Symplectic field theory and its applications

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Abstract. Symplectic field theory (SFT) attempts to approach the theory of holomorphic curves in symplectic manifolds (also called Gromov-Witten theory) in the spirit of a topological field theory. This naturally leads to new algebraic structures which seems to have interesting applications and connections not only in symplectic geometry but also in other areas of mathematics, e.g. topology and integrable PDE. In this talk we sketch out the formal algebraic structure of SFT and discuss some current work towards its applications.

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1. Formal algebraic structure of SFT

The project of SFT was initiated by A. Givental, H. Hofer and the author in [15]. Since its inception, it has branched in different directions and now involves a large number of authors working on the foundation and the different parts of the project. SFT is closely related to the *relative Gromov–Witten theory*, see e.g. [21], [30], [35], [34], [31], as well the work of Yu. Chekanov [11] and Fukaya–Oh–Ohta–Ono project [20].

Symplectic field theory can be viewed as a functor SFT from a geometric category GEOM_{SFT} of framed Hamiltonian structures and framed cobordisms between them to an algebraic category ALG_{SFT} of certain differential D-modules and Fourier integral operators between them. We describe these categories in the next two sections.

1.1. The category ALG_{SFT}. Roughly speaking, the objects in the category ALG_{SFT} are certain D-modules over a graded Weyl algebra with an operator $\mathbb{H}$ which satisfies the "master equation" $\mathbb{H} \circ \mathbb{H} = 0$. Before listing the algebraic structures involved, let us make a couple of general remarks. First, we will be dealing in this paper with graded objects. To simplify the exposition we will usually mean by grading a $\mathbb{Z}/2$-grading, unless it is noted otherwise. Usually, with extra work it can be upgraded to an integer grading. Second, we will systematically use $\mathbb{C}$ as the coefficient ring. In some situations it has to be changed to a certain Novikov ring, see Remark 2.1 below.

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Given an integer vector $\boldsymbol{d} = (d_1, \ldots, d_N)$ let us denote by $\mathsf{CT}_{\boldsymbol{d}} = \mathbb{C}[T]$ a $\mathbb{Z}$-graded (super-)commutative algebra with complex coefficients, generated by graded elements of an infinite $N \times \infty$ matrix $T = (t_{ij})$, $i = 1, \ldots, N$, $j = 0, \ldots$; the $\mathbb{Z}/2$-grading of t_{ij} coincides with the parity of d_i for each $j \geq 0$ and $i = 1, \ldots, N$.

An *object* in the category ALG_{SFT} is a collection of the following structures O1–O5, which satisfy axioms AO1 and AO2.

O1. A possibly infinite-dimensional space $\mathcal{P}$ with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a fixed basis Γ. To simplify the notation we will assume that there exists an involution $\gamma \mapsto \bar{\gamma}$, $\gamma \in \Gamma$, such that $\langle \gamma, \gamma' \rangle = \delta_{\bar{\gamma}\gamma'}$, $\gamma, \gamma' \in \Gamma$.

O2. A $\mathbb{Z}$-graded, possibly infinite-dimensional vector space $\mathbb{V} = \oplus_j \mathbb{V}_j$ over $\mathbb{C}$, called the *phase space*, with a degree 1 differential $d \colon \mathbb{V} \to \mathbb{V}$, such that $d^2 = 0$ (e.g. a space of differential forms on a manifold with de Rham differential d). For $\boldsymbol{d} = (d_1, \ldots, d_N)$ we denote by $V^{\boldsymbol{d}}$ the space $\oplus_i^N \mathbb{V}_{d_i}$.

O3. An associative algebra $\widetilde{\mathsf{W}}$ over $\mathbb{C}$ generated by graded elements $p_{k,\gamma}$, $q_{k,\gamma}$, $\gamma \in \Gamma$, $k \geq 1$, and an even graded element $\hbar$, with the following commutation relations: all elements commute (in the graded sense) except that the (graded) commutator $[p_{k,\gamma}, q_{k,\bar{\gamma}}]$ equals $k\hbar$ for any $\gamma \in \Gamma$ and $k \geq 1$.

O4. A completion W of $\widetilde{\mathsf{W}}$, called the *Weyl algebra*, which consists of formal power series of $\hbar$ and p-variables with coefficients which are polynomials of q-variables.

O5. A smooth function $\mathbb{H} \colon \mathbb{V}^{\boldsymbol{d}} \to \frac{1}{\hbar}\mathsf{W} \otimes \mathsf{CT}_{\boldsymbol{d}}$, which associates with any $\Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{V}^{\boldsymbol{d}}$ an odd (in fact, of degree 1 if the integer grading is used) element $\mathbb{H}\langle\Theta\rangle \in \frac{1}{\hbar}\mathsf{W} \otimes \mathsf{CT}_{\boldsymbol{d}}$, called the *Hamiltonian*. Here smoothness is understood in the formal sense: all coefficients of the corresponding power expansions are smooth.

Before formulating the axioms let us introduce some notation. Given two vectors $\boldsymbol{d} = (d_1, \ldots, d_N)$ and $\boldsymbol{d}' = (d'_1, \ldots, d'_{N'})$ we denote by $\boldsymbol{d} \sqcup \boldsymbol{d}'$ the vector $(d_1, \ldots, d_N, d'_1, \ldots, d'_{N'})$. Similarly, for $\Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{V}^{\boldsymbol{d}}$ and $\Theta' = (\theta'_1, \ldots, \theta'_{N'}) \in \mathbb{V}^{\boldsymbol{d}'}$ we write

$$\Theta \sqcup \Theta' = (\theta_1, \ldots, \theta_N, \theta'_1, \ldots, \theta'_{N'}) \in \mathbb{V}^{\boldsymbol{d} \sqcup \boldsymbol{d}'}.$$

We will also denote by t_{ij} and t'_{kl}, $i = 1, \ldots, N$, $k = 1, \ldots, N'$, $j, l = 1 \ldots$, the generators of the algebra $\mathsf{CT}_{\boldsymbol{d} \sqcup \boldsymbol{d}'}$, and by $\pi : \mathsf{CT}_{\boldsymbol{d} \sqcup \boldsymbol{d}'} \to \mathsf{CT}_{\boldsymbol{d}}$ the projection.

The following axioms should be satisfied:

AO1.
$$\mathbb{H}\langle\Theta \sqcup d\Theta\rangle \circ \mathbb{H}\langle\Theta \sqcup d\Theta\rangle = \partial\mathbb{H}\langle\Theta \sqcup d\Theta\rangle, \tag{1}$$

where

$$\partial = \sum_{i,j} t_{ij} \frac{\partial}{\partial t'_{ij}} \tag{2}$$

is an odd differential on $\mathsf{CT}_{\boldsymbol{d}\sqcup\boldsymbol{d}'}$, $\boldsymbol{d}' = (d_1+1, \dots, d_N+1)$.

Note that $\partial^2 = 0$.

AO2.

$$\mathbb{H}\langle\Theta\rangle = \pi\left(\mathbb{H}\langle\Theta \sqcup 0\rangle\right), \tag{3}$$

for $\Theta \in \mathbb{V}^{\boldsymbol{d}}$ and $0 \in \mathbb{V}^{\boldsymbol{d}'}$.

Let us note an important corollary of the axioms. Suppose that $d\Theta = 0$. Then we have

$$\mathbb{H}\langle\Theta\rangle \circ \mathbb{H}\langle\Theta\rangle = 0. \tag{4}$$

Hence, if $d\Theta = 0$ then $\left(\mathsf{W} \otimes \mathsf{CT}^{\boldsymbol{d}}, \mathbb{D}\langle\Theta\rangle\right)$ is a differential Weyl algebra with a differential

$$\mathbb{D}\langle\Theta\rangle(x) = [\mathbb{H}\langle\Theta\rangle, x], \quad x \in \mathsf{W} \otimes \mathsf{CT}^{\boldsymbol{d}}.$$

More generally, for *any* $\Theta \in \mathbb{V}^{\boldsymbol{d}}$ we can define a differential Weyl algebra

$$\left(\mathsf{W} \otimes \mathsf{CT}^{\boldsymbol{d}\sqcup\boldsymbol{d}'}, \mathbb{D}\langle\Theta \sqcup d\Theta\rangle\right)$$

with a differential

$$\mathbb{D}\langle\Theta \sqcup d\Theta\rangle(x) = \partial x + [\mathbb{H}\langle\Theta\rangle, x], \quad x \in \mathsf{W} \otimes \mathsf{CT}^{\boldsymbol{d}\sqcup\boldsymbol{d}'}.$$

Let us also consider

- a space F of formal Fourier series

$$\sum_{k=1}^{\infty} P_k e^{ikx} + Q_k e^{-ikx}, \tag{5}$$

  where $P_k = \{p_{k,\gamma}\}_{\gamma\in\Gamma}$, $Q_k = \{q_{k,\gamma}\}_{\gamma\in\Gamma}$, $k = 1, \dots,$ are ordered strings of graded variables, indexed by elements of Γ; the space F is canonically *polarized*, i.e. split $\mathsf{F} = \mathsf{F}_+ \oplus \mathsf{F}_-$, where F_+ (resp. F_-) is formed by Fourier series with only positive (resp. negative) coefficients;

- a space Fock which consists of formal series $\sum_{k\ge 0} f_k \hbar^k$, where f_k are functionals on the space F_- which can be expressed as polynomials of Fourier coefficients $q_{k,\gamma}$.

Note that the space Fock can be viewed as a D-module over $\frac{1}{\hbar}\mathsf{W}$. Indeed, the quantization

$$p_{k,\gamma} \mapsto k\hbar \frac{\partial}{\partial q_{k,\bar{\gamma}}}$$

provides a representation of $\frac{1}{\hbar}\mathsf{W}$ as the algebra of differential operators acting from the left on elements of the space Fock. Given an element $A \in \frac{1}{\hbar}\mathsf{W}$ we will denote by $\lfloor A \rfloor$ the corresponding differential operator if we need to distinguish it from A.

For each Θ with $d\Theta = 0$ the equation (4) implies that the operator $\mathbb{H}(\Theta)$ is a differential on $\mathsf{Fock} \otimes \mathsf{CT}_d$. Indeed, if we define

$$D_\Theta = \lfloor \mathbb{H}\langle\Theta\rangle \rfloor f, \quad f \in \mathsf{Fock} \otimes \mathsf{CT}_{\boldsymbol{d}},$$

then $D_\Theta^2 = 0$.

Consider now two objects $\mathcal{O}^+, \mathcal{O}^- \in \mathrm{ALG}_{SFT}$. We will label with $\pm$ all the structures associated with these objects. *Morphisms* $\mathcal{O}^+ \to \mathcal{O}^-$ are formed by the following structures:

- A graded space $\mathbb{V} = \sum_j \mathbb{V}_j$ over $\mathbb{C}$ with a differential $d \colon \mathbb{V} \to \mathbb{V}$, $d^2 = 0$, and two grading-preserving *restriction homomorphisms* $R^\pm \colon \mathbb{V} \to \mathbb{V}^\pm$;

- A graded commutative algebra A_-^+ over $\mathbb{C}$ which consists of formal power series of $\hbar$ and p^+-variables whose coefficients are polynomials of q^--variables. Note that the Weyl algebra $\frac{1}{\hbar}\mathsf{W}^-$ acts on A_-^+ on the left by quantizing

$$p^-_{k,\gamma} \mapsto k\hbar \frac{\overrightarrow{\partial}}{\partial q^-_{k,\bar\gamma}}, \tag{6}$$

  while the Weyl algebra W^+ acts on A_-^+ on the right by quantizing

$$q^+_{k,\gamma} \mapsto k\hbar \frac{\overleftarrow{\partial}}{\partial p^+_{k,\bar\gamma}}; \tag{7}$$

  given an element $D \in \frac{1}{\hbar}\mathsf{W}^\pm$ we will denote by $\lfloor D \rfloor$ and $\lceil D \rceil$ its quantizations defined by formulas (6) and (7), respectively.

- A smooth function

$$\widehat{\mathbb{V}}^{\boldsymbol{d}} \to \frac{1}{\hbar}\mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}}$$

  which associates with a string $\Theta = (\theta_1, \dots, \theta_N) \in \widehat{\mathbb{V}}^{\boldsymbol{d}}$ an even (of degree 0 in the situation of $\mathbb{Z}$-grading) element

$$\boldsymbol{\Phi}\langle\Theta\rangle = \frac{1}{\hbar}\boldsymbol{\Phi}\langle\Theta\rangle(T, q^-, p^+, \hbar) \in \mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}},$$

  called the *potential*.

The following axioms should be satisfied:

AM1.

$$\lfloor \mathbb{H}^-\langle R^-(\Theta \sqcup d\Theta)\rangle \rfloor e^{\mathbf{\Phi}\langle\Theta\sqcup d\Theta\rangle} + e^{\mathbf{\Phi}\langle\Theta\sqcup d\Theta\rangle}\lceil \mathbb{H}^+\langle R^+(\Theta \sqcup d\Theta)\rangle\rceil = \partial e^{\mathbf{\Phi}\langle\Theta\rangle}, \tag{8}$$

where the linear differential operator $\partial = \sum_{i,j} t_{ij}\frac{\partial}{\partial t'_{ij}}$ is defined above in (2).

AM2.

$$\mathbf{\Phi}\langle\Theta\rangle = \pi\left(\mathbf{\Phi}\langle\Theta \sqcup 0\rangle\right), \tag{9}$$

where $\Theta \in \mathbb{V}^{\boldsymbol{d}}$, $0 \in \mathbb{V} \in \mathbb{V}^{\boldsymbol{d}'}$, and π is the projection $\mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}\sqcup\boldsymbol{d}'} \to \mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}}$.

An important partial case is when $d\Theta = 0$. In this case the axioms imply

$$\lfloor \mathbb{H}^-\langle R^-(\Theta)\rangle \rfloor e^{\mathbf{\Phi}\langle\Theta\rangle} + e^{\mathbf{\Phi}\langle\Theta\rangle}\lceil \mathbb{H}^+\langle R^+(\Theta)\rangle\rceil = 0. \tag{10}$$

Note that $\mathbf{\Phi}$ defines for each Θ a formal Fourier integral operator

$$\widetilde{\Phi}\langle\Theta\rangle\colon \mathsf{Fock}_+ \otimes \mathsf{CT}_{\boldsymbol{d}} \to \mathsf{Fock}_- \otimes \mathsf{CT}_{\boldsymbol{d}}$$

by the formula

$$\widetilde{\Phi}\langle\Theta\rangle(f)(T, q^-, \hbar) = \left(e^{\frac{1}{\hbar}\mathbf{\Phi}\langle\Theta\rangle(T,q^-,p^+,\hbar)}\left\lceil f(T, q^+, \hbar)\right\rceil\right)\Big|_{p^+=0}. \tag{11}$$

If $d\Theta = 0$ then the equation (10) translates into the fact that

$$\widetilde{\Phi}\langle\Theta\rangle\colon \left(\mathsf{Fock}^+ \otimes \mathsf{CT}_{\boldsymbol{d}}, D_{R^+(\Theta)}\right) \to \left(\mathsf{Fock}^- \otimes \mathsf{CT}_{\boldsymbol{d}}, D_{R^-(\Theta)}\right)$$

is a *chain map*.

Suppose now that we are given three objects $\mathcal{O}_0$, $\mathcal{O}_1$ and $\mathcal{O}_2$, and morphisms $\mathbf{\Phi}_{01}\colon \mathcal{O}_0 \to \mathcal{O}_1$ and $\mathbf{\Phi}_{12}\colon \mathcal{O}_1 \to \mathcal{O}_2$. Then their composition $\mathbf{\Phi}_{02}\colon \mathcal{O}_0 \to \mathcal{O}_2$ is defined as follows. First, we define the phase space $\mathbb{V}_{02}$ as the fiber product

$$\mathbb{V}_{02} = \{(\theta_{01}, \theta_{12});\ \theta_{01} \in \mathbb{V}_{01}, \theta_{12} \in \mathbb{V}_{12}, R_{01}^-(\theta_{01}) = R_{12}^+(\theta_{12})\}.$$

Given $\Theta_{02} = (\Theta_{01}, \Theta_{12}) \in \mathbb{V}_{02}^{\boldsymbol{d}}$ we define an element

$$\mathbf{\Phi}_{02}\langle\Theta_{02}\rangle = \frac{1}{\hbar}\Phi_{02}\langle\Theta_{02}\rangle(T, q_2, p_0, \hbar) \in \frac{1}{\hbar}\mathsf{A}_2^0$$

by the formula

$$\begin{aligned} e^{\frac{1}{\hbar}\mathbf{\Phi}_{02}\langle\Theta_{02}\rangle(T,q_2,p_0,\hbar)} &= \left(\left\lfloor e^{\frac{1}{\hbar}\mathbf{\Phi}_{12}\langle\Theta_{12}\rangle(T,q_2,p_1,\hbar)}\right\rfloor e^{\frac{1}{\hbar}\mathbf{\Phi}_{01}\langle\Theta_{01}\rangle(T,q_1,p_0,\hbar)}\right)\Big|_{q_1=0} \\ &= \left(e^{\frac{1}{\hbar}\mathbf{\Phi}_{12}\langle\Theta_{12}\rangle(T,q_2,p_1,\hbar)}\left\lceil e^{\frac{1}{\hbar}\mathbf{\Phi}_{01}\langle\Theta_{01}\rangle(T,q_1,p_0,\hbar)}\right)\right\rceil\Big|_{p_1=0}. \end{aligned} \tag{12}$$

Note that the corresponding operator $\widetilde{\Phi}_{02}(\Theta_{02})$ is the composition:

$$\widetilde{\Phi}_{02}\langle\Theta_{02}\rangle = \widetilde{\Phi}_{12}\langle\Theta_{12}\rangle \circ \widetilde{\Phi}_{01}\langle\Theta_{12}\rangle\colon \mathsf{Fock}_0 \otimes \mathsf{CT}_{\boldsymbol{d}} \to \mathsf{Fock}_2 \otimes \mathsf{CT}_{\boldsymbol{d}}.$$

1.2. The category $\mathbf{GEOM}_{SFT}$. The exposition in this section is essentially taken from Section 4.1 in [17]. A *Hamiltonian structure* is a pair (V, Ω), where V is an oriented manifold of dimension $2n-1$ endowed with a closed 2-form Ω of maximal rank ($= 2n-2$). The tangent line field $\ell = \operatorname{Ker}\Omega$ is called the *characteristic* line field. The field ℓ integrates to a 1-dimensional *characteristic foliation* of Ω. Note that Ω defines a fiber-wise symplectic structure (and hence an orientation) on the bundle TV/ℓ. Thus the line bundle ℓ is equipped with an orientation. We will call *characteristic* any vector field R which generates ℓ and respects its orientation.

Any co-orientable hypersurface V in a symplectic manifold $(W, \widetilde{\Omega})$ inherits a Hamiltonian structure $\widetilde{\Omega}|_V$. Conversely, any Hamiltonian structure (V, Ω) embeds as a hypersurface in a symplectic manifold $(V \times (-\varepsilon, \varepsilon), \widetilde{\Omega})$ where the form $\widetilde{\Omega}$ can be constructed as follows. Let λ be any 1-form which is not vanishing on ℓ, and s the coordinate along the second factor. Then we set $\widetilde{\Omega} = \Omega + d(s\lambda)$. Note that by Darboux's theorem the Hamiltonian structure (V, Ω) determines its symplectic extension to a neighborhood of the hypersurface $V = V \times 0 \subset V \times (-\varepsilon, \varepsilon)$ uniquely up to a diffeomorphism fixed on V. We call (a germ along V of) the symplectic structure $\widetilde{\Omega}$ on $V \times (-\varepsilon, \varepsilon)$ the *symplectic extension* of (V, Ω).

A Hamiltonian structure $\mathcal{H} = (V, \Omega)$ is called *stable* (see [24]) if its symplectic extension can be realized by a form $\widetilde{\Omega}$ on $V \times (-\varepsilon, \varepsilon)$ such that the Hamiltonian structures induced on hypersurfaces $V \times s$, $s \in (-\varepsilon, \varepsilon)$, all have the same characteristic line field ℓ. It is easy to check (see [17]) that

Proposition 1.1. *A Hamiltonian structure $\mathcal{H} = (V, \Omega)$ is stable if and only if there exists a 1-form λ and a characteristic vector field R such that*

$$\lambda(R) = 1 \quad \textit{and} \quad i_R d\lambda = 0. \tag{13}$$

Note that in view of Cartan's formula we have $L_R\lambda = d(\lambda(R)) + i_R d\lambda$, and hence the second condition can be restated as invariance of λ under the flow of R.

A *framing* of a stable Hamiltonian structure is a pair (λ, J) where

- λ is as in (13); the form λ automatically defines the hyperplane field $\xi = \{\lambda = 0\}$, called a *cut* of the Hamiltonian structure, and the vector field R, called its *Reeb* field;
- J is an almost complex structure on ξ (also called a CR-structure on V) compatible with the symplectic form Ω.

Here are three major examples of stable framed Hamiltonian structures.

Example 1.2. (1) *Contact forms.* Let ξ be a contact structure on V, i.e. a completely non-integrable tangent hyperplane field, and λ a contact form for ξ, i.e. $\xi = \{\lambda = 0\}$. Let an almost complex structure $J \colon \xi \to \xi$ be compatible with $d\lambda|_\xi$. Then $\mathcal{H} = (\Omega = d\lambda, \lambda, J)$ is a framed stable Hamiltonian structure on V with the cut ξ, and R is the usual Reeb field of the contact form λ. We say in this case that the Hamiltonian structure $\mathcal{H}$ is of *contact* type.

(2) *Hamiltonian functions.* Let (M, ω) be a symplectic manifold and $H_t : M \to \mathbb{R}$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$, a 1-periodic time-dependent Hamiltonian function. Set $V = M \times S^1$, $\Omega = -\omega + H_t dt$ and $\lambda = dt$. Let J be an almost complex structure on M compatible with ω. Then $\mathcal{H} = (V, \Omega, \lambda, J)$ is a framed stable Hamiltonian structure. Its Reeb vector field is given by $R = \frac{\partial}{\partial t} + \mathrm{sgrad}\, H_t$, where sgrad H_t is the Hamiltonian vector field defined by H_t. We say in this case that $\mathcal{H}$ is of *Floer* type.

(3) S^1*-bundles.* Let (M, ω) be a symplectic manifold and $p: V \to M$ any S^1-bundle over it. Set $\Omega = p^*\omega$. Then Ω is a stable Hamiltonian structure on V. Indeed, one can choose any S^1-connection form λ as its framing. The corresponding Reeb vector field R is the infinitesimal generator of the S^1-action, and the cut of ξ is formed by the horizontal spaces of the connection. Let J_M be an almost complex structure on M compatible with ω, and J be the pull-back of J_M to ξ via the projection $V \to M$. We say that a framed Hamiltonian structure $\mathcal{H} = (V, \Omega, \lambda, J)$ is of *fibration* type. Note that if the cohomology class $[\omega]$ of the symplectic form is integral, then one could take as V the corresponding *pre-quantization space* , i.e. the principal S^1-bundle $p: V \to M$ with the first Chern class $[\omega]$. In this case the lift $\Omega = p^*\omega$ of the symplectic form is exact and one can choose λ to be a primitive of Ω. Hence, in this case (V, Ω, λ, J) is also of contact type.

All Hamiltonian structures which we consider in this paper will be assumed stable.

Framed Hamiltonian structures are *objects in the category* GEOM_{SFT}, while morphisms are *framed symplectic cobordisms* which we describe below.

A *symplectic cobordism* between two Hamiltonian structures $\mathcal{H}_+ = (V_+, \Omega_+)$ and $\mathcal{H}_- = (V_-, \Omega_-)$ is a symplectic manifold (W, Ω) such that $\partial W = V_+ \cup (-V_-)$ and $\Omega|_{V_\pm} = \Omega_\pm$. Note that "symplectic cobordism" is a partial order, and not an equivalence relation, because it is not symmetric. A *framed* symplectic cobordism between two framed Hamiltonian structures $\vec{\mathcal{H}}_+ = (V_+, \Omega_+, \lambda_+, J_+)$ and $\vec{\mathcal{H}}_- = (V_-, \Omega_-, \lambda_-, J_-)$ is a cobordism (W, Ω) between $\mathcal{H}_+$ and $\mathcal{H}_-$ equipped with an almost complex structure J which is compatible with Ω, and such that $J(\xi_\pm) = \xi_\pm$; here $\xi_\pm$ denotes the cut $\{\lambda_\pm = 0\}$ of the framed Hamiltonian $\vec{\mathcal{H}}_\pm$. *Morphisms* in the category GEOM_{SFT} are *multi-storied framed symplectic cobordisms*, i.e. sequences $(C_{0,1}, C_{1,2}, \dots, C_{k-1,k})$ where $C_{j-1,j} = (W_{j-1,j}, \Omega_{j-1,j}, J_{j-1,j})$ is a framed symplectic cobordism between framed Hamiltonian structures $\vec{\mathcal{H}}_{j-1}$ and $\vec{\mathcal{H}}_j$, $j = 1, \dots, k$. An associative operation of composition of morphisms is defined in an obvious way as concatenation of such sequences.

1.3. 2-categories. Both categories, GEOM_{SFT} and ALG_{SFT}, can be upgraded to 2-categories which are respected by the functor *SFT*.

On the geometric side, a 2-morphism is a fixed on the boundary homotopy of symplectic cobordisms and their framings. More precisely, a 2-*morphism* is a pair (Ω_s, J_s), $s \in [0, 1]$, where Ω_s is a family of symplectic forms on a cobordism W

such that $\Omega_s|_{\partial W} = \Omega_0|_{\partial W}$, $\Omega_s = d\Xi_s$, $\Xi_s|_{\partial W} = 0$, $s \in [0,1]$, and J_s is a fixed on ∂W deformation of almost complex structures compatible with Ω_s. The notion of homotopy can be extended to morphisms represented by multi-storied cobordisms via the process, called *splitting* or *stretching the neck*. We refer the reader to [15], [8] for the precise definition.

Let us move now to the algebraic side of the story. Let $\mathbf{\Phi}^{(0)}, \mathbf{\Phi}^{(1)}\colon \mathcal{O}^+ \to \mathcal{O}^-$ be two morphisms, where for $\Theta \in \mathbb{V}^{\boldsymbol{d}}$. We have

$$\mathbf{\Phi}^{(s)}(\Theta) = \frac{1}{\hbar}\Phi^{(s)}\langle\Theta\rangle(T, q^-, p^+, \hbar) \in \frac{1}{\hbar}\mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}}, \quad s = 0, 1.$$

A 2-*morphism* between $\mathbf{\Phi}^{(0)}, \mathbf{\Phi}^{(1)}$ is a function which associates with $\Theta \in \mathbb{V}^{\boldsymbol{d}}$ a family

$$\boldsymbol{K}^{(s)}\langle\Theta\rangle = \frac{1}{\hbar}\boldsymbol{K}^{(s)}\langle\Theta\rangle(T, q^-, p^+, \hbar) \in \frac{1}{\hbar}\mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}}, \quad s \in [0,1].$$

When $d\Theta = 0$ then $\boldsymbol{K}^{(s)}$ generates a homotopy $\mathbf{\Phi}^{(s)} = \frac{1}{\hbar}\mathbf{\Phi}^{(s)}\langle\Theta\rangle(T, q^-, p^+, \hbar)$, $s \in [0,1]$, defined by the following differential equation:

$$\frac{d\mathbf{\Phi}^{(s)}}{ds} = e^{-\mathbf{\Phi}^{(s)}}\big([\lfloor\mathbb{H}^-\rfloor, \boldsymbol{K}^{(s)}]e^{\mathbf{\Phi}^{(s)}} + e^{\mathbf{\Phi}^{(s)}}[\lceil\mathbb{H}^+\rceil, \boldsymbol{K}^{(s)}]\big), \quad s \in [0,1], \tag{14}$$

where we identify $\boldsymbol{K}^{(s)}$ with an operator of multiplication by $\boldsymbol{K}^{(s)}$ acting on the algebra $\mathsf{A}_-^+ \otimes \mathsf{CT}_{\boldsymbol{d}}$. More generally, for any Θ we define a homotopy $\mathbf{\Phi}^{(s)}\langle\Theta \sqcup d\Theta\rangle$ by the equation

$$\frac{d\mathbf{\Phi}^{(s)}}{ds}\langle\Theta\sqcup d\Theta\rangle = e^{-\mathbf{\Phi}^{(s)}}\big([\lfloor\mathbb{H}^-\rfloor+\partial, \boldsymbol{K}^{(s)}]e^{\mathbf{\Phi}^{(s)}}+e^{\mathbf{\Phi}^{(s)}}[\lceil\mathbb{H}^+\rceil+\partial, \boldsymbol{K}^{(s)}]\big), \quad s \in [0,1], \tag{15}$$

where the differential operator $\partial = \sum_{i,j} t_{ij}\frac{\partial}{\partial t'_{ij}}$ is defined in (2). Let us point out an important corollary of (14) and (15). Suppose that $\mathbb{H}^\pm\langle R^\pm(\Theta)\rangle = 0$. Then *any homotopy leaves* $\mathbf{\Phi}\langle\Theta\rangle$ *unchanged.*

The category ALG_{SFT} and the functor *SFT* can be further significantly enriched. As we explain below, the construction of our Hamiltonian $\mathbb{H}$, potential $\mathbf{\Phi}$, etc., is based on the study of appropriate moduli spaces of holomorphic curves and their compactifications. In fact, all these objects, as they are described above, are analogs of the so-called *descendent potential* in the Gromov–Witten theory. A more systematic use of the topology of the moduli spaces allows one to define further enrichments of the theory (e.g. see the discussion of *satellites* in [15]).

1.4. Quasi-classical approximation. Let us consider the "quasi-classical" limit (when $\hbar \to 0$) of the structures entering the definition of the category ALG_{SFT}. This leads to the category ALG^0_{SFT} which is formed by the following structures. Let us first describe the objects.

- The Weyl algebra W is replaced by a graded *Poisson algebra* P over $\mathbb{C}$ which is formed by power series in p-variables with polynomial coefficients in q-variables. All variables Poisson commute except that the $\{p_{k,\gamma}, q_{k,\bar{\gamma}}\} = k$. It is useful, in fact, to think about P as an algebra of functions on a symplectic (super-)space S with coordinates $p_{k,\gamma}$ and $q_{k,\gamma}$, and the symplectic form

$$\sum_{\gamma\in\Gamma,k\geq 1} \frac{1}{k} dp_{k,\gamma} \wedge dq_{k,\bar{\gamma}}.$$

- Given $\Theta \in \mathbb{V}^d$, the element

$$\mathbb{H}\langle\Theta\rangle = \sum_{g=0}^{\infty} \mathbb{H}^{(g)}\langle\Theta\rangle\hbar^{g-1} \in \frac{1}{\hbar}\mathsf{W}\otimes\mathsf{CT}_{\boldsymbol{d}}$$

is replaced by $\boldsymbol{h}\langle\Theta\rangle = \mathbb{H}^{(0)}\langle\Theta\rangle \in \mathsf{P}\otimes\mathsf{CT}_{\boldsymbol{d}}$.

- Axiom AO1 takes the form

$$\frac{1}{2}\{\boldsymbol{h}\langle\Theta\sqcup d\Theta\rangle, \boldsymbol{h}\langle\Theta\sqcup d\Theta\rangle\} = \partial\boldsymbol{h}\langle\Theta\sqcup d\Theta\rangle, \tag{16}$$

where ∂ is defined in (2).

In particular, if $d\Theta = 0$ the we have

$$\{\boldsymbol{h}\langle\Theta\rangle, \boldsymbol{h}\langle\Theta\rangle\} = 0. \tag{17}$$

In the definition of a morphism we replace the algebra A_{-}^{+} by a_{-}^{+} formed by formal power series in p^{+}-variables with polynomials coefficients in q^{-}-variables. An element $\boldsymbol{\Phi}\langle\Theta\rangle = \sum_{g=0}^{\infty}\boldsymbol{\Phi}^{(g)}\langle\Theta\rangle\hbar^{g-1} \in \frac{1}{\hbar}\mathsf{A}_{-}^{+}\otimes\mathsf{CT}_{\boldsymbol{d}}$ reduces to $\boldsymbol{\phi}\langle\Theta\rangle = \boldsymbol{\Phi}^{(0)}\langle\Theta\rangle \in \mathsf{a}_{-}^{+}\otimes\mathsf{CT}_{\boldsymbol{d}}$. It is convenient to think about the function $\boldsymbol{\phi}\langle\Theta\rangle = \boldsymbol{\phi}\langle\Theta\rangle(T, q^{-}, p^{+}) \in \mathsf{a}_{-}^{+}\otimes\mathsf{CT}_{\boldsymbol{d}}$ as a Lagrangian submanifold

$$L_{\boldsymbol{\phi}} \subset \Big(\mathsf{S}_{+}, \sum_{\gamma\in\Gamma^{+},k\geq 1}\frac{1}{k}dq_{k,\gamma}^{+}\wedge dp_{k,\bar{\gamma}}^{+}\Big) \oplus \Big(\mathsf{S}_{-}, \sum_{\gamma\in\Gamma^{-},k\geq 1}\frac{1}{k}dp_{k,\gamma}^{-}\wedge dq_{k,\bar{\gamma}}\Big),$$

or rather a family of Lagrangian submanifolds parameterized by T:

$$L_{\boldsymbol{\phi}} = \Big\{p_{k,\gamma}^{-} = k\frac{\partial\boldsymbol{\phi}}{\partial q_{k,\bar{\gamma}}^{-}},\ q_{k,\gamma}^{+} = k\frac{\partial\boldsymbol{\phi}}{\partial p_{k,\bar{\gamma}}^{+}};\ \gamma\in\Gamma, k\geq 1\Big\}. \tag{18}$$

Axiom AM1 reduces to the following equation for $\boldsymbol{\phi}$:

$$\big(\boldsymbol{h}^{+}\langle R^{+}(\Theta\sqcup d\Theta)\rangle + \boldsymbol{h}^{-}\langle R^{-}(\Theta\sqcup d\Theta)\rangle\big)\big|_{L_{\boldsymbol{\phi}}} = \partial\boldsymbol{\phi}\langle\Theta\rangle. \tag{19}$$

In particular, when $d\Theta = 0$ we have:

$$\big(\boldsymbol{h}^{+}\langle R^{+}(\Theta)\rangle + \boldsymbol{h}^{-}\langle R^{-}(\Theta)\rangle\big)\big|_{L_{\boldsymbol{\phi}}} = 0. \tag{20}$$

Similarly, the composition rule (12) becomes the Legendre transform formula

$$\phi_{02}(q^{(2)}, p^{(0)}) = \Big(\phi_{12}(q^{(2)}, p^{(1)}) + \phi_{01}(q^{(1)}, p^{(0)}) - \sum_{\gamma\in\Gamma^{(1)}, k\geq 1} k^{-1} q^{(1)}_{k,\gamma} p^{(1)}_{k,\gamma}\Big)\Big|_L, \tag{21}$$

where

$$L = \begin{cases} p^{(1)}_{k,\gamma} = k\frac{\partial\phi_{01}}{\partial q^{(1)}_{k,\bar\gamma}}, \\ q^{(1)}_{k,\gamma} = k\frac{\partial\phi_{12}}{\partial p^{(1)}_{k,\bar\gamma}}. \end{cases}$$

We denote here by ϕ_{01}, ϕ_{12} and ϕ_{02} the coefficient of $\hbar^{-1}$ in the $\hbar$-expansion of Φ_{01}, Φ_{12} and Φ_{02}, respectively.

The "chain-homotopy" equation (14) takes (assuming $d\Theta = 0$) the form of a Hamilton–Jacobi equation:

$$\frac{d\boldsymbol{\phi}^{(s)}}{ds} = \left\{(\boldsymbol{h}^+ + \boldsymbol{h}^-), \boldsymbol{k}^{(s)}\right\}\Big|_{L_{\boldsymbol{\phi}^{(s)}}}. \tag{22}$$

1.5. SFT and differential equations. We explain in this section that the axioms of ALG_{SFT} (e.g. equations (1), (8), (14)) associate with each object an infinite system of commuting differential operators. In the quasi-classical approximation these operators reduce to systems of Poisson commuting integrals. On the other hand, morphisms provide (formal) solutions of evolution (Schrödinger) equations corresponding to these operators. In the quasi-classical version ALG^0_{SFT} morphisms provide solutions to Hamilton–Jacobi equations corresponding to the hierarchies of the commuting Hamiltonian functions.

Commuting differential operators. Consider an object in ALG_{SFT} with the Hamiltonian $\mathbb{H}$. Take $\Theta \in \mathbb{V}^{\boldsymbol{d}}$ with $d\Theta = 0$. Then $\mathbb{H}\langle\Theta\rangle \in \frac{1}{\hbar}\mathsf{W}\otimes\mathsf{CT}_{\boldsymbol{d}}$ satisfies the equation $\mathbb{H}\langle\Theta\rangle \circ \mathbb{H}\langle\Theta\rangle = 0$ for all values of the parameter $T = (t_{ij})$. Let us write $\mathbb{H}\langle\Theta\rangle = G(T, \hbar, q, p)$ and differentiate the identity $G\circ G = 0$ in T-variables. We get

$$\begin{aligned} &\left[\frac{\partial G}{\partial t_{ij}}, G\right] = 0, \\ &\left[\frac{\partial G}{\partial t_{ij}}, \frac{\partial G}{\partial t_{kl}}\right] + \left[\frac{\partial^2 G}{\partial t_{ij}\partial t_{kl}}, G\right] = 0, \end{aligned} \tag{23}$$

where the commutators are taken according to the sign rules in the graded world. The first equation means that the elements $G_{ij} = \frac{\partial G}{\partial t_{ij}} \in \frac{1}{\hbar}\mathsf{W}\otimes\mathsf{CT}_{\boldsymbol{d}}$ commute with the Hamiltonian, while the second one says that they commute among themselves *after passing to homology of* $\frac{1}{\hbar}\mathsf{W}\otimes\mathsf{CT}_{\boldsymbol{d}}$ *with the differential* $DA = [A, G]$, $A \in \frac{1}{\hbar}\mathsf{W}\otimes\mathsf{CT}_{\boldsymbol{d}}$. Moreover, in many interesting examples we have $G|_{T=0} = 0$, and hence in this case

$$[G_{ij}|_{T=0}, G_{kl}|_{T=0}] = 0 \tag{24}$$

for all i, j, k, l. Recall that elements of $\frac{1}{\hbar}\mathsf{W}$ have a representation as differential operators on the Fock space

$$\mathsf{Fock} \otimes \mathsf{CT}_{\boldsymbol{d}} = \Big\{ \sum_{k \geq 1} f_k(T)\hbar^k \Big\}.$$

Hence, we get an *infinite sequence of commuting differential operators* $\lfloor G_{ij}|_{T=0} \rfloor$ acting on $\mathsf{Fock} \otimes \mathsf{CT}_{\boldsymbol{d}}$.

Let us write $G = \sum_0^\infty G^{(g)}\hbar^{g-1}$, and, respectively, $G_{ij} = \sum_0^\infty G_{ij}^{(g)}\hbar^{g-1}$, where $G_{ij}^{(g)} \in \frac{1}{\hbar}\mathsf{P} \otimes \mathsf{CT}_{\boldsymbol{d}}$. We also denote $\boldsymbol{g} := G^{(0)}$ and $\boldsymbol{g}_{ij} := G_{ij}^{(0)}$. Then in the quasi-classical approximation we get

$$\big\{ \boldsymbol{g}_{ij}|_{T=0},\ \boldsymbol{g}_{kl}|_{T=0} \big\} = 0, \tag{25}$$

provided that $\boldsymbol{g}|_{T=0} = 0$. In other words, $\boldsymbol{g}_{ij}|_{T=0} \in \mathsf{P}$, $i = 1, \dots, N$, $j \geq 0$, are Poisson commuting integrals.

Hence, *the sequence of commuting differential operators* $G_{ij}|_{T=0} \in \frac{1}{\hbar}\mathsf{W}$ *is the* (*deformation*) *quantization of Poisson commuting Hamiltonians* $\boldsymbol{g}_{ij}|_{T=0} \in \mathsf{P}$.

Morphisms in $\mathbf{ALG}_{SFT}$ and evolution equations. Let us consider a morphism between two objects, $\boldsymbol{\Phi}\colon \mathcal{O}^+ \to \mathcal{O}^-$. Let $\mathbb{V}$ be the phase space associated with the morphism. For $\Theta \in \mathbb{V}^{\boldsymbol{d}}$, $\boldsymbol{d} = (d_1, \dots, d_N)$, such that $R^{\pm}(d\Theta) = 0$, we denote

$$\begin{aligned} G^{\pm}(S, q^{\pm}, p^{\pm}, \hbar) &:= \mathbb{H}^{\pm}\langle R^{\pm}(\Theta)\rangle, \\ \Phi(S, T, q^-, p^+, \hbar) &:= \boldsymbol{\Phi}\langle \Theta \sqcup d\Theta \rangle, \end{aligned}$$

where the variables S, T generate $\mathsf{CT}_{\boldsymbol{d} \sqcup \boldsymbol{e}}$, $\boldsymbol{e} = (d_1 + 1, \dots, d_N + 1)$.

Then according to (8) we have for $\Phi = \Phi(S, T, q^-, p^+, \hbar)$ that

$$\sum_{i,j} s_{ij} \frac{\partial}{\partial t_{ij}} \Phi = e^{-\Phi} \left(\lfloor G^-(S, q^-, p^-, \hbar) \rfloor e^{\Phi} + e^{\Phi} \lceil G^+(S, q^+, p^+, \hbar) \rceil \right). \tag{26}$$

By differentiating both sides of (8) in variables s_{ij} and then setting $S = 0$ we get

Proposition 1.3. *Suppose that* $G^{\pm}|_{S=0} = 0$. *Then* $\Phi(S, T, q^-, p^+, \hbar)$ *satisfies the system of commuting evolution equations*

$$\begin{aligned} &\frac{\partial \Phi}{\partial t_{ij}}(S, T, q^-, p_+, \hbar) \\ &\quad = e^{-\Phi(S,T,q^-,p^+,\hbar)} \big(\lfloor G_{ij}^- \rfloor e^{\Phi(S,T,q^-,p^+,\hbar)} + e^{\Phi(S,T,q^-,p^+,\hbar)} \lceil G_{ij}^- \rceil \big), \end{aligned} \tag{27}$$

where $G_{ij}^{\pm} := \frac{\partial G^{\pm}}{\partial s_{ij}}\big|_{S=0}$, $i = 1, \dots, N$, $j \geq 0$.

In the quasi-classical approximation the system (27) reduces to a system Hamilton–Jacobi equations for the evolution of the corresponding Lagrangian submanifold under the system of commuting Hamiltonian flows:

$$\frac{d\phi}{dt_{ij}}(S,T,q^-,p^+) = \left(\boldsymbol{g}^-_{ij}(q^-,p^-) + \boldsymbol{g}^+_{ij}(q^+,p^+)\right)\Big|_{L_{\phi(S,T,q^-,p^+)}}, \tag{28}$$

$i = 1,\dots,k$, $j \geq 0$.

2. Construction of the functor *SFT*

2.1. Beginning of the construction. The description of the functor *SFT* which we present here is very sketchy, and only gives a very general picture of the structures involved in the construction. It also omits many very important points. In particular, in order to actually define the functor *SFT* we need to restrict the geometric category by imposing certain genericity constraints. The actual construction of *SFT* is a large project which is currently well under way (e.g. see [8], [25]), but not yet fully completed.

Let $\mathcal{O} = (V,\Omega,\lambda,J)$ be an object in GEOM_{SFT}, i.e. a framed Hamiltonian structure, and R the corresponding Reeb field. Let us begin building the corresponding object $SFT(\mathcal{O}) \in \mathrm{ALG}_{SFT}$.

Denote by $\mathcal{P}$ the space of *simple* periodic orbits of the Reeb field R. Generically, periodic orbits are non-degenerate, i.e. the linearized Poincaré return map along each orbit has no eigenvalues equal to 1. If this is the case, then the number of orbits in $\mathcal{P}$ of bounded period is finite. We will assume either that R satisfies this non-degeneracy assumption, or the so-called *Morse–Bott* condition (see [5] for the precise definition) when periodic orbits are organized in submanifolds, and the flow of R satisfies a certain non-degeneracy condition in the direction complementary to critical submanifolds.

Let $H^*(\mathcal{P})$ be the (de Rham) cohomology space of $\mathcal{P}$. Choose a basis of Γ represented by a finite or countable system of differential forms on $\mathcal{P}$, such that the matrix of the Poincaré pairing has in this basis the form $\delta_{\gamma,\bar{\gamma}}$ for a certain involution $\gamma \mapsto \bar{\gamma}$ on Γ. Of course, in the non-degenerate case the space $\mathcal{P}$ is discrete, and hence in this case there is a canonical basis of 0-forms, dual to individual orbits. In this case the involution $\gamma \mapsto \bar{\gamma}$ is the identity map.

In the non-degenerate case each γ can be identified with an orbit from $\mathcal{P}$. We then associate the variables $p_{\gamma,k}$ and $q_{\gamma,k}$ with the k-multiple cover of the orbit γ. Their $\mathbb{Z}/2$-grading is determined as follows. Let $A_{\gamma,k}$ be the linearized Poincaré return map for this k-multiple orbit. Then the variables $p_{\gamma,k}$ and $q_{\gamma,k}$ are even or odd graded depending on whether the Lefschetz number $\det(1 - A_{\gamma,k})$ is positive or negative. If some extra choices are made one can define the integral grading of the variables $p_{\gamma,k}$ and $q_{\gamma,k}$ but we will not discuss it in this paper. With the graded variables $p_{\gamma,k}$ and $q_{\gamma,k}$ introduced, we can then define the Weyl algebra W and the space Fock.

We will not discuss here the Morse–Bott case in full generality and only consider its extreme case described above in Example 1.2 (3), when (V, Ω, λ, J) is of fibration type. All orbits of R are closed in this case and the space $\mathcal{P}$ of simple periodic orbits coincides with M. There exists a basis Γ of $H^*(M) = H^*(\mathcal{P})$, and an involution $\gamma \to \bar{\gamma}$ such that the Poincaré pairing in this basis is given by the matrix

$$(\gamma, \gamma') = \delta_{\gamma, \bar{\gamma}'}.$$

The $\mathbb{Z}/2$-degrees of the variables $p_{\gamma,k}$ and $q_{\gamma,k}$ coincide in this case with the degree of the corresponding cohomology classes $\gamma \in H^*(M)$. The phase space $\mathbb{V}$ associated with $SFT(\mathcal{O})$ is the space of differential forms on V with the de Rham differential.

The main part of $SFT(\mathcal{O})$, the Hamiltonian $\mathbb{H}$, is defined in terms of moduli spaces of certain holomorphic curves in the cylinder $V \times \mathbb{R}$ with an almost complex structure, still denoted by J, which is defined by the following conditions.

- J is invariant with respect to translations $(x, t) \mapsto (x, t + c)$, $(x, t) \in V \times \mathbb{R}$;
- $J \frac{\partial}{\partial t} = R$;
- the CR-structure induced on each slice $V \times t$ coincides with the given CR-structure J.

H. Hofer (see [27]) was the first who studied holomorphic curves in almost complex cylindrical manifolds of this type in his work on the Weinstein conjecture. He followed the pioneering work of M. Gromov (see [23]) who essentially created the new field of symplectic topology by introducing the technique of (pseudo-)holomorphic curves. Before considering the general case we sketch the construction in the very special, but already highly non-trivial case when $V = S^1$.

2.2. The circle. Consider Example 1.2 (3) for the special case when M is the point. In this case $V = S^1 = \mathbb{R}/\mathbb{Z}$ and $R = \frac{\partial}{\partial s}$, $s \in \mathbb{R}/\mathbb{Z}$. The complex structure J defined on the cylinder $C = S^1 \times \mathbb{R}$ at the end of the previous section coincides in this case with the standard complex structure on the cylinder $C = \mathbb{C}/\{z \sim z + 1\}$.

The space $\mathcal{P}$ consists of only one simple orbit, and hence Γ is just a point. Therefore, we have two infinite series of even variables $p_k, q_k, k = 1, \ldots,$ and the space F is the space of "scalar" Fourier series $u(x) = \sum_{k=1}^{\infty} p_k e^{ikx} + q_k^{-ikx}$. The spaces F_+ and F_- are formal analogs of spaces of holomorphic functions in the unit disc and its complement, which are equal to 0 at the origin or ∞, respectively. The Weyl algebra W is generated by even elements p_k, q_k with $k = 1, \ldots,$ and an even element $\hbar$, and consists of formal power series

$$\sum_{n=0}^{\infty} \sum_{I} g_{I,n}(q) \hbar^n p^I,$$

where $g_{I,n}(q)$ are polynomials, the second sum is taken over all infinite multi-indices $I = (i_1, i_2, \dots)$ with finitely many non-zero entries, and $p^I = p_1^{i_1} p_2^{i_2} \dots$. All variables commute except that $[p_k, q_k] = k\hbar$.

By quantizing $\lfloor p_k \rfloor = k\hbar \frac{\partial}{\partial q_k}$ we represent elements of $\frac{1}{\hbar}\mathsf{W}$ as linear differential operators on the space Fock formed by power series $\sum_{k\geq 0} f_k \hbar^k$, whose coefficients f_k are functionals on the space F_- (of "equal to 0 at ∞ holomorphic functions u in the complement of the unit disc") which can be expressed as polynomials of Fourier coefficients of u.

Next, we describe the Hamiltonian $\mathbb{H}$. Let (S, j) be a closed Riemann surface of genus g and $F: S \to \mathbb{C}P^1$ a meromorphic function with r_+ poles $(x_1, \dots, x_{r_+})$ and r_- zeroes $(y_1, \dots, y_{r_-})$ of multiplicities $c = (c_1, \dots, c_{r_+})$ and $b = (b_1, \dots, b_{r_-})$, respectively. By identifying $\mathbb{C}P^1 \setminus \{0, \infty\}$ with the cylinder

$$C = \mathbb{C}/\{z \sim z+1\} = S^1 \times \mathbb{R}, \quad S^1 = \mathbb{R}/\mathbb{Z},$$

we can equivalently view the function F as a map

$$F = (f, a)\colon S \setminus \big(\{x_1, \dots, x_{r_+}\} \cup \{y_1, \dots, y_{r_-}\}\big) \to C. \tag{29}$$

With this interpretation we will call $X = \{x_1, \dots, x_{r_+}\}$ and $Y = \{y_1, \dots, y_{r_-}\}$ the sets of *positive* and *negative punctures*, respectively. If $z = e^{-\rho + i\varphi}$ is a local coordinate on S near a puncture $x_i \in X$ where $\rho \in (0, \infty)$, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, then the map F near this puncture can be written as

$$\begin{aligned} s &= f(\rho, \varphi), \\ t &= a(\rho, \varphi), \end{aligned}$$

where $f(\rho, \varphi) \underset{\rho\to\infty}{\longrightarrow} \frac{c_i \varphi}{2\pi}$ and $\frac{a(\rho,\varphi)}{\rho} \underset{\rho\to\infty}{\longrightarrow} c_i$. In other words, at x_i the map F is asymptotic to the c_i-multiple circle $S^1 = \mathbb{R}/\mathbb{Z}$ at $+\infty$ of the coordinate t. Similarly, at a puncture $y_j \in Y$ the map F is asymptotic to the $-b_j$-multiple circle $S^1 = \mathbb{R}/\mathbb{Z}$ at $-\infty$ of the coordinate t. For a fixed genus and fixed multiplicity vectors $c = (c_1, \dots, c_{r_+})$ and $b = (b_1, \dots, b_{r_-})$ we denote by $\mathcal{M}_g(C; c, b)$ the moduli space of equivalency classes of meromorphic functions defined in (29). The integer vectors c and b are called the *positive* and *negative ramification data*. We will also denote by $\mathcal{M}_{g,k}(C; c, b)$ a similar moduli space with k additional marked points (disjoint from X and Y and each other) $z_1, \dots, z_k$. The stability condition: $g + 2k + r_+ + r_- \geq 3$, is required to be satisfied. Notice that we do not fix a conformal structure on the surface and the configurations of punctures and marked points. Two maps are called equivalent if they differ by a conformal map $(S_g, j) \to (S_g, j')$ which preserves all punctures and marked points. We will also consider the quotient $\mathcal{M}_{g,k}(C; c, b)/\mathbb{R}$ by translations of $C = S^1 \times \mathbb{R}$ along the $\mathbb{R}$-factor.

The moduli space $\mathcal{M}_{g,k}(C; c, b)/\mathbb{R}$ can be compactified by adding *stable holomorphic buildings*, see [8]. A stable building of height 1 is a stable nodal holomorphic curve in the sense of Kontsevich, i.e. an equivalency class of holomorphic maps defined on a possibly disconnected Riemann surface with certain pairs of marked points (called *special*) required to be mapped to one point on C. The stability condition should be satisfied for each connected component, and the source surface must become connected after identifying points of each special pair. As above, the equivalence relation identifies buildings which differ by translation of C along the $\mathbb{R}$-factor. A stable building F of height $l > 1$ is a collection of stable buildings $F_1, \dots, F_l$ of height 1, with the condition that the positive ramification data of the building F_i, $i = 1, \dots, l-1$, coincides with the negative ramification data of F_{i+1}. By definition, the negative ramification data of F_1 is the negative ramification data of F, and the positive ramification data of F_l is the positive ramification data of F. The genus of F is the genus of the surface obtained by gluing for each $i = 1, \dots, l-1$ the source surfaces of buildings F_i and F_{i+1} along their respective ends. The compactified moduli space will be denoted by $\overline{\mathcal{M}/\mathbb{R}}_{g,k}(C; c, b)$.

The evaluation map at the j-th marked points z_j defines a map

$$\mathrm{ev}_j : \overline{\mathcal{M}/\mathbb{R}}_{g,k}(C; c, b) \to S^1 = \mathbb{R}/\mathbb{Z}.$$

To define the Hamiltonian $\mathbb{H}$ we need to pick a system of forms. For our case of S^1 let us take $\Theta = (\theta_0 = 1, \theta_1 = ds)$. Then the corresponding algebra CT is generated by elements of the matrix $T = (t_{ij})$, $i = 0, 1$, $j \geq 0$, with even variables t_{0j} and odd variables t_{1j}.

As it is customary in Gromov–Witten theory, we define *correlators*

$$\langle T, \dots, T\rangle_{g,k,c,b} = \int\limits_{\overline{\mathcal{M}/\mathbb{R}}_{g,k}(C;c,b)} \Big(\sum t_{ij}\, \mathrm{ev}_1^*(\theta_i) c_1(L_1)^j\Big) \wedge \dots \wedge \Big(\sum t_{ij}\, \mathrm{ev}_k^*(\theta_i) c_1(L_k)^j\Big), \tag{30}$$

where L_j, $j = 1, \dots, k$, is a tautological line bundle over $\overline{\mathcal{M}/\mathbb{R}}_{g,k}(C; c, b)$ which associates with each holomorphic curve (building) the cotangent line at the j-th marked point z_j.[1]

Consider now the generating function $\mathbb{H} \in \frac{1}{\hbar}\mathsf{W} \otimes \mathsf{CT}_{\boldsymbol{d}}$,

$$\mathbb{H} = \sum_{g \geq 0,\, k \geq 0, b, d} \frac{\langle T, \dots, T\rangle_{g,k,c,b}}{k!(r_-)!(r_+)!} \hbar^{g-1} q^b p^c, \tag{31}$$

where $q^b = q_{b_1} \dots q_{b_{r_-}}$, $p^c = p_{c_1} \dots p_{c_{r_+}}$.

[1] The integration in this and other similar formulas should be understood either in the sense of the virtual cycle theory if one works in the algebro-geometric context, or literally but after an appropriate generic perturbation, see [23], [39]. In fact, to achieve transversality one needs sometimes to perturb in a class of objects more general than holomorphic curves. The relevant transversality theorem was first proven by K. Fukaya and K. Ono in [19] in the context of Floer homology theory. Following their work the transversality issues in Gromov–Witten theory were studied by several authors. H. Hofer, jointly with K. Wysocki and E. Zehnder, has recently developed a new functional analytic theory of polyfolds, which provides the most suitable set-up for handling transversality problems arising in SFT, see [25], [26]. One also needs to use *coherent orientations* of different moduli spaces, as it is described in [15].

The Hamiltonian $\mathbb{H}$ can be quite explicitly written in this case, thanks to the results of A. Okounkov and R. Pandharipande, see [41].

First of all, it follows from the parity arguments that $\mathbb{H}|_{T_1=0}=0$, where we denote $T_i=(t_{ij})$, $j\geq 0, i=0,1$. Hence,

$$\mathbb{H}=\sum_{j\geq 0} t_{1j}\mathbb{H}_j+o(T) \tag{32}$$

Let us introduce a new variable y and define a *generating function* for the sequence of operators $G_j=\mathbb{H}|_{T_0=0}$:

$$G(y)=\sum_0^\infty G_j y^j.$$

Take $u(x)=\sum_{k=1}^\infty p_k e^{ikx}+q_k e^{-ikx}\in \mathsf{F}$ and denote by $\phi(x)$ the function determined by equations

$$\phi'(x)=u(x),\ \phi(0)=0.$$

In other words,

$$\phi(x)=-i\sum_{k=1}^\infty\left(\frac{p_k}{k}e^{ikx}-\frac{q_k}{k}e^{-ikx}\right).$$

Let us also set $\hbar=\lambda^2$. Then we have

$$G(y)[u]=\frac{1}{2\pi\lambda^2y^2\,\mathrm{s}(\lambda y)}\int_0^{2\pi}dx\left(e^{\frac{i}{\lambda}\left(\phi\left(x-\frac{i\lambda y}{2}\right)-\phi\left(x+\frac{i\lambda y}{2}\right)\right)}-1\right), \tag{33}$$

where

$$\mathrm{s}(u)=\frac{2\sinh\frac{u}{2}}{u}.$$

Let us write explicitly a few first terms G_k:

$$\begin{aligned}
G_0&=\frac{1}{2\pi\hbar}\int_0^{2\pi}\frac{u^2}{2}dx\,;\\
G_1&=\frac{1}{2\pi\hbar}\int_0^{2\pi}\frac{u^3}{6}dx\,;\\
G_2&=\frac{1}{2\pi}\int_0^{2\pi}\left(\hbar^{-1}\frac{u^4}{24}+\frac{u^2}{12}-\frac{uu''}{6}\right)dx.
\end{aligned} \tag{34}$$

It is interesting to note that the genus 0 term of G_k, $k \geq 0$, i.e. the coefficient of $\hbar^{-1}$, is equal to

$$G_k^{(0)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{u^{k+2}}{(k+2)!} \, dx.$$

These are commuting integrals of the dispersionless KdV, or Burgers integrable hierarchy, and hence the operators $\lfloor G_k \rfloor$ acting on Fock, provides the deformation quantization of this hierarchy.

2.3. Case of a general Hamiltonian structure. In order to define $\mathbb{H}$ for a general Hamiltonian structure (V, Ω, λ, J) we consider moduli spaces of J-holomorphic curves in the cylindrical almost complex manifold $(W = V \times \mathbb{R}, J)$.

Notice that for our choice of J the cylinder $\gamma \times \mathbb{R} \subset W$ over a trajectory γ of the Reeb field R is always a J-holomorphic curve. Given a J-holomorphic map F of a punctured disk $D^2 \setminus 0 \to W$ with the coordinate $z = e^{-\rho + i\varphi}$, we say that the map $F = (f, a)$ is *asymptotically cylindrical* over a periodic orbit γ of the Reeb field R at $+\infty$ (resp. at $-\infty$) if $\lim_{\rho\to\infty} a(z) = +\infty$ (resp. $= -\infty$), and $\lim_{\rho\to\infty} f(z) = \bar{f}\big(\pm \frac{T\varphi}{2\pi}\big)$, where the map $\bar{f} \colon [0, T] \to V$ parameterizes the trajectory γ in such a way that R is its velocity vector, and T is the period of γ.

Let $S = S_g$ be a compact Riemann surface of genus g with a conformal structure j, with r_+ punctures $\boldsymbol{x} = \{x_1, \dots, x_{r_+}\}$, called positive, r_- punctures $\boldsymbol{y} = \{y_1, \dots, y_{r_-}\}$, called negative, and also k marked points $z_1, \dots, z_k$, disjoint from each other and the punctures.

Given two vectors $c = (c_1, \dots, c_{r_+})$ and $b = (b_1, \dots, b_{r_-})$ of positive integers we consider moduli spaces $\mathcal{M}_{g,k}(W, J; c, b)$ of (j, J)-holomorphic curves

$$(S_g \setminus (\boldsymbol{x} \cup \boldsymbol{y}), j) \to (W, J)$$

with k marked points $z_1, \dots, z_k$, which are asymptotically cylindrical over a c_i-multiply covered periodic orbit from $\mathcal{P}$ at the positive end at the puncture x_i, and asymptotically cylindrical over a $(-b_j)$-multiply covered periodic orbit at the negative end at the puncture y_j. We will also consider the quotient $\mathcal{M}(W, J; c, b)/\mathbb{R}$ of the space $\mathcal{M}_{g,k}(W, J; c, b)$ by translations along the $\mathbb{R}$-factor.

For our distinguished structure J, the holomorphic curve equation takes the form

$$\begin{aligned} \pi \circ df \circ j &= J \circ \pi \circ df \\ da &= (f^*\lambda) \circ j. \end{aligned} \tag{35}$$

Notice that the second equation just means that the form $f^*\lambda \circ j$ is exact on S and that the function a is a primitive of the 1-form $f^*\lambda \circ j$. Thus the holomorphicity condition for $F = (f, a)$ is essentially just a condition on its V-component f. If f satisfies the first of the equations (35) and the form $(f^*\lambda) \circ j$ is exact then the coordinate a can be reconstructed uniquely up to an additive constant on each connected component of S. Therefore, an element $F \in \mathcal{M}_{g,k}(W, J; c, b)$ is uniquely determined by its V-component f, which is a surface bounded by multiply covered orbits from $\mathcal{P}$.

Given $\alpha \geq 0$, let us denote by $\mathcal{M}^{\alpha}_{g,k}(W, J, c, b)$ the subspace

$$\mathcal{M}_{g,k}(W, J, c, b) \cap \left\{ \int\limits_{S_g} F^*\omega \leq \alpha \right\}.$$

The quotient space $\mathcal{M}^{\alpha}_{g,k}(W, J, c, b)$ has a compactification $\overline{\mathcal{M}/\mathbb{R}}^{\alpha}_{g,k}(W, J, c, b)$ by holomorphic buildings[2] (see [8]), similar to the one considered above for the case $V = S^1$. We denote

$$\overline{\mathcal{M}/\mathbb{R}}_{g,k}(W, J, c, b) = \bigcup_{\alpha \geq 0} \overline{\mathcal{M}/\mathbb{R}}^{\alpha}_{g,k}(W, J, c, b)$$

and

$$\overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J) = \bigcup_{\substack{k,c,b \\ k+r_+ + r_- = m}} \overline{\mathcal{M}/\mathbb{R}}_{g,k}(W, J, c, b)$$

The space $\overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J)$ may consists of different components,

$$\overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J) = \bigcup C_i.$$

Given $F \in C_i$, we denote by μ_i its symplectic area $\int_{S_g} F^*\omega$, which depends only on the component C_i.

By using the notation $\overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J)$ we put the punctures and the marked points on the equal footing. Keeping up with this point of view, let us consider the disjoint union

$$X = \coprod_{-\infty}^{\infty} \mathcal{P}_j,$$

where

$$\mathcal{P}_j = \begin{cases} \mathcal{P} & \text{if } j \neq 0, \\ V & \text{if } j = 0, \end{cases}$$

and supply each $\mathcal{P}_j$, $j \neq 0$, with an identical copy $\Gamma^{(j)}$ of the basis Γ of $H^*(\mathcal{P})$.

Consider an evaluation map

$$\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_m) \colon \overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J) \to \underbrace{X \times \cdots \times X}_{m}$$

which associates

[2] In the Morse–Bott case one also needs to add to the compactification the so-called *generalized holomorphic buildings*, see [5] and [8].

– with each marked point z_i its value $f(z_i) \in V = \mathcal{P}_0$,

– with each positive puncture x_i the corresponding periodic orbit in the $\mathcal{P}_k$-copy of $\mathcal{P}$, where $k = c_i$ is its multiplicity,

– with each negative puncture y_i the corresponding periodic orbit in the $\mathcal{P}_{-k}$-copy of $\mathcal{P}$, where $k = b_i$ is its multiplicity.

Choose a system of closed forms $\Theta = (\theta_1, \dots, \theta_N)$ and associate with it a matrix $T = (t_{ij})$ of graded variables. Consider the following formal expression ("general cohomology class of X with descendents")

$$Z = \sum_{i=1}^{n} \sum_{j=0}^{\infty} t_{ij} \theta_i c^j + \sum_{k=1}^{\infty} \sum_{\gamma \in \Gamma} p_{\gamma,k} \gamma^{(k)} + q_{\gamma,k} \gamma^{(-k)},$$

where $\gamma^{(j)}$ denotes the copy of $\gamma \in \Gamma$ in $\Gamma^{(j)}$, $j \neq 0$, and set

$$\mathrm{ev}_l^* Z = \sum_{i=1}^{n} \sum_{j=0}^{\infty} t_{ij} \, \mathrm{ev}_l^* \theta_i \, (c_1(L_l))^j + \sum_{k=1}^{\infty} \sum_{\gamma \in \Gamma} p_{\gamma,k} \, \mathrm{ev}_l^* \, \gamma^{(k)} + q_{\gamma,k} \, \mathrm{ev}_l^* \, \gamma^{(-k)},$$

where the line bundles L_l over $\overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J)$ have the same meaning as in Section 2.2 above. Define the correlator

$$\langle \underbrace{Z, \dots, Z}_{m} \rangle_g = \sum_{j=1}^{\infty} z^{\mu_j} \int_{C_j} \mathrm{ev}^* (\underbrace{Z \otimes \cdots \otimes Z}_{m}), \tag{36}$$

where the sum is taken over all components of $\overline{\mathcal{M}/\mathbb{R}}_{g,m}(W, J)$.

Remark 2.1. Note that by introducing exponents z^{μ_j} in the definition of the correlators we effectively extended the coefficient ring from $\mathbb{C}$ to a certain Novikov ring (of Puiseux power series $\sum_j a_j z^{\mu_j}$). This was done to avoid infinities in (36). However, it is not absolutely necessary to do that, and one can ignore these weights by setting $z = 1$ in most of the cases. For instance, for Hamiltonian structures of contact type there are always only finitely many terms in the sum which contribute in (36) to the coefficient of a fixed monomial of q, p and $\hbar$ variables. But even in the most general situation one can alternatively deal with this problem by requiring the string of forms Θ to contain closed 2-forms which form a basis of $H^2(V)$ (this approach is similar to the divisor equation in the Gromov–Witten theory).

Finally, we define the Hamiltonian

$$\mathbb{H}(\Theta) = \sum_{g=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \langle \underbrace{Z, \dots, Z}_{m} \rangle_g \hbar^{g-1}. \tag{37}$$

Note that all terms of $\mathbb{H}$ have the same *odd* degree (and, in fact, degree 1 if the grading is upgraded to $\mathbb{Z}$ from $\mathbb{Z}/2$), because we integrate over the moduli spaces quotiented by the $\mathbb{R}$-action.

The "master equations" (1) and (16) follow from Stokes' formula combined with the description of the boundary of the corresponding moduli spaces.

All the other necessary constructions to build the functor *SFT* are done in the same spirit. Consider, for instance, a framed cobordism (W, Ω, J) which realizes a morphism $\mathbf{\Phi}\colon \mathcal{O}^+ \to \mathcal{O}^-$ between two framed Hamiltonian structures

$$\mathcal{O}^\pm = (V^\pm, \Omega^\pm, \lambda^\pm, J^\pm).$$

The phase space $\mathbb{V}$ associated with this cobordism is the space of differential forms on W, and $R^\pm$ are the restriction homomorphisms to $V^\pm$.

Take $\Theta \in \widehat{\mathbb{V}}^d$ and associate with it the corresponding graded algebra CT_d. To define the potential $\Phi \in \frac{1}{\hbar}\mathsf{A}^+_- \otimes \mathsf{CT}_d$ we attach to the cobordism cylindrical ends corresponding to framed Hamiltonian structures $\mathcal{O}^\pm$,

$$\widehat{W} = \left(V^- \times (-\infty, 0]\right) \cup W \cup \left(V^+ \times [0, \infty)\right),$$

and consider the compactified moduli space of holomorphic curves in $\widehat{W}$ asymptotically cylindrical to periodic orbits of the Reeb field R_+ at the positive end, and the orbits of R_- at the negative one. Then the correlators and the potential are defined by the formulas similar to (36) and (37) with one very important difference: in this situation there is no $\mathbb{R}$-action on the moduli space, and hence the integrals should be evaluated on the moduli space itself, rather than its quotient by the $\mathbb{R}$-action, as was done for the Hamiltonian. The implication of this is that the potential, unlike the Hamiltonian, has an *even* degree (in fact, degree 0 if the grading is upgraded to $\mathbb{Z}$ from $\mathbb{Z}/2$). As in the case of the Hamiltonian, the structural equation (8) is a consequence of Stokes' formula and the description of the boundary of the corresponding moduli space.

Note that if the symplectic manifold W is closed, i.e. it is a cobordism between empty Hamiltonian structures, then the corresponding SFT-potential $\mathbf{\Phi}(\Theta) \in \mathsf{CT}_{\boldsymbol{d}}$ is just the descendent potential of the Gromov–Witten theory.

2.4. The 3-sphere. Let us consider here an example when $V = S^3$, λ is the standard contact form whose Reeb field generates the Hopf fibration, J is the CR-structure induced from $\mathbb{C}^2$ on the round sphere. This is a pre-quantization space, so it fits into both, the contact and the fibration cases in the sense of Example 1.2.

The manifold (W, J) can be equivalently described here either as $\mathbb{C}^2 \setminus 0$, or the total space of the canonical degree 1 complex line bundle L over $\mathbb{C}P^1$ minus the 0-section. In the second interpretation a holomorphic curve from $\mathcal{M}(W, J, c, b)$ can be viewed as a pair (h, ψ), where $h\colon S_g \to \mathbb{C}P^1$ is a holomorphic curve, and ψ is a meromorphic section of the induced complex line bundle h^*L over S_g. The punctures from $\boldsymbol{x}$ and $\boldsymbol{y}$ correspond to zeroes and poles of this section, respectively, while the vectors c and b appear as the multiplicities of zeroes and poles.

Take a basis of $H^*(\mathbb{C}P^1)$ which consists of $\gamma_0 = 1$ and the harmonic form γ_2 with $\int_{\mathbb{C}P^1} \gamma_2 = 1$. The Poincaré duality involution acts as $\bar{\gamma}_0 = \gamma_2$. Thus the Weyl algebra W is generated by even graded variables $p_{0k} = p_{\gamma_0,k}$, $p_{2k} = p_{\gamma_2,k}$, $q_{0k} = q_{\gamma_0,k}$, and $q_{2k} = q_{\gamma_2,k}$, $k \geq 1$. We organize them into formal Fourier series

$$u_0(x) = \sum_1^\infty p_{0k}e^{ikx} + q_{0k}e^{-ikx},\ u_2(x) = \sum_1^\infty p_{2k}e^{ikx} + q_{2k}e^{-ikx},\ u = (u_0, u_2).$$

Let us choose a basis $(\theta_0 = 1, \theta_3)$ of $H^*(S^3)$, where θ_3 is a harmonic 3-form with $\int_{S^3} \theta_3 = 1$, as the required string Θ of differential forms. The algebra $\mathsf{CT}_{\boldsymbol{d}}$ in this case is generated by $T = (T_0, T_3)$, where $T_i = (t_{ij})$, $i = 0, 3$; $j \geq 0$. The variables t_{0j} are even, while t_{3j} are odd.

As was shown in [15], the genus 0 part $\mathbb{H}^{(0)}$ of the Hamiltonian $\mathbb{H}$ can be explicitly reconstructed in terms of the genus 0 descendent Gromov–Witten potential of $\mathbb{C}P^1$ (in fact, this is a general phenomenon for all Hamiltonian structures of fibration type). In particular, we get

$$G_0 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{(t_0 + u_0(x))^2}{2} + e^{u_2(x)-ix}\right)dx, \tag{38}$$

and the Hamiltonian equations for the Hamiltonian G_0 can be written as

$$\begin{aligned} \dot{u}_0(x) &= -i\frac{d}{dx}\left(e^{u_2(x)-ix}\right), \\ \dot{u}_2(x) &= -i\frac{du_0}{dx}(x), \end{aligned} \tag{39}$$

or $\ddot{u}_2 = -\frac{d^2}{dx^2}(e^{u_2-ix})$, where the dot denotes the time derivative.

As was pointed out to me by B. Dubrovin, this is the continuous limit of the Toda lattice. The other G_i are Poisson commuting integrals of this integrable hierarchy. Hence, if one were to explicitly write for this example the terms of the expansion of the full Hamiltonian $\mathbb{H}$ (and not only of its genus 0 term $\mathbb{H}^{(0)}$) then this would provide the quantum commuting integrals for the quantization of the Toda system (39).[3]

Let us now use the Hamilton–Jacobi equation (28) to compute the genus 0 potential of the round 4-ball $B \subset \mathbb{C}^2$. Take a 4-form θ supported in Int B^4 with $\int_{B^4} \theta = 1$ and set $\Theta = \{\theta\}$. Let $T = (t_j)$, $j \geq 0$ be the corresponding string of even graded variables. Take the genus 0 potential $\boldsymbol{\Phi}^{(0)}(\Theta) = \Phi^{(0)}(T, p) \in \mathsf{P} \otimes \mathbb{C}[T]$, and consider its restriction $\phi(t, p)$ to the subspace $T = \{(t, 0, 0, \dots)\}$. Note that $\phi(t, p)$

[3] One can extract from the work [38] an explicit, though quite complicated recurrent procedure for writing down the expansion of the full Hamiltonian $\mathbb{H}$ in terms of the descendent Gromov–Witten potential of $\mathbb{C}P^1$.

is, in fact, a certain relative genus 0 Gromov–Witten invariant. Coefficients in its expansion in t and p variables count the numbers of rational curves in $\mathbb{C}P^2$ which pass through a given number of fixed points and have a prescribed tangency pattern to a fixed complex line $C \subset \mathbb{C}P^2$. According to (28) $\phi(t, p)$ can be computed as a solution of a Hamilton–Jacobi equation associated with the Hamiltonian flow (39). Let $E^t\colon \mathsf{F} \to \mathsf{F}$ be the (formal, i.e. understood in terms of formal power series) Hamiltonian flow defined by the equation (39). Take the Lagrangian subspace $\mathsf{F}_+ = \{(u_+, 0)\} = \{q = 0\} \subset \mathsf{F}$ and denote by L^t its image $E^t(\mathsf{F}_+)$ under the flow E^t. Then $\phi(t, p)$ is the generating function for $L^{(t)}$ in the sense of (18), i.e.

$$L^{(t)} = L_\phi = \left\{ q_{k,0} = k\frac{\partial \phi}{\partial p_{k,2}},\ q_{k,2} = k\frac{\partial \phi}{\partial p_{k,0}},\ k \geq 1 \right\}.$$

Let us switch to the (u_-, u_+)-notation, i.e. write u_- for q and u_+ for p, and apply a standard symplectic-geometric procedure for computing the generating function in terms of the Lagrangian submanifold which it defines. Let us define L_t by an explicit equation $u_- = f^t(u_+)$ (i.e. exclude v from the parametric equations $(u_+, u_-) = E^t(v, 0),\ v \in \mathsf{F}_+$). Then we have

$$\phi(t, u_+) = -\frac{i}{2\pi}\int_0^1\int_0^{2\pi} \left\langle f^t(su_+(x)), \frac{du_+(x)}{dx}\right\rangle dx\,ds, \tag{40}$$

where $\langle \cdot, \cdot\rangle$ is a bilinear form on $\mathbb{C}^2$ with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$.

It is interesting to note that the value of the functional $\phi(t, u_+)$ at the point $\overline{u}_+ = (ze^{ix}, 0)$, i.e. the function

$$g(t, z) = \phi(t, \overline{u}_+) = z\int_0^1 f^t_{(2,1)}(sze^{ix})\,ds, \tag{41}$$

where we write

$$f^t = (f^t_0, f^t_2) = \Big(\sum_1^\infty f^t_{0,k}e^{-ikx}, \sum_1^\infty f^t_{2,k}e^{-ikx}\Big),$$

is the generating function

$$g(t, z) = \sum_{d=1}^\infty \sum_{m=1}^\infty N_{d,k}t^m z^d$$

for the numbers $N_{d,k}$ of rational curves of degree d which pass through m points in general position in the complex projective plane.[4] In order to get (41) from (40) one needs to split $\mathbb{C}P^2$ along a boundary of a tubular neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ and apply the gluing formula (21), see [15].

[4]As it is well known, the coefficients $N_{d,k}$ vanish unless $k = 3d-1$, and we have $N_{1,2} = 1$, $N_{2,5} = 1$, $N_{3,8} = 12$, $N_{4,11} = 620, \dots$. Several recursion relations, beginning from the one discovered by M. Kontsevich in [32], are known for computing the coefficients $N_{d,k}$.

3. Invariants of contact manifolds and other applications of SFT

3.1. Invariants of contact manifolds. Defining invariants of contact manifolds was one of the primary motivations for the SFT project.

Let (V, λ, J) be a framed Hamiltonian structure of contact type. Choose a system of closed forms $\Theta = (\theta_1, \dots, \theta_N)$ which represents a basis of the cohomology $H^*(V)$ and consider the corresponding Hamiltonian

$$\mathbb{H}\langle\Theta\rangle = \sum_{g=0}^{\infty} H^{(g)}(T, q, p)\hbar^{g-1}.$$

Consider the following SFT objects which can be associated with (V, λ, J):

1. The Weyl differential algebra $(\mathsf{W} \otimes \mathsf{CT}_{\boldsymbol{d}}, \boldsymbol{D})$, where
$$\boldsymbol{D}A = [A, \mathbb{H}\langle\Theta\rangle], \quad \mathbb{H}\langle\Theta\rangle \in \frac{1}{\hbar}\mathsf{W} \otimes \mathsf{CT}_{\boldsymbol{d}}.$$
2. The space $\mathsf{Fock} \otimes \mathsf{CT}_{\boldsymbol{d}}$ with the differential[5]
$$Df = \lfloor \mathbb{H}\langle\Theta\rangle \rfloor f, \quad f \in \mathsf{Fock} \otimes \mathsf{CT}_{\boldsymbol{d}}.$$
3. The Poisson differential algebra $(\mathsf{P} \otimes \mathsf{CT}_{\boldsymbol{d}}, d)$ with the differential
$$dA = \{A, \mathbb{H}^{(0)}\langle\Theta\rangle\}, \quad A \in \mathsf{P} \otimes \mathsf{CT}_{\boldsymbol{d}}.$$
4. The differential algebra $(\mathsf{fock} \otimes \mathsf{CT}_{\boldsymbol{d}}, \mathfrak{d})$ where $\mathsf{Fock} = \mathsf{fock}[[\hbar]]$ and the differential $\mathfrak{d}$ is defined as follows. Consider the expansion
$$\mathbb{H}^{(0)}\langle\Theta\rangle = \sum_{\gamma\in\Gamma, k\geq 1} h_{k,\gamma}(q, T)p_{k,\gamma} + o(p). \tag{42}$$
Then we define $\mathfrak{d}q_{k,\gamma} = kh_{k,\bar{\gamma}}(q, T)$ and extend $\mathfrak{d}$ to the whole algebra using the Leibnitz rule.

In all the above cases the corresponding homology, together with all the inherited algebraic structures, is an invariant of the contact manifold $(V, \xi = \{\lambda = 0\})$ (see [15]), and thus independent of the choice of J, the contact form λ, and the representatives θ_j of the corresponding cohomology classes of $H^*(V)$. Moreover, the homotopy types of the corresponding differential algebras are also invariants of (V, ξ).

However, sometimes it is possible to define a simpler, easier computable contact invariant. Let us restrict the discussion to the case when the set of forms Θ is empty or, equivalently, set $T = 0$. The differential $\mathfrak{d}$ in Case 4 can be viewed as a vector field

$$\mathfrak{d}(q) = \sum_{k,\gamma} kh_{k,\bar{\gamma}}(q)\frac{\partial}{\partial q_{k,\gamma}}$$

[5]The algebraic structure of $(\mathsf{Fock} \otimes \mathsf{CT}_{\boldsymbol{d}}, D)$ can be described in terms of the BV_∞-formalism, see [9].

on the space with coordinates $q_{k,\gamma}$. Suppose there are constants $a_{k,\gamma} \in \mathbb{C}$ such that $\mathfrak{d}(a) = 0$, where $a = \{a_{k,\gamma}\}$. Then one can define the linearized homology of the algebra $(\mathsf{fock}, \mathfrak{d})$ at the point a. More precisely, following Yu. Chekanov [11] we define an *augmentation* of the algebra $(\mathsf{fock}, \mathfrak{d})$ as a graded chain homomorphism $\varepsilon\colon (\mathsf{fock}, \mathfrak{d}) \to (\mathbb{C}, 0)$. In other words, this means that $\mathfrak{d}(a) = 0$ where $a = \{a_{k,\gamma} = h(q_{\gamma,k})\}$ and $a_{k,\gamma} = 0$ unless $a_{k,\gamma}$ has grading 0. The linearized complex is defined as

$$\left(\mathsf{fock}_{d\geq 1}/\mathsf{fock}_{d\geq 2}, d_\varepsilon = \phi_\varepsilon \circ \mathfrak{d} \circ \phi_\varepsilon^{-1}\right), \tag{43}$$

where $\mathsf{fock}_{d\geq m}$ denotes the ideal in fock generated by monomials of degree $\geq m$, and the algebra homomorphism $\phi_\varepsilon\colon \mathsf{fock} \to \mathsf{fock}$ is defined on the generators $q_{k,\gamma}$ as the shift $q_{k,\gamma} \mapsto q_{k,\gamma} + a_{k,\gamma}$. It turns out that if the algebra $(\mathsf{fock}, \mathfrak{d})$ admits a unique augmentation for a certain choice of λ and J, then for any other choice the corresponding algebra admits an augmentation ε, and the homology of the complex (43) is independent of choices of λ, J and ε, and hence it is an invariant of the contact structure ξ, see [11] and [7]. This homology, denoted $CH_*(V, \xi)$ is usually called *cylindrical contact homology* because in all known cases when this homology is defined, there exists a class of forms for which $\mathfrak{d}(0) = 0$, and hence the differential of the linearized complex (43) is determined by holomorphic cylinders. If the cylindrical contact homology is defined then all the other algebraic structures described in examples 1–4 can be interpreted as certain (co-)homological operations on $\mathrm{CH}_*(V, \xi)$. Here are some examples when cylindrical contact homology is well defined and can be computed:

a) *Subcritical Stein-fillable contact manifolds.* (V, ξ) is called Stein fillable if it appears as a strictly pseudo-convex boundary of a Stein domain W. The subcriticality means that W has a homotopy type of a CW-complex of dimension $< \dim_{\mathbb{C}} W$. Under an additional assumption $c_1(\xi) = 0$, M.-L. Yau (see [50]) proved that the cylindrical contact homology is well defined. She also computed it in terms of $H_*(W)$. It seems likely that the condition $c_1(\xi) = 0$ can be removed.

b) *Prequantization spaces.* Cylindrical contact homology of a prequantization space (V, ξ) of a symplectic manifold (M, ω) is well defined and can be expressed through the homology $H_*(M)$, see [15] and [5]. Note that by juxtaposing the computations in a) and b) one gets non-trivial restrictions on the topology of symplectic manifolds with subcritical polarizations in the sense of [4] (e.g. complex projective manifolds admitting a hyperplane section whose complement is a subcritical Stein manifold).

c) *Spaces of co-oriented contact manifolds.* Given an oriented n-dimensional closed M, the cylindrical contact homology of its unit cotangent bundle ST^*M is always well defined, and we have $\mathrm{CH}_*(ST^*M) = H_*^{S^1}(\Lambda(M), M)$ where $H_*^{S^1}(\Lambda(M), M)$ is the equivariant homology of the free loop space modulo constant loops. See [49], [43], [1], [9] for related results.

d) *Brieskorn varieties.* I. Ustilovsky (see [48]) computed contact homology of certain Brieskorn spheres. His computation implied existence of infinitely many non-isomorphic contact structures on spheres of dimension $4k+1$. F. Bourgeois ([5]) and O. van Koert ([33]) extended Ustilovsky's computations to a large class of other Brieskorn varieties.

e) *Toroidal* 3*-manifolds.* It was shown in [15] and [5] that cylindrical contact homology distinguish all the contact structures on T^3 (there are infinitely many of them according to E. Giroux, see [22]). F. Bourgeois and V. Colin, see [6], generalized this computation to toroidal (i.e. containing an incompressible torus) irreducible 3-manifolds and as a consequence showed that such manifolds have infinitely many non-isotopic (universally) tight contact structures. This result should be contrasted with a theorem of V. Colin, E. Giroux and Ko Honda, see [12], which states that atoroidal irreducible 3-manifolds may admit only finitely many non-isotopic tight contact structures.

f) *Exact triangle for Legendrian surgery.* In [7] F. Bourgeois, T. Ekholm and the author found an exact triangle which relates cylindrical contact homology before and after surgery along a Legendrian sphere, and a certain cyclic complex associated to the differential algebra of the Legendrian sphere, see discussion of relative SFT in Section 3.3 below. This exact triangle is tightly related to Seidel's exact triangle describing an effect of a symplectic Dehn twist on Floer homology, see [44] and [45].

F. Bourgeois computed in his dissertation [5] cylindrical contact homology for a number of other interesting examples (e.g. for T^k-invariant contact structures constructed by R. Lutz in [37] on some $(2k+1)$-manifolds). Most recently, V. Colin and K. Honda, see [28], announced a result that the cylindrical contact homology is defined and not trivial for a large class of tight contact 3-manifolds. This theorem implies the Weinstein conjecture (i.e. existence of periodic orbits of the Reeb flow) for this class of contact 3-manifolds. It seems likely that cylindrical contact homology is well defined at least for all Stein fillable, or maybe even more generally, symplectically fillable contact manifolds. Note that the algebra $(\mathsf{fock}, \mathfrak{d})$ for symplectically fillable contact manifolds always admits an augmentation (see [7]), which is unique in all known cases for an appropriate choice of λ and J.

3.2. Topological invariants via SFT. There are several canonical constructions which associate with smooth manifolds and their submanifolds symplectic and contact manifolds and their Lagrangian and Legendrian submanifolds. Here are a few examples:

(1) Given a smooth closed n-manifold, one can associate with it its cotangent bundle T^*M with its canonical symplectic form $\omega = dp \wedge dq$, or its unit cotangent bundle (the space of co-oriented contact elements) ST^*M with its canonical contact structure ξ given by the contact form $pdq|_{ST^*M}$.

(2) Given a submanifold $K \subset M$ one can associate with K its Lagrangian conormal bundle $L_K \subset T^*M$, or its Legendrian lift $\Lambda_K \subset T^*M$, formed by co-oriented hyperplanes tangent to K.

(3) Here is another interesting variant of this construction. Let M be a compact manifold with boundary N. Choose a metric on M and take a smooth function $\rho\colon M \to \mathbb{R}_+$ which is positive on the interior of M and such that $\rho(q) = \operatorname{dist}(q, N)$ for $q \in M$ close to the boundary $N = \partial M$. Let $U \subset T^*M$ be a neighborhood of M in T^*M defined by

$$U = \{(q, p) \in T^*M;\ \|p\|^2 \le \rho(q)\}.$$

Take the function $H(q, p) = p(\nabla\rho(q))$. Then $d(pdq - dH) = \omega$, and it is straightforward to check that the form $\lambda = (pdq - dH)|_{\partial U}$ is a contact form. In other words, $V = \partial U$ is a contact type hypersurface and the contact manifold $(V, \zeta = \{\lambda = 0\})$ depends only on the smooth manifold M, up to an isotopic to the identity contactomorphism. Then $N \subset V$ is a Legendrian submanifold in V whose Legendrian isotopy class is another smooth invariant of M.

(4) Moreover, note that the involution $\operatorname{inv}(p) = -p$ interacts well with all the above structures. For instance, it induces an anti-symplectic involution of T^*M, a contact, co-orientation reversing involution of the space of co-oriented contact elements ST^*M and of the contact manifold V in (3). In that example the Legendrian manifold N is the fixed point set of inv, while inv induces an involution of the Lagrangian L_K and Legendrian Λ_K in (2).

The author believes that all the above canonical symplectic and contact constructions retain a lot of information about the differential topology of the manifold M, or the pair (M, K). For instance, let Σ be a homotopy n-sphere with an exotic smooth structure.

Are the cotangent bundles $T^*\Sigma$ and T^*S^n symplectomorphic?[6]

Are the spaces of contact elements $ST^*\Sigma$ and ST^*S^n contactomorphic?

Can any gauge-theoretic invariants of a 4*-manifold M* (*and maybe even its smooth type*) *be recovered from the symplectic and contact information about T^*M and ST^*M?*

Note that as smooth manifolds, T^*M and ST^*M depend only on the (tangential) homotopy type of M, and hence all the subtle differential-topological information gets lost this way.

Recently M. Abouzaid and P. Seidel [2] developed a program for proving that certain homotopy spheres do not admit Lagrangian embeddings into T^*S^n. This would answer negatively to the first question for this class of homotopy spheres. In the Legendrian version of example (2) one can try to use the differential algebra of

[6]This question I first heard 18 years ago from G. Mess.

the Legendrian submanifold Λ_K as a tool to detect topological invariants of the knot $K \subset M$. L. Ng successfully used this construction for knots in $\mathbb{R}^3$ and recovered this way a wealth of invariants. For instance, he proved (see [40]) that even the simplest linearized version of this algebra homology already encodes the Alexander polynomial, and also, essentially, the so-called A-polynomial. In particular, this linearized homology distinguishes the unknot – any knot which has the same Ng invariant as the unknot is actually the unknot.

It is interesting to apply construction (4) to a 3- or 4-manifold whose boundary N is a sphere, and then compute the equivariant homology of the differential algebra of the Legendrian submanifold $N \subset V$. It seems plausible (and this is a current joint project of T. Ekholm and the author) that the $\mathbb{Z}/2$-equivariant homology of this algebra carries non-trivial information about the differential topology of the manifold M.

3.3. Other SFT-related development. We briefly mention in this section some recent development relating SFT with hot topics in topology.

Embedded contact homology. As it was already pointed out by M. Gromov in his pioneering paper [23], the holomorphic curve technique is especially powerful in 4-dimensional symplectic topology, because the adjunction formula allows one to control singularities and intersections of holomorphic curves by topological means. The work of C. Taubes [46] emphasized further a special role played by holomorphic curves in 4-dimensional topology. A current project of M. Hutchings, M. Sullivan and C. Taubes attempts to define a contact homology theory in the spirit of SFT, but based on embedded holomorphic curves, see [29] and [47] for partial results in this direction. When fully completed, this theory is expected to provide a unified approach to Ozsváth–Szabó homology theory for 3-manifolds ([42] and also [36]), and to a (yet to be developed) theory of holomorphic curves in near-symplectic manifolds (see [46]).

SFT and string topology. The relation between the topology of the loop space of a manifold M and the Floer homology theory of its cotangent bundle T^*M was first revealed by C. Viterbo [49], and then further developed by D. Salamon and J. Weber [43]. A. Abbondandolo and M. Schwartz [1]) related string topological operations introduced by M. Chas and D. Sullivan [10] with cohomological operations in the Floer homology of T^*M. Based on the fundamental study of Lagrangian intersection Floer homology theory in [20], K. Fukaya [18] observed that the relation between Chas–Sullivan string operations and the theory of holomorphic curves can be used to obtain new restrictions on the topology of Lagrangian submanifolds. In an ongoing project K. Cieliebak and J. Latchev [9] have further developed these ideas, and related the BV_∞-version of contact homology of ST^*M, discussed above in Example 2 of Section 3.1, with Chas–Sullivan string operations in the manifold M.

Relative SFT. Conjecturally, relative SFT is a functor defined on the geometric category of pairs (V, Λ), where V is a contact manifold and Λ its Legendrian submanifold, with morphisms realized by pairs (W, L) of symplectic cobordisms W between contact manifolds and Lagrangian cobordisms L between Legendrian submanifolds. The target algebraic category should consist of non-commutative analogs of structures considered in Section 1.1. However, in this full form the relative SFT-functor has not yet been constructed. Yu. Chekanov (see also [16]) defined in [11] an associative differential algebra of a Legendrian link in the standard contact $\mathbb{R}^3$. This algebra (already mentioned above in Section 3.2) is a relative analog of the differential contact homology algebra in Example 4 of Section 3.1. Following a sketch in [16] and [15], T. Ekholm, J. Etnyre and M. Sullivan (see [14]) constructed an analogue of Chekanov's algebra in a context of high-dimensional Legendrian submanifolds. Currently there are two promising approaches which may lead to the construction of the full relative version of SFT. One is based on O. Cornea and F. Lalonde [13] theory of cluster Floer homology, and the other one tries to exploit the discussed above relation with string topology along the lines of [18], [20] and [9].

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