SYMPLECTIC HYPERSURFACES IN THE COMPLEMENT OF AN ISOTROPIC SUBMANIFOLD

DENIS AUROUX, DAMIEN GAYET, AND JEAN-PAUL MOHSEN

Abstract. Using Donaldson's approximately holomorphic techniques, we construct symplectic hypersurfaces lying in the complement of any given compact isotropic submanifold of a compact symplectic manifold. We discuss the connection with rational convexity results in the Kähler case and various applications.

1. Introduction

It was first observed by Duval (see e.g. [Du]) that, in Kähler geometry, the notions of isotropy and rational convexity are tightly related to each other. Recall that a compact subset N of $\mathbb{C}^n$ or more generally of a complex algebraic manifold is said to be *rationally convex* if there exists a complex algebraic hypersurface passing through any given point in the complement of N and avoiding N. Among the results motivating the interest in this notion, one can mention the classical theorem of Oka and Weil (further improved by subsequent work) stating that every holomorphic function over a neighborhood of a rationally convex compact subset $N \subset \mathbb{C}^n$ can be uniformly approximated over N by rational functions.

It was shown in 1995 by Duval and Sibony that, if a smooth compact submanifold of $\mathbb{C}^n$ is isotropic with respect to some Kähler structure on $\mathbb{C}^n$, then it is rationally convex [DS]. This result was extended in 1999 by Guedj to the context of complex projective manifolds :

Theorem 1 (Guedj [Gu]). *Let (X, ω, J) be a closed Kähler manifold, such that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral. Then any smooth compact isotropic submanifold $\mathcal{L} \subset X$ (possibly with boundary) is rationally convex, i.e. there exist complex hypersurfaces in X passing through any given point in the complement of $\mathcal{L}$ and avoiding $\mathcal{L}$.*

Because the concept of isotropic submanifold originates in symplectic geometry, it is natural to seek an analogue of this result for symplectic manifolds. Although the lack of an integrable almost-complex structure prevents the existence of holomorphic hypersurfaces in a general symplectic manifold, a suitable analogue may be found in Donaldson's construction of approximately holomorphic symplectic hypersurfaces.

Let (X, ω) be a closed compact symplectic manifold of real dimension $2n$. Unless otherwise stated, we will always assume that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{R})$ is integral ; this does not restrict the diffeomorphism type of X in any way. A compatible almost-complex structure J on X and the corresponding Riemannian metric g are also fixed.

Let L be a complex line bundle on X with first Chern class $c_1(L) = \frac{1}{2\pi}[\omega]$, endowed with a Hermitian structure and a Hermitian connection ∇^L whose curvature 2-form is $-i\omega$. It was shown by Donaldson in [D1] that, when the integer k is large enough, the line bundles $L^{\otimes k}$ admit many approximately J-holomorphic sections, some of which possess remarkable transversality properties ensuring that their zero sets are smooth symplectic submanifolds in X. Many interesting constructions in symplectic topology have recently been obtained by using the same techniques (see e.g. [A2], [D2] and [S]).

Let us recall the following definitions. The almost-complex structure J and the Hermitian connection on $L^{\otimes k}$ induced by that on L yield ∂ and $\bar\partial$ operators on $L^{\otimes k}$. Since the connection on $L^{\otimes k}$ has curvature $-ik\omega$, we introduce the rescaled metric $g_k = k\,g$ on X, in order to be able to consider uniform bounds for covariant derivatives of sections of $L^{\otimes k}$. As a consequence of this rescaling, the diameter of X is multiplied by $k^{1/2}$, and all derivatives of order p are divided by $k^{p/2}$.

Definition 1. *Let $(s_k)_{k\gg 0}$ be a sequence of sections of $L^{\otimes k}$ over X. The sections s_k are said to be* asymptotically holomorphic *if there exists a constant $C > 0$ such that, for all k and at every point of X, $|s_k| + |\nabla s_k| + |\nabla\nabla s_k| \leq C$ and $|\bar\partial s_k| + |\nabla\bar\partial s_k| \leq Ck^{-1/2}$, where the norms of the derivatives are evaluated with respect to the metrics $g_k = k\,g$.*

The sections s_k are said to be uniformly transverse to 0 *if there exists a constant $\eta > 0$ (independent of k) such that the sections s_k are η*-transverse to 0*, i.e. such that, for any k and at any point $x \in X$ where $|s_k(x)| < \eta$, the covariant derivative $\nabla s_k(x) : T_xX \to L_x^{\otimes k}$ is surjective and satisfies the bound $|\nabla s_k(x)|_{g_k} > \eta$.*

With these definitions, Donaldson's construction amounts to showing the existence of a sequence of sections s_k of $L^{\otimes k}$ which are at the same time asymptotically holomorphic and uniformly transverse to 0 [D1]. It then follows easily from these properties that, for large enough k, the zero sets W_k of s_k are smooth symplectic hypersurfaces in X.

Let $\mathcal{L}$ be a compact isotropic submanifold in X, not necessarily connected : we wish to show that one can get the symplectic hypersurfaces W_k to lie in $X - \mathcal{L}$. The fundamental reason why it is reasonable to expect such a result is that, since ω vanishes over $\mathcal{L}$, the line bundle $L_{|\mathcal{L}}$ comes equipped with a flat connection. However $L^{\otimes k}$ admits non-vanishing sections over $\mathcal{L}$ only when its restriction to $\mathcal{L}$ is topologically trivial ; if $\mathcal{L}$ is not simply connected, this can restrict the admissible values of the parameter k. For example, if $X = \mathbb{CP}^2$ and $\mathcal{L} = \mathbb{RP}^2$, an easy calculation in homology with $\mathbb{Z}/2$ coefficients shows that any symplectic submanifold of odd degree must intersect $\mathcal{L}$. Our main result is the following :

Theorem 2. *Let $\mathcal{L}$ be a compact isotropic submanifold in X, and let N be the order of the torsion part of $H_1(\mathcal{L}, \mathbb{Z})$. Then, for all large enough values of k, there exist asymptotically holomorphic sections s_k of $L^{\otimes k}$ over X whose zero sets W_k are smooth symplectic submanifolds, disjoint from $\mathcal{L}$ whenever k is a multiple of N. Moreover, W_k can be assumed to pass through any given point $x_0 \in X - \mathcal{L}$.*

This result is mildly surprising when one considers the results obtained in [D1] and [A1] indicating that, when k increases, the submanifolds W_k tend to fill all of X. There is no contradiction, though, as the distance by which the submanifolds W_k given by Theorem 2 stay away from $\mathcal{L}$ actually decreases like $k^{-1/2}$.

Remark 1. (*a*) *Theorem 2 remains valid when $\mathcal{L}$ has non-empty boundary ; see* [M] *for details.*

(*b*) *When X is a Kähler manifold, one can perform the construction in such a way that the sections s_k are holomorphic. The submanifolds W_k are then complex hypersurfaces ; this provides a new proof of Guedj's rational convexity result.*

(*c*) *When the cohomology class $\frac{1}{2\pi}[\omega]$ is no longer assumed to be integral, the line bundle L is no longer defined, but it is still possible to obtain symplectic hypersurfaces in X which avoid the submanifold $\mathcal{L}$ and pass through any given point in $X - \mathcal{L}$.*

Additional motivation for these results can be found in the work of Biran [B], where the notion of *Lagrange skeleton* of a symplectic manifold of Kähler type with respect to a hypersurface of Donaldson type is defined. As will be explained in §3, Theorem 2 can be interpreted in this context as a flexibility result for Lagrange skeleta in large degrees.

More importantly, it was observed by Seidel and Viterbo that Theorem 2 implies that if $\mathcal{L}$ is Lagrangian then its homology class is a primitive element of $H_n(X - W_k)$ (see §3) ; this remark might lead to obstructions to the existence of certain Lagrangian embeddings.

Note. Different proofs of Theorem 2 were obtained independently by the three authors ; the curious reader is referred to [M] and [Ga] for various alternate arguments and generalizations.

The authors wish to thank Claude Viterbo, Paul Seidel and Paul Biran for motivating discussions and for suggesting applications of Theorem 2. The authors are respectively thankful to Ivan Smith, Julien Duval, Bruno Sévennec and Emmanuel Giroux for discussions and advice.

2. Proof of Theorem 2

We first define the notion of concentrated sections of $L^{\otimes k}$:

Definition 2. *Asymptotically holomorphic sections s_k of $L^{\otimes k}$ are said to be* concentrated *over a subset $N \subset X$ if there exist positive constants λ, c and C (independent of k) such that for all $y \in N$, $|s_k(y)| \geq c$, and, for all $y \in X$, $|s_k(y)| \leq C \exp(-\lambda\, d(y,N)^2)$, where $d(.,.)$ is the distance induced by g_k. When the subset N consists of a single point $x \in X$, we say that the sections s_k are concentrated at x.*

With this terminology, recall the following result (Proposition 11 of [D1]) :

Lemma 1 (Donaldson)**.** *For all large enough k the line bundles $L^{\otimes k}$ admit asymptotically holomorphic sections $\sigma_{k,x}$ concentrated at any given point $x \in X$.*

As the properties of the sections $\sigma_{k,x}$ play an important role in the argument, let us recall briefly their construction.

Remember that, at any point $x \in X$, it is possible to find a local approximately holomorphic Darboux coordinate chart, i.e. a local symplectomorphism $\psi : (X, x, \omega) \to (\mathbb{C}^n, 0, \omega_0)$ such that, with respect to J and the standard complex structure of $\mathbb{C}^n$, $\bar\partial\psi(x) = 0$ and $|\nabla\bar\partial\psi|_g$ is bounded uniformly by a constant C. The compactness of X implies that the size of the neighborhood over which ψ is defined and the value of the constant C can be assumed not to depend on the chosen point x.

In our case, we will moreover require that, whenever the point x belongs to the given isotropic submanifold $\mathcal{L}$, the coordinate map ψ locally sends $\mathcal{L}$ to a linear subspace in $\mathbb{C}^n$ (obviously isotropic). The existence of Darboux coordinate charts with this property is a very classical result of Weinstein ([W], see also [McS]) ; it is an immediate observation that the coordinate map can still be chosen to satisfy $\bar\partial\psi(x) = 0$, and the compactness of $\mathcal{L}$ implies the existence of uniform estimates on $|\nabla\bar\partial\psi|$ and on the size of the coordinate chart.

In a Darboux coordinate chart, a suitable unitary gauge transformation leads to a local trivialization of $L^{\otimes k}$ in which the connection 1-form is given by $\frac{k}{4}\sum(z_j d\bar z_j - \bar z_j dz_j)$. The local section defined by $f_k(z) = \exp(-k|z|^2/4)$ is then holomorphic over a neighborhood of 0 in $\mathbb{C}^n$. Pulling back f_k via the coordinate chart ψ, one obtains sections $\hat\sigma_{k,x}$ of $L^{\otimes k}$ over a neighborhood of x in X, and it easily follows from the estimates on $\bar\partial\psi$ that these sections are asymptotically holomorphic.

Finally, multiplying $\hat\sigma_{k,x}$ by a smooth cut-off function vanishing at distance $k^{-1/6}$ from x yields the desired asymptotically holomorphic sections $\sigma_{k,x}$, easily shown to be concentrated at the point x (see [D1]).

Recall from [D1] (see also [A1]) that asymptotically holomorphic sections with uniform transversality estimates are constructed by an iterative process, where one starts with any given asymptotically holomorphic sections s_k of $L^{\otimes k}$ (e.g. $s_k = 0$) and perturbs them over small open subsets of X in order to achieve transversality over those subsets ; successive smaller and smaller perturbations are performed in such a way that the transversality property gained at each step is preserved by all subsequent perturbations, until transversality holds over all of X. In particular, given any constant $C > 0$ it is possible to ensure that the constructed sections $\tilde s_k$ differ from the given sections s_k by less than C in C^1 norm (i.e., at every point of X we have $|\tilde s_k - s_k| + |\nabla\tilde s_k - \nabla s_k|_{g_k} \le C$) [A1].

Therefore, in order to prove Theorem 2 (without requiring yet the submanifolds to pass through a given point of $X - \mathcal{L}$), it is sufficient to construct asymptotically holomorphic sections $\sigma_{k,\mathcal{L}}$ of $L^{\otimes k}$, concentrated over $\mathcal{L}$ for k ranging over all large enough multiples of $N = |\mathrm{Tor}\, H_1(\mathcal{L}, \mathbb{Z})|$. By definition these sections satisfy a uniform lower bound over $\mathcal{L}$ by some constant $c > 0$, and perturbing them by less than $c/2$ we get (for large enough k) uniformly transverse sections which do not vanish over $\mathcal{L}$. Our next ingredient is the following observation :

Lemma 2. *Given any compact isotropic submanifold $\mathcal{L} \subset X$, there exists a constant $C_{\mathcal{L}} > 0$ such that, whenever k is a multiple of $N = |\mathrm{Tor}\, H_1(\mathcal{L}, \mathbb{Z})|$, the restriction of $L^{\otimes k}$ to $\mathcal{L}$ admits a section τ_k such that $|\tau_k(x)| = 1$ and $|\nabla\tau_k(x)|_g \le C_{\mathcal{L}}$, i.e. $|\nabla\tau_k(x)|_{g_k} \le C_{\mathcal{L}}\, k^{-1/2}$, at every point $x \in \mathcal{L}$.*

Proof. Since $\mathcal{L}$ is isotropic, the restriction to $\mathcal{L}$ of the connection ∇^L on L is flat ; therefore the first Chern class $c_1(L_{|\mathcal{L}})$, although not necessarily trivial, belongs to the kernel of the natural map $\iota : H^2(\mathcal{L}, \mathbb{Z}) \to H^2(\mathcal{L}, \mathbb{R})$. By the universal coefficients theorem (see e.g. [BT], page 194), $\mathrm{Ker}(\iota) = \mathrm{Tor}\, H^2(\mathcal{L}, \mathbb{Z}) \simeq \mathrm{Tor}\, H_1(\mathcal{L}, \mathbb{Z})$. It follows that the order of $c_1(L_{|\mathcal{L}})$ divides N, so that the complex line bundle $L^{\otimes k}_{|\mathcal{L}}$ has zero first Chern class and hence is topologically trivial whenever k is a multiple of N.

Fix a trivialization of $L^{\otimes k}$ over $\mathcal{L}$, and consider the 1-form $\alpha_k \in \Omega^1(\mathcal{L}, i\mathbb{R})$ representing the connection on $L^{\otimes k}$ induced by ∇^L. We work with the metric on $\mathcal{L}$ induced by g, and observe that a suitable choice of trivialization of $L^{\otimes k}$ ensures that the 1-form α_k and its derivatives satisfy uniform bounds which depend only on the geometry of $\mathcal{L}$ and not on k.

Indeed, it is well-known that the moduli space of flat unitary connections on the trivial complex line bundle over $\mathcal{L}$ up to $U(1)$ gauge transformations is compact and isomorphic to $H^1(\mathcal{L}, \mathbb{R})/H^1(\mathcal{L}, \mathbb{Z})$. Therefore, a well-chosen gauge transformation makes it possible to obtain uniform bounds on the 1-form α_k and its derivatives, independently of k. More precisely, a first gauge transformation in the identity component can be used to make the closed 1-form α_k harmonic, while the flexibility coming from the connected components of the gauge group makes it possible to ensure that α_k lies in a fixed bounded subset of $H^1(\mathcal{L}, \mathbb{R})$.

Let τ_k be the section of $L^{\otimes k}$ over $\mathcal{L}$ which identifies with the constant function 1 in the chosen trivialization : clearly, $|\tau_k| = 1$ at every point of $\mathcal{L}$ and the derivatives of τ_k are bounded by uniform constants independently of k with respect to the metric g. □

Remark. The bounds satisfied by α_k and $\nabla \tau_k$ depend on the minimum g-length $\delta(\mathcal{L})$ of a homotopically non-trivial loop in $\mathcal{L}$; in fact $C_{\mathcal{L}}$ must be at least of the order of $\delta(\mathcal{L})^{-1}$. This is one of the reasons why the submanifold $\mathcal{L}$ cannot be allowed to vary with k, another one being that we need to control the size of the balls centered at points of $\mathcal{L}$ which can be trivialized by Weinstein's theorem.

Throughout the remainder of this section we assume that k is a multiple of N. For each such k, let P_k be a finite set of points of $\mathcal{L}$ such that the balls of g_k-radius 1 centered at the points of P_k cover $\mathcal{L}$ and any two points of P_k are at g_k-distance at least $\frac{2}{3}$ from each other. Such a set can be constructed by covering $\mathcal{L}$ by finitely many balls of g_k-radius $\frac{1}{3}$ and iteratedly removing the points that are too close to each other (see also [D1]).

Define the sections

$$\sigma_{k,\mathcal{L}} = \sum_{p \in P_k} \frac{\tau_k(p)}{\sigma_{k,p}(p)}\, \sigma_{k,p}$$

of $L^{\otimes k}$ over X. The sections $\sigma_{k,\mathcal{L}}$ are linear combinations of the asymptotically holomorphic sections $\sigma_{k,p}$, with coefficients unitary complex numbers (recall that $|\tau_k(p)| = |\sigma_{k,p}(p)| = 1$). Therefore, because any two points of P_k are mutually g_k-distant of at least $\frac{2}{3}$ and because the sections $\sigma_{k,p}$ are concentrated at points, a standard argument ([D1],[S]) shows that the sections $\sigma_{k,\mathcal{L}}$ are uniformly bounded and asymptotically holomorphic.

We now show that the sections $\sigma_{k,\mathcal{L}}$ are concentrated over $\mathcal{L}$. The decay properties of $\sigma_{k,\mathcal{L}}$ away from $\mathcal{L}$ follow from the following lemma :

Lemma 3. *Let $P_k \subset X$ be a finite set of points whose mutual g_k-distance is bounded from below by a constant $\delta > 0$. Let $(\alpha_{k,p})_{p\in P_k}$ be a family of complex numbers such that $|\alpha_{k,p}| \le 1\ \forall p \in P_k$, and let $s_k = \sum_{p\in P_k} \alpha_{k,p}\sigma_{k,p}$. Then there exist constants C_δ and λ_δ, independent of k and P_k, such that $|s_k(x)| \le C_\delta \exp(-\lambda_\delta d_{g_k}(x, P_k)^2)$ at every point of X.*

Proof. Because $\sigma_{k,p}$ is supported in $B_g(p, 2k^{-1/6})$, we can restrict ourselves to only considering points in a fixed ball around the given point $x \in X$; since the g_k-distance between any two points of P_k is greater than δ, this implies that the number of points $p \in P_k$ lying within a given fixed g_k-distance ρ of x is bounded by $Q(\rho)$, where Q is a polynomial depending only on δ. Therefore, using the existence of a bound $|\sigma_{k,p}(x)| \le C' \exp(-\lambda' d(x,p)^2)$ for $\sigma_{k,p}$ and ordering the points of P_k according to their distance from x, we get the desired bound on $|s_k(x)|$ by summing over concentric slices. □

We immediately conclude that $|\sigma_{k,\mathcal{L}}(x)| \le C_{2/3} \exp(-\lambda_{2/3} d_{g_k}(x,\mathcal{L})^2)$. It remains to be shown that the norm of $\sigma_{k,\mathcal{L}}$ at a point of $\mathcal{L}$ admits a uniform lower bound. For this, we first prove the following result :

Lemma 4. *If k is large enough, and if p and x are two points of $\mathcal{L}$ such that $d_{g_k}(p,x) \le k^{1/10}$, then $\sigma_{k,p}(x) \ne 0$ and*

$$\left| \arg\left(\frac{\sigma_{k,p}(x)}{\tau_k(x)}\right) - \arg\left(\frac{\sigma_{k,p}(p)}{\tau_k(p)}\right) \right| \le \frac{\pi}{4}.$$

Proof. Since the g-distance between x and p is less than $k^{-2/5}$, the cut-off function used to define $\sigma_{k,p}$ is equal to 1 at x, and therefore $\sigma_{k,p}(x) \ne 0$.

We work in the same local coordinate chart ψ and local trivialization of $L^{\otimes k}$ that were used to define $\sigma_{k,p}$; we write $\psi(x) = u$, and consider the radial path $\gamma(t) = \psi^{-1}(tu)$ from p to x. Recall that the connection on $L^{\otimes k}$ is expressed as $d + A_k = d + \frac{k}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j)$, while $\sigma_{k,p}$ is locally given by the function $\exp(-\frac{k}{4}|z|^2)$. Therefore one easily checks that

$$(1)\qquad \int_0^1 \left(\frac{\nabla\sigma_{k,p}}{\sigma_{k,p}}\right)_{\gamma(t)} \cdot \gamma'(t)\,dt = \int_0^1 d(-\frac{k}{4}|z|^2)_{(tu)} \cdot u\,dt = -\frac{k}{4}|u|^2 \in \mathbb{R}.$$

Recall that by construction we require that ψ locally maps $\mathcal{L}$ to a linear subspace of $\mathbb{C}^n$. Therefore the radial path γ is contained in $\mathcal{L}$, and we can use the bound on $\nabla\tau_k$ given by Lemma 2 to obtain that

$$(2)\qquad \left|\int_0^1 \left(\frac{\nabla\tau_k}{\tau_k}\right)_{\gamma(t)} \cdot \gamma'(t)\,dt\right| \le \int_0^1 |(\nabla\tau_k)_{\gamma(t)}| \cdot |\gamma'(t)|\,dt = O(k^{-2/5}).$$

Therefore,

$$\arg\left(\frac{\sigma_{k,p}(x)}{\tau_k(x)}\right) - \arg\left(\frac{\sigma_{k,p}(p)}{\tau_k(p)}\right) = \mathrm{Im}\left[\int_0^1 \left(\frac{\nabla\sigma_{k,p}}{\sigma_{k,p}} - \frac{\nabla\tau_k}{\tau_k}\right)_{\gamma(t)} \cdot \gamma'(t)\,dt\right]$$

is bounded by a constant times $k^{-2/5}$, which gives the result. □

Lemma 4 implies the existence of a uniform lower bound on $\sigma_{k,\mathcal{L}}$ at any point of $\mathcal{L}$. Indeed, consider a point $x \in \mathcal{L}$, and let p be the point of P_k closest to x. By construction $d_{g_k}(x,p) \leq 1$, and therefore there exists a constant $c > 0$ (independent of x, p and k) such that $|\sigma_{k,p}(x)| \geq c$. By Lemma 4 we know that the contributions of the various points $q \in P_k$ whose g_k-distance to x is less than $k^{1/10}$ cannot cancel each other, and we have

$$\left| \sum_{\substack{q\in P_k \\ d(x,q)\leq k^{1/10}}} \frac{\tau_k(q)}{\sigma_{k,q}(q)}\, \sigma_{k,q}(x) \right| \geq |\sigma_{k,p}(x)| \geq c.$$

On the other hand, Lemma 3 implies that the contribution of the remaining points of P_k decreases exponentially with k. Therefore, when k is large enough we get that $|\sigma_{k,\mathcal{L}}(x)| \geq c/2$ at any point x of $\mathcal{L}$; in fact, we even get that $\sup_{x\in\mathcal{L}} |\arg(\sigma_{k,\mathcal{L}}(x)/\tau_k(x))|$ becomes arbitrarily small for large k.

We conclude that the asymptotically holomorphic sections $\sigma_{k,\mathcal{L}}$ are concentrated over $\mathcal{L}$, which ends the argument : perturbing $\sigma_{k,\mathcal{L}}$ by less than $c/4$ we obtain asymptotically holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$ satisfying a uniform transversality property, and by construction their zero sets are (asymptotically holomorphic) symplectic submanifolds which do not intersect $\mathcal{L}$.

The final step to complete the proof of Theorem 2 is to show that these asymptotically holomorphic hypersurfaces can be made to pass through a given point $x_0 \in X - \mathcal{L}$. Considering the sections $u_{k,x_0} = k^{1/2} z_1\, \sigma_{k,x_0}$, where z_1 is a local approximately holomorphic coordinate function at x_0, the idea is to work with $\sigma_{k,\mathcal{L}} + u_{k,x_0}$ instead of $\sigma_{k,\mathcal{L}}$. Indeed, observing that for large k the support of u_{k,x_0} is disjoint from $\mathcal{L}$, a small perturbation of $\sigma_{k,\mathcal{L}} + u_{k,x_0}$ yields asymptotically holomorphic hypersurfaces W_k avoiding $\mathcal{L}$ and passing through a point x within unit g_k-distance of x_0. It is then possible to find a Hamiltonian diffeomorphism ϕ preserving $\mathcal{L}$, mapping x to x_0, and sufficiently close to the identity in order to ensure the asymptotic holomorphicity of $\phi(W_k)$. □

Remark. When $\mathcal{L}$ is Lagrangian, Theorem 2 can also be proved by arguing along the following lines. By Weinstein's Lagrangian neighborhood theorem, a neighborhood V of $\mathcal{L}$ in X is symplectomorphic to a neighborhood of the zero section in $T^*\mathcal{L}$ with its standard symplectic structure $dp \wedge dq$; the fibers of $\pi : T^*\mathcal{L} \to \mathcal{L}$ can be chosen g-orthogonal to $\mathcal{L}$ at every point of $\mathcal{L}$. Consider the trivialization of $L^{\otimes k}$ over $\mathcal{L}$ given by the section τ_k of Lemma 2, and extend it over V in such a way that the connection 1-form is given by $\beta_k = \pi^*\alpha_k - ik\, p\, dq$, where α_k is the same 1-form on $\mathcal{L}$ as in Lemma 2. It can then be checked that the sections of $L^{\otimes k}$ over V defined by $s_k = \exp(-\frac{1}{2}k|p|_g^2)$ (where $|\cdot|_g$ is the metric induced by $g_{|\mathcal{L}}$ on the fibers of $T^*\mathcal{L}$) are asymptotically holomorphic ; multiplying s_k by a suitable cut-off function we obtain asymptotically holomorphic sections concentrated over $\mathcal{L}$, from where Theorem 2 is easily obtained.

3. Remarks and applications

3.1. **The Kähler case.** We consider the case where (X, ω, J) is a Kähler manifold, and show how the construction can be performed in the holomorphic category (Remark 1 (b)) using the ideas of Donaldson (see pp.

696–700 of [D1]). The first observation is that near any point $x \in X$ there exists a local holomorphic section of L which, in the same local trivialization of L as in the proof of Lemma 1, is given by a function f such that $f(z) = 1 - \frac{1}{4}|z|^2 + O(|z|^3)$ and $df(z) = -\frac{1}{4}\sum_j (z_j d\bar{z}_j + \bar{z}_j dz_j) + O(|z|^2)$; see the proof of Lemma 36 of [D1].

Multiplying $f(z)^k$ by a smooth cut-off function at distance $k^{-1/6}$ from x yields asymptotically holomorphic sections $\sigma_{k,x}$ of $L^{\otimes k}$, concentrated at x as in Lemma 1 ; moreover, as observed by Donaldson in [D1], there exist holomorphic sections $\tilde{\sigma}_{k,x}$ of $L^{\otimes k}$ such that $\sup |\tilde{\sigma}_{k,x} - \sigma_{k,x}| \le C \exp(-ak^{1/3})$, with a and C positive constants (independent of k and x).

We now proceed as in §2.1, using the new sections $\sigma_{k,x}$ instead of those obtained in Lemma 1. The argument remains the same, the only difference being in the proof of Lemma 4 where the l.h.s. of (1) becomes equal to

$$\int_0^1 \frac{(d+A_k)f(z)^k_{(tu)}}{f(tu)^k} \cdot u\,dt = \int_0^1 k\Big(\frac{df}{f}\Big)_{(tu)} \cdot u\,dt = -\frac{k}{4}|u|^2 + O(k|u|^3).$$

Since $|u|$ is at most of the order of $k^{-2/5}$ the imaginary part of this quantity is bounded by $O(k^{-1/5})$, which is enough to prove Lemma 4 and hence construct $\sigma_{k,\mathcal{L}}$ as in §2.1.

Replacing $\sigma_{k,x}$ by $\tilde{\sigma}_{k,x}$ in the definition of $\sigma_{k,\mathcal{L}}$, we obtain holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$ which differ from $\sigma_{k,\mathcal{L}}$ by at most $C\exp(-ak^{1/3})\,\mathrm{card}(P_k)$ and therefore also satisfy a uniform lower bound over $\mathcal{L}$. It is then possible to conclude as usual, by adding a linear combination of the sections $\tilde{\sigma}_{k,x}$ to $\tilde{\sigma}_{k,\mathcal{L}}$ in order to achieve uniform transversality.

Alternately, given a point $x_0 \in X - \mathcal{L}$, one can add a multiple of $\tilde{\sigma}_{k,x_0}$ to $\tilde{\sigma}_{k,\mathcal{L}}$ in order to obtain holomorphic sections $\tilde{\sigma}_{k,\mathcal{L},x_0}$ which vanish at x_0 while remaining bounded away from zero over $\mathcal{L}$. In terms of the projective embeddings $i : X \to \mathbb{P}H^0(L^{\otimes k})^*$, these sections correspond to hyperplanes passing through $i(x_0)$ while avoiding $i(\mathcal{L})$. A small generic perturbation yields a hyperplane passing through $i(x_0)$ which intersects $i(X)$ transversely and still avoids $i(\mathcal{L})$; this gives smooth complex hypersurfaces passing through x_0 and avoiding $\mathcal{L}$, giving a new proof of Guedj's result.

3.2. **The non-integral case.** In this section we no longer assume that the cohomology class $\frac{1}{2\pi}[\omega]$ is integral, as in Remark 1 (c). As in [D1] the idea is to perturb the symplectic form ω into a symplectic form ω' such that $\frac{1}{2\pi}[\omega']$ is proportional to an integral class, and work with a multiple of ω'. It is however necessary to ensure that $\mathcal{L}$ remains isotropic.

Because $\frac{1}{2\pi}[\omega]$ lies in the kernel of the restriction map from $H^2(X,\mathbb{R})$ to $H^2(\mathcal{L},\mathbb{R})$, it is the image of a class $\alpha \in H^2(X,\mathcal{L};\mathbb{R})$. Moreover, $H^2(X,\mathcal{L};\mathbb{Q})$ contains elements lying arbitrarily close to α in $H^2(X,\mathcal{L};\mathbb{R})$. Therefore, by adding to ω an arbitrarily small closed 2-form vanishing over $\mathcal{L}$, we obtain a symplectic form ω' such that $\frac{1}{2\pi}[\omega']$ is the image of a class in $H^2(X,\mathcal{L};\mathbb{Q})$ and hence belongs to $H^2(X,\mathbb{Q})$. By construction, ω' satisfies up to multiplication by a constant factor the required integrality condition, and $\mathcal{L}$ is ω'-isotropic.

The symplectic form ω' admits a compatible almost-complex structure J', C^0-close to J ; since $\omega(v, J'v) > 0\ \forall v \in TX$, any J'-complex subspace is ω-symplectic. So, if a sequence of submanifolds $W_k \subset X$ is asymptotically

J'-holomorphic, then W_k is a symplectic submanifold of (X, ω) for large enough k. One then concludes by applying Theorem 2 to (X, ω', J').

3.3. **Uniqueness up to isotopy.** It was shown in [A1] that the symplectic submanifolds constructed by Donaldson in [D1] are, for each large enough value of k, canonical up to symplectic isotopy, independently of the almost-complex structure J. One may ask whether in our case the submanifolds W_k are canonical up to a symplectic isotopy of X preserving $\mathcal{L}$; such a uniqueness property does not hold in general, because the homotopy class of the non-vanishing section s_k of $L^{\otimes k}$ over $\mathcal{L}$ plays a determining role.

Let γ be a non-contractible loop in $\mathcal{L}$ bounding a disc D in X : the homotopy class of the non-vanishing section $(s_k)_{|\gamma}$ over γ determines the number of zeroes of s_k over D, i.e. the linking number of W_k with γ, which can be modified by choosing different trivializations of $L^{\otimes k}$ over $\mathcal{L}$. Still, when $\mathcal{L}$ is simply connected the homotopy classes of the nowhere vanishing sections $(s_k)_{|\mathcal{L}}$ are uniquely determined.

Even though it seems reasonable to expect that the isotopy class of asymptotically holomorphic hypersurfaces in $X - \mathcal{L}$ should only depend on the homotopy class of $(s_k)_{|\mathcal{L}}$, our techniques do not allow us to prove so strong a statement ; we are only able to prove that the submanifolds constructed in §2 (using either the given proof or the alternate argument sketched at the end) are canonical up to symplectic isotopy in $X - \mathcal{L}$. For this, we use the control on the complex argument of $(s_k)_{|\mathcal{L}}$ given by the construction : it follows directly from Lemma 4 and the subsequent discussion that for large k the argument of s_k/τ_k remains small at every point of $\mathcal{L}$.

Proposition 1. *Let τ_k^0 and τ_k^1 be sections of $L^{\otimes k}$ over $\mathcal{L}$ belonging to the same homotopy class and such that $|\tau_k^i| \equiv 1$ and $|\nabla \tau_k^i|_g = O(1)$. Let s_k^0 and s_k^1 be asymptotically holomorphic sections of $L^{\otimes k}$ over X, uniformly transverse to 0, uniformly bounded from below over $\mathcal{L}$, and such that the bound $|\arg(s_k^i/\tau_k^i)| \leq \frac{\pi}{3}$ holds at every point of $\mathcal{L}$. Then for large enough k their zero sets W_k^0 and W_k^1 differ by a symplectic isotopy preserving $\mathcal{L}$.*

Proof. We use the same one-parameter argument as in [A1] in order to construct for large k a one-parameter family of asymptotically holomorphic sections s_k^t, bounded from below on $\mathcal{L}$, interpolating between s_k^0 and s_k^1. First, choosing a trivialization of $L^{\otimes k}$ over $\mathcal{L}$ to express τ_k^i in the form $\exp(\phi_k^i)$ for $i \in \{0, 1\}$, we define sections τ_k^t of $L^{\otimes k}_{|\mathcal{L}}$ for $t \in [0, 1]$ by $\tau_k^t = \exp((1-t)\phi_k^0 + t\phi_k^1)$. Observing that $|\tau_k^t| \equiv 1$ and $|\nabla \tau_k^t|_g = O(1)$ for all t, we can define sections $\sigma_{k,\mathcal{L}}^t = \sum_{p \in P_k} (\tau_k^t(p)/\sigma_{k,p}(p))\, \sigma_{k,p}$ of $L^{\otimes k}$ over X which are asymptotically holomorphic and concentrated over $\mathcal{L}$.

Define s_k^t to be equal to $(1-3t)s_k^0 + 3t\sigma_{k,\mathcal{L}}^0$ for $t \in [0, \frac{1}{3}]$, to $\sigma_{k,\mathcal{L}}^{3t-1}$ for $t \in [\frac{1}{3}, \frac{2}{3}]$ and to $(3-3t)\sigma_{k,\mathcal{L}}^1 + (3t-2)s_k^1$ for $t \in [\frac{2}{3}, 1]$. All these sections are asymptotically holomorphic ; observing that for $i \in \{0, 1\}$ the arguments of s_k^i and $\sigma_{k,\mathcal{L}}^i$ both remain within $\frac{\pi}{3}$ of that of τ_k^i at every point of $\mathcal{L}$, they also satisfy a uniform lower bound by some constant $c > 0$ at every point of $\mathcal{L}$.

Let $\gamma > 0$ be the uniform transversality estimate satisfied by s_k^i for $i \in \{0, 1\}$. Applying the main theorem of [A1], we obtain, provided that

k is large enough, uniformly transverse sections $\tilde{s}_k^t$ of $L^{\otimes k}$ depending continuously on t and differing from s_k^t by at most $\frac{1}{2}\inf(c,\gamma)$ in C^1 norm ; slightly modifying this 1-parameter family near its extremities we can safely assume that $\tilde{s}_k^0 = s_k^0$ and $\tilde{s}_k^1 = s_k^1$ (see Corollary 2 in [A1]). The zero sets of $\tilde{s}_k^t$ are then symplectic hypersurfaces $W_k^t \subset X - \mathcal{L}$ realizing a smooth isotopy between W_k^0 and W_k^1. The argument in §4.2 of [A1] then shows that this smooth isotopy can be turned into a symplectic isotopy preserving $\mathcal{L}$ (observe that all the quantities appearing in the argument can be chosen to vanish over a neighborhood of $\mathcal{L}$). □

A final remark about the homotopy class of the sections we construct in the non simply connected case : the homotopy class of $(s_k)_{|\mathcal{L}}$ as given by our construction is in fact related to the evaluation of ω on elements of $\pi_2(X,\mathcal{L})$. More precisely, given a loop $\gamma \subset \mathcal{L}$ bounding a disc D in X, the trivialization of $L^{\otimes k}$ over γ which minimizes the norm of the connection 1-form differs from the one which extends over D by an amount of twisting approximately equal to $\frac{1}{2\pi}\int_D k\omega$; therefore, in the construction of W_k we obtain a linking number differing from this amount by at most a bounded quantity.

3.4. **Behavior of concentrated sections along normal slices.** For any point $x \in \mathcal{L}$, let N_x be the image by the exponential map of the metric g of a small disc in the normal space to $\mathcal{L}$ at x. Let $\sigma_{k,\mathcal{L}}$ be the asymptotically holomorphic sections concentrated over $\mathcal{L}$ constructed in §2. The following Lemma will be useful for applications.

Lemma 5. *There exist constants $\delta > 0$ and $\gamma > 0$, independent of k, such that the restriction of $|\sigma_{k,\mathcal{L}}|^2$ to the intersection of N_x with $B_{g_k}(x,\delta)$ is strictly concave, with second derivatives bounded from above by $-\gamma$ w.r.t. g_k, and reaches its maximum at a point within g_k-distance $o(1)$ from x. The set of all these maxima is a smooth submanifold $\mathcal{L}'_k$, C^0-converging towards $\mathcal{L}$ as k increases. Moreover, when X is Kähler the same properties remain true for the holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$ constructed in* §3.1.

Proof. Fix a value of k and a point $p \in P_k$ such that $d_{g_k}(x,p) \leq k^{1/10}$, and work in the approximately holomorphic Darboux coordinate chart used to define $\sigma_{k,p}$; recalling that $\mathcal{L}$ is locally mapped to a linear subspace, let N'_x be the affine subspace through x orthogonal to $\mathcal{L}$ in these coordinates. Since x lies at g-distance less than $k^{-2/5}$ from p where the coordinate map is an isometry, N_x and N'_x are very close to each other (their angle at x is at most $O(k^{-2/5})$). Moreover, the restriction to N'_x of the function $f(z) = \exp(-\frac{1}{4}|z|^2)$ is strictly concave (with a uniform upper bound on its second derivatives) and admits a maximum at x ; therefore, $f_{|N_x}$ is also strictly concave and admits a maximum within g-distance $O(k^{-4/5})$ from x. Since $\sigma_{k,p}$ coincides with f^k near x, the same property holds for $|\sigma_{k,p}|^2$, except that the upper bound on second derivatives depends on $d_{g_k}(p,x)$ and only holds over a ball of fixed g_k-radius around x.

Next, recall from the proof of Lemma 4 that the contributions to $\sigma_{k,\mathcal{L}}$ coming from the various points of P_k lying within g_k-distance $k^{1/10}$ from x do not cancel each other at x, and more precisely their complex arguments at x differ from each other by at most $O(k^{-2/5})$. Of course this no longer remains true as soon as one moves away from $\mathcal{L}$; still, by a computation similar to the proof of Lemma 4 we can obtain control on the manner in which the complex arguments of the various contributions to $\sigma_{k,\mathcal{L}}$ differ from each other at a point close to x.

More precisely, consider a geodesic arc γ joining x to a nearby point y in N_x, and let p be a point of P_k within g_k-distance $k^{1/10}$. Then

$$\mathrm{Im} \int_0^1 \left(\frac{\nabla \sigma_{k,p}}{\sigma_{k,p}} \right)_{\gamma(t)} \cdot \gamma'(t)\, dt = \int_0^1 -\tfrac{ik}{4} \textstyle\sum z_j d\bar{z}_j - \bar{z}_j dz_j \cdot \gamma'(t)\, dt$$

is equal to $-\frac{k}{2}\omega_0(x-p, y-x) + O(k\, d_g(x,p)^2 d_g(x,y))$, where ω_0 is the standard symplectic form on $\mathbb{C}^n$ and the error term comes from the non-linearity of N_x in the Darboux coordinate chart. In particular, if p, p' and y are at bounded g_k-distance from x then the difference of complex arguments between the contributions of $\sigma_{k,p}$ and $\sigma_{k,p'}$ to $\sigma_{k,\mathcal{L}}(y)$ is given by $\phi_{p,p'}(y) = \frac{k}{2}\omega_0(p-p', y-x) + O(k^{-2/5})$, where the first term is bounded by a fixed constant times $d_{g_k}(y,x)$.

Fix a large constant $D > 0$ (independent of k and x), and let us first restrict ourselves to the sum $\sigma_{k,\mathcal{L},x,D}$ of the contributions of the points of P_k within g_k-distance D from x. It follows from the above remarks that there exists a constant $\delta(D) > 0$ (of the order of D^{-1}) such that $|\sigma_{k,\mathcal{L},x,D}|^2$ is a strictly concave function at every point of $N_x \cap B_{g_k}(x, \delta(D))$, with a uniform upper bound (independent of k, D and x) on its second derivatives. Indeed,

$$|\sigma_{k,\mathcal{L},x,D}(y)|^2 = \sum_p |\sigma_{k,p}(y)|^2 + \sum_{p \neq p'} |\sigma_{k,p}(y)|\, |\sigma_{k,p'}(y)| \, \cos\phi_{p,p'}(y).$$

When $d_{g_k}(y,x)$ is not too large, $\cos\phi_{p,p'}$ has second derivatives bounded from above by $o(1)$ (by the above expression of $\phi_{p,p'}$ and the corresponding bounds on its first and second derivatives) ; therefore, using the lower bounds on $|\sigma_{k,p}|$, $|\sigma_{k,p'}|$ and $\cos\phi_{p,p'}$, the upper bounds on their second derivatives and the estimates on their first derivatives near x, we obtain that all the terms in the sum are strictly concave functions, thus yielding the desired concavity property for $|\sigma_{k,\mathcal{L},x,D}|^2$.

Moreover, since the total contribution of the remaining points of P_k to the section $\sigma_{k,\mathcal{L}}$ decreases exponentially fast as a function of D, it cannot affect the concavity property provided that D is chosen large enough.

The contributions of the points within distance $k^{1/10}$ from x reach their maxima over N_x within g-distance $O(k^{-4/5})$ from x and their arguments at x differ by $O(k^{-2/5})$, while the remaining terms decrease exponentially fast with k. Therefore, the value of $|\sigma_{k,\mathcal{L}}(x)|^2$ is sufficiently close to the maximal possible one in order to guarantee that the maximum of $|\sigma_{k,\mathcal{L}}|^2$ over N_x is reached within g_k-distance $o(1)$ from x.

Finally, the smoothness of the set $\mathcal{L}'_k$ of all maxima is an immediate consequence of the smoothness of $\sigma_{k,\mathcal{L}}$ and of the uniform concavity property.

In the Kähler case, recall from §3.1 that the sections $\sigma_{k,p}$ are now constructed using the local holomorphic section $f(z) = 1 - \frac{1}{4}|z|^2 + O(|z|^3)$, for which the maximum over N'_x is reached not necessarily at x but at an arbitrary point within g-distance $O(k^{-4/5})$ from x ; however this does not affect the properties of $|\sigma_{k,p}|^2_{|N_x}$ that we have used. Similarly, the fact that f is no longer real-valued affects the complex arguments of the various contributions to $\sigma_{k,\mathcal{L}}$, both at a point $x \in \mathcal{L}$ (bound by $O(k^{-1/5})$ instead of $O(k^{-2/5})$ in Lemma 4, see §3.1) and outside $\mathcal{L}$ (but it turns out that these extra contributions do not affect the estimates) ; still, the argument remains valid without modification. Finally, since the holomorphic section $\tilde{\sigma}_{k,\mathcal{L}}$ differs from $\sigma_{k,\mathcal{L}}$ by an amount decreasing exponentially fast with k, it enjoys the same concavity and maximum properties as $\sigma_{k,\mathcal{L}}$, so that the conclusion remains valid in this case as well. □

Remark. The assertions of Lemma 5 are also trivially satisfied by the concentrated sections obtained in the alternate proof of Theorem 2 outlined at the end of §2.

3.5. **Relations with Lagrange skeleta.** Let X be a compact Kähler manifold, let s be a holomorphic section of $L^{\otimes k}$, transverse to 0, and consider the smooth hypersurface $W = s^{-1}(0)$. It is a result of Biran [B] that the section s determines a splitting $X = B \sqcup \Delta$, where B is a "standard" symplectic disc bundle over W and Δ is an isotropic CW-complex called the *Lagrange skeleton* of (X, W). The skeleton Δ is obtained as the union of the ascending varieties of all the critical points of the plurisubharmonic function $\log |s|^2$; it is well-known that these critical points are all of index at least n. Combined with standard results in Lagrangian intersection theory, this result provides powerful restrictions on Lagrangian embeddings. For example, any simply connected embedded Lagrangian submanifold in X must intersect either W or Δ (otherwise it could be disjointed from itself by a Hamiltonian flow in $B - W$).

Biran's result is generally expected to remain valid in the more general case of a symplectic manifold and a symplectic hypersurface "of Donaldson type". However, to be on the safe side we will assume throughout this section that X is Kähler, considering only the construction of §3.1.

Proposition 2. *Let $\mathcal{L}$ be a compact isotropic submanifold of X. Then for large k there exist holomorphic sections s_k of $L^{\otimes k}$, transverse to 0 and non-vanishing over $\mathcal{L}$, such that $\mathcal{L}$ is contained in arbitrarily small neighborhoods of the Lagrange skeleta Δ_k corresponding to their zero sets W_k.*

Proof. We use the notations of §3.1, and consider the local behavior near $\mathcal{L}$ of the transverse sections s_k constructed as small perturbations of the concentrated holomorphic sections $\tilde{\sigma}_{k,\mathcal{L}}$. By Lemma 5 we know that the restriction of $|\tilde{\sigma}_{k,\mathcal{L}}|^2$ to each normal slice N_x is locally concave and reaches its maximum close to $\mathcal{L}$. Therefore, choosing the transverse sections s_k close enough to $\tilde{\sigma}_{k,\mathcal{L}}$ we conclude that the restriction of $h_k = \log |s_k|^2$ to N_x admits a unique local maximum at g_k-distance less than $\frac{1}{2}\delta$ from x ; as in Lemma 5, the set of these local maxima is a smooth submanifold $\mathcal{L}''_k$ in X, obtained from $\mathcal{L}$ by an arbitrarily small deformation.

Observe that, by construction, every critical point of $h_{k|\mathcal{L}''_k}$ is also a critical point of h_k, with index increased by $\operatorname{codim} \mathcal{L}$. Moreover, although the union Λ_k of the ascending varieties of these critical points is not exactly $\mathcal{L}''_k$, one expects it to be a small deformation of $\mathcal{L}$ as well. More precisely, observe that the gradient of h_k is directed inwards at every point of the boundary of the δ-tubular neighborhood $T_\delta(\mathcal{L})$ of $\mathcal{L}$ (w.r.t. g_k). This implies, first, that every point of Λ_k lies at g_k-distance less than δ from $\mathcal{L}$, since all ascending trajectories remain in $T_\delta(\mathcal{L})$. Conversely, consider the disc $D_x = N_x \cap B_{g_k}(x, \delta)$ and its image by the downward gradient flow of h_k : since no trajectory can re-enter $T_\delta(\mathcal{L})$, the algebraic intersection number of the disc with $\mathcal{L}''_k$ constantly remains equal to 1, which implies that $D_x \cap \Lambda_k$ is non-empty. In particular $\mathcal{L}$ is contained in the δ-neighborhood of Λ_k, which is itself contained in the Lagrange skeleton. □

3.6. **Obstructions to Lagrangian embeddings.** In this section, we no longer assume that X is Kähler, but we assume that $\mathcal{L}$ is Lagrangian (i.e., $\dim \mathcal{L} = n$). It was suggested to us by Seidel, Viterbo and Biran that Theorem 2 might provide obstructions to the existence of certain Lagrangian embeddings by arguing along the following lines.

Consider the asymptotically holomorphic sections s_k of $L^{\otimes k}$, bounded from below over $\mathcal{L}$ and uniformly transverse to 0, given by Theorem 2, and their zero sets W_k. It follows from Lemma 5 that, if the sections constructed in §2 are chosen sufficiently close to the concentrated sections $\sigma_{k,\mathcal{L}}$, their norms reach local maxima over the transverse slices N_x along smooth submanifolds $\mathcal{L}''_k$ obtained by slightly deforming $\mathcal{L}$. Moreover, after an arbitrarily small perturbation we can assume that $h_k = \log |s_k|^2$ is a generic Morse function over $X - W_k$, without affecting the other properties.

Consider a point $x \in \mathcal{L}''_k$ where the restriction of h_k to $\mathcal{L}''_k$ reaches a local minimum : it is a critical point of index n of h_k. However the sections s_k are asymptotically holomorphic and uniformly transverse to 0, so it follows from a result of Donaldson [D1] that the critical points h_k are all of index at least n. Therefore, the genericity condition on h_k implies that the stable manifold Δ_x is a topological disc in $X - W_k$, with boundary mapped to W_k, and intersecting $\mathcal{L}''_k$ transversely at x. Observe that Δ_x is the image by the downward gradient flow of h_k of the small disc $\Delta_x \cap T_\delta(\mathcal{L})$, where $T_\delta(\mathcal{L})$ is the δ-tubular neighborhood of $\mathcal{L}$. However, the downward gradient flow is pointing outwards at every point of the boundary of $T_\delta(\mathcal{L})$, so that x is the only intersection between $\mathcal{L}''_k$ and Δ_x, and the intersection pairing between these two cycles evaluates to 1. This implies that the homology class $[\mathcal{L}''_k] \in H_n(X - W_k)$ is a primitive element. Since $\mathcal{L}''_k$ is isotopic to $\mathcal{L}$, we obtain the following

Proposition 3. *The element $[\mathcal{L}] \in H_n(X - W_k)$ is primitive.*

Moreover, when $\mathcal{L}$ is not connected we can apply the same argument to the minima of h_k over each component individually, obtaining that the fundamental classes of the various components of $\mathcal{L}$ are linearly independent primitive classes in $H_n(X - W_k)$.

When X is a complex projective manifold, working with the holomorphic sections of §3.1 and assuming moreover that $\mathcal{L}$ is simply connected, it is an interesting question to ask whether the smooth complex hypersurfaces W_k are always isotopic in $X - \mathcal{L}$ to hypersurfaces H_k arbitrarily close to a given hyperplane section H of X avoiding $\mathcal{L}$. A positive answer would imply that

$[\mathcal{L}]$ is primitive in $H_n(X - H)$ as well, providing a new proof of a theorem of Gromov.

However, even though no problem with homotopy classes of sections over $\mathcal{L}$ is to be feared in the simply connected case, the isotopy result of §3.3 does not apply in this context, as we have no control over the complex argument of the holomorphic section of $L^{\otimes k}$ defining H_k. Whether a refinement of Proposition 1 can handle this case or not remains an open question.

References

[A1] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[A2] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbb{CP}^2$, Invent. Math. **139** (2000), 551–602.

[B] P. Biran, *Lagrangian barriers and symplectic embeddings*, preprint.

[BT] R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.

[D1] S.K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. **44** (1996), 666–705.

[D2] S.K. Donaldson, *Lefschetz pencils on symplectic manifolds*, preprint.

[Du] J. Duval, *Convexité rationnelle des tores lagrangiens de* $\mathbb{C}^2$, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 943–948.

[DS] J. Duval, N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. **79** (1995), 487–513.

[Ga] D. Gayet, Ph.D. thesis, Université Toulouse III, in preparation.

[Gu] V. Guedj, *Approximation of currents on complex manifolds*, Math. Ann. **313** (1999), 437–474.

[McS] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, 1995.

[M] J.-P. Mohsen, Ph.D. thesis, ENS Lyon, in preparation.

[S] J.-C. Sikorav, *Construction de sous-variétés symplectiques (d'après S. K. Donaldson et D. Auroux)*, Séminaire Bourbaki. Vol. 1997/98. Astérisque **252** (1998), 231–253.

[W] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Math. **6** (1971), 329–346.

Department of Mathematics, M.I.T., Cambridge MA 02139-4307, USA
E-mail address: auroux@math.mit.edu

Laboratoire Emile Picard, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France
E-mail address: gayet@picard.ups-tlse.fr

UMPA, ENS-Lyon, UMR 5669, 46 allée d'Italie, 69364 Lyon Cedex 7, France
E-mail address: jpmohsen@umpa.ens-lyon.fr