

Symplectic Forms on Moduli Spaces of Flat Connections on 2-Manifolds

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Abstract

Let G be a compact connected semisimple Lie group. We extend the techniques of Weinstein [W] to give a construction in group cohomology of symplectic forms ω on ‘twisted’ moduli spaces of representations of the fundamental group π of a 2-manifold Σ (the smooth analogues of $\text{Hom}(\pi_1(\Sigma), G)/G$) and on relative character varieties of fundamental groups of 2-manifolds. We extend this construction to exhibit a symplectic form on the extended moduli space [J1] (a Hamiltonian G -space from which these moduli spaces may be obtained by symplectic reduction), and compute the moment map for the action of G on the extended moduli space.

1 Introduction

Let Σ be a closed oriented 2-manifold of genus $g \geq 2$; the fundamental group of Σ will be denoted π . Let G be a compact connected Lie group with Lie algebra \mathfrak{g} . This paper concerns the moduli space $\mathcal{M} = \text{Hom}(\pi, G)/G$ of conjugacy classes of representations of π into G , and certain more general analogues of \mathcal{M} . The space \mathcal{M} has an open dense set on which the structure of a smooth symplectic manifold is defined. In addition to the definition we have given in terms of representations of π , the space \mathcal{M} has two alternative descriptions. The first of these is the gauge theory description: via the holonomy map, \mathcal{M} is identified with the space of gauge equivalence classes of flat connections on a trivial principal G bundle over Σ . The second alternative description appears once one fixes a complex structure on the 2-manifold Σ , so that Σ becomes a Riemann surface; \mathcal{M} is then identified with the space of equivalence classes of semistable holomorphic $G^{\mathbb{C}}$ bundles over Σ .

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The purpose of this paper is to extend work of Karshon [K] and Weinstein [W] to a more general setting; this paper follows [W] closely and should be read in conjunction with it. These two papers complement the work of Goldman [G]. Goldman originally gave a construction in group cohomology of the symplectic form ω on the space \mathcal{M} ; in order to prove ω is closed, Goldman used the gauge theory description of the symplectic form. Karshon [K] gave a proof of the closedness of the symplectic form using group cohomology; Weinstein [W] reinterpreted Karshon's construction in the setting of the de Rham-bar bicomplex [B,Sh]. In the present work we extend Weinstein's work to construct symplectic forms on relative character varieties of surface groups, and on 'twisted' moduli spaces \mathcal{M}_β of bundles on Riemann surfaces (see Section 6 of [AB]) associated to an element $\beta \in Z(G)$. The spaces \mathcal{M}_β share many properties with \mathcal{M} but in general have less singularities. Indeed, \mathcal{M}_β is smooth when $G = SU(n)$ and β is a generator of the center of $SU(n)$: in contrast, even when $G = SU(2)$, the space \mathcal{M} is smooth only in the very special case $g = 2$.

We also give a group cohomology construction of symplectic forms on the extended moduli spaces X and X_β [J1]: these are finite dimensional symplectic G -spaces from which \mathcal{M} and \mathcal{M}_β may be obtained by symplectic reduction. In [J1], symplectic structures on X_β and X (which are in fact the same as the symplectic forms we recover below) were specified using gauge theoretic techniques. Here we compute the moment maps for the G action on X and X_β ; up to a normalization factor, these coincide with the moment maps found in [J1].

The purpose of the construction of X_β in [J1] was to exhibit \mathcal{M}_β as the result of finite dimensional symplectic reduction (in contrast to the infinite dimensional quotient construction given in [AB]). We make further use of this finite dimensional quotient construction in [JK2], where we extend the techniques of [JK1] (which gives a formula for intersection pairings in the cohomology ring of the symplectic quotient \mathcal{M}_X of a finite dimensional Hamiltonian G -space X , in terms of the G -equivariant cohomology $H_G^*(X)$) to treat the intersection pairings in \mathcal{M}_β , starting from the equivariant cohomology of X_β . By this means we give proofs of formulas (found originally by Witten [Wi] using physical methods) for the intersection pairings in $H^*(\mathcal{M}_\beta)$.

In this paper, the symplectic forms on X and X_β are constructed explicitly in terms of the Maurer-Cartan form on G and the chain homotopy operator that occurs in the standard proof of the Poincaré lemma. This explicit description of the symplectic form will be important in [JK2], where we make use of explicit equivariantly closed differential forms representing the relevant classes in de Rham cohomology. In [J2] we extend the methods of this paper to give explicit representatives in De Rham cohomology for all the generators of the cohomology ring of \mathcal{M}_β (one of which is the cohomology class of the symplectic form); for the applications in [JK2], we shall need the de Rham representatives for all the generators.

After this work was completed, we received the paper of Huebschmann [H], in which he has obtained similar results independently.

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2 Group cohomology

Let $\mathbb{F} = \mathbb{F}_{2g}$ be the free group on $2g$ generators x_1, \dots, x_{2g} . We introduce a relation $R = \prod_{i=1}^g [x_{2i-1}, x_{2i}]$, where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. The fundamental group π of a closed 2-manifold of genus g is then given by

$$\pi = \mathbb{F}/\mathcal{R}$$

where \mathcal{R} is the normal subgroup generated by R .

We shall work with Eilenberg-Mac Lane group chains (see [G] section 3.8), and shall denote the differential on the group chain complex by ∂ . The p -chains $C_p(\Gamma)$ on a group Γ are \mathbb{Z} -linear combinations of elements of Γ^p . In particular, associated to the relation R there is a distinguished 2-chain $c \in C_2(\mathbb{F})$ given by

$$c = \sum_{i=1}^{2g} (\partial R / \partial x_i, x_i). \quad (2.1)$$

(Here, $\partial/\partial x_i$ refers to the differential in the Fox free differential calculus: see [G], sections 3.1-3.3.) Goldman ([G], above Proposition 3.9) shows

$$\partial c = 1 - R. \quad (2.2)$$

3 The de Rham-bar bicomplex

Weinstein [W] introduces a bicomplex $(C^{*,*}(G), \delta, d)$ whose p, q term is $C^q(G^p)$. The second coboundary is the exterior differential d , while the first is the differential δ appearing in group cohomology:

$$(\delta\beta)(g_0, \dots, g_p) = (-1)^{p+1}\beta(g_0, \dots, g_{p-1}) + \sum_{i=1}^p (-1)^i \beta(g_0, \dots, g_{i-1}g_i, \dots, g_p) + \beta(g_1, \dots, g_p). \quad (3.1)$$

Let Y denote $\text{Hom}(\mathbb{F}, G) = G^{2g}$. Then there is a second bicomplex $(\tilde{C}^{*,*}(G), \delta, d)$ whose p, q term is $\Omega^q(\mathbb{F}^p \times Y) = \Omega^0(\mathbb{F}^p) \otimes \Omega^q(Y)$. (The differential δ in the bicomplex $\tilde{C}^{*,*}(G)$ is the adjoint of the differential ∂ in the Eilenberg-Mac Lane group chain complex.) As in [W], the evaluation maps

$$E_p : \mathbb{F}^p \times Y \rightarrow G^p$$

give rise to maps

$$E_p^* : \Omega^q(G^p) \rightarrow \Omega^0(\mathbb{F}^p) \otimes \Omega^q(Y)$$

which combine to form a map of bicomplexes $E^* : C^{*,*}(G) \rightarrow \tilde{C}^{*,*}(G)$.

We recall the following elements of $C^{*,*}(G)$ defined in [W]. Let $\alpha \in \Omega^1(G) \otimes \mathfrak{g}$ denote the (left-invariant) Maurer-Cartan form on G , and $\bar{\alpha}$ the corresponding right-invariant form. Define projection maps $\pi_i : G^p \rightarrow G$ ($i = 1, \dots, p$) to the i 'th copy of G , and let $\alpha_i = \pi_i^* \alpha$ and $\bar{\alpha}_i = \pi_i^* \bar{\alpha}$. In terms of this notation, the following are introduced in [W]:

$$\lambda = \frac{1}{6} \alpha \cdot [\alpha, \alpha] \in \Omega^3(G), \quad (3.2)$$

$$\Omega = \alpha_1 \cdot \bar{\alpha}_2 \in \Omega^2(G^2). \quad (3.3)$$

Given $\eta \in \mathfrak{g}$, Weinstein also introduces

$$\theta_\eta = \eta \cdot (\alpha + \bar{\alpha}) \in \Omega^1(G). \quad (3.4)$$

(Here, \cdot denotes an invariant inner product on \mathfrak{g} .) These forms satisfy the following properties ([W], Lemmas 3.1, 3.3, 4.1 and 4.4):

Proposition 3.1 *We have*

$$d\lambda = 0; \quad (3.5)$$

$$d\Omega = \delta\lambda; \quad (3.6)$$

$$\iota_{\tilde{\eta}}\lambda = d\theta_\eta; \quad (3.7)$$

$$\iota_{\tilde{\eta}}\Omega = -\delta\theta_\eta, \quad (3.8)$$

where $\tilde{\eta}$ is the vector field generated by η .

4 Two-forms on moduli spaces

Let us now introduce

$$\omega = \langle c, E^*\Omega \rangle \in \Omega^2(Y). \quad (4.1)$$

We then have

Proposition 4.1 *In terms of the identification of $Y = \text{Hom}(\mathbb{F}, G)$ with G^{2g} , we have*

$$d\omega = -\epsilon_R^*\lambda \quad (4.2)$$

where $\epsilon_R : \text{Hom}(\mathbb{F}, G) \rightarrow G$ is the map given by evaluation on the element $R \in \mathbb{F}$:

$$\epsilon_R(g_1, \dots, g_{2g}) = \prod_{i=1}^g [g_{2i-1}, g_{2i}]. \quad (4.3)$$

Proof: We have

$$\begin{aligned} d\langle c, E^*\Omega \rangle &= \langle c, dE^*\Omega \rangle \\ &= \langle c, E^*d\Omega \rangle = \langle c, E^*\delta\lambda \rangle \\ &= \langle c, \delta E^*\lambda \rangle = \langle \partial c, E^*\lambda \rangle \\ &= \langle 1 - R, E^*\lambda \rangle, \end{aligned}$$

where the last step follows from (2.2). \square

If t is any element of G , define $Y_t = \epsilon_R^{-1}(t) \subset Y$. The following is an immediate consequence of Proposition 4.1:

Proposition 4.2 *The 2-form ω restricts on Y_t to a closed form.*

Definition 4.3 *If t is an element of G , let $Z_t \subset G$ be the centralizer of t in G .*

Notice that Y_t carries an action of Z_t by conjugation.

Definition 4.4 *If $t \in G$, the relative character variety associated to t is the space $\mathcal{M}_t = Y_t/Z_t$, where Z_t acts on Y_t by conjugation.*

Some properties of the symplectic geometry of relative character varieties were given in [JW1] and [JW2]. Relative character varieties also arise in algebraic geometry where (under appropriate circumstances) they are identified with moduli spaces of semistable parabolic vector bundles on Riemann surfaces: see [MS] (where the identification between relative character varieties and moduli spaces of parabolic bundles is given) or [Se]. A gauge theory construction of a symplectic form on relative character varieties is given in [J1]: it will follow from the Remark at the end of Section 5 that this is essentially the same as the symplectic form we construct here (i.e., the two are the same under a natural map identifying the relevant Zariski tangent spaces).

By extending the calculations in [W], we obtain the following two propositions.

Proposition 4.5 *The form ω is invariant under the action of G on Y by conjugation.*

Proof: Let $\eta \in \mathfrak{g}$; we will show that the Lie derivative of ω with respect to the vector field $\tilde{\eta}$ generated by η is zero, in other words $\mathcal{L}_{\tilde{\eta}}\omega = (d\iota_{\tilde{\eta}} + \iota_{\tilde{\eta}}d)\omega = 0$. Now

$$\iota_{\tilde{\eta}}d\omega = -\iota_{\tilde{\eta}}\epsilon_R^*\lambda = -\epsilon_R^*\iota_{\tilde{\eta}}\lambda. \quad (4.4)$$

Also $d\iota_{\tilde{\eta}}\omega = d\langle c, E^*\iota_{\tilde{\eta}}\Omega \rangle$. But $\iota_{\tilde{\eta}}\Omega = -\delta\theta_\eta$ by (3.8), so we get

$$\begin{aligned} d\iota_{\tilde{\eta}}\omega &= -d\langle c, E^*\delta\theta_\eta \rangle \\ &= -d\langle c, \delta E^*\theta_\eta \rangle \\ &= -d\langle \partial c, E^*\theta_\eta \rangle = -d\langle 1 - R, E^*\theta_\eta \rangle \\ &= d\epsilon_R^*\theta_\eta = \epsilon_R^*d\theta_\eta. \end{aligned}$$

But by (3.7) we have $d\theta_\eta = \iota_{\tilde{\eta}}\lambda$, so $d\iota_{\tilde{\eta}}\omega = \epsilon_R^*\iota_{\tilde{\eta}}\lambda$, and $\mathcal{L}_{\tilde{\eta}}\omega = 0$. \square

Proposition 4.6 *We have the following identification of 1-forms on G^{2g} :*

$$\iota_{\tilde{\eta}}\omega = \epsilon_R^*\theta_\eta. \quad (4.5)$$

Thus, the restriction of ω to Y_t is horizontal with respect to the action of Z_t . (In other words, if $\eta \in \text{Lie}(Z_t)$ generates the vector field $\tilde{\eta}$ on Y , then $\iota_{\tilde{\eta}}\omega|_{Y_t} = 0$, where $\iota_{\tilde{\eta}}$ denotes the interior product with respect to $\tilde{\eta}$.)

Proof: We have

$$\begin{aligned}\iota_{\bar{\eta}}\omega &= \langle c, \iota_{\bar{\eta}}E^*\Omega \rangle = \langle c, E^*\iota_{\bar{\eta}}\Omega \rangle = -\langle c, E^*\delta\theta_\eta \rangle, \\ &= -\langle \partial c, E^*\theta_\eta \rangle = -\langle 1 - R, E^*\theta_\eta \rangle = \epsilon_R^*\theta_\eta.\end{aligned}$$

This form necessarily restricts to zero on the level sets of ϵ_R . \square

Propositions 4.5 and 4.6 imply that the form ω descends to a 2-form $\bar{\omega}$ on the space \mathcal{M}_t , which is closed by Proposition 4.2. Nondegeneracy will be established in Corollary 5.5.

In particular if t is a central element $\beta \in Z(G)$ then Z_β is the full group G . If $G = U(n)$ and $\beta = e^{2\pi id/n} \text{diag}(1, \dots, 1)$ then the space \mathcal{M}_β appears in algebraic geometry (see [AB]) as the moduli space of semistable holomorphic vector bundles of rank n and degree d . When β is as above but $G = SU(n)$, the algebraic geometry interpretation of the space \mathcal{M}_β is as the moduli space of semistable holomorphic vector bundles of rank n and degree d with fixed determinant.¹ The constructions in this section exhibit a group cohomology construction of a symplectic form on the ‘twisted’ moduli spaces \mathcal{M}_β associated to central elements β of G . It is shown in [J2] that the form $\bar{\omega}$ is in the cohomology class of (a constant multiple of) the standard generator f_2 of $H^*(\mathcal{M}_\beta; \mathbb{R})$ (in the notation of Sections 2 and 9 of [AB]). In fact in [J2] we extend the construction which gives rise to the symplectic form $\bar{\omega}$, to give representatives in de Rham cohomology for all the generators of the ring $H^*(\mathcal{M}_\beta; \mathbb{R})$ given in [AB]. Our applications in [JK2] will rely at least as heavily on the fact that $\bar{\omega}$ is a de Rham representative of the cohomology class f_2 as on its being nondegenerate: in any case, many results of the type we shall invoke for Hamiltonian G -manifolds generalize (cf. [KT]) to manifolds where the symplectic form degenerates on a locus of measure 0.

5 The symplectic form on the extended moduli space

Let β be an element of the center $Z(G)$. The associated *extended moduli space* X_β constructed in [J1] may be described as a fibre product

$$X_\beta = (\epsilon_R \times e_\beta)^{-1}(\Delta) \subset G^{2g} \times \mathfrak{g}. \quad (5.1)$$

Here, Δ is the diagonal in $G \times G$ and $\epsilon_R : G^{2g} \rightarrow G$ was defined above, while $e : \mathfrak{g} \rightarrow G$ is the exponential map and $e_\beta = \beta \cdot e$. The space X_β is equipped with two canonical projection maps $\text{pr}_1 : X_\beta \rightarrow G^{2g}$ and $\text{pr}_2 : X_\beta \rightarrow \mathfrak{g}$, for which there is the following commutative diagram:

$$\begin{array}{ccc} X_\beta & \xrightarrow{\text{pr}_2} & \mathfrak{g} \\ \text{pr}_1 \downarrow & & \downarrow e_\beta \\ G^{2g} & \xrightarrow{\epsilon_R} & G \end{array} \quad (5.2)$$

A straightforward argument using the regular value theorem endows X_β with a smooth structure on an open dense set X_β^s including $\text{pr}_2^{-1}(0)$: see [J1], Proposition 5.4, where an explicit characterization of the singular locus of X_β is given.

¹When $G = U(n)$ or $SU(n)$ and d is coprime to n , the spaces \mathcal{M}_β are smooth manifolds. This is in contrast to the spaces \mathcal{M} , which are singular except in a few very special cases.

The space X_β carries an action of the group G . A gauge theoretic construction of a G -invariant closed 2-form on X_β was given in [J1], and it was shown that this form is nondegenerate on an open dense set in X_β and that the action of G is Hamiltonian where the 2-form is nondegenerate. There is thus an open dense set in X_β which is a smooth finite dimensional Hamiltonian G -space such that the space \mathcal{M}_β is given by symplectic reduction of this G -space at 0.

By extending the techniques of [W], we construct here a G -invariant closed 2-form on X_β which is nondegenerate on an open dense set, and show that the moment map μ is given by a constant multiple of pr_2 , as was shown in [J1]. The symplectic form will in fact turn out to be the same as the one constructed using gauge theory (see the Remark at the end of this section): in other words, one of these symplectic forms is the pullback of the other under a natural map. It follows from our construction that the symplectic quotient (at 0) with respect to the action of G on X_β is the twisted moduli space \mathcal{M}_β described above. For $t \in G$, the relative character variety \mathcal{M}_t from Section 4 is the symplectic reduction of X at the orbit $\mathcal{O}_\Lambda \subset \mathfrak{g}$ (under the adjoint action) that corresponds to an element $\Lambda \in \mathfrak{g}$ for which $\exp(\Lambda) = t$.

To construct the symplectic form, we first construct a form $\sigma \in \Omega^2(\mathfrak{g})$ for which

$$e^*\lambda = d\sigma. \quad (5.3)$$

The existence of such a form follows from the following standard result (see e.g. [Wa], Lemma 4.18):

Proposition 5.1 [*Poincaré Lemma*]

(a) If $\gamma \in \Omega^{p+1}(V)$ where V is a vector space, and $d\gamma = 0$, then there is a form $\sigma \in \Omega^p(V)$ with $\gamma = d\sigma$.

(b) Denote by I the map $\Omega^{p+1}(V) \rightarrow \Omega^p(V)$ sending γ to σ . Then $dI + Id = \text{id}$.

For $\beta \in \Omega^*(V)$, $I\beta$ is given at $v \in V$ by

$$(I\beta)_v = \int_0^1 F_t^*(\iota_{\bar{v}}\beta) dt \quad (5.4)$$

where \bar{v} is the vector field on V which takes the constant value v , and F_t is the map $V \rightarrow V$ given by multiplication by t . In our case the form

$$\sigma = I(e^*\lambda)$$

is G -invariant because λ is G -invariant and e is a G -equivariant map.

We now restrict to the fibre product $X_\beta \subset G^{2g} \times \mathfrak{g}$. For $(h, \Lambda) \in X_\beta$ we have $\epsilon_R(h) = e_\beta(\Lambda)$, so if $(H, \zeta) \in T_h G^{2g} \times \mathfrak{g}$ represents an element in the tangent space to X_β , we have

$$\epsilon_{R*}H = e_{\beta*}\zeta. \quad (5.5)$$

We define a 2-form on X_β by

$$\tilde{\omega} = \text{pr}_1^*\omega + \text{pr}_2^*\sigma, \quad (5.6)$$

where ω was defined in (4.1). We find that

$$d(\text{pr}_1^*\omega + \text{pr}_2^*\sigma) = \text{pr}_1^*d\omega + \text{pr}_2^*d\sigma$$

so

$$\begin{aligned} d\tilde{\omega}(H, \zeta) &= -\epsilon_R^*\lambda(H) + d\sigma(\zeta) \quad (\text{by (4.2)}) \\ &= -\lambda(\epsilon_{R^*}H) + \lambda(e_*\zeta) = -\lambda(\epsilon_{R^*}H) + \lambda(e_{\beta^*}\zeta) = 0. \end{aligned} \quad (5.7)$$

(Here, we have used the fact that λ is invariant under multiplication by β , so $e^*\lambda = e_{\beta^*}\lambda$.) So we have

Proposition 5.2 *The 2-form $\tilde{\omega}$ on $G^{2g} \times \mathfrak{g}$ restricts on X_β to a closed form.*

The G -invariance of $\tilde{\omega}$ follows because σ and ω are G -invariant and the projection maps pr_1 and pr_2 are G -equivariant maps.

We may now identify the moment map for the action of G on X_β : in other words, we find a function $\mu : X \rightarrow \mathfrak{g}$ such that $\iota_{\tilde{\eta}}\tilde{\omega} = \eta \cdot d\mu$ where $\tilde{\eta}$ is the vector field on X_β generated by $\eta \in \mathfrak{g}$.² First we recall from (4.5) that $(\iota_{\tilde{\eta}}\omega) = \epsilon_R^*\theta_\eta$. Now since $\sigma = I(e^*\lambda) = I(e_{\beta^*}\lambda)$, we have

$$\begin{aligned} \iota_{\tilde{\eta}}\sigma &= \iota_{\tilde{\eta}}(Ie_{\beta^*}\lambda) = -I(\iota_{\tilde{\eta}}e_{\beta^*}\lambda) = -I(e_{\beta^*}\iota_{\tilde{\eta}}\lambda) \\ &= -I(de_{\beta^*}\theta_\eta) \quad (\text{by (3.7)}) . \end{aligned} \quad (5.8)$$

Combining (5.8) with Proposition 5.1 (b) we find

$$\iota_{\tilde{\eta}}\sigma = -e_{\beta^*}\theta_\eta + d(Ie_{\beta^*}\theta_\eta) \quad (5.9)$$

Adding (4.5) and (5.9) we have

$$\iota_{\tilde{\eta}}(\tilde{\omega}) = d(Ie_{\beta^*}\theta_\eta)$$

so that a moment map $\mu : X_\beta \rightarrow \mathfrak{g}$ for the action of G on X_β is given by

$$\eta \cdot \mu = Ie_{\beta^*}\theta_\eta. \quad (5.10)$$

Now

$$\begin{aligned} (Ie_{\beta^*}\theta_\eta)_\Lambda &= \int_0^1 F_t^*(e_{\beta^*}\theta_\eta(\Lambda)) \quad \text{by (5.4)} \\ &= \int_0^1 (e_{\beta^*}\theta_\eta)_{\Lambda t}(\Lambda) = \int_0^1 (\theta_\eta)_{e_{\beta^*}(\Lambda t)}(e_{\beta^*}\Lambda) \\ &= \int_0^1 \eta \cdot (\alpha + \bar{\alpha})_{e_{\beta^*}(\Lambda t)}(e_{\beta^*}\Lambda) \\ &= 2\eta \cdot \Lambda. \end{aligned} \quad (5.11)$$

Thus we have explicitly specified a moment map for the action of G , which is equivariant with respect to the adjoint action of G :

²Ordinarily the moment map is specified as a map into \mathfrak{g}^* . Here we have used the inner product to identify \mathfrak{g}^* with \mathfrak{g} .

Proposition 5.3 *A moment map for the action of G on X_β is given by the map $\mu = 2\text{pr}_2 : (h, \Lambda) \mapsto 2\Lambda$.*

To complete the identification of the form $\tilde{\omega}$ as a symplectic form on an open dense set in X_β , one needs the following:

Proposition 5.4 *The form $\tilde{\omega}$ is a nondegenerate bilinear form on the Zariski tangent space $T_{(h,\Lambda)}X_\beta$, for any $(h, \Lambda) \in X_\beta$ for which $(d\epsilon_R)_h$ is surjective.*

Our proof of this Proposition parallels the gauge theory argument given in [J1] (see Proposition 3.1 of [J1] for the case $G = SU(2)$). This material is treated in [H] (Theorem 4.4 and Section 5) and [K] (Theorem 4): the proof we sketch is essentially the one given by Huebschmann [H], to whom the group cohomology proof of the nondegeneracy of the symplectic form on an open neighbourhood of the zero locus of the moment map in the extended moduli space is due.³

Proof of Proposition 5.4: To establish nondegeneracy of $\tilde{\omega}$ we proceed as follows. Proposition 5.3 establishes that if $\tilde{\eta}$ is the vector field associated to the action of η on X_β , and if $(H, \zeta) \in T_{(h,\Lambda)}X_\beta$ for $(h, \Lambda) \in X_\beta$ (in other words, $(d\epsilon_R)_h H = (d\epsilon_\beta)_\Lambda \zeta$), then

$$\tilde{\omega}_{(h,\Lambda)}(\tilde{\eta}, (H, \zeta)) = 2\eta \cdot \zeta. \quad (5.12)$$

Thus to establish nondegeneracy of $\tilde{\omega}$ at those (h, Λ) for which $(d\epsilon_R)_h$ is surjective (see (5.5)) it suffices to establish it on the orthocomplement in $\text{Ker}(d\text{pr}_2) \subset T_{(h,\Lambda)}X_\beta$ of the image of the action of \mathfrak{g} . This means we must establish that ω is nondegenerate restricted to

$$\frac{T_{(h,\Lambda)}(\text{pr}_2^{-1}(\Lambda) \subset X_\beta)}{\{\tilde{\eta} : \eta \in \text{Stab}(\Lambda)\}}. \quad (5.13)$$

We have commutative diagrams

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_B & \longrightarrow & B^1(\mathbb{F}; \mathfrak{g}_h) & \xrightarrow{\delta} & B^1(\mathbb{Z}; \mathfrak{g}_{\epsilon_R(h)}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_C & \longrightarrow & C^1(\mathbb{F}; \mathfrak{g}_h) & \xrightarrow{\delta} & C^1(\mathbb{Z}; \mathfrak{g}_{\epsilon_R(h)}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_H & \longrightarrow & H^1(\mathbb{F}; \mathfrak{g}_h) & \xrightarrow{\delta} & H^1(\mathbb{Z}; \mathfrak{g}_{\epsilon_R(h)}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \quad (5.14)$$

in which the columns are short exact sequences. Here, if Γ is a discrete group (where $\Gamma = \mathbb{Z}$ or $\Gamma = \mathbb{F}$) equipped with a representation $\rho : \Gamma \rightarrow G$, the notation $C^*(\Gamma; \mathfrak{g}_\rho)$ refers to the

³The proof given in [H] applies to a suitable neighbourhood of the zero locus of the moment map in X_β which is contained in $\text{pr}_2^{-1}(\mathcal{O}_{\text{reg}})$, where \mathcal{O}_{reg} is the subset of \mathfrak{g} where the exponential map is regular: the subset $\text{pr}_2^{-1}(\mathcal{O}_{\text{reg}})$ is a proper subset of the smooth locus of X_β . We have adapted the proof so it applies to the Zariski tangent space $T_{(h,\Lambda)}X_\beta$ for all points $(h, \Lambda) \in X_\beta$ for which $(d\epsilon_R)_h$ is surjective.

Eilenberg-Mac Lane group cochain complex with coefficients in the Γ -module \mathfrak{g}_ρ specified by ρ under the adjoint action of G on \mathfrak{g} . For $\Gamma = \mathbb{F}$ or $\Gamma = \mathbb{Z}$, we have $C^1(\Gamma; \mathfrak{g}_\rho) = Z^1(\Gamma; \mathfrak{g}_\rho)$. The maps δ are induced by $d\epsilon_R$, while K_B, K_C and K_H are the kernels of the maps δ on B^1, C^1 and H^1 . Then the vector space $T_{(h, \Lambda)}(\text{pr}_2^{-1}(\Lambda))$ is identified with $K_C \subset C^1(\mathbb{F}; \mathfrak{g}_h)$, while $\{\tilde{\eta} : \eta \in \text{Stab}(\Lambda)\}$ is identified with $K_B = K_C \cap B^1(\mathbb{F}; \mathfrak{g}_h)$. Then we see from (5.14) that the quotient K_C/K_B is canonically identified with $K_H = \text{Ker}(\delta : H^1(\mathbb{F}; \mathfrak{g}_h) \rightarrow H^1(\mathbb{Z}; \mathfrak{g}_{\epsilon_R(h)}))$.

We have the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{F}; \mathfrak{g}_h) \rightarrow H^0(\mathbb{Z}; \mathfrak{g}_{\epsilon_R(h)}) \xrightarrow{\delta^*} H^1(\mathbb{F}, \mathbb{Z}; \mathfrak{g}_h) \rightarrow H^1(\mathbb{F}; \mathfrak{g}_h) \rightarrow \\ \xrightarrow{\delta} H^1(\mathbb{Z}; \mathfrak{g}_{\epsilon_R(h)}) \rightarrow H^2(\mathbb{F}, \mathbb{Z}; \mathfrak{g}_h) \rightarrow 0. \end{aligned} \quad (5.15)$$

The vector spaces and maps in this sequence satisfy Poincaré duality. Furthermore, the pairing $\cdot : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ gives rise to a cup product pairing

$$H^1(\mathbb{F}, \mathbb{Z}; \mathfrak{g}_h) \otimes H^1(\mathbb{F}; \mathfrak{g}_h) \rightarrow H^2(\mathbb{F}, \mathbb{Z}; \mathbb{R}) \cong \mathbb{R}, \quad (5.16)$$

and Poincaré duality in (5.15) implies that the restriction of this pairing to $(H^1(\mathbb{F}, \mathbb{Z}; \mathfrak{g}_h)/\text{Im}(\delta^*)) \otimes (\text{Ker}(\delta) \subset H^1(\mathbb{F}; \mathfrak{g}_h))$ is nondegenerate.

Now one may show (see for instance [K] Theorem 4 or [H] Theorem 4.4) that this cup product is the restriction of the form ω to $\text{Ker}(\delta) \subset H^1(\mathbb{F}; \mathfrak{g}_h) \cong T_h(\epsilon_R^{-1}(C_{\epsilon_R(h)})/G)$. (Here, $C_{\epsilon_R(h)}$ denotes the conjugacy class of $\epsilon_R(h)$ in G .) The nondegeneracy of the pairing arising from the cup product thus completes the proof of nondegeneracy. \square

Remark: Goldman ([G], proof of Proposition 3.7) has shown that

$$\text{Im}(d\epsilon_R)_h = (\text{Lie}(\text{Stab}(h)))^\perp.$$

Suppose $G = SU(n)$ and β is a generator of $Z(G)$. The subset of X_β where $(d\epsilon_R)_h$ is surjective then contains the zero level set of the moment map. Thus the following is an immediate consequence of Proposition 5.4:

Corollary 5.5 *Let $G = SU(n)$ and suppose β is a generator of $Z(G)$. Then 0 is a regular value of the moment map $\mu = 2\text{pr}_2$. Further, \mathcal{M}_β is a smooth manifold and the form $\bar{\omega}$ on \mathcal{M}_β is nondegenerate.⁴*

Remark: At the end of the last Proof, we alluded to the identification of ω (on quotients of level sets of pr_2) with the bilinear form given by the cup product (5.16). It is easy to see that the vector spaces $H^1(\mathbb{F}; \mathfrak{g}_h)$ and $H^1(\mathbb{F}, \mathbb{Z}; \mathfrak{g}_h)$ (arising from group cohomology with coefficients in the \mathbb{F} -module \mathfrak{g}_h specified by $h \in \text{Hom}(\mathbb{F}, G)$) are the same as the vector spaces $H^1(\Sigma - D^2; d_A)$ and $H^1(\Sigma - D^2, \partial D^2; d_A)$ arising in the gauge theory description of X_β (cf. Section 2.2 of [J1]). Here, A is a flat connection on the punctured surface $\Sigma - D^2$ whose holonomy gives rise to the representation h of the fundamental group \mathbb{F} . This identification

⁴When zero is a regular value of the moment map on X_β , standard arguments establish the smoothness of the reduced space \mathcal{M}_β .

comes from the identification between gauge equivalence classes of flat G connections and conjugacy classes of representations of the fundamental group into G , which arises from the map sending a flat connection to the representation given by its holonomy.⁵ Furthermore, the pairing $H^1(\mathbb{F}; \mathfrak{g}_h) \otimes H^1(\mathbb{F}, \mathbb{Z}; \mathfrak{g}_h) \rightarrow \mathbb{R}$ arising from the group cohomology cup product is the same as the pairing $(\alpha, \beta) \mapsto \int_{\Sigma - D^2} \alpha \cdot \wedge \beta$ that gives the symplectic form in the gauge theory description. (Here, α and β are d_A -closed \mathfrak{g} -valued 1-forms on $\Sigma - D^2$, and $\alpha \cdot \wedge \beta$ is the element of $\Omega^2(\Sigma)$ that arises from the wedge product combined with the pairing $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ given by the invariant inner product $\cdot \cdot$.) Hence the symplectic form we have constructed on X_β is in fact the same as the one constructed in [J2] using gauge theory.

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⁵Using the holonomy representation, a homeomorphism from the extended moduli space (as defined gauge theoretically in Section 2.1 of [J1]) to X_β (as defined above) is given in Section 2.3 of [J1]. This map identifies the relevant Zariski tangent spaces and gives rise to the identification of the symplectic forms.

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