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SYMPLECTIC GEOMETRY AND HILBERT'S FOURTH PROBLEM

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Abstract

Inspired by Hofer's definition of a metric on the space of compactly supported Hamiltonian maps on a symplectic manifold, this paper exhibits an *area-length duality* between a class of metric spaces and a class of symplectic manifolds. Using this duality, it is shown that there is a twistor-like correspondence between Finsler metrics on $\mathbb{R}P^n$ whose geodesics are projective lines and a class of symplectic forms on the Grassmannian of 2-planes in $\mathbb{R}^{n+1}$.

... es quizá un error suponer que puedan inventarse metáforas. Las verdaderas, las que formulan íntimas conexiones entre una imagen y otra, han existido siempre ... .

Jorge Luis Borges.

1. Introduction

Hilbert's fourth problem ([15, 19]) asks to construct and study all metrics on $\mathbb{R}^n$ such that the straight line segment is the shortest curve joining two points. Busemann, who called these metrics *projective*, proposed a geometric construction which is inspiringly simple. Let us say that a (possibly) signed measure μ on the set of hyperplanes is *quasi-positive* if whenever x, y, and z are three non-collinear points, the measure of the set of hyperplanes that intersect both the segment xy and the segment yz is strictly positive. Given a quasi-positive measure such that the set of all hyperplanes passing through a point has measure zero, we define the distance between two points as the measure of all hyperplanes intersecting the line segment joining them.

The quasi-positivity of the measure can be reformulated as follows: if Π is a (two-dimensional) plane in $\mathbb{R}^n$ and μ is a signed measure on the space of hyperplanes, we may define a measure μ_π on the space $G(\Pi)$ of lines lying on Π by equating the measure of a Borel subset $U \subset G(\Pi)$

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to the measure of the set of all hyperplanes that intersect Π along a line belonging to U. The measure μ is quasi-positive if the induced measure μ_π is positive on open sets for any choice of plane $\Pi \subset \mathbb{R}^n$.

This reformulation makes it easy to characterize the Finsler metrics (see Definition 2.2) that arise from Busemann's construction. The metric is Finsler if and only if the measure μ is smooth and the induced (smooth) measure μ_π is positive for any choice of plane Π. Moreover, as was proved by Pogorelov in [**23**], every projective Finsler metric on $\mathbb{R}^n$ can be constructed in this way (see also [**25**] and [**9**] for alternate presentations).

While it is possible to take Busemann's construction and Pogorelov's theorem as the solution (and the end) of Hilbert's fourth problem, there are strong reasons for not doing so. For one, a large number of projective metrics cannot be constructed by taking positive measures on the space of hyperplanes and it is not clear how to construct all quasi-positive measures. A second objection is that Busemann's construction, by itself, does not shed much light on the properties of projective metrics. Therefore, it is useful to complement the results of Busemann and Pogorelov with other characterizations of projective metrics that either yield new explicit examples, or shed more light on the general properties of these metric spaces.

The main result of this paper is a new characterization of projective Finsler metrics in terms of a class of symplectic forms on the space of lines.

Definition 1.1. Following [**17**], we say that a differential two-form ω on the space of oriented lines in $\mathbb{R}^n$ is *admissible* if

1) it is closed;
2) the pull-back of ω to the submanifold of all oriented lines passing through an arbitrary point in $\mathbb{R}^n$ is identically zero;
3) it is *odd* (i.e., if a is the involution than reverses the orientation of the lines, then $a^*\omega = -\omega$).

Let ω be a differential two-form on the space of oriented lines in $\mathbb{R}^n$. If x and y are two points in $\mathbb{R}^n$ and Π is a plane containing them, set $d_\omega(x, y; \Pi)$ to be the integral of $|\omega|$ over the set of all oriented lines on Π intersecting the segment xy.

Theorem 1. *If ω is an admissible symplectic form, then the function d_ω is independent of the choice of 2-dimensional subspace and defines a projective Finsler metric on $\mathbb{R}^n$. Moreover, any projective Finsler metric can be obtained from this construction.*

The usefulness of this new characterization is threefold: together with a detailed analysis of Crofton-type formulas, it allows us to extend the classical Crofton formulas to all projective Finsler spaces ([**7**, **8**]); it uncovers the relation between Hilbert's fourth problem and the "black and white" twistor theory of Guillemin and Sternberg ([**18**, **3**]); and it opens the way for symplectic and contact techniques *à l'Arnold* in the study of submanifolds in projective Finsler spaces ([**4**]).

Theorem 1, first announced in [**9**], is a relatively easy consequence of the following symplectic result.

Theorem 2. *Let ω be an admissible two-form on the space of oriented lines in $\mathbb{R}^n$, $n > 2$. The form ω is symplectic if and only if for every plane $\Pi \subset \mathbb{R}^n$, the pull-back of ω to the (two-dimensional) submanifold of all oriented lines lying on Π never vanishes.*

When $n = 3$, the above result has a simple topological proof (see [**3**]). Unfortunately, this proof completely breaks down for $n > 3$. The present proof of Theorem 2 relies on a certain duality between metric and symplectic geometry which extends Arnold's *area-length duality* for spherical curves (see [**10**]). Curiously enough, the area-length duality by itself almost provides an independent proof of Theorem 1. However, *a priori*, the metrics obtained through the construction in Section 4 are not Finsler metrics, but just G-space metrics in the sense of Busemann [**16**]. On the other hand, the construction is a very general criterion for the metrization of path geometries that is interesting in its own right.

Let M be an n-dimensional manifold together with a prescribed system of smooth curves, called *paths,* such that through every point and every direction there is a unique path passing through this point in the given direction. Such a geometric structure is called a *path geometry* on M. A path geometry is *tame* if it satisfies the following two properties:

1) any two distinct points in M that are sufficiently close determine a unique path;
2) the system of oriented curves on M obtained from the path geometry by providing each curve with its two possible orientations is parameterized by a smooth $2n-2$ dimensional manifold Γ.

Note that, in turn, the manifold M parameterizes a system of submanifolds in the parameter space Γ: to each $x \in M$, we associate the submanifold $\hat{x}$ of all paths passing through x. When the path geometry is the system of geodesics of a Finsler metric on M, it is known (see [**2**], [**13**] and Section 2) that the space of geodesics carries a natural symplectic form and that the submanifolds $\hat{x}$, $x \in M$, are Lagrangian. In Section 4, the reader will find the following partial converse:

Theorem 3. *Let M be smooth manifold of dimension greater than two endowed with a tame path geometry and let Γ denote its parameter space. If there exists a symplectic form on Γ such that the submanifold of all paths passing through an arbitrary point in M is Lagrangian, then M can be given a G-space metric such the path geometry coincides with the system of geodesics in M.*

Roughly, the idea of the proof is to consider the map $x \mapsto \hat{x}$ as an embedding of M into the space of all Lagrangian submanifolds of Γ and use the Hofer metric ([**20**], [**21**], and [**22**]) in this infinite-dimensional space to induce a metric in M. More precisely, infinitesimal Hamiltonian deformations of a Lagrangian submanifold L in the symplectic manifold (Γ, ω) are described as the quotient of all smooth functions on L by those that are constant. This space, $C^\infty(L)/\text{constants}$, carries the infinitesimal Hofer norm

$$\|f\| = \text{variation}(f) = \max f - \min f.$$

Since the family of Lagrangian spheres $\hat{x}$, $x \in M$, induces canonical linear maps

$$T_x M \longrightarrow C^\infty(\hat{x})/\text{constants},$$

we can pull-back the Hofer norm to $T_x M$. This gives us, a priori, a low-regularity version of a Finsler structure on M. Actually, in order to avoid technical difficulties (e.g., it is not known when a low-regularity Finsler metric on a manifold M gives rise to a G-space structure) and to make the proof more accessible to metric and Finsler geometers, Section 4 presents an integrated version of the above construction that makes no explicit use of the Hofer metric.

The breakdown of the paper — and the global structure of the proof of Theorem 1 — is as follows: Section 2 is a short review of Finsler manifolds and their spaces of geodesics that ends with a proof of the (easy) second part of Theorem 1. In Section 3, it is shown that the first part of Theorem 1 follows from Theorem 2 and the work of Pogorelov ([**23**]) on Hilbert's fourth problem in two-dimensions.

Section 4 contains the proofs of Theorem 3 and of the following weaker version of Theorem 2:

Theorem 4. *If ω is an admissible symplectic form on the space of oriented lines in $\mathbb{R}^n$, $n > 2$, then for every plane $\Pi \subset \mathbb{R}^n$ the pull-back of ω to the (two-dimensional) submanifold of all oriented lines lying on Π never changes sign, nor vanishes on an open subset.*

In Section 5, Theorem 4 and a detailed analysis of the generalized conformal structure on the Grassmannian of oriented planes in $\mathbb{R}^{n+1}$,

$n > 2$, are used to prove Theorem 2, thus completing the proof of our main result.

2. Finsler manifolds and spaces of geodesics

A Finsler manifold is a manifold together with the choice of a norm on each tangent space. The precise definition requires us to restrict the class of norms to those where the unit sphere is smooth and *quadratically convex:* the principal curvatures are positive for some (and therefore any) auxiliary Euclidean structure. These norms are intrinsically defined as follows:

Definition 2.1. A norm on a vector space V is said to be a *Minkowski norm* if the unit sphere in V and its polar in V^* are smooth. A finite-dimensional vector space provided with a Minkowski norm will be called a *Minkowski space*.

Definition 2.2. A *Finsler metric* on a smooth manifold M is a continuous function on its tangent bundle that is smooth outside the zero-section and such that its restriction to each tangent space is a Minkowski norm.

Given a Finsler metric φ on a manifold M, the length of a smooth curve $\gamma : [a, b] \to M$ is defined by the equation

$$\text{length of } \gamma := \int_a^b \varphi(\dot{\gamma}(t))dt \ ,$$

and the distance between two points x and y in M is defined as the infimum of the lengths of all smooth curves joining x and y.

Later in the paper, we shall be studying *projective Finsler metrics:* Finsler metrics defined on open convex subsets of $\mathbb{R}P^n$ for which the geodesics lie on projective lines. Among the earliest examples of projective Finsler spaces are the Hilbert geometries.

Hilbert geometries. Let $D \subset \mathbb{R}^n$ be a bounded, convex, open subset. Let x and y denote two distinct points on D and let a and b denote the two points of intersection of the boundary of D with the line passing through x and y (Figure 1).

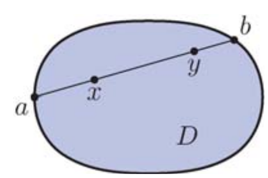

Figure 1.

The distance between x and y is defined by the formula:

$$d(x,y) := \frac{1}{2}\ln\left(\frac{\|y-a\|}{\|x-a\|}\frac{\|x-b\|}{\|y-b\|}\right).$$

In many instances, including when the boundary of D is smooth and quadratically convex, the Hilbert metric on D comes from a Finsler metric (see [**24**]).

Finsler geometry from the Hamiltonian viewpoint. Let M be a Finsler manifold and let $\|\cdot\|_x$ denote the norm at a tangent space T_xM. The *Hamiltonian* of a Finsler metric is the function $H : T^*M \to \mathbb{R}$ whose value at a covector p_x is $\|p_x\|_x^*$, where $\|\cdot\|_x^*$ is the norm dual to $\|\cdot\|_x$. The set $\{p \in T^*M : H(p) < 1\}$ and its boundary are respectively called the *unit co-disc bundle* and the *unit co-sphere bundle* of M, and are denoted by D^*M and S^*M.

Definition 2.3. If we denote by α the pullback of the canonical one-form to S^*M, we define the *Reeb vector field*, X, by the equations

$$d\alpha(X,\cdot) = 0 \quad \text{and} \quad \alpha(X) = 1.$$

The image of an integral curve of X under the *Legendre transform*—the map that sends a unit covector $p_x \in T_x^*M$ to the unique unit vector $v_x \in T_xM$ such that $p_x(v_x) = 1$ —is the curve of unit tangent vectors of a geodesic parameterized with unit speed. Moreover, the length of any segment of this geodesic is given as the integral of α over the corresponding segment of the integral curve of the Reeb vector field.

Spaces of geodesics. If (M,φ) is a Finsler manifold such that its space of oriented geodesics is a manifold $G(M)$, then this manifold carries a natural symplectic structure. Let S^*M denote the unit co-sphere bundle of M and let $\pi : S^*M \to G(M)$ be the canonical projection that sends a given unit covector to the geodesic which has this covector as initial condition. If $i : S^*M \to T^*M$ denotes the canonical inclusion into T^*M and ω_0 is the standard symplectic form on T^*M, there is a unique symplectic form ω on $G(M)$ that satisfies the equation $\pi^*\omega = i^*\omega_0$.

Notice that the space of oriented projective lines in $\mathbb{R}P^n$ is the Grassmannian $G_2^+(\mathbb{R}^{n+1})$ of oriented planes in $\mathbb{R}^{n+1}$, while the space of oriented lines in $\mathbb{R}^n$ is easily identifiable with the cotangent bundle of S^{n-1}.

Let M be a Finsler manifold with a smooth manifold of geodesics $G(M)$. The double fibration

$$\begin{array}{ccc} & S^*M & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ M & & G(M), \end{array}$$

where π_1 and π_2 are the canonical projections, allows us to relate the metric geometry of M, the symplectic geometry of $G(M)$, and the contact geometry of S^*M. We will now review some simple aspects of the geometry of the fibrations $\pi_1 : S^*M \to M$ and $\pi_2 : S^*M \to G(M)$.

Definition 2.4. Let M be an n-dimensional Finsler manifold with unit co-sphere bundle S^*M and canonical form α. The *contact structure* on S^*M is the field of hyperplanes in TS^*M defined by the equation $\alpha = 0$. An immersed manifold $L \subset S^*M$ is said to be *Legendrian* if its dimension is $n-1$ and the form α vanishes on every one of its tangent spaces.

For example, the fibers of the projection π_1 are Legendrian. In the terminology of Arnold (see [**12**]), the fibration $\pi_1 : S^*M \to M$ is a *Legendrian fibration.*

We now turn our attention to the fibration $\pi_2 : S^*M \to G(M)$. Its most important property is that it is a line or circle bundle over $G(M)$ with connection form α and curvature $d\alpha = \pi_2^*\omega$. In other words, it is a *prequantization* of the symplectic manifold $(G(M), \omega)$. Other simple and useful properties are given in the following propositions:

Proposition 2.1. *The map $a : G(M) \to G(M)$ that sends each oriented geodesic to the same geodesic with its opposite orientation is an anti-symplectic involution.*

Proof. Note that the involution A that sends every unit covector p_x to its opposite $-p_x$ takes the form α to $-\alpha$ and that $\pi_2 \circ A = a \circ \pi_2$. It follows that

$$\pi_2^* a^* \omega = A^* \pi_2^* \omega = A^* d\alpha = -d\alpha.$$

This implies that $\pi_2^*(-a^*\omega) = d\alpha$, which in turn implies that $-a^*\omega = \omega$.
q.e.d.

Proposition 2.2. *If $i : L \to S^*M$ is a Legendrian immersion, then the composite map $\pi_2 \circ i : L \to G(M)$ is a Lagrangian immersion. In particular, the submanifold of all geodesics passing through a point in M is Lagrangian.*

Proof. Note that the kernel of the differential of the projection π_2 is transversal to the contact hyperplanes given by the equation $\alpha = 0$. Since the tangent spaces of any Legendrian submanifold are contained in the contact hyperplanes, it follows that if $i : L \to S^*M$ is a Legendrian immersion, then $\pi_2 \circ i$ is an immersion. To see that it is a Lagrangian immersion just compute:

$$(\pi_2 \circ i)^* \omega = i^* \pi_2^* \omega = i^* \omega_M = d i^* \alpha = 0.$$

q.e.d.

Proposition 2.3. *If $N \subset M$ is a totally geodesic submanifold, then the set of all geodesics lying on N is a symplectic submanifold of $G(M)$.*

Proof. If $i : N \to M$ denotes the inclusion, the dual of its differential $i_x^* : T_x^*M \to T_x^*N$ at a point $x \in N$ is a surjective linear map. The restriction of this map to the unit co-sphere S_x^*M has a fold-type singularity along a submanifold Σ_x. Since N is isometrically embedded, i_x^* defines a diffeomorphism between Σ_x and the unit co-sphere S_x^*N.

Performing this construction at each point of N, we construct an embedding $j : S^*N \to S^*M$ that can be characterized by saying that if p_x is a unit covector on N based at x, then $j(p_x)$ is the unique unit covector on M based at x for which

$$j(p_x) \cdot i_{*x}(v_x) = p_x \cdot v_x, \text{ for all } v_x \in T_xN.$$

If α_N and α_M denote, respectively, the canonical one-forms on S^*N and S^*M, it is easy to check that $j^*\alpha_M = \alpha_N$. Moreover, when N is totally geodesic, the map j is equivariant with respect to the actions defined by the geodesic flows. As a result, we have the following commutative diagram:

$$\begin{array}{ccc} S^*N & \xrightarrow{j} & S^*M \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_M} \\ G(N) & \xrightarrow{\tilde{j}} & G(M), \end{array}$$

where the map $\tilde{j} : G(N) \to G(M)$ is the natural embedding (i.e., a geodesic on N is a geodesic on M). This embedding is symplectic. Indeed, if ω_N and ω_M are, respectively, the symplectic forms on $G(N)$ and $G(M)$, we have that $\tilde{j}^*\omega_M = \omega_N$. To verify this, note that

$$\pi_N^*(\tilde{j}^*\omega_M) = j^*(\pi_M^*\omega_M) = j^*(d\alpha_M) = d\alpha_N.$$

Since ω_N is defined uniquely by the equality $\pi_N^*\omega_N = d\alpha_N$, we must have that $\tilde{j}^*\omega_M = \omega_N$. q.e.d.

For the rest of the section, we restrict our attention to projective Finsler metrics in $\mathbb{R}^n$ and the symplectic forms they induce on the space of oriented lines of $\mathbb{R}^n$. The reader will have no difficulty in extending the statements and proofs to projective Finsler metrics defined on general open convex subsets of $\mathbb{R}P^n$, including $\mathbb{R}P^n$ itself.

Proposition 2.4. *If ω is the symplectic form on the space of oriented lines induced by a projective Finsler metric on $\mathbb{R}^n$, then ω is admissible (see Definition 1.1) and its pull-back to the (two-dimensional) submanifold of oriented lines lying on an arbitrary plane never vanishes.*

Proof. The admissibility of ω follows at once from Propositions 2.1 and 2.2. By Proposition 2.3, the pull-back of ω to the submanifold of lines lying on a plane (or any affine subspace of $\mathbb{R}^n$) is a symplectic form and, therefore, never vanishes. q.e.d.

It is not hard to recover the projective Finsler metric from the symplectic form ω. If x and y are two points in $\mathbb{R}^n$ and Π is a plane containing them, set $d_\omega(x, y; \Pi)$ to be the integral of $|\omega|$ over the set of all oriented lines on Π intersecting the segment xy.

Proposition 2.5. *Given two points x and y in $\mathbb{R}^n$ and any plane Π in $\mathbb{R}^n$ that contains them, the quantity $d_\omega(x, y; \Pi)$ equals four times the Finsler distance between x and y. In particular, $d_\omega(x, y; \Pi)$ is independent of the choice of plane Π.*

Proof. Notice that since the metric is projective, the plane Π is totally geodesic and the distance between x and y in Π equals the distance between them in $\mathbb{R}^n$. Moreover, by Proposition 2.3, the natural symplectic form on $G(\Pi)$ is the pull-back of ω to the submanifold of all oriented lines lying on Π. We can now use the Crofton formula for Finsler surfaces due to Blaschke (see [**14**] and [**2**]) to write the distance between x and y as one-fourth the integral of $|\omega|$ over the set of all oriented lines intersecting the segment xy. q.e.d.

This settles the second part of Theorem 1.

3. Reduction to two-dimensions

In view of Proposition 2.5, to complete the proof of Theorem 1, we need to show that any admissible symplectic form on the space of oriented lines of $\mathbb{R}^n$ is induced by some projective Finsler metric on its space of geodesics. In this section, we prove the following partial result—the converse of Proposition 2.4.

Theorem 3.1. *Let ω be an admissible symplectic form on the space of oriented lines of $\mathbb{R}^n$. If the pull-back of ω to the space of oriented lines lying on an arbitrary plane never vanishes, then ω is the symplectic form induced by some projective Finsler metric on its space of geodesics.*

The proof of this theorem is largely based on Pogorelov's solution of Hilbert's fourth problem in two dimensions in [**23**] (cf. [**5**] and [**25**]).

Let μ be a Borel measure defined on the space $G(\mathbb{R}^2)$ of oriented lines in $\mathbb{R}^2$ satisfying the following conditions:

1) The set of all oriented lines passing through an arbitrary point has measure zero.

2) The measure is even (i.e., the involution $a : G(\mathbb{R}^2) \to G(\mathbb{R}^2)$ that reverses the orientation of lines is measure-preserving).
3) The measure of any open set is positive.

Define the distance $d_\mu(x, y)$ between two points x and y as the measure of the set of all oriented lines intersecting the segment that joins them.

Theorem 3.2 (Pogorelov, [**23**]). *The function d_μ is the distance function of a projective Finsler metric on the plane if and only if μ is smooth and positive. Moreover, every projective Finsler metric on the plane can be constructed this way.*

Proof of Theorem 3.1. Since the pull-back of ω to the space $G(\Pi)$ of oriented lines lying on a plane Π never vanishes, the pull-back of $|\omega|$ to $G(\Pi)$ of is a smooth positive measure. By Theorem 3.2, the function $d_\omega(x, y; \Pi)$ defined as the integral of $|\omega|$ over the set of all oriented lines on Π intersecting the segment $xy \subset \Pi$ is the distance function of a projective Finsler metric φ_π on Π.

Using the admissibility of ω and Stokes's formula, it is easy to show (see [**3**, p. 18]) that $d_\omega(x, y; \Pi)$ is independent of the choice of the plane Π. We may thus define a function φ on $T\mathbb{R}^n$ by setting $\varphi(v) = \varphi_\pi(v)$ for any choice of plane Π containing v in its tangent bundle.

Since a norm on a vector space is a Minkowski norm if and only if all its restrictions to two-dimensional subspaces are Minkowski norms, the function φ defines a Finsler metric on $\mathbb{R}^n$. To verify that φ is a projective Finsler metric, we remark that a continuous distance function d (and, a fortiori, a Finsler metric) defines a projective metric on $\mathbb{R}^n$ if and only if

1) whenever x, y, and z are collinear points with y contained in the segment xz, $d(x, z) = d(x, y) + d(y, z)$;
2) whenever x, y, and z are not collinear, $d(x, z) < d(x, y) + d(y, z)$.

It follows at once that a metric on $\mathbb{R}^n$ is projective if and only if its restriction to any plane is projective. It follows that φ is projective. q.e.d.

Proposition 2.5 and Theorem 3.1 show that Theorem 1 follows from Theorem 2. The proof of this last result involves some geometric constructions of independent interest that will be presented in the next two sections.

4. From symplectic forms to distance functions

In this section, we study a classical inverse problem in variational calculus.

Problem. Given a system of smooth curves on a manifold M such that through every point and every tangent direction there is a unique curve passing through this point in the given direction, find necessary and sufficient conditions for this system to be the geodesics of some Finsler metric on M.

In order to describe the main theorem in this section, we first give formal definitions of the systems of curves and of the generalization of Finsler metrics that we will be considering.

Definition 4.1. Let M be a smooth manifold and let $\pi : PTM \to M$ be its projectivized tangent bundle. A *path geometry* on M is a smooth foliation of PTM by one-dimensional submanifolds that are transverse to the fibers PT_xM, $x \in M$.

The projection onto M of a leaf of the path geometry will be called a *path.* Notice that the paths form a system of smooth curves such that through every point and every direction, there is a unique curve passing through this point in the given direction. In what follows, we will also make use of *oriented paths* and *path segments.*

Definition 4.2. We shall say that a path geometry on a manifold M is *tame* if it satisfies the following two properties:

1) any two distinct points in M that are sufficiently close determine a unique path;
2) the set of oriented paths on M obtained from the path geometry by providing each path with its two possible orientations is parameterized by a smooth $2n-2$ dimensional manifold Γ.

Examples of tame path geometries are the geodesic foliations of rank-one symmetric spaces and Hadamard manifolds (i.e., simply connected Riemannian manifolds of non-positive sectional curvature). However, most geodesic foliations are not tame. For example, it will be shown later that if a path has a self-intersection, the path geometry cannot be tame.

Whenever we have a tame path geometry, the incidence relation

$$\{(x,\gamma) \in M \times \Gamma : x \in \gamma\}$$

is naturally diffeomorphic to the spherized (or homogeneous) tangent bundle STM of the manifold M, and we have the double fibration

$$\begin{array}{ccc} & STM & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ M & & \Gamma. \end{array}$$

In his researches on geodesics and inverse problems in variational calculus, Busemann (see [16]) abstracted the properties of Finsler spaces and defined a more general class of metric spaces which he called G-spaces or geodesic spaces.

Definition 4.3. A metric space (M, d) is said to be a *G-space* if it satisfies the following properties:

1) The space M is locally compact.
2) Given two distinct points x and z, there exists a third distinct point y for which $d(x, y) + d(y, z) = d(x, z)$.
3) For every point $p \in M$, there is a metric ball B centered at p such that if x and y are in B, then there exists a point z distinct from x and y such that $d(x, y) + d(y, z) = d(x, z)$.
4) If x, y, z_1, and z_2 are points such that

$$d(x, y) + d(y, z_1) = d(x, z_1) = d(x, y) + d(y, z_2) = d(x, z_2),$$

then $z_1 = z_2$.

Property (2) states that the metric is *intrinsic:* the infimum of the lengths of all curves joining a pair of points equals the distance between them. Property (3) implies that geodesic segments can be locally extended, while property (4) implies that the extension of a geodesic segment is unique.

Roughly speaking, the main result in this section is a sufficient condition for a tame path geometry to be the "geodesic foliation" of a G-space.

Theorem 4.1. *Let M be smooth manifold of dimension greater than two endowed with a tame path geometry and let Γ denote its parameter space. If there exists a symplectic form on Γ such that the submanifold of all paths passing through an arbitrary point in M is Lagrangian, then the system of paths in M is the system of geodesics for a G-space metric on M.*

Later in this section, it will be shown that if M is a Finsler manifold with manifold of geodesics $G(M) = \Gamma$ and ω is the induced symplectic form on $G(M)$, then the constructed metric is the Finsler metric on M.

The motivation for the proof of Theorem 4.1 in terms of the Hofer metric on the space of Lagrangian submanifolds of a symplectic manifold has already been given in the introduction. However, the proof below is completely elementary and only makes use of the following two classical results of Whitehead and Weinstein:

Definition 4.4. Let M be a manifold provided with a path geometry. An open set $V \subset M$ is said to be *convex* if for any two points in V, there is a unique path segment joining them and lying entirely inside V.

Theorem 4.2 (Whitehead, [**27**])**.** *If M is a manifold provided with a path geometry, then every point in M has a convex neighborhood.*

Theorem 4.3 (Weinstein, [**26**])**.** *If L is an embedded Lagrangian manifold of a symplectic manifold (Γ, ω), then there is a symplectomorphism between a tubular neighborhood of L and a tubular neighborhood of the zero-section in T^*L that takes L to the zero-section.*

Using Whitehead's theorem, we can see that tame path geometries are really quite special.

Proposition 4.1. *If Γ is the parameter space of a tame path geometry on M, then*

1) *all paths are simple;*
2) *for every point $x \in M$, the submanifold $\hat{x}$ of all oriented paths passing through x is an embedded sphere in Γ;*
3) *whenever two points x and y are sufficiently close, then $\hat{x}$ and $\hat{y}$ intersect in precisely two points.*

Proof. Suppose a path γ has a double point x. If V is a sufficiently small convex neighborhood around x, there are pairs of points on γ that lie on two paths. One of the paths is γ and the other contains the unique path segment in V that passes through both points (Figure 2). Since this violates the first condition in Definition 4.2, every path must be simple.

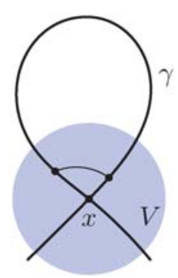

Figure 2.

To prove (2), we first notice that $\hat{x}$ is immersed. Indeed, if π_1 and π_2 are the canonical projections of $\{(x, \gamma) \in M \times \Gamma : x \in \gamma\}$ onto M and Γ, then $\hat{x}$ is the projection of $\pi_1^{-1}(x)$ onto Γ. Since the kernels of the differentials of π_1 and π_2 are transversal, it follows that $\hat{x}$ is immersed. Statement (1) implies that this immersion is injective.

Statement (3) is merely a reformulation of the hypothesis. q.e.d.

For the rest of the section, we assume that M and Γ satisfy the hypotheses of Theorem 4.1. The first step in the construction of the G-space metric on M is to define an open covering of M and an intrinsic metric on each open subset V_x of the covering such that the geodesics for this metric coincide with the path segments lying in V_x.

Lemma 4.1. *There exists an open covering of M that associates to every point $x \in M$ an open neighborhood V_x with the following properties:*

1) *the set V_x is convex;*
2) *any two distinct points in V_x determine a unique path;*
3) *there exists a symplectomorphism Ψ_x between a tubular neighborhood U_x of $\hat{x}$ in Γ and a tubular neighborhood of the zero-section in $T^*\hat{x}$ that takes $\hat{x}$ to the zero-section and such that if y belongs to the closure of V_x, then $\hat{y} \subset U$ and $\Psi_x(\hat{y}) \subset T^*\hat{x}$ is the graph of the exterior differential of some smooth function f_y^x on $\hat{x}$.*

Proof. From Theorem 4.2 and the tameness of the path geometry, it is clear that around every point x, we can find an open set V_x satisfying the first two properties. To see that we can choose V_x to satisfy the third property as well, we use Theorem 4.3 and notice that if a point y is sufficiently close to x, the Lagrangian sphere $\hat{y}$ is C^1 close to $\hat{x}$. Therefore, $\Psi_x(\hat{y}) \subset T^*\hat{x}$ is the graph of a closed 1-form. Since $\hat{x}$ is a sphere of dimension greater than one, it is simply connected and the closed 1-form is the exterior differential of some smooth function. q.e.d.

On each V_x, we define a metric by setting

$$\delta_x(y,z) := \frac{1}{2}\left(\max(f_y^x - f_z^x) - \min(f_y^x - f_z^x)\right).$$

In other words, we consider the map $y \mapsto f_y^x$ as a mapping from V_x to the space of smooth functions on $\hat{x}$ modulo constants and pull-back the distance function $\Delta(f,g) := (\max(f-g) - \min(f-g))/2$. The metric Δ comes from the norm $\|f\| = (\max(f) - \min(f))/2$ that symplectic geometers will recognize as the local model for the Hofer norm. Like in many other normed spaces, the geodesic segment joining two given points is far from being unique. However, the following simple lemma makes it easy to verify if three points belong to a geodesic segment.

Lemma 4.2. *Let f and g be two continuous functions on a compact topological space X. The equality*

$$\max(f+g) - \min(f+g) = \max(f) - \min(f) + \max(g) - \min(g)$$

holds if and only if there exist points x and y in X for which $\max(f) = f(x)$, $\max(g) = g(x)$, $\min(f) = f(y)$, *and* $\min(g) = g(y)$.

Proposition 4.2. *The metric space* (V_x, δ_x) *is a length space where the geodesics coincide with the prescribed system of curves on* $V_x \subset M$.

Proof. A simple way of verifying whether a locally compact metric space is a length space is to see whether for any two points u and v there exists a third point w such that the distance between u and v equals the distance between u and w plus the distance between w and v (see [**16**, p. 29]).

In our case, let u and v be two points in V_x and let w be some point in the unique curve segment joining u and v. Because the spheres $\hat{u}$, $\hat{v}$, and $\hat{w}$ intersect at the same two points, the functions $f_v^x - f_u^x$, $f_v^x - f_w^x$, and $f_w^x - f_u^x$ have the same two critical points a and b in $\hat{x}$. Note that unless w coincides with u or v, none of these functions is constant.

Without loss of generality, assume that $\max(f_v^x - f_u^x) = f_v^x(a) - f_u^x(a)$. By continuity, and because there are only two critical points, if w is close to u, we have that $\max(f_v^x - f_w^x) = f_v^x(a) - f_w^x(a)$. Moreover, as we move w from u to v along the path segment that joins them, this cannot change since $f_v^x - f_w^x$ is never constant. The same argument applies to $f_w^x - f_u^x$ and we have that for any w between u and v

$$\max(f_v^x - f_u^x) = \max(f_v^x - f_w^x) + \max(f_w^x - f_u^x).$$

Since we have the analogous statement for the minima, we conclude that $\delta_x(u, v) = \delta_x(u, w) + \delta_x(w, v)$.

This argument not only proves that (V_x, δ_x) is a length space, but also that the path segment joining u and v is a geodesic segment for the metric. To complete the proof, we must show that any geodesic segment lies on a path. In order to do this, we prove that if w is not in the curve segment joining u and v, then $\delta_x(u, v) < \delta_x(u, w) + \delta_x(w, v)$.

We distinguish two cases: either w is not on the path passing through u and v or w lies on this path, but does not belong to the segment that joins u and v. In the first case, the Lagrangian spheres $\hat{u}$, $\hat{v}$, and $\hat{w}$ do not intersect in the same two points and, by Lemma 4.2, we have the strict triangle inequality. In the second case, the Lagrangian spheres $\hat{u}$, $\hat{v}$, and $\hat{w}$ do intersect in the same two points, but it is easily ascertained that while $f_v^x - f_u^x$ reaches its maximum at $a \in \hat{x}$, either $f_w^x - f_u^x$ or $f_v^x - f_w^x$ reaches its maximum at b. Again, Lemma 4.2 implies that we have the strict triangle inequality. q.e.d.

This finishes the first step of the proof of Theorem 4.1. The second step consists in piecing together the metrics we just constructed on the elements of the open cover of M to obtain, first, a length structure and, then, a G-space metric on M.

Proposition 4.3. *If x and y are two points on M for which $V_x \cap V_y$ is not empty, then δ_x and δ_y agree on the intersection.*

The proof follows from a useful symplectic interpretation of the distance function δ_x. If u and v are in V_x, the Lagrangian spheres $\hat{u}$ and $\hat{v}$ are in a tubular neighborhood U_x of $\hat{x}$ that is symplectomorphic to a neighborhood of the zero-section of the cotangent of the $(n-1)$-dimensional sphere. Remark that since $n > 2$, the set U_x is simply connected.

Lemma 4.3. *Let σ be any simple closed curve in U_x formed by concatenating a curve σ_u in $\hat{u}$ and a curve σ_v in $\hat{v}$ that join the two points of intersection of these Lagrangian spheres. The distance $\delta_x(u,v)$ is equal to one-half the absolute value of the symplectic area of any disc in U_x whose boundary is σ.*

Proof. Let $\Psi_x : U_x \to T^*\hat{x}$ be the symplectic embedding of Lemma 4.1. The map Ψ_x takes $\hat{u}$ and $\hat{v}$ into Lagrangian submanifolds in $T^*\hat{x}$ that are, moreover, graphs of the exterior differentials of smooth functions f_u^x and f_v^x on $\hat{x}$. The two intersection points of $\Psi_x(\hat{u})$ and $\Psi_x(\hat{v})$ correspond to the two critical points of $f_u^x - f_v^x$. Let us denote by a the critical point where the minimum is attained and by b, the critical point where the maximum is attained.

Let D be a disc in U_x with boundary σ and oriented so that its symplectic area is non-negative. In the computation below, σ is oriented as the boundary of D, α_0 and ω_0 are the canonical one and two-forms on $T^*\hat{x}$, γ_u and γ_v are the images of σ_u and σ_v under Ψ_x, and π is the canonical projection from $T^*\hat{x}$ to $\hat{x}$.

$$\begin{aligned}\int_D \omega &= \int_{\Psi_x(D)} \omega_0 = \int_{\gamma_u \cup \gamma_v} \alpha_0 \\ &= \int_{\pi(\gamma_u)} df_u^x + \int_{\pi(\gamma_v)} df_v^x \\ &= f_u^x(b) - f_u^x(a) - (f_v^x(b) - f_v^x(a)) = 2\delta_x(u,v).\end{aligned}$$

q.e.d.

Proof of Proposition 4.3. Let u and v be two points $V_x \cap V_y$, and let U_x and U_y be the tubular neighborhoods of $\hat{x}$ and $\hat{y}$ that appear in Lemma 4.1. By Lemma 4.3, if σ is any simple closed curve formed by concatenating a curve in $\hat{u}$ and a curve in $\hat{v}$ that join the two points of intersection of these Lagrangian spheres, the distances $\delta_x(u,v)$ and $\delta_y(u,v)$ are both equal to one-half the absolute value of the symplectic area of any disc in $U_x \cap U_y$ whose boundary is σ. q.e.d.

By Proposition 4.2, we have an open covering of M together with a *length structure*—a way of measuring the length of curves—ℓ_x defined on each set V_x of the covering, and by Proposition 4.3, the length structures coincide on the intersections $V_x \cap V_y$. Therefore, there is a unique length structure ℓ on M which restricts to ℓ_x on V_x for all $x \in M$. This allows us to define a pseudo-metric d on M by setting

$$d(x, y) := \inf\{\ell(\sigma) : \sigma \text{ is a smooth curve joining } x \text{ and } y\}.$$

The following proposition concludes the proof of Theorem 4.1.

Proposition 4.4. *The pseudo-metric d defined above is actually a metric, and (M, d) is a G-space. Moreover, the geodesics in (M, d) coincide with the prescribed system of curves on M.*

Proof. Let us first prove that d is a metric. Let x and y be two distinct points on M and let $V'_x \subset V_x$ be a neighborhood of x whose closure is compact and does not contain y. Since δ_x is continuous on $V_x \times V_x$ and the boundary of V'_x is compact, we have that the distance between it and x is a positive number ϵ. Since any continuous curve between x and y must cut the boundary of V'_x, the length of this curve must be at least equal to $\epsilon > 0$ and hence, the distance between x and y is strictly positive.

Now, we show that if two points u and v are sufficiently close to a point x, then $d(u, v) = \delta_x(u, v)$. In view of Proposition 4.2 and the locality of the definition of geodesics, this implies that the geodesics of (M, d) are precisely the paths on M.

Let $V'_x \subset V_x$ be a neighborhood of x such that, for the metric δ_x, its diameter is strictly less than twice the distance between its boundary and that of V_x. If u and v are in V'_x, then the length of any continuous curve joining these two points and leaving V_x is greater than $\delta_x(u, v)$. Since (V_x, δ_x) is a path metric space, we have that the infimum of the lengths of all curves joining u and v is $\delta_x(u, v)$ and hence, $d(u, v) = \delta_x(u, v)$.

Since Definition 4.3 immediately implies that any locally compact length space whose geodesics agree with the paths of a path geometry is a G-space, it follows that (M, d) is a G-space. q.e.d.

Application to Finsler geometry. We now show that when the path geometry is the system of geodesics of a Finsler metric on M and ω is the induced symplectic form on the space of geodesics $G(M) = \Gamma$, then the metric d given by Theorem 3 is the Finsler metric on M. This is a generalization of Blaschke's Crofton formula for two-dimensional Finsler spaces (see [**14**] and [**2**]).

Theorem 4.4. *Let M be a Finsler manifold whose space of geodesics is a smooth manifold $G(M)$. If every two points of M that are sufficiently close determine exactly one geodesic and if ω is the natural symplectic form on $G(M)$, then the metric d constructed from ω in Theorem 4.1 is the original Finsler metric on M.*

Proof. It is sufficient to show that on each V_x, the Finsler metric agrees with δ_x. This implies that the length structure agrees with that of the Finsler metric and, therefore, the metrics agree.

Recall the symplectic interpretation of δ_x given in Lemma 4.3. If σ is a loop made up of two simple curves on $\hat{u}$ and $\hat{v}$ joining their points of intersection, and D is any disc with boundary equal to σ, then $\delta_x(u, v)$ is the absolute value of the integral of ω over D. We can lift σ to a loop σ' on the unit co-sphere bundle of V_x by concatenating the following curves (Figure 3):

1) the Legendre transforms of the unit tangent vectors of all geodesics passing through u and belonging to σ;
2) the Legendre transforms of the unit vectors tangent to the unique oriented geodesic going from u and v;
3) the Legendre transforms of the unit tangent vectors of all geodesics passing through v and belonging to σ;
4) the Legendre transforms of the unit vectors tangent to the unique oriented geodesic going from v and u.

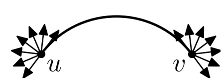

Figure 3.

If D' is a disc on the unit co-sphere bundle of V_x spanning σ' and π is the standard projection onto the space of geodesics, then

$$2\delta_x(u, v) = \int_{\pi(D')} \omega = \int_{D'} \pi^* \omega = \int_{D'} d\alpha = \int_{\sigma'} \alpha.$$

Notice that the integral of α over the curves (1) and (3) is zero, while its integral over the curves (2) and (4) equals the length of the geodesic segment. We have then that $\delta_x(u, v)$ is the Finsler distance between u and v. q.e.d.

Positivity properties of admissible two-forms. To finish the section, we apply Theorem 4.1 to the case where $M = \mathbb{R}^n$ and $\Gamma = G(\mathbb{R}^n)$ is the space of oriented lines in $\mathbb{R}^n$. In this case, the metric associated to an admissible symplectic form on $G(\mathbb{R}^n)$ is projective.

The next result is a key element in proving that the metric is also a Finsler metric.

Theorem 4.5. *If ω is an admissible symplectic form on $G(\mathbb{R}^n)$, $n > 2$, and $\Pi \subset \mathbb{R}^n$ is a plane, then the pull-back of ω to the (two-dimensional) submanifold of all oriented lines lying on Π never changes sign nor vanishes on an open set.*

Note that the set of all oriented lines on Π is a cylinder and that the set of all lines intersecting a line segment is the union of two closed topological discs that intersect in precisely two points (see Figure 4).

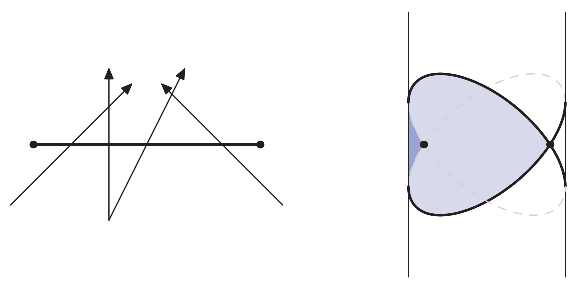

Figure 4.

In what follows, we will use the term *line-disc* to denote each of these topological discs, and we will say that a line-disc is *small* if the line segment corresponding to it lies in one of the sets of the cover V_x guaranteed by Lemma 4.1.

Lemma 4.4. *There is an orientation on $G(\Pi)$ such that the integral of ω over any line-disc is positive.*

Proof. It is enough to look at small line-discs. Indeed, any line segment can be partitioned into arbitrarily small line segments and the integral of ω over the original line-disc equals the sum of the integrals of ω over the small line-discs corresponding to these segments.

Lemma 4.3 implies that the integral of ω over a small line-disc corresponding to a segment xy is, up to a sign, the quantity $2d(x, y)$, and is thus different from zero. Since any two small line-discs can be embedded in a continuous one-parameter family of line-discs, the orientation may be fixed so that the integral of ω over any two small line-discs is positive. q.e.d.

From now on, $G(\Pi)$ is oriented so that the integral of ω over all line-discs is positive. However, not every open subset of $G(\Pi)$ contains a line-disc and, therefore, this is not enough to prove Theorem 4.5. To remedy the situation, we introduce *line-triangles:* sets of lines intersecting two sides of a given triangle in Π in a given order (see Figure 5). A line-triangle in $G(\Pi)$ is said to be *small* if its corresponding triangle in Π lies in one of the sets of the open covering V_x, $x \in \Pi$.

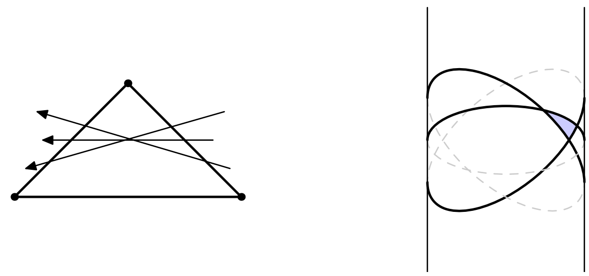

Figure 5.

It is easy to see that the pull-back of ω to $G(\Pi)$ never changes sign nor vanishes in an open set if and only if the integral of ω over any small line-triangle is different from zero. Therefore, the following lemma concludes the proof of Theorem 4.5.

Lemma 4.5. *If ω is an admissible symplectic form, its integral over any sufficiently small line-triangle in $G(\Pi)$ is positive.*

Proof. Let uvw be a triangle in $\Pi \cap V_x$ for some $x \in \Pi$ and consider the four small line-triangles T_1, T_2, T_3, and T_4 represented by Figure 6.

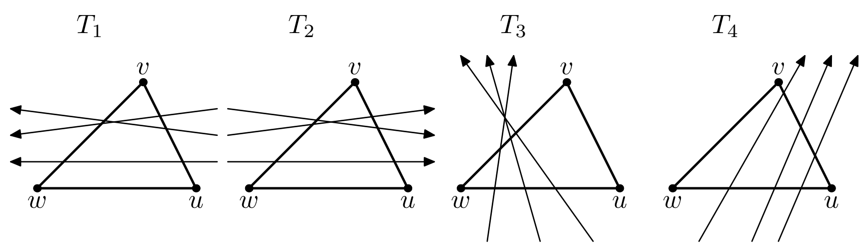

Figure 6.

By Lemma 4.3,

$$\begin{aligned} 2d(u,v) &= \int_{T_2} \omega + \int_{T_4} \omega, \\ 2d(v,w) &= \int_{T_1} \omega + \int_{T_3} \omega, \\ 2d(u,w) &= \int_{T_3} \omega + \int_{T_4} \omega. \end{aligned}$$

Using the strict triangle inequality for d, we have that

$$\begin{aligned} 0 < 2d(x,y) + 2d(y,z) - 2d(x,z) &= \int_{T_1} \omega + \int_{T_2} \omega \\ &= 2\int_{T_1} \omega. \end{aligned}$$

In the last equality, we used that ω is odd and that the triangle T_2 is obtained from T_1 by reversing the orientation of the lines. q.e.d.

5. A characterization of admissible symplectic forms

In this section, a detailed study of the generalized conformal structure of the Grassmannian $G_2^+(\mathbb{R}^{n+1})$ of planes in $\mathbb{R}^{n+1}$ will allow us show that the pull-back of an admissible symplectic form to the space of all lines lying on a plane never vanishes. This completes the proof of Theorem 2.

Throughout this section, we consider $\mathbb{R}^n$ as $\mathbb{R}P^n$ minus a hyperplane at infinity, and the space of oriented lines in $\mathbb{R}^n$ as an open subset of $G_2^+(\mathbb{R}^{n+1})$.

Recall that the tangent space to $G_2^+(\mathbb{R}^{n+1})$ at a plane P can be identified with the space of linear transformations from P to its orthogonal complement $P^\perp$. In fact, a whole neighborhood of P in $G_2^+(\mathbb{R}^{n+1})$ can be identified with this vector space since any plane sufficiently close to P is the graph in $P \oplus P^\perp = \mathbb{R}^{n+1}$ of some linear transformation from P to $P^\perp$. By choosing bases for P and $P^\perp$, we may represent the tangent space $T_P G_2^+(\mathbb{R}^{n+1})$ as the space of $2 \times (n-1)$ matrices

$$\begin{pmatrix} q_1 & \dots & q_{n-1} \\ p_1 & \dots & p_{n-1} \end{pmatrix}.$$

Definition 5.1. The *generalized conformal structure* on the Grassmannian $G_2^+(\mathbb{R}^{n+1})$ assigns to every plane $P \in G_2^+(\mathbb{R}^{n+1})$ the *incidence cone* in $T_P G_2^+(\mathbb{R}^{n+1})$ that consists of all rank-one linear transformations from P to $P^\perp$.

The incidence cones are made up of two types of linear subspaces, the *α-planes* and the *β-planes.*

Given real numbers λ_1 and λ_2, we define the n-dimensional subspace

$$H_{(\lambda_1,\lambda_2)} := \left\{ \begin{pmatrix} \lambda_1 q_1 & \dots & \lambda_1 q_{n-1} \\ \lambda_2 q_1 & \dots & \lambda_2 q_{n-1} \end{pmatrix} : q_1, \dots, q_{n-1} \in \mathbb{R} \right\} \subset T_P G_2^+(\mathbb{R}^{n+1}).$$

These are the α-planes. Note that $H_{(\lambda_1,\lambda_2)}$ depends only on the quotient of λ_1 and λ_2. In other words, the $H_{(\lambda_1,\lambda_2)}$ define a projective line of n-dimensional subspaces in $T_P G_2^+(\mathbb{R}^{n+1})$. Since all the matrices in $H_{(\lambda_1,\lambda_2)}$ have rank one, the subspaces are all inside the incidence cone.

Describing the β-planes is equally simple: for real numbers $\lambda_1, \dots, \lambda_{n-1}$, define the two-dimensional subspace

$$K_{(\lambda_1, \dots \lambda_{n-1})} := \left\{ \begin{pmatrix} \lambda_1 q & \dots & \lambda_{n-1} q \\ \lambda_1 p & \dots & \lambda_{n-1} p \end{pmatrix} : q, p \in \mathbb{R} \right\} \subset T_P G_2^+(\mathbb{R}^{n+1}) \ .$$

Note that the set of β-planes in $T_P G_2^+(\mathbb{R}^{n+1})$ is parameterized by a projective space of dimension $n-2$.

An alternate, projective, description of α-planes and β-planes is as follows: if x is a point in $\mathbb{R}P^n$ and l is an oriented line passing through x, then $T_l \hat{x}$ is an α-plane. Similarly, if $\Pi \subset \mathbb{R}P^n$ is a two-dimensional subspace and l is an oriented line lying on Π, then $T_l G(\Pi)$ is a β-plane.

Note that we can now characterize admissible two-forms on the space of oriented lines of $\mathbb{R}^n$ (or, more generally, $\mathbb{R}P^n$) as differential two-forms that are odd, closed, and that vanish on every α-plane.

Two-forms vanishing on α-planes. We shall now describe all two-forms $\omega \in \Lambda^2(T_P G_2^+(\mathbb{R}^{n+1}))$ which vanish on the subspaces of the form $H_{(\lambda_1, \lambda_2)}$, $\lambda_1, \lambda_2 \in \mathbb{R}$. In the computations that follow, we identify the space of $2 \times (n-1)$ matrices with $\mathbb{R}^{2n-2}$ by using the coordinates $q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}$ as we have been doing in the previous paragraphs.

Proposition 5.1. *A 2-form $\omega \in \Lambda^2(T_P G_2^+(\mathbb{R}^{n+1}))$ vanishes on all α-planes if and only if it is of the form $\omega = \sum b_{ij} dq_i \wedge dp_j$ with $b_{ij} = b_{ji}$. In particular, such ω is non-degenerate if and only if the matrix (b_{ij}) is invertible.*

Proof. A general 2-form $\omega \in \Lambda^2(T_P G_2^+(\mathbb{R}^{n+1}))$ has the form

$$\omega = \sum a_{ij} dq_i \wedge dq_j + \sum b_{ij} dq_i \wedge dp_j + \sum c_{ij} dp_i \wedge dp_j \ ,$$

with $a_{ij} = -a_{ji}$ and $c_{ij} = -c_{ji}$. If ω vanishes on the subspaces of the form

$$H_{(\lambda_1, 0)} := \left\{ \begin{pmatrix} \lambda_1 q_1 & \dots & \lambda_1 q_{n-1} \\ 0 & \dots & 0 \end{pmatrix} : q_1, \dots, q_{n-1} \in \mathbb{R} \right\} \ ,$$

then $\sum a_{ij} dq_i \wedge dq_j = 0$ and, therefore, the coefficients a_{ij} are all equal to zero. Similarly, if ω vanishes on the subspaces of the form $H_{(0, \lambda_2)}$, then the coefficients c_{ij} are all equal to zero.

Finally, if ω is to vanish on a subspace of the form $H_{(\lambda_1, \lambda_2)}$ with neither λ_1 nor λ_2 equal to zero, then $\sum b_{ij} dq_i \wedge dp_j = 0$. This implies that $b_{ij} = b_{ji}$. q.e.d.

We shall denote the subspace of $\Lambda^2(T_P G_2^+(\mathbb{R}^{n+1}))$ consisting of those 2-forms which vanish on all α-planes by Λ_P^-. The previous proposition shows that the dimension of this subspace equals $n(n-1)/2$. The bundle over $G_2^+(\mathbb{R}^{n+1})$ whose fiber over every point P is Λ_P^- will be denoted by Λ^-. Clearly, an admissible 2-form is a section of this bundle. If the admissible 2-form is symplectic, then it is a nowhere-vanishing section.

We shall now prove that an admissible 2-form which does not vanish on β-planes is necessarily symplectic.

Proposition 5.2. *The form $\omega := \sum b_{ij} dq_i \wedge dp_j \in \Lambda_P^-$ never vanishes on a β-plane if and only if the matrix (b_{ij}) is definite. Therefore, if an admissible two-form never vanishes on a β-plane, it is symplectic.*

Proof. Let us evaluate the form ω on a basis of the β-plane $K_{(\lambda_1,\ldots,\lambda_{n-1})}$.

$$\begin{aligned}\omega\left(\begin{pmatrix}\lambda_1 q & \ldots & \lambda_{n-1} q\\ \lambda_1 p & \ldots & \lambda_{n-1} p\end{pmatrix}, \begin{pmatrix}\lambda_1 q' & \ldots & \lambda_{n-1} q'\\ \lambda_1 p' & \ldots & \lambda_{n-1} p'\end{pmatrix}\right) &= \sum b_{ij}(\lambda_i q \lambda_j p' - \lambda_i p \lambda_j q') \\ &= (qp' - pq') \sum b_{ij} \lambda_i \lambda_j \ .\end{aligned}$$

Since the vectors are linearly independent the quantity $qp' - pq'$ is not zero and the quantity $\sum b_{ij}\lambda_i\lambda_j$ is non-zero for all non-zero values of $(\lambda_1, \ldots, \lambda_{n-1})$ if and only if the matrix (b_{ij}) is definite. q.e.d.

The index of the matrix (b_{ij}) does not depend on the choice of bases on the plane P and its orthogonal complement $P^\perp$. In particular, if ω is a non-degenerate 2-form that is also a section of Λ^- over some connected open subset $U \subset G_2^+(\mathbb{R}^{n+1})$, then the index of the matrix (b_{ij}) is the same at all points $P \in U$. It is thus natural to propose the following definition:

Definition 5.2. Let ω be a non-degenerate 2-form that is also a section of Λ^- over a connected open subset $U \subset G_2^+(\mathbb{R}^{n+1})$. The *index* of ω is the index of the matrix (b_{ij}) at any point $P \in U$. The two-form ω will be called *definite* or *indefinite* depending on whether the matrix (b_{ij}) is definite or not.

It is easy to construct admissible symplectic forms on open subsets of $G_2^+(\mathbb{R}^{n+1})$ that are not definite. However, this is not possible if the open subset is connected and contains the set of all lines passing through a given point in $\mathbb{R}P^n$.

Theorem 5.1. *Let $U \subset G_2(\mathbb{R}^{n+1})$ be a connected open subset that contains the set of all oriented lines passing through a point $x \in \mathbb{R}P^n$. If ω is an admissible symplectic form defined on U, then ω is definite.*

Proof. Without loss of generality, we may assume that U is the set of oriented lines passing through an open convex subset M of $\mathbb{R}P^n$. The strategy of the proof consists in showing that if ω is indefinite, then there exists a plane Π passing through M and such that the pull-back of ω to the manifold of all lines passing through M and lying on Π either changes sign or vanishes on an open set. According to Theorem 4.5, this cannot be the case if ω is symplectic.

Consider the double fibration

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ U & & \mathcal{B}, \end{array}$$

where $\mathcal{B}$ is the set of all projective planes in $\mathbb{R}P^n$ that contain a line in U, and $\mathcal{F}$ denotes the incidence relation $\{(l,\Pi) \in U \times \mathcal{B} : l \subset \Pi\}$. Remark that $\mathcal{B}$ is an open subset of the Grassmannian of projective planes in $\mathbb{R}P^n$, $G_3(\mathbb{R}^{n+1})$.

If ω is an indefinite 2-form on U, we define $\mathcal{N} \subset \mathcal{F}$ to be the set of all pairs (l,Π) such that ω vanishes on the β-plane $T_l G(\Pi)$. From the proof of Proposition 5.2, it follows that $\mathcal{N}$ is a smooth hypersurface and that $\mathcal{N} \cap \pi_1^{-1}(l)$ is a non-degenerate projective quadric for each $l \in U$.

Notice that if $\Pi \in \mathcal{B}$ and $\pi_2^{-1}(\Pi)$ intersects $\mathcal{N}$ transversely at some point, then the pull-back of ω to $G(\Pi) \cap U$ changes sign. If $\pi_2^{-1}(\Pi)$ is contained in $\mathcal{N}$, then the pull-back of ω vanishes on $G(\Pi) \cap U$. Therefore, in order to prove the theorem, it is enough to show that if no fiber of π_2 intersects $\mathcal{N}$ transversely, then there exists at least one that is wholly contained in $\mathcal{N}$.

If no fiber is transverse, we may consider the distribution of tangent planes on $\mathcal{N}$ given by $(l,\Pi) \mapsto T_{(l,\Pi)}\pi_2^{-1}(\Pi)$. Since this distribution is integrable in $\mathcal{F} \supset \mathcal{N}$ (its leaves are the fibers of π_2), it is integrable in $\mathcal{N}$. It follows that there is a fiber of π_2 that lies completely on $\mathcal{N}$ and this finishes the proof. q.e.d.

Proof of Theorem 2. Let ω be an admissible symplectic form on the space $G(\mathbb{R}^n)$ of oriented lines of $\mathbb{R}^n$. By the previous theorem, ω is a definite form. Applying Proposition 5.2, we have that ω never vanishes on a β-plane and, therefore, the pull-back of ω to the submanifold of oriented lines lying on a plane never vanishes.

On the other hand, if ω is an admissible two-form on $G(\mathbb{R}^n)$ such that its pull-back to the submanifold of oriented lines lying on a plane never vanishes, then it never vanishes on a β-plane and, by Proposition 5.2, it is symplectic. q.e.d.

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