right) + b\,\frac{\epsilon'}{a} \\
&= b\left(1 - \frac{u\left(\alpha(z_1)\right)}{a+\epsilon}\right) + \epsilon - \frac{\epsilon}{a+\epsilon}\left(a + \epsilon'\right) \\
&\le b\left(1 - \frac{u\left(\alpha(z_1)\right)}{a+\epsilon}\right) + \epsilon - \frac{\epsilon}{a+\epsilon}\,u\left(\alpha(z_1)\right) \\
&= (b+\epsilon)\left(1 - \frac{u(\alpha(z_1))}{a+\epsilon}\right).
\end{aligned}$$

It follows that

$$(\alpha(z_1), z_2) \in T(a+\epsilon, b+\epsilon) = \mathfrak{F}\left(R(a+\epsilon), (b+\epsilon)\left(1 - \frac{u}{a+\epsilon}\right)\right)$$

as claimed.

In order to prove (ii) we shall construct an area preserving diffeomorphism ω from a rectangular neighbourhood of $R(a)$ having smooth corners and area $a+\epsilon$ to $D(a+\epsilon)$ such that

$$(8) \qquad \pi|\omega(z_1)|^2 \le u+\epsilon \quad \text{for all } z_1=(u,v)\in R(a).$$

Such a map ω can again be obtained with the help of Lemma 2.5. In an "optimal world" we would choose the loops $\hat{L}_u$ in the domain $R(a)$ as before. This time, we perturb this non admissible family to an admissible family $\mathcal{L}$ of loops as illustrated in Figure 4. If the smooth corners of all those loops in $\mathcal{L}$ which enclose an area greater than $\epsilon/2$ lie outside $R(a)$ and if the upper, left and lower edges of all these loops are close enough, then the induced map ω will satisfy (8).

Restricting ω to $R(a)$ we obtain a symplectic embedding $\omega\times id\colon T^4(a)\hookrightarrow \mathbb{R}^4$. For $(z_1,z_2)\in T^4(a)$ we have $\pi\,|z_2|^2 < a\,(1-u/a)$, where $z_1=(u,v)\in R(a)$. In view of (8) we conclude that

$$\begin{aligned} \pi\left(|\omega(z_1)|^2+|z_2|^2\right) &< u+\epsilon+a\left(1-\frac{u}{a}\right)\\ &= u+\epsilon+a-u\\ &= a+\epsilon, \end{aligned}$$

and so $(\omega\times id)(z_1,z_2)\in B^4(a+\epsilon)$ for all $(z_1,z_2)\in T^4(a)$. □

Lemma 2.8 allows us to reformulate Proposition 2.2 as follows.

Proposition 2.9. *Assume $a>2\pi$. Given $\epsilon>0$, there exists a symplectic embedding*

$$\Psi\colon T(a,\pi)\hookrightarrow T^4\left(\frac{a}{2}+\pi+\epsilon\right),\quad (z_1,z_2)\mapsto(z_1',z_2'),$$

$z_1=(u,v)$ and $z_1'=(u',v')$, satisfying

$$(9)\quad u'+\pi|z_2'|^2 < \frac{a}{2}+\epsilon+\frac{\pi u}{a}+\pi|z_2|^2 \quad \text{for all } (u,v,z_2)\in T(a,\pi).$$

Postponing the proof, we first show that Proposition 2.9 implies Proposition 2.2.

Corollary 2.10. *Assume the statement of Proposition 2.9 holds true. Then there exists a symplectic embedding $\Phi\colon E(a,\pi)\hookrightarrow B^4\left(\frac{a}{2}+\pi+\epsilon\right)$ satisfying*

(10)

$$\pi|\Phi(z_1,z_2)|^2 < \frac{a}{2}+\epsilon+\frac{\pi^2|z_1|^2}{a}+\pi|z_2|^2 \quad \text{for all } (z_1,z_2)\in E(a,\pi).$$

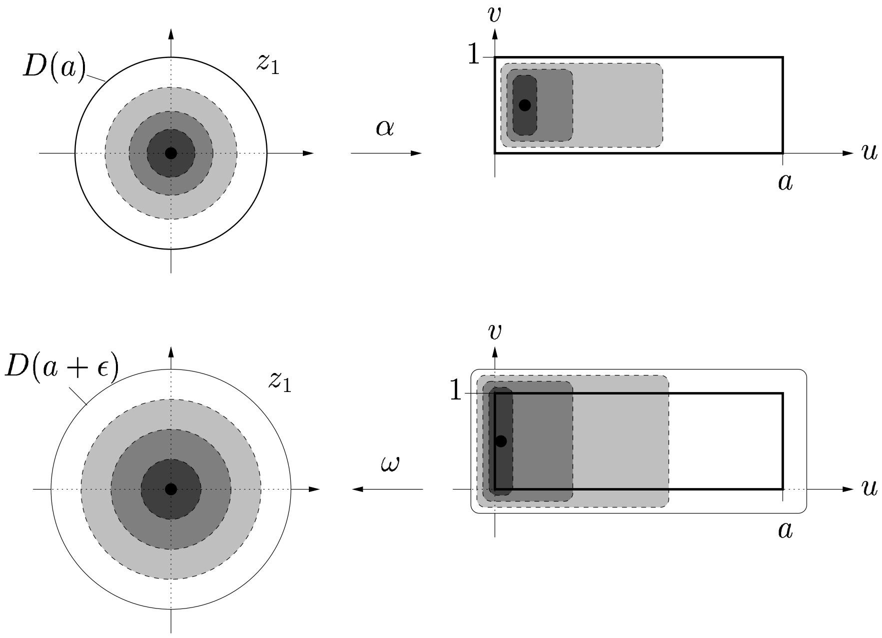

FIGURE 4. The first and the last base deformation.

Proof. Let $\epsilon' > 0$ be so small that $ca + \epsilon' > 2\pi$, where $c = 1 - \epsilon'/\pi$. As in the proof of Lemma 2.8 we can construct a symplectic embedding

$$\alpha \times id \colon E(ca, c\pi) \hookrightarrow T(ca + \epsilon', c\pi + \epsilon') = T(ca + \epsilon', \pi)$$

satisfying the estimate

$$u(\alpha(z_1)) \leq \pi|z_1|^2 + \frac{a(\epsilon')^2}{ca\pi + a\epsilon' + \pi\epsilon'} \quad \text{for all } z_1 \in D(ca) \tag{11}$$

and another symplectic embedding

$$\omega \times id \colon T^4\left(\frac{ca}{2} + \pi + \epsilon'\right) \hookrightarrow B^4\left(\frac{ca}{2} + \pi + 2\epsilon'\right)$$

satisfying

$$\pi|\omega(z_1)|^2 \leq u + \epsilon' \quad \text{for all } z_1 = (u, v) \in R\left(\frac{ca}{2} + \pi + \epsilon'\right). \tag{12}$$

Since $ca + \epsilon' > 2\pi$, Proposition 2.9 applied to $ca + \epsilon'$ replacing a and $\epsilon'/2$ replacing ϵ guarantees a symplectic embedding

$$\Psi \colon T(ca + \epsilon', \pi) \hookrightarrow T^4\left(\frac{ca}{2} + \pi + \epsilon'\right),$$

$(z_1, z_2) \mapsto (\Psi_1(z_1, z_2), \Psi_2(z_1, z_2))$, satisfying

$$u\left(\Psi_1(\alpha(z_1), z_2)\right) + \pi\left|\Psi_2\left(\alpha(z_1), z_2\right)\right|^2 < \frac{ca}{2} + \epsilon' + \frac{\pi u(\alpha(z_1))}{ca + \epsilon'} + \pi\left|z_2\right|^2 \tag{13}$$

for all $(u(\alpha(z_1)), v, z_2) \in T(ca+\epsilon', \pi)$. Set $\widehat{\Phi} = (\omega \times id) \circ \Psi \circ (\alpha \times id)$. Then $\widehat{\Phi}$ symplectically embeds $E(ca, c\pi)$ into $B^4\left(\frac{ca}{2} + \pi + 2\epsilon'\right)$. Moreover, if $(z_1, z_2) \in E(ca, c\pi)$, then

$$
\begin{aligned}
\pi\left|\widehat{\Phi}(z_1,z_2)\right|^2 &= \pi\left|\omega\left(\Psi_1(\alpha(z_1),z_2)\right)\right|^2 + \pi\left|\Psi_2(\alpha(z_1),z_2)\right|^2 \\
&\overset{(12)}{\leq} u(\Psi_1(\alpha(z_1),z_2)) + \epsilon' + \pi\left|\Psi_2(\alpha(z_1),z_2)\right|^2 \\
&\overset{(13)}{<} \frac{ca}{2} + 2\epsilon' + \frac{\pi u(\alpha(z_1))}{ca+\epsilon'} + \pi|z_2|^2 \\
&\overset{(11)}{\leq} \frac{ca}{2} + 2\epsilon' + \frac{\pi^2|z_1|^2}{ca+\epsilon'} + \frac{\pi}{ca+\epsilon'}\frac{a(\epsilon')^2}{ca\pi + a\epsilon' + \pi\epsilon'} + \pi|z_2|^2 \\
&< \frac{ca}{2} + 3\epsilon' + \frac{\pi^2|z_1|^2}{ca} + \pi|z_2|^2
\end{aligned}
$$

where in the last step we again used $ca + \epsilon' > 2\pi$. Now choose $\epsilon' > 0$ so small that $\frac{\pi+3\epsilon'}{c} < \pi + \epsilon$. We denote the dilatation by $\sqrt{c}$ in $\mathbb{R}^4$ also by $\sqrt{c}$, and define $\Phi\colon E(a,\pi) \to \mathbb{R}^4$ by $\Phi = \left(\sqrt{c}\right)^{-1} \circ \widehat{\Phi} \circ \sqrt{c}$. Then Φ symplectically embeds $E(a,\pi)$ into $B^4\left(\frac{a}{2} + \frac{\pi+2\epsilon'}{c}\right) \subset B^4\left(\frac{a}{2} + \pi + \epsilon\right)$, and since $\pi\left|z_1\right|^2 < a$ for all $(z_1,z_2) \in E(a,\pi)$ and by the choice of ϵ',

$$
\begin{aligned}
\pi\left|\Phi(z_1,z_2)\right|^2 &= \frac{\pi}{c}\left|\widehat{\Phi}\left(\sqrt{c}\,z_1, \sqrt{c}\,z_2\right)\right|^2 \\
&< \frac{1}{c}\left(\frac{ca}{2} + 3\epsilon' + \frac{\pi^2|z_1|^2}{a} + \pi c|z_2|^2\right) \\
&= \frac{a}{2} + \frac{3\epsilon'}{c} + \frac{1}{c}\frac{\pi^2|z_1|^2}{a} + \pi|z_2|^2 \\
&< \frac{a}{2} + \epsilon + \frac{\pi^2|z_1|^2}{a} + \pi|z_2|^2
\end{aligned}
$$

for all $(z_1,z_2) \in E(a,\pi)$. This proves the required estimate (10), and so the proof of Corollary 2.10 is complete. □

It remains to prove Proposition 2.9. This is done in the following two sections.

2.2. **The folding construction.** The idea in the construction of an embedding Ψ as in Proposition 2.9 is to separate the small fibers from the large ones and then to fold the two parts on top of each other. As in the previous section we denote the coordinates in the base and the fiber by $z_1 = (u,v)$ and $z_2 = (x,y)$, respectively.

Step 1. Following [9, Lemma 2.1] we first separate the "low" regions over $R(a)$ from the "high" ones. We may do this using Lemma 2.5. We prefer, however, to give an explicit construction.

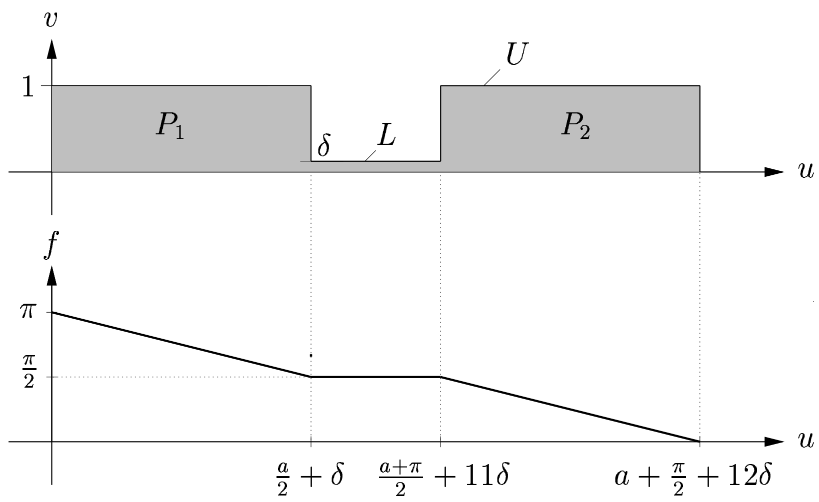

FIGURE 5. Separating the low fibers from the large fibers.

Let $\delta > 0$ be small. Set $\mathcal{F} = \mathcal{F}(U, f)$, where U and f are described in Figure 5, and write

$$\begin{aligned} P_1 &= U \cap \left\{ u \leq \frac{a}{2} + \delta \right\}, \\ P_2 &= U \cap \left\{ u \geq \frac{a+\pi}{2} + 11\delta \right\}, \\ L &= U \setminus (P_1 \cup P_2). \end{aligned}$$

Hence, U is the disjoint union

$$U = P_1 \coprod L \coprod P_2.$$

Choose a smooth function $h\colon [0, a+\delta] \to \,]0,1]$ as in Figure 6, i.e.

(i) $h(w) = 1$ for $w \in \left[0, \frac{a}{2}\right]$,
(ii) $h'(w) < 0$ for $w \in \left]\frac{a}{2}, \frac{a}{2} + \delta^2\right[$,
(iii) $h\left(\frac{a}{2} + \delta^2\right) = \delta$,
(iv) $h(w) = h(a-w)$ for all $w \in [0, a+\delta]$.

By (ii), (iii) and (iv) we have that

$$\int_{\frac{a}{2}}^{\frac{a}{2}+\delta^2} \frac{1}{h(w)}\, dw < \delta \quad \text{and} \quad \int_{\frac{a}{2}+\delta-\delta^2}^{\frac{a}{2}+\delta} \frac{1}{h(w)}\, dw < \delta. \tag{14}$$

We may thus further require that

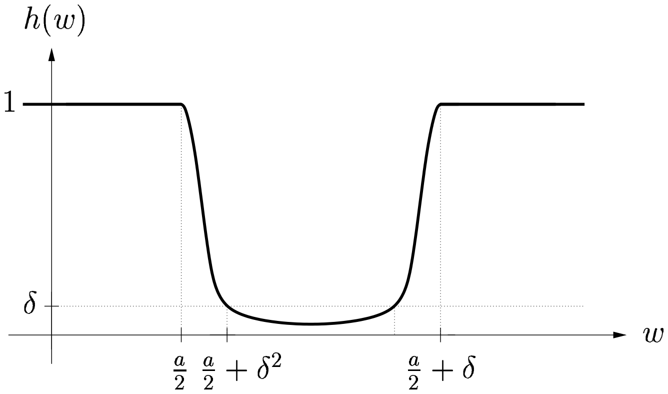

FIGURE 6. The function h.

(v) $h(w) < \delta$ for $w \in \left]\frac{a}{2}+\delta^2, \frac{a}{2}+\delta-\delta^2\right[$,

(vi) $\int_{\frac{a}{2}}^{\frac{a}{2}+\delta} \frac{1}{h(w)}\, dw = \frac{\pi}{2} + 12\delta$.

Consider the map

$$\beta\colon R(a) \to \mathbb{R}^2, \quad (u,v) \mapsto \left(\int_0^u \frac{1}{h(w)}\, dw, h(u)v \right).$$

Clearly, β is a symplectic embedding. We see from (i), (iv) and (vi) that

$$\beta\big|_{\left\{u<\frac{a}{2}\right\}} = id \quad \text{and} \quad \beta\big|_{\left\{u>\frac{a}{2}+\delta\right\}} = id + \left(\frac{\pi}{2}+11\delta, 0\right). \tag{15}$$

These identities and the estimates (14) and (v) imply that β embeds $R(a)$ into U (cf. Figure 7, where the black region in $R(a)$ is mapped to the black region in U, and so on). Finally, by construction, $\beta \times id$ symplectically embeds $T(a,\pi)$ into $\mathcal{F}$.

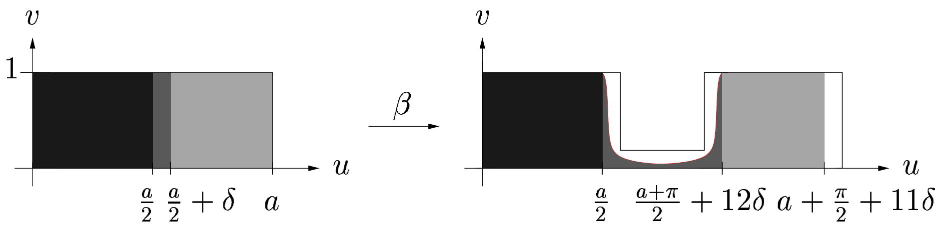

FIGURE 7. The embedding $\beta\colon R(a) \hookrightarrow U$.

Step 2. We next map the fibers into a convenient shape. Using Lemma 2.5 in a similar way as it was used in the proof of Lemma 2.8 we find a symplectomorphism σ mapping $D(\pi)$ to the rectangle

R_e and $D\left(\frac{\pi}{2}\right)$ to the rectangle with smooth corners R_i as specified in Figure 8. We require that for $z_2 \in D\left(\frac{\pi}{2}\right)$

$$\pi|z_2|^2 + 2\delta \,>\, y(\sigma(z_2)) - \left(-\frac{\pi}{2} - 2\delta\right),$$

i.e.

$$y(\sigma(z_2)) \,<\, \pi|z_2|^2 - \frac{\pi}{2} \quad \text{for } z_2 \in D\left(\frac{\pi}{2}\right). \tag{16}$$

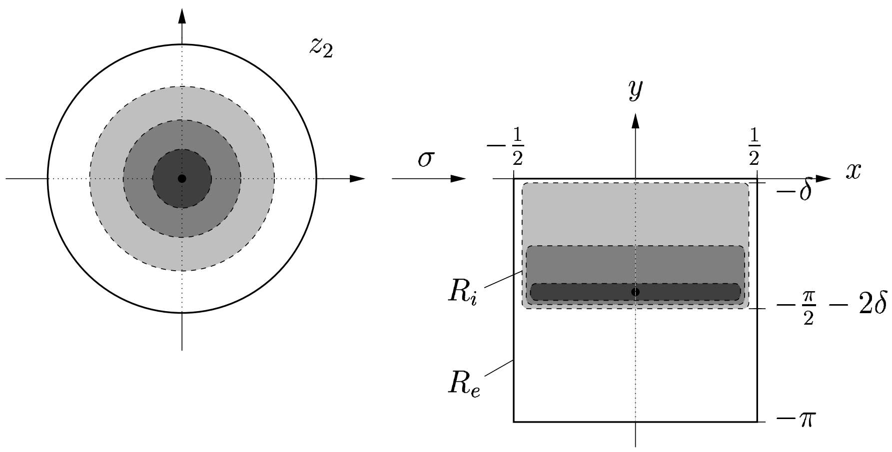

FIGURE 8. Preparing the fibers.

Write for the resulting bundle $(id \times \sigma)\mathcal{F}$ of rectangles with smooth corners

$$(id \times \sigma)\mathcal{F} \,=\, \mathcal{S} \,=\, \mathcal{S}(P_1) \coprod \mathcal{S}(L) \coprod \mathcal{S}(P_2).$$

In order to fold $\mathcal{S}(P_2)$ over $\mathcal{S}(P_1)$ we first move $\mathcal{S}(P_2)$ along the y-axis and then turn it in the z_1-direction over $\mathcal{S}(P_1)$.

Step 3. In order to move $\mathcal{S}(P_2)$ along the y-axis we follow again [8, p. 355]. Let $c\colon \mathbb{R} \to [0, 1-2\delta]$ be a smooth cut off function as in Figure 9:

$$c(t) \,=\, \begin{cases} 0, & t \le \frac{a}{2} + 2\delta \text{ and } t \ge \frac{a+\pi}{2} + 10\delta, \\ 1-2\delta, & \frac{a}{2} + 3\delta \le t \le \frac{a+\pi}{2} + 9\delta, \end{cases}$$

Set $I(t) = \int_0^t c(s)\,ds$ and define the diffeomorphism $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\varphi(u,x,v,y) \,=\, \left(u, x, v + c(u)\left(x + \frac{1}{2}\right), y + I(u)\right). \tag{17}$$

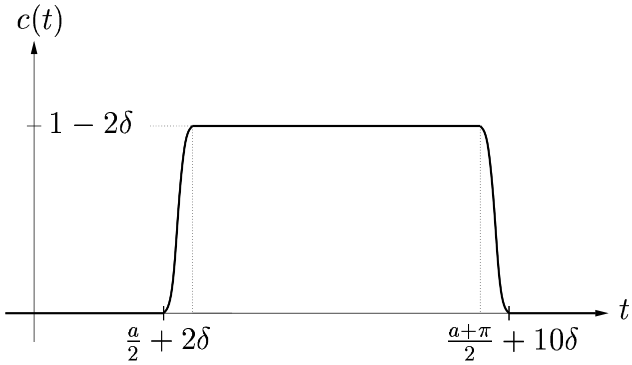

FIGURE 9. The cut off c.

We then find for the derivative

$$d\varphi(u,x,v,y) \;=\; \begin{bmatrix} \mathbb{I}_2 & 0 \\ A & \mathbb{I}_2 \end{bmatrix} \quad \text{with } A \;=\; \begin{bmatrix} * & c(u) \\ c(u) & 0 \end{bmatrix},$$

whence φ is a symplectomorphism in view of the criterion in [7, p. 5]. Moreover, defining the number I_∞ by $I_\infty = I\left(\frac{a+\pi}{2}+10\delta\right)$, we find

$$\varphi\big|_{\left\{u\le \frac{a}{2}+2\delta\right\}} \;=\; id \quad \text{and} \quad \varphi\big|_{\left\{u\ge \frac{a+\pi}{2}+10\delta\right\}} \;=\; id+(0,0,0,I_\infty), \tag{18}$$

and assuming that $\delta < \frac{1}{15}$ we compute with the help of Figure 9 that

$$\frac{\pi}{2}+2\delta \;<\; I_\infty \;<\; \frac{\pi}{2}+5\delta. \tag{19}$$

The first inequality in (19) implies

$$\varphi(P_2\times R_i)\cap(\mathbb{R}^2\times R_e) \;=\; \emptyset. \tag{20}$$

Remark 2.11. The map φ is the crucial map of the folding construction. Indeed, φ is the only map in the construction which does not split as a product of 2-dimensional maps. It is the map which sends the lines $\{v,x,y \text{ constant}\}$ to the characteristics of the hypersurface

$$(u,x,y) \;\mapsto\; \left(u,x,c(u)\left(x+\frac{1}{2}\right),y\right)$$

which generates (the cut off of) the obvious flow separating R_i from R_e. $\diamond$

Step 4. In this step we turn $\varphi\left(\mathcal{S}(P_2)\right)$ over $\mathcal{S}(P_1)$ by folding in the base. From the definition (17) of the map φ and Figure 5 and Figure 8 we read off that the projection of $\varphi(\mathcal{S})$ onto the (u,v)-plane is contained in the union $\mathcal{U}$ of U with the open set bounded by the graph of $u\mapsto \delta+c(u)$, the u-axis and the two lines $\{u=a/2+\delta\}$ and $\{u=(a+$

$\pi)/2+11\delta\}$, cf. Figure 10. Observe that $\delta+c(u) \le 1-\delta$. Define a local symplectic embedding γ of $\mathcal{U}$ into $\{ (u,v) \mid 0 < u < (a+\pi)/2+11\delta,\ 0 < v < 1 \}$ as follows: On $P_1 = \mathcal{U} \cap \{u \le a/2+\delta\}$ the map γ is the identity, and on $\mathcal{U} \cap \{u \ge a/2+2\delta\}$ it is the orientation preserving isometry which maps the right edge of $P_2 = \mathcal{U} \cap \{u \ge (a+\pi)/2+11\delta\}$ to the left edge of P_1. In particular, we have for $z_1 = (u,v) \in P_2$,

$$u'(\gamma(z_1)) \;=\; a + \frac{\pi}{2} + 12\delta - u. \tag{21}$$

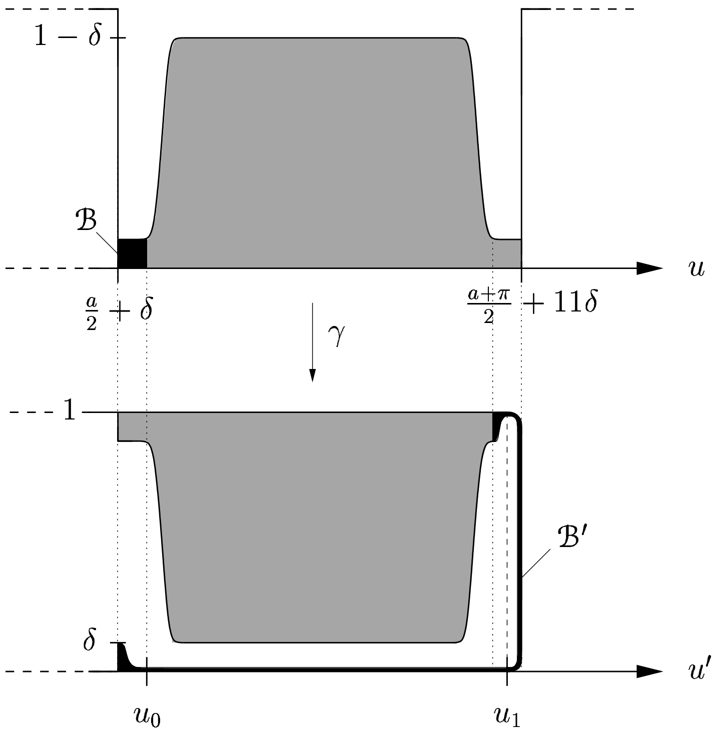

FIGURE 10. Folding in the base.

On the remaining black square $\mathcal{B} = \mathcal{U} \cap \{a/2+\delta < u < a/2+2\delta\}$ the map γ looks as shown in Figure 10. We then have for $(u,v) \in \mathcal{U} \setminus (P_1 \cup P_2)$,

$$u'(\gamma(u,v)) - \left(\frac{a}{2}+\delta\right) \;<\; \frac{\pi}{2} + 10\delta - \left(u - \left(\frac{a}{2}+\delta\right)\right) + \delta,$$

i.e.

$$u'(\gamma(u,v)) \;<\; -u + \frac{\pi}{2} + a + 13\delta. \tag{22}$$

By (20) the map $\gamma \times id$ is one-to-one on $\varphi(\mathcal{S})$.

The existence of an area and orientation preserving embedding as proposed in Figure 10 can be proved as follows: Set $u_0 = a/2 + 2\delta$ and $u_1 = (a+\pi)/2 + 21\delta/2$. Moreover, set $l = \pi/2 + 1 + 39\,\delta/4$ and choose $\lambda_3 > 0$ so small that $\lambda_3 l \le \delta^2/3$. Similar to Figure 6 we choose a smooth function $h\colon \left[\frac{a}{2}+\delta, \frac{a}{2}+2\delta\right] \to]0,1]$ such that

(i) $h(u) = 1$ for u near $\frac{a}{2}+\delta$ and u near $\frac{a}{2}+2\delta$,

(ii) $h(u) = \frac{\lambda_3}{\delta}$ for $u \in \left[\frac{a}{2}+\frac{3\delta}{2}, \frac{a}{2}+\frac{3\delta}{2}+\frac{\lambda_3 l}{\delta}\right]$,

(iii) $\int_{\frac{a}{2}+\delta}^{\frac{a}{2}+\frac{3\delta}{2}} \frac{1}{h(w)}\,dw = \delta$ and $\int_{\frac{a}{2}+\frac{3\delta}{2}+\frac{\lambda_3 l}{\delta}}^{\frac{a}{2}+2\delta} \frac{1}{h(w)}\,dw = \frac{\delta}{2}$.

The embedding $\gamma_\delta\colon \mathcal{B} \to \left[\frac{a}{2}+\delta, u_0+l+\frac{\delta}{2}\right] \times [0,\delta]$ defined by

$$(u,v) \mapsto \left(\frac{a}{2}+\delta+\int_{\frac{a}{2}+\delta}^{u} \frac{1}{h(w)}\,dw,\; h(u)v\right)$$

and illustrated in Figure 11 is symplectic.

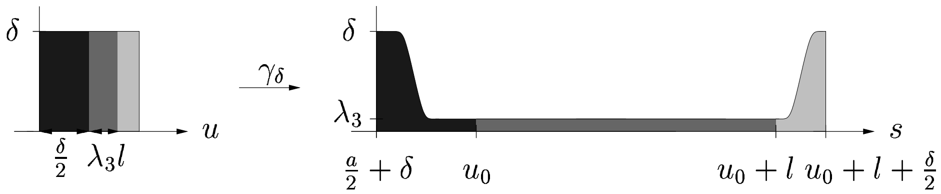

FIGURE 11. The map γ_δ.

We now map the image of γ_δ to a domain $\mathcal{B}'$ in the (u',v')-plane as painted in Figure 10: By the choice of l we may require that the part of the "outer" boundary of $\mathcal{B}'$ between $(u_0,0)$ and $(u_1,1)$, which contains $(u_1,0)$, is smooth, has length l, and is parametrized by $\zeta(s)$, where the parameter $s \in I := [u_0, u_0+l]$ is arc length and

$$\begin{array}{lcll} \zeta(s) & = & (s,0) & \text{on } [u_0,u_1]\,, \\ \zeta(s) & = & (u_1+u_0+l-s,1) & \text{on } \left[u_0+l-\frac{\delta}{4}, u_0+l\right]. \end{array} \tag{23}$$

Denote the inward pointing unit normal vector field along ζ by ν. We choose $\lambda_1 > 0$ so small that

$$\eta\colon I \times [0,\lambda_1] \to \mathbb{R}^2, \quad (s,t) \mapsto \zeta(s) + t\,\nu(s)$$

is an embedding. In order to make the map area preserving, we consider the initial value problem

$$\left.\begin{array}{lcl} \frac{\partial f}{\partial t}(s,t) & = & 1/\det d\eta(s,f(s,t)) \\ f(s,0) & = & 0 \end{array}\right\} \tag{24}$$

in which $s \in I$ is a parameter. The existence and uniqueness theorem for ordinary differential equations with parameters yields a smooth solution f on $I \times [0, \lambda_2]$ for some $\lambda_2 > 0$. Then $f(s,t) < \lambda_1$ for all $(s,t) \in I \times [0, \lambda_2]$. This and the second equation in (24) imply that the composition

$$\gamma_\zeta \colon (s,t) \mapsto (s, f(s,t)) \overset{\eta}{\mapsto} (u', v')$$

is a diffeomorphism of $I \times [0, \lambda_2]$ onto half of a tubular neighbourhood of ζ. Moreover, by the first equation in (24),

$$\det \gamma_\zeta(s,t) = \frac{\partial f}{\partial t}(s,t) \det d\eta(s, f(s,t)) = 1,$$

i.e., γ_ζ is area preserving. In view of the identities (23) for ζ, the map γ_ζ is the identity in $\mathbb{R}^2$ for s near u_0 and $t \in [0, \lambda_2]$, and γ_ζ is an isometry for s near $u_0 + l$ and $t \in [0, \lambda_2]$.

We now choose the parameter $\lambda_3 > 0$ in the construction of γ_δ smaller than λ_2. Restrict γ_ζ to the gray region $I \times]0, \lambda_3[$ in the image of γ_δ, and let $\overline{\gamma}_\zeta$ be the smooth extension of γ_ζ to the image of γ_δ which is the identity on $\{u \le u_0\}$ and an isometry on $\{u \ge u_0 + l\}$. By (i), the composition $\overline{\gamma}_\zeta \circ \gamma_\delta$ is the identity near $u = a/2 + \delta$ and an isometry near $u = a/2 + 2\delta$. It thus smoothly fits with the map $\gamma|_{U \setminus \mathcal{B}}$ already defined at the beginning of this step.

Step 5. We finally adjust the fibers. In view of the constructions in Step 2 and Step 3, the projection of the image $\varphi(\mathcal{S})$ onto the z_2-plane is contained in a tower shaped domain $\mathcal{T}$ (cf. Figure 12), and by the second inequality in (19) we have $\mathcal{T} \subset \left\{(x,y) \mid y < \frac{\pi}{2} + 4\delta\right\}$. Using once more our Lemma 2.5 we construct a symplectomorphism τ from a neighbourhood of $\mathcal{T}$ to a disc such that the preimages of the concentric circles in the image are as in Figure 12. We require that for $z_2 = (x,y)$,

$$\pi |\tau(z_2)|^2 < y + \frac{\pi}{2} + 3\delta \quad \text{for } y \ge -\frac{\pi}{2} - 2\delta, \tag{25}$$

$$\pi |\tau(z_2)|^2 < \pi |\sigma^{-1}(z_2)|^2 + \frac{\pi}{2} + 8\delta \quad \text{for } z_2 \in R_e, \tag{26}$$

where $\sigma \colon D(\pi) \to R_e$ is the diffeomorphism constructed in Step 2.

Step 1 to Step 5 are the ingredients of our folding construction. The folding map $\Psi \colon T(a, \pi) \hookrightarrow \mathbb{R}^4$ is defined as the composition of maps

$$\Psi = (id \times \tau) \circ (\gamma \times id) \circ \varphi \circ (id \times \sigma) \circ (\beta \times id) = (\gamma \times \tau) \circ \varphi \circ (\beta \times \sigma). \tag{27}$$

2.3. **End of the proof.** Recall that it remains to prove Proposition 2.9. So let $\epsilon > 0$ be as in Proposition 2.9 and set $\delta = \min\{\frac{1}{15}, \frac{\epsilon}{15}\}$. We define the desired map Ψ as in (27). It remains to verify that Ψ meets the required estimate (9). So let $z = (z_1, z_2) = (u, v, x, y) \in T(a, \pi)$

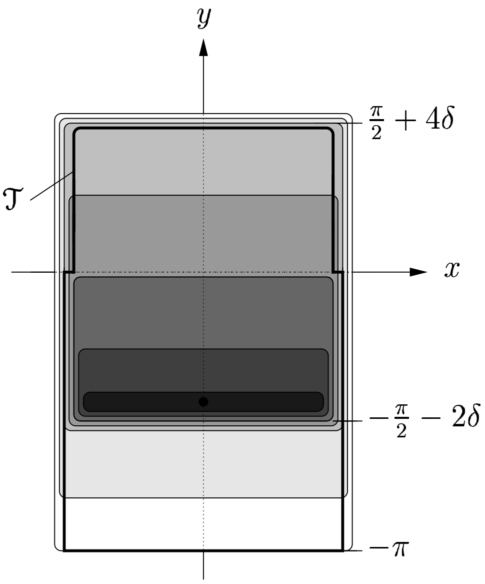

FIGURE 12. Mapping the tower to a disc.

and write $\Psi(z) = (u', v', z_2')$. By the choice of δ it suffices to show that for all $(u, v, z_2) \in T(a, \pi)$

$$u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 \; < \; \frac{a}{2} + 15\delta. \tag{28}$$

We distinguish three cases according to the locus of the image $\beta(z_1)$ in the set $U = P_1 \coprod L \coprod P_2$ (see Figure 5 and Figure 7). We denote the u-coordinate of $\beta(z_1) = \beta(u, v)$ by $u''\,(\beta(u, v))$.

Case 1. $\beta(z_1) \in P_1$. The first identity in (18) implies $\varphi|_{\mathcal{S}(P_1)} = id$, and Step 4 implies $\gamma|_{\mathcal{S}(P_1)} = id$. Therefore, $u' = u''(\beta(u, v))$. Moreover, $u''(\beta(u, v)) < u + \delta$. Indeed, the definition of the map β illustrated in Figure 7 shows that if $u''(\beta(u, v)) \le \frac{a}{2}$, then $u''(\beta(u, v)) = u$, and if $u''(\beta(u, v)) \in \left]\frac{a}{2}, \frac{a}{2} + \delta\right]$, then $u > \frac{a}{2}$. Summarizing, we have

$$u' \; < \; u + \delta.$$

Using again $\varphi|_{\mathcal{S}(P_1)} = id$ we find $\sigma(z_2) \in R_e$ and $z_2' = \tau(\sigma(z_2))$. Hence, the estimate (26) for the map τ yields

$$\pi|z_2'|^2 \; = \; \pi|\tau(\sigma(z_2))|^2 \; < \; \pi|\sigma^{-1}(\sigma(z_2))|^2 + \frac{\pi}{2} + 8\delta \; = \; \pi|z_2|^2 + \frac{\pi}{2} + 8\delta.$$

Finally, we have $u \leq \frac{a}{2} + \delta$. Indeed, if $u > \frac{a}{2} + \delta$, then the second identity in (15) implies $\beta(u, v) \in P_2$. Altogether we can estimate

$$\begin{aligned} u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 &< u\left(1 - \frac{\pi}{a}\right) + \delta + \frac{\pi}{2} + 8\delta \\ &< \frac{a}{2}\left(1 - \frac{\pi}{a}\right) + \frac{\pi}{2} + 10\delta \\ &= \frac{a}{2} + 10\delta. \end{aligned}$$

Case 2. $\beta(z_1) \in P_2$. By the second identity in (18) we have $\varphi|_{\mathcal{S}(P_2)} = id + (0, 0, 0, I_\infty)$, and so, in view of the identity (21), $u' = u'(\gamma(\beta(z_1))) = a + \frac{\pi}{2} + 12\delta - u''(\beta(u, v))$. Moreover, $u''(\beta(u, v)) > u + \frac{\pi}{2} + 10\delta$. Indeed, the definition of β shows that if $u''(\beta(u, v)) \geq \frac{a+\pi}{2} + 12\delta$, then $u''(\beta(u, v)) = u + \frac{\pi}{2} + 11\delta$, and if $u''(\beta(u, v)) \in \left[\frac{a+\pi}{2} + 11\delta, \frac{a+\pi}{2} + 12\delta\right[$, then $u < \frac{a}{2} + \delta$. Summarizing, we have

$$u' < a - u + 2\delta.$$

Step 2 shows $\sigma(z_2) \in R_i$, and so $y\big(\sigma(z_2) + (0, I_\infty)\big) \geq -\frac{\pi}{2} - 2\delta$. Hence, the estimates (25), (16) and (19) imply

$$\begin{aligned} \pi|z_2'|^2 &= \pi\left|\tau\big(\sigma(z_2) + (0, I_\infty)\big)\right|^2 \\ &< y(\sigma(z_2)) + I_\infty + \frac{\pi}{2} + 3\delta \\ &< \left(\pi|z_2|^2 - \frac{\pi}{2}\right) + \left(\frac{\pi}{2} + 5\delta\right) + \frac{\pi}{2} + 3\delta \\ &= \pi|z_2|^2 + \frac{\pi}{2} + 8\delta. \end{aligned}$$

Finally, we have $u \geq \frac{a}{2}$. Indeed, if $u < \frac{a}{2}$, then the first identity in (15) implies $\beta(u, v) \in P_1$. Altogether we can estimate

$$\begin{aligned} u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 &< a - u\left(1 + \frac{\pi}{a}\right) + 2\delta + \frac{\pi}{2} + 8\delta \\ &\leq a - \frac{a}{2}\left(1 + \frac{\pi}{a}\right) + \frac{\pi}{2} + 10\delta \\ &= \frac{a}{2} + 10\delta. \end{aligned}$$

Case 3. $\beta(z_1) \in L$. Using the definition of φ, the estimate (22) implies

$$u' < -u''(\beta(u, v)) + \frac{\pi}{2} + a + 13\delta.$$

Since $\pi|z_2|^2 < \frac{\pi}{2}$, we have $\sigma(z_2) \in R_i$, cf. Figure 8. In particular, $y\big(\sigma(z_2) + (0, I(u''(\beta(u, v))))\big) \geq -\frac{\pi}{2} - 2\delta$. Hence, the estimates (25) and (16) and the estimate $I(t) < (1-2\delta)(t - (\frac{a}{2}+2\delta))$ read off from Figure 9 yield

$$\begin{aligned}
\pi|z_2'|^2 &= \pi\big|\tau\big(x(\sigma(z_2)), y(\sigma(z_2)) + I(u''(\beta(u,v)))\big)\big|^2 \\
&< y(\sigma(z_2)) + I(u''(\beta(u,v))) + \frac{\pi}{2} + 3\delta \\
&< \left(\pi|z_2|^2 - \frac{\pi}{2}\right) + (1-2\delta)\left(u''(\beta(u,v)) - \frac{a}{2} - 2\delta\right) + \frac{\pi}{2} + 3\delta \\
&= \pi|z_2|^2 + u''(\beta(u,v)) - \frac{a}{2} - 2\delta - 2\delta u''(\beta(u,v)) + \delta a + 4\delta^2 + 3\delta.
\end{aligned}$$

Finally, we have $u''(\beta(u,v)) > \frac{a}{2} + \delta$ by the definition of L, and $u \geq \frac{a}{2}$ by the first identity in (15). Altogether we can estimate

$$\begin{aligned}
u' - \frac{\pi u}{a} + \pi|z_2'|^2 - \pi|z_2|^2 &< -u''(\beta(u,v)) + \frac{\pi}{2} + a + 13\delta - \frac{\pi}{a}\frac{a}{2} \\
&\quad + u''(\beta(u,v)) - \frac{a}{2} - 2\delta - 2\delta\left(\frac{a}{2} + \delta\right) \\
&\qquad\qquad + \delta a + 4\delta^2 + 3\delta \\
&= \frac{a}{2} + 14\delta + 2\delta^2 \\
&< \frac{a}{2} + 15\delta,
\end{aligned}$$

where in the last step we have used that $2\delta^2 < \delta$ which follows from $\delta < \frac{1}{15}$.

We have verified that the estimate (28) holds for all $(u, v, z_2) \in T(a,\pi)$, and the proof of Proposition 2.9 is complete. □

Recall that by Corollary 2.10, Proposition 2.9 implies Proposition 2.2, and so, in view of Corollary 2.3, the proof of Theorem 2.1 is complete.

Remarks 2.12.

1. As the verifications done in this section showed, the specific choice of the maps β, σ, φ, γ and τ constructed in the previous section is crucial for obtaining the required estimate (9).

2. We recall that the embedding $\Phi\colon E(a,\pi) \hookrightarrow B^4(\frac{a}{2}+\pi+\epsilon)$ in our construction is the composition

$$\begin{aligned}
\Phi &= c^{-1} \circ (\omega \times id) \circ \Psi \circ (\alpha \times id) \circ c \\
&= c^{-1} \circ (\omega \times id) \circ (id \times \tau) \circ (\gamma \times id) \circ \varphi \circ (id \times \sigma) \circ (\beta \times id) \circ (\alpha \times id) \circ c,
\end{aligned}$$

where c is the dilatation by a number close to 1.

3. The folding map $\Psi\colon T(a,\pi) \hookrightarrow T^4(A)$ can be visualized as in Figure 13, in which the pictures are to be understood in the same sense as the picture in Figure 2: The horizontal direction is the u-direction and refers to the base, while the vertical direction indicates the locus of the fibers. In the first two pictures and in the last one, the fibers are (contained in) discs, and in the other three pictures they are (contained in) rectangles. As illustrated in Figure 14, the map Ψ

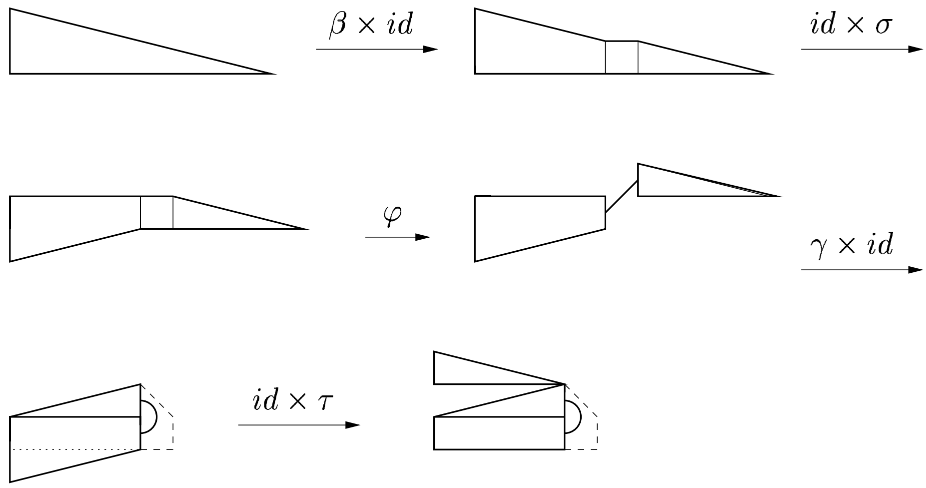

FIGURE 13. Folding an ellipsoid into a ball.

essentially restricts to the identity on the black rectangle and maps the triangle $\{u > \frac{a}{2}\}$ to the light triangle and the triangle $\{\pi|z_2|^2 > \frac{\pi}{2}\}$ to the dark triangle.

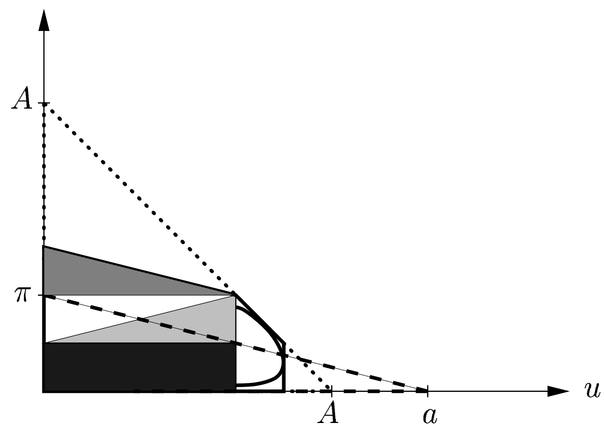

FIGURE 14. How the parts of the ellipsoid are mapped.

3. Proof of Theorem 3

Recall from the introduction that in order to prove Theorem 3 it is enough to show that for each $a > 2\pi$ and each $\epsilon > 0$ there exists a symplectic embedding $\Phi\colon E(\pi, a) \hookrightarrow B^4\left(s_2(a) + \epsilon\right)$ where

$$s_2(a) \;=\; 2\pi + (a - 2\pi)\frac{a + \pi}{3a + \pi}.$$

Using Lemma 2.8 we see as in the proof of Corollary 2.10 that to this end it is enough to prove

Proposition 3.1. *Assume $a > 2\pi$. Given $\epsilon > 0$, there exists a symplectic embedding*

$$\Psi\colon T(a, \pi) \;\hookrightarrow\; T^4\left(s_2(a) + \epsilon\right).$$

Proof. In order not to disturb the exposition unnecessarily with the arbitrarily small δ-terms (arising from "rounding off corners" and so on) we shall skip them. Since all the sets under consideration will be bounded and all constructions will involve only finitely many steps, we will not lose control of the δ-terms.

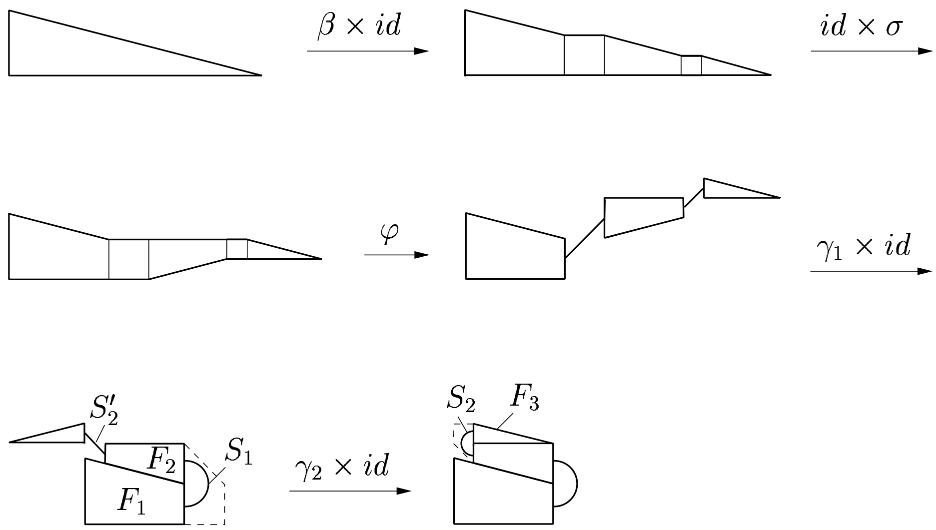

Figure 15. Folding twice.

We shall prove Proposition 3.1 by folding $T(a, \pi)$ twice. Up to the final fiber adjusting map τ, the folding map Ψ is the composition of maps explained in Figure 15, in which the pictures are to be understood as in Figure 2: The horizontal direction refers to the base and the vertical direction to the fibers. Here are the details: Recall that $T(a, \pi)$

fibers over the rectangle $R(a) = \{(u,v) \mid 0 < u < a,\, 0 < v < 1\}$. We set

$$u_1 = \frac{a^2 + a\pi}{3a+\pi} \quad \text{and} \quad u_2 = u_3 = \frac{a^2}{3a+\pi}. \tag{29}$$

Then $u_1 + u_2 + u_3 = a$. Define the heights h_1 and h_2 by

$$h_1 = \pi - \frac{\pi}{a} u_1 \quad \text{and} \quad h_2 = \pi - \frac{\pi}{a}(u_1 + u_2). \tag{30}$$

Using the definitions (29) of u_1 and u_2 we find that $h_2 = u_1 - u_2$.

Step 1 (Separating smaller fibers from larger ones)**.** Let U and f be as in Figure 16. Proceeding as in Step 1 of the folding construction in Section 2.2 we find a symplectic embedding $\beta\colon R(a) \hookrightarrow U$ such that $(\beta \times id)(T(a,\pi)) \subset \mathcal{F}(U,f)$.

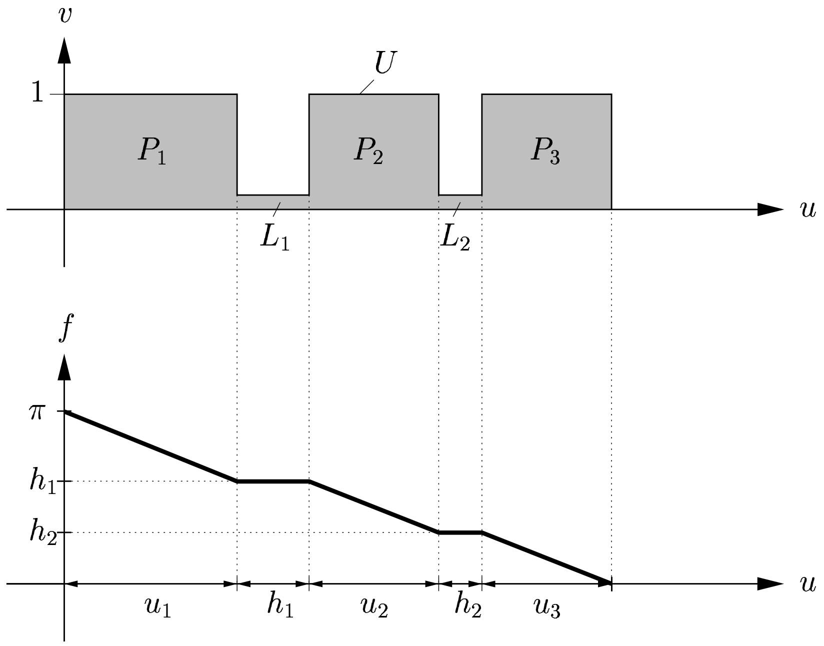

FIGURE 16. $\mathcal{F}(U,f)$.

Step 2 (Preparing the fibers)**.** The map σ is explained in Figure 17. More precisely, σ maps the central black disc to the black disc D, and up to some neglected δ-terms we have

$$y(\sigma(z_2)) = \begin{cases} h_1 - h_2 + \pi |z_2|^2 & \text{for most } z_2 \in D(h_2) \setminus D, \\ h_1 - \pi |z_2|^2 & \text{for most } z_2 \in D(h_1) \setminus D(h_2), \\ \pi |z_2|^2 & \text{for most } z_2 \in D(\pi) \setminus D(h_1). \end{cases}$$

Here, $D(h)$ denotes again the open disc of area h centered at the origin.

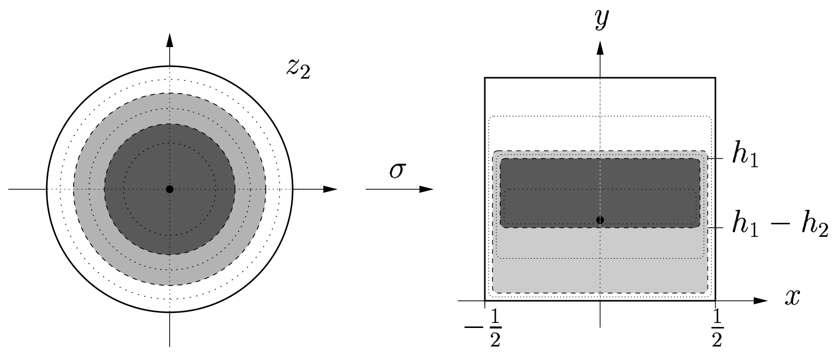

FIGURE 17. The map σ.

Step 3 (Lifting the fibers)**.** Choose cut off functions c_i over L_i, $i = 1, 2$, and abbreviate $c(t) = c_1(t)+c_2(t)$ and $I(t) = \int_0^t c(s)\, ds$. The symplectic embedding $\varphi\colon (\beta \times \sigma)(T(a,\pi)) \hookrightarrow \mathbb{R}^4$ is defined as in (17) by

$$\varphi(u,x,v,y) \;=\; \left(u, x, v + c(u)\left(x + \frac{1}{2}\right), y + I(u)\right).$$

Step 4 (Folding)**.** Step 4 in Section 2.2 now requires two steps.

1. The folding map γ_1 is essentially the map γ of Section 2.2: On the part of the base denoted by P_1 it is the identity, for $u_1 < u < u_1 + h_1$ it looks like the map in Figure 10, and for $u \geq u_1 + h_1$ it is an isometry. By construction, the stairs S_1 are contained in the "trapezoid" over $\{(u,v) \mid u_1 < u < u_1 + h_1,\, 0 < v < 1\}$ with left edge of length $2h_1$ and right edge of length h_1, cf. Figure 19. Moreover, the identity $h_2 = u_1 - u_2$ implies that the stairs S_2' lie over $\{(u,v) \mid 0 < u < h_2,\, 0 < v < 1\}$. By construction, the slope of the stairs S_2' is 1, while the slope of the upper edge of the floor F_1 is $\pi/a < 1$, and so the sets S_2' and F_1 are disjoint.

2. The map $\gamma_2 \times id$ is not really a global product map, but restricts to a product on certain pieces of its domain: It is the identity on $F_1 \coprod S_1 \coprod F_2$, and it is the product $\gamma_2 \times id$ on the remaining domain, where γ_2 is explained in Figure 18: It is the identity on the gray part of its domain, maps the black square to the black part of its image, and is an isometry on $\{u \leq 0\}$. The map γ_2 is constructed the same way as the map γ in Section 2.2.

By construction, the stairs S_2 are contained in a "trapezoid" over the set $\{(u,v) \mid 0 < u < h_2,\, 0 < v < 1\}$ with horizontal upper edge, left

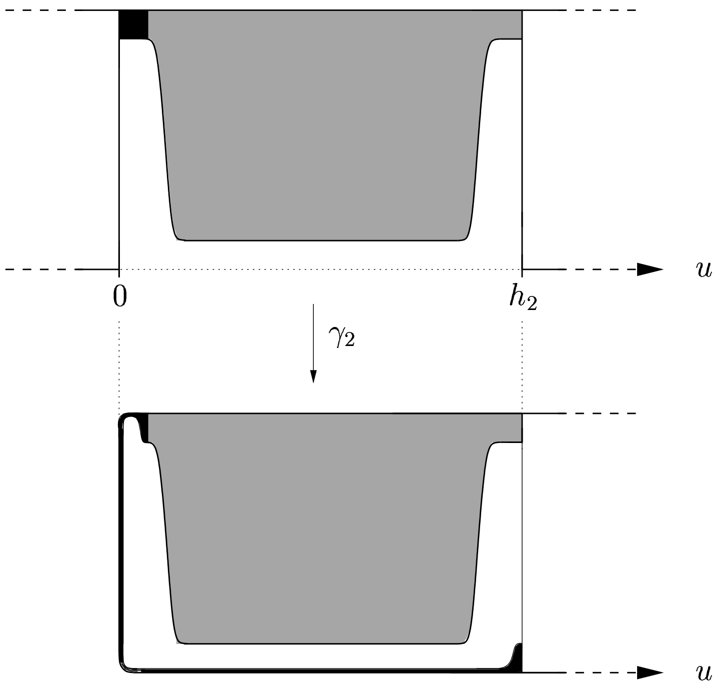

FIGURE 18. Folding on the left.

edge of height h_2 and right edge of height $2h_2$, and since $u_2 = u_3$, the lower edge of the floor F_3 coincides with the upper edge of F_2, cf. Figure 19.

Step 5 (Adjusting the fibers)**.** The z_2-projection of the image of φ is a tower shaped domain $\mathcal{T}$. The final map τ is a symplectomorphism from a small neighbourhood of $\mathcal{T}$ to a disc. We choose τ in such a way that up to some neglected δ-term we have for any $z_2 = (x, y)$, $z_2' = (x', y') \in \mathcal{T}$,

$$y < y' \implies |\tau(z_2)|^2 < |\tau(z_2')|^2.$$

This finishes the 2-fold folding construction. We define the embedding $\Psi \colon T(a, \pi) \hookrightarrow \mathbb{R}^4$ as the composition

$$\Psi \;=\; (id \times \tau) \circ (\gamma_2 \times id) \circ (\gamma_1 \times id) \circ \varphi \circ (\beta \times \sigma)\,.$$

If all the δ's were chosen appropriately, then

$$\Psi\left(T(a, \pi)\right) \;\subset\; T^4\left(u_1 + 2h_1 + \epsilon\right),$$

cf. Figure 19. Using the definitions (29) and (30) of u_1 and h_1 we find that

$$u_1 + 2h_1 \;=\; 2\pi + \frac{a + \pi}{3a + \pi} \;=\; s_2(a).$$

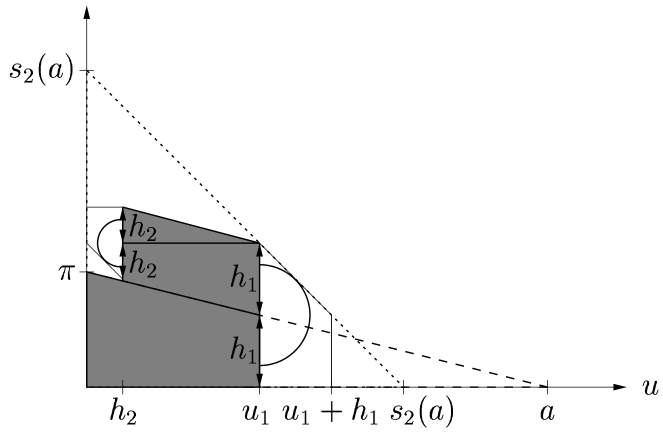

FIGURE 19. The image $\Psi\left(T(a,\pi)\right) \subset T^4\left(s_2(a)+\epsilon\right)$ for $a = 4\pi$.

This completes the proof of Proposition 3.1. □

Proceeding in a similar way as in the above proof, one can associate to each $m \geq 3$ and to $a > 2\pi$ and $\epsilon > 0$ an m-fold folding procedure which symplectically embeds $E(\pi, a)$ into $B^4\left(s_m(a)+\epsilon\right)$. We claim that

$$s_3(a) \;=\; 2\pi + (a-2\pi)\,\frac{(a+\pi)(a+2\pi)}{4\left(a^2+a\pi+\pi^2\right)}.$$

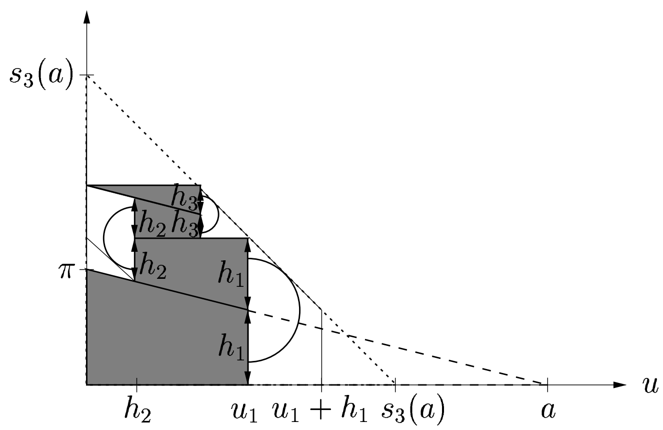

FIGURE 20. The image of $E(\pi, a)$ in $B^4\left(s_3(a)+\epsilon\right)$ for $a = 4\pi$.

Indeed, we read off from Figure 20 that

$$\begin{aligned} u_2 &= u_1 - h_2 \\ u_3 &= u_2 - 2h_3 \\ u_4 &= u_3 + h_2 \end{aligned}$$

where we abbreviated

$$h_i \;=\; \pi - \frac{\pi}{a}\sum_{j=1}^{i} u_j, \quad j = 1, 2, 3.$$

Substituting these equations into the equation

$$u_1 + u_2 + u_3 + u_4 = a$$

we find that

$$u_1 \;=\; \frac{a(a+\pi)(a+2\pi)}{4\left(a^2+a\pi+\pi^2\right)},$$

and so

$$s_3(a) \;=\; u_1 + 2h_1 \;=\; 2\pi + \left(1 - \frac{2\pi}{a}\right) u_1 \;=\; 2\pi + (a-2\pi)\,\frac{(a+\pi)(a+2\pi)}{4\left(a^2+a\pi+\pi^2\right)},$$

as claimed.

4. Sketch of a proof of Theorem 4 in dimension 4

The proof of Theorem 4 given in [11] combines a non-elementary result of [10] with an elementary but intricate filling procedure based on the symplectic folding method. In dimension 4, however, there is an elementary way of proving Theorem 4. The reason is that in dimension 4 a thin cuboid can almost be filled by a thin ellipsoid.

Define the rectangle $R(a,b)$ by

$$R(a,b) \;=\; \{(x,y) \mid 0 < x < a,\ 0 < y < b\}.$$

We denote the 4-dimensional cuboid $R(a,1) \times R(1,b)$ by

$$C(a,b) \;=\; R(a,1) \times R(1,b).$$

If $b = a$, we abbreviate the cube $C^4(a) = C(a,a)$.

Proposition 4.1. *Assume $a > \pi$. Given $\epsilon > 0$, there exists a symplectic embedding*

$$E(\pi,a) \hookrightarrow C\left(\frac{a+\pi}{2} + \epsilon, \pi + \epsilon\right).$$

Proof. As in the previous sections we can replace the ellipsoid $E(\pi, a)$ by the trapezoid $T(a, \pi)$. We shall embed $T(a, \pi)$ into $C\left(\frac{a+\pi}{2} + \epsilon, \pi + \epsilon\right)$ by folding once. We choose β as in Step 1 of the folding construction given in Section 2.2, replace the map σ of Step 2 by the map σ given by Figure 21, define φ as in (17), and choose γ as in Step 4.

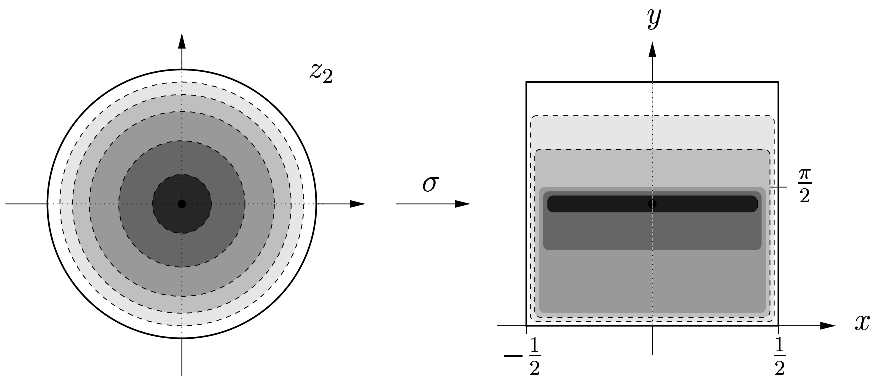

FIGURE 21. The map σ.

If all the δ's were chosen appropriately, the map Ψ defined by

$$\Psi = (\gamma \times id) \circ \varphi \circ (\beta \times \sigma)$$

embeds $T(a, \pi)$ into $C\left(\frac{a+\pi}{2} + \epsilon, \pi + \epsilon\right)$.

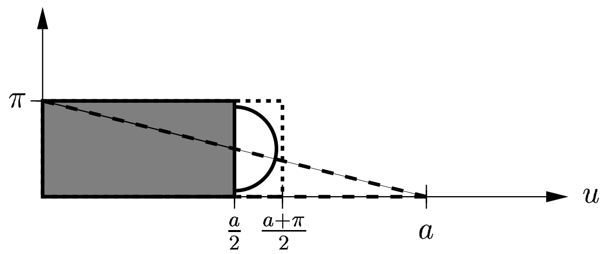

FIGURE 22. The image $\Psi\left(T(a, \pi)\right) \subset C\left(\frac{a+\pi}{2} + \epsilon, \pi + \epsilon\right)$.

The map Ψ can be visualized as in Figure 22. $\square$

Fix now a connected 4-dimensional symplectic manifold (M, ω) of finite volume $\mathrm{Vol}(M, \omega) = \frac{1}{2}\int_M \omega^2$. For $a \geq \pi$ we define

$$c_a(M, \omega) = \sup_\lambda \frac{\mathrm{Vol}\left(\lambda C(\pi, a)\right)}{\mathrm{Vol}\left(M, \omega\right)}$$

where the supremum is taken over all those λ for which $\lambda C(\pi, a)$ symplectically embeds into (M, ω). The desired identity

$$\lim_{a\to\infty} e_a(M,\omega) \;=\; 1$$

is a consequence of Proposition 4.1 and

Theorem 4.2. *We have* $\lim_{a\to\infty} c_a(M,\omega) \;=\; 1$.

Sketch of proof. In the case that (M, ω) is a cube, Theorem 4.2 follows from

Proposition 4.3. *For every $\epsilon > 0$ and every natural number N there exists a symplectic embedding*

$$C\left(\pi, \left(N^2+1\right)\pi\right) \;\hookrightarrow\; C^4\left((N+1)\pi + \epsilon\right).$$

Proof. We fold $C\left(\pi, (N^2+1)\pi\right)$ alternatingly on the right at $N\pi$ and on the left at π, and fold altogether N times, cf. Figure 23. □

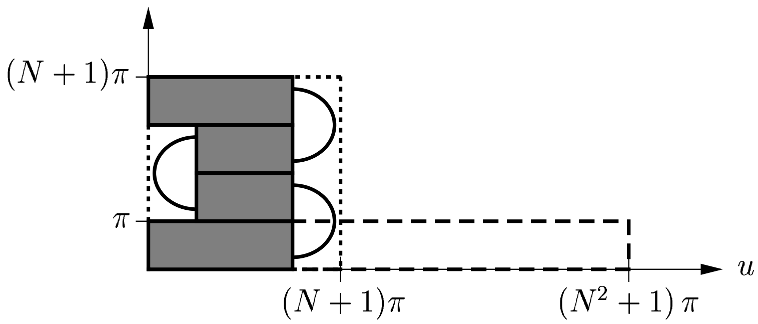

FIGURE 23. The embedding $C\left(\pi, (N^2+1)\pi\right) \hookrightarrow C^4\left((N+1)\pi+\epsilon\right)$ for $N = 3$.

In the general case, Theorem 4.2 can be proved along the following lines: First, fill almost all of M with finitely many symplectically embedded cubes whose closures are disjoint, and connect these cubes by neighbourhoods of lines. In view of Proposition 4.3, the cubes can almost be filled with symplectically embedded thin cuboids, and the neighbourhoods of the lines can be used to pass from one cube to another, cf. Figure 24.
We refer to [11, Section 5.1] for a complete proof. □

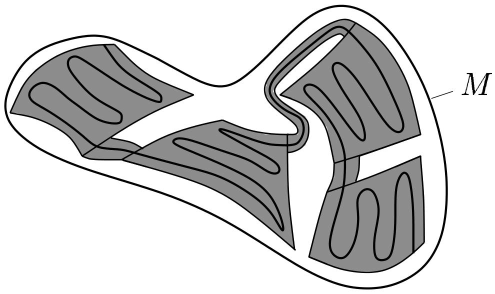

FIGURE 24. Filling M with a thin cuboid.

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