Symplectic Geometry of Entanglement

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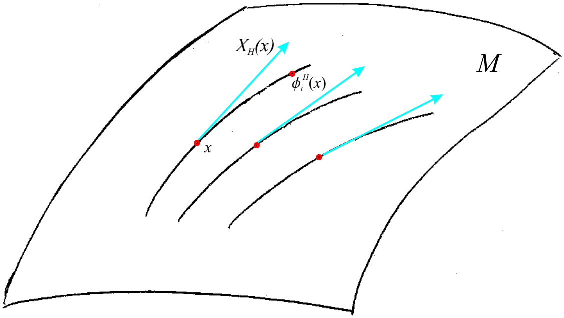
Description of Quantum Entanglement

- Quantum Entanglement = "... a correlation that is stronger then any classical correlation" (J. Bell)
- System composed of L subsystems
- Problem 1: Construction of Entanglement measure which enables to distinguish between entangled and not entangled states + ' hierarchy'
- Problem 2: Local transformations
- Symplectic Geometry of Entanglement = An approach to these problems via tools of Symplectic Geometry

Symplectic geometry and classical mechanics

- Dynamics



- Hamilton function

$$H : M \to \mathbb{R}$$

- Flow (classical dynamics)

$$M \ni x \mapsto \phi_t^H(x) \in M$$

- Vector field

$$X_H(x) = \frac{d}{dt}\Big|_{t=0} \phi_t^H(x)$$

Symplectic geometry and classical mechanics

- (M, ω) - symplectic manifold,
- $d\omega = 0$, ω - nondegenerate
- To find X_H for a given H we need ω:

$$dH = \omega(X_H, \cdot)$$

- Dynamics (flow)

$$\frac{d}{dt}x(t) = X_H(x(t))$$

- The Poisson bracket of two functions F, G on M

$$\{F, G\} = \omega(X_F, X_G) = X_F(G)$$

- Homomorphism of Lie algebras

$$(\mathcal{F}(M)\,,\,\{\cdot,\cdot\}) \to (\chi(M)\,,\,[\cdot,\cdot])$$

Symplectic group actions

- K – compact semisimple Lie group + symplectic action on M

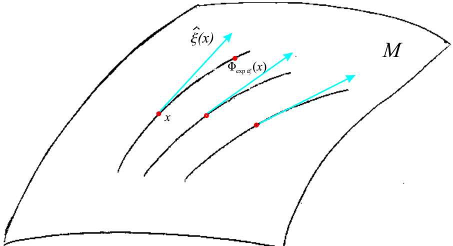
$$K \times M \ni (g,x) \mapsto \Phi_g(x) \in M, \quad \Phi_{g_1 g_2} = \Phi_{g_1}(\Phi_{g_2}(x)), \quad \Phi_g^* \omega = \omega$$

- $\mathfrak{k}$ – Lie algebra of K
- Let $\xi \in \mathfrak{k}$, then
  - $\exp t\xi$ - a one parameter subgroup of K
  - $\Phi_{\exp t\xi}$ - a one parameter subgroup of symplectomorphisms of M
- Define a fundamental vector field $\hat{\xi}$

$$\hat{\xi}(x) = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp t\xi}(x).$$

- Lie algebra of fundamental vector fields

$$[\hat{\xi}_1, \hat{\xi}_2] = \widehat{[\xi_1, \xi_2]}$$



Symplectic group actions

- Lie derivative of ω in the $\hat{\xi}$ direction

$$0 = \mathcal{L}_{\hat{\xi}}\omega = d\omega(\hat{\xi}, \cdot) + i_{\hat{\xi}} d\omega = d\omega(\hat{\xi}, \cdot)$$

- Locally there exist Hamilton function $\mu_\xi : M \to \mathbb{R}$ for $\hat{\xi}$, i.e.

$$d\mu_\xi(\cdot) = \omega(\hat{\xi}, \cdot)$$

- This function can be chosen to be linear in ξ, i.e.

$$\mu_\xi(x) = \langle \mu(x), \xi \rangle, \quad \mu(x) \in \mathfrak{k}^*,$$

  where $\langle \, , \rangle$ is the pairing between $\mathfrak{k}$ and $\mathfrak{k}^*$.

- The maps μ_ξ defines the moment/momentum map $\mu : M \to \mathfrak{k}^*$.

Coadjoint action. Symplectic structure on coadjoint orbits

- The group K acts in a natural way on its algebra $\mathfrak{k}$

$$\mathrm{Ad}_g X = gXg^{-1}.$$

- The coadjoint action Ad_g^* on $\mathfrak{k}^*$ is the dual one

$$\langle \mathrm{Ad}_g^* \alpha, X \rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}} X \rangle = \langle \alpha, g^{-1} X g \rangle,$$

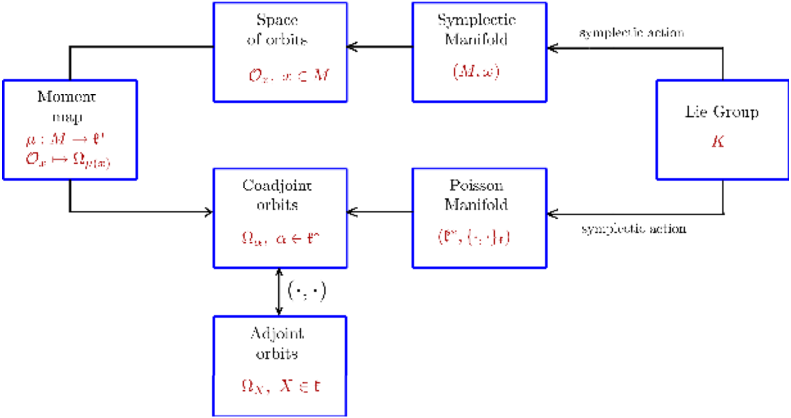
- $\mathfrak{k}^*$ is Poisson manifold with the Lie-Poisson bracket $\{\cdot, \cdot\}_{\mathfrak{k}}$.
- Coadjoint orbits $\Omega_\alpha = \{\mathrm{Ad}_g^* \alpha, g \in K\}$, are symplectic leaves of $\{\cdot, \cdot\}_{\mathfrak{k}}$
- The symplectic form ω at $\beta \in \Omega_\alpha$ is

$$\omega(\tilde{X}, \tilde{Y})(\beta) = \langle \beta, [X, Y] \rangle$$

- μ can be chosen equivariant with respect to the coadjoint action of K, i.e.

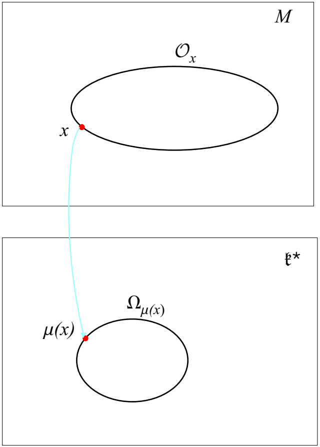
$$\mu\left(\Phi_g(x)\right) = \mathrm{Ad}_g^* \mu(x),$$

Geometric Structure



Orbits of group actions and the momentum map

- Two symplectic structures connected by the momentum map:
  - on M
  - on coadjoint orbits
- An orbit $\mathcal{O}_x$ of K action on M is mapped onto a coadjoint orbit $\Omega_{\mu(x)}$ w $\mathfrak{k}^*$ via momentum map μ.



Orbits of group actions and the momentum map

- In general, however, it is not a diffeomorphism between $\mathcal{O}_x$ and $\Omega_{\mu(x)}$.
- There exist two interesting subgroups of K to consider
  - stabilizer of the points on the orbit of the action on M (i.e. a subgroup which does not move them), $\mathrm{Stab}(x)$
  - stabilizer of the elements of the coadjoint action on the corresponding coadjoint orbit, $\mathrm{Stab}(\mu(x))$
- An orbit $\mathcal{O}_x$ on M is diffeomorphic to the corresponding coadjoint orbit $\Omega_{\mu(x)}$ iff both stabilizers are equal.
- If this is the case the orbit on M is symplectic (since the corresponding coadjoint orbit is), i.e. the symplectic form on M restricted to this orbit is nondegenerate (Kostant-Sternberg).
- Otherwise the orbit $\mathcal{O}_x$ is not symplectic, the symplectic form is degenerate and the dimension of the degeneracy can be a useful characterization of orbits.

$$\mathrm{D}(x) = \dim(\mathcal{O}_x) - \dim(\Omega_{\mu(x)}) = \dim(\mathrm{Stab}(\mu(x))) - \dim(\mathrm{Stab}(x))$$

- In general coadjoint orbits encode only partial information about orbits in M.

Quantum mechanics

- Pure states - points in the projective space $\mathbb{P}(\mathcal{H})$, where $\mathcal{H}$ - underlying Hilbert space
- The projective space $M = \mathbb{P}(\mathcal{H})$ is a symplectic manifold
  - Any vector from $T_{[v]}\mathbb{P}(\mathcal{H})$ can be written as $[Av]$, where $A \in \mathfrak{su}(\mathcal{H})$ and

$$\omega([Av], [Bv]) = -\frac{i\langle [A\,,\,B]v|v\rangle}{2\langle v|v\rangle}$$

- The unitary group $SU(\mathcal{H})$ acts on $M = \mathbb{P}(\mathcal{H})$ via symplectomorphisms.
- The momentum map for this action

$$\mu\big([v]\big)(X) = -\frac{i}{2}\frac{\langle v|X|v\rangle}{\langle v|v\rangle},$$

Quantum mechanics. Separable and entangled states

- Composite (L-partite) systems

$$\mathcal{H}_c = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L$$

  (for simplicity, let's assume all $\mathcal{H}_i$ are the same, say $\mathcal{H} \simeq \mathbb{C}^N$)
- Quantum state is separable (not entangled) iff it is simple tensor

$$v = v_1 \otimes v_2 \cdots \otimes v_L, \quad v_j \in \mathcal{H}_j$$

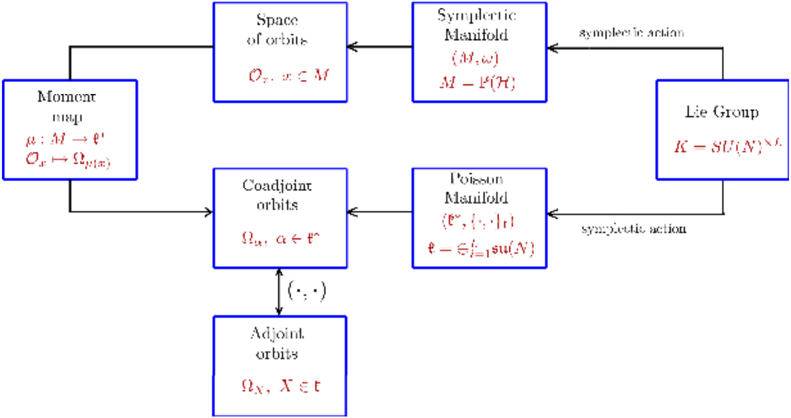
  otherwise it is entangled.

- Separability (entanglement) of states is invariant under the direct product (local unitary operations)

$$K = SU(N) \times \cdots \times SU(N)$$

- Using our machinery we can attack some interesting problems
  - How the degeneracy of the symplectic form of an orbit is connected with entanglement properties of the states on it?
  - How to check that two states are locally unitary equivalent (i.e., they belong to the same orbit of K)
  - How to define quantum correlations if there is no tensor product structure (indistinguishable particles)

Geometric structure - Quantum Entanglement



Two identical but distinguishable particles

- Hilbert space $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ with the natural action of $K = SU(N) \times SU(N)$, $\mathfrak{g} = \mathrm{Span}_{\mathbb{R}}\{X_1 \otimes I,\, I \otimes X_2\}$, where $X_1,\, X_2 \in \mathfrak{su}(N)$

- Quantum state $|\Psi\rangle$

$$|\Psi\rangle = \sum_{i,j=1}^{N} C_{ij}|i\rangle \otimes |j\rangle, \quad C_{ij} \in \mathbb{C}.$$

can be transformed by the K-action to the canonical ('Schmidt') form

$$|u\rangle = \sum_i \lambda_i |i\rangle \otimes |i\rangle$$

- Let m_i = the multiplicity of distinct λ_i, and m_0 corresponds to $\lambda_0 = 0$
- Dimension of degeneracy

$$D(|\Psi\rangle) = \dim(\mathcal{O}_{|\Psi\rangle}) - \dim(\mu(\mathcal{O}_{|\Psi\rangle})) = \sum_{n=1}^{r} m_n^2 - 1.$$

- Separable states form the only symplectic orbit $D(|\Psi\rangle) = 0$

- The degree of degeneracy is well defined entanglement measure

Maximally entangled state

- Maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle.$$

- The restriction of symplectic form ω to $\mathcal{O}_{|\Psi\rangle}$ is zero and

$$\dim \mathcal{O}_{|\Psi\rangle} = \frac{1}{2} \dim \mathbb{P}(\mathcal{H})$$

- The orbit through the maximally entangled state is lagrangian (maximally non symplectic)

Symplectic geometry of entanglement - many particles

- L-particles, $\mathcal{H} = \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N$, $K = SU(N)^{\times L}$
- The orbit of separable states is symplectic (the degeneracy of ω equals 0, $D(|\Psi\rangle) = 0$). Moreover it is the only symplectic orbit
- It is an orbit of the Perelomov coherent states for the irreducible representation of K on $\mathcal{H}$ (the orbit through the highest weight vector of the representation).
- The degeneracy is maximal for the maximally entangled states
- The degree of degeneracy is well defined entanglement measure

$|GHZ_L\rangle$ state

- The $|GHZ_L\rangle$ state of L-qubits

$$|GHZ_L\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes L} + |1\rangle^{\otimes L})$$

- For $|GHZ_L\rangle$ we have

$$\omega|_{\mathcal{O}_{|GHZ_L\rangle}} = 0$$

- Dimension $\dim \mathcal{O}_{|GHZ_L\rangle} = 2L + 1$, when $L > 2$ and $\dim \mathcal{O}_{|GHZ_2\rangle} = 3$
- When $L = 2$ or $L = 3$ then an orbit $\mathcal{O}_{|GHZ_L\rangle}$ is lagrangian
- When $L > 3$ the orbit $\mathcal{O}_{|GHZ_L\rangle}$ is isotropic (its dimension is too small to be lagrangian)

Local Unitary Equivalence

- Two states are Locally Unitary equivalent (LU-equivalent) iff

$$|v_1\rangle = U_1 \otimes \ldots \otimes U_L |v_2\rangle.$$

- Or equivalently

$$[v_1] = [U_1 \otimes \ldots \otimes U_L v_2].$$

- A pure state of a bipartite system

$$|\Psi\rangle = \sum_{ij} c_{ij} |e_i\rangle \otimes |f_j\rangle$$

  can be transformed by local unitary transformations to the canonical ('Schmidt') form

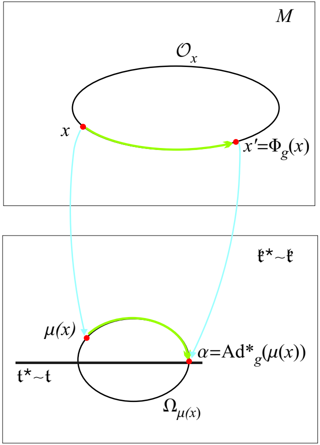
$$|u\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle$$

- Two states are locally unitary equivalent iff they have the same Schmidt decomposition

- There is no direct analog of the Schmidt decomposition for systems with more than 2 components.

Canonical forms

- In general an orbit $\mathcal{O}_x$ does not contain any distinguished point, but corresponding coadjoint orbit $\Omega_{\mu(x)}$ has such a point
- Each orbit of the coadjoint action intersects the subspace $\mathfrak{t}^*$ in $\mathfrak{k}^*$ which is dual to the maximal commutative subalgebra of $\mathfrak{k}$
- Let $x \in M$, then $\mu(x) \in \mathfrak{k}^*$ and there is $g \in K$ such that $\mathrm{Ad}_g^*(\mu(x)) \in \mathfrak{t}^*$.
  Let us call $x' = \Phi_g(x)$ - canonical form of x



Canonical form

Properties of canonical form

- It is given up to the action of $G = \mathrm{Stab}(\mu(x'))/\mathrm{Stab}(x')$
- Points on the same orbit have the same (modulo G) canonical forms
- For x, y on the same orbit $\mu(x') = \mu(y')$ (x', y' are in the same fiber of μ)
- In other words, if $|v\rangle$ and $|u\rangle$ are LU-equivalent then their canonical forms belong to the same fiber of the moment map, $\mu([v']) = \mu([u'])$, but this can happen also for LU-nonequivalent states
- The problem arises since there might be different orbits in M which are mapped by μ on the same coadjoint orbit

LU-equivalence and fibers of momentum map

- but if the fiber of the moment map lies entirely in the orbit (i.e., the tangent space to the fiber is a subspace of the tangent space of the orbit) then all states in the fiber are LU-equivalent
- Fact: the tangent space to the fiber of μ over $[v]$ is contained in the ω-orthogonal complement of the tangent space at $[v]$ to the orbit through $[v]$
- If the orbit is coisotropic (i.e., contains its ω - orthogonal complement) then the states in the same fiber are LU-equivalent
- Example: The orbit through $|GHZ_3\rangle$ is coisotropic - using this fact we find simple solution of LU-equivalence problem for any three qubit states
- It turns out that even in two-partite case not all orbits are coisotropic although μ fibers are contained in them.

Summary and other applications

The presented construction works for

- arbitrary number of subsystems
- arbitrary (finite) dimensions of subsystems
- ... but also for systems of identical (undistinguishable) particles (fermions, bosons) where we have to consider only the antisymmetric or symmetric part of the full Hilbert space, or when there is no tensor structure
- We have only to adjust appropriately our definition of 'locality' of transformations

- Distinguishable particles
  - space $\mathcal{H} = \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \cdots \otimes \mathbb{C}^{N_n}$
  - group of local transformations $K = SU(N_1) \times SU(N_2) \cdots \times SU(N_n)$

- Fermions (Schliemann et al. 2001)
  - space $\mathcal{H} = \mathbb{C}^N \wedge \mathbb{C}^N \wedge \cdots \wedge \mathbb{C}^N$
  - group of local transformations $K = SU(N)$

- Bosons (Eckert et al. 2002)
  - space $\mathcal{H} = \mathbb{C}^N \vee \mathbb{C}^N \vee \cdots \vee \mathbb{C}^N$
  - group of local transformations $K = SU(N)$

- A Sawicki, A Huckleberry, M Kuś Symplectic Geometry of Entanglement Commun. Math. Phys Volume 305, Number 2, 441-468 (2011)
- A Sawicki, M Kuś Geometry of the local equivalence of states J. Phys. A: Math. Theor. 44 495301 (2011),