

## Symplectic Rational Blow-down Along Seifert Fibered 3-Manifolds

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We verify that the rational blow down schemes along certain Seifert fibered 3-manifolds found by Stipsicz, Szabó and Wahl are, in fact, symplectic operations.

### 1 Introduction

The rational blow-down procedure (introduced by Fintushel and Stern [4] and generalized by Park [25]) turned out to be one of the most effective operations in constructing smooth 4-manifolds with interesting topological properties, cf. [5, 26–28]. Recall that when performing the rational blow-down operation we simply replace the tubular neighbourhood of a string of 2-spheres (intersecting each other according to the linear plumbing tree with framings specified by the continued fraction coefficients of  $-\frac{p^2}{pq-1}$  for some  $p, q$  relatively prime) with a rational homology disk. The success of this operation might be explained by the fact that—as Symington showed [30, 31]—it can be performed symplectically. More precisely, if the ambient 4-manifold is symplectic and the spheres are symplectic submanifolds intersecting each other orthogonally then the neighbourhood

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can be chosen so that the symplectic structure (when restricted to the complement of this neighbourhood) extends over the glued-in rational homology disk. Symington's argument used a beautiful application of toric geometry by constructing both the right symplectic neighbourhood and the appropriate symplectic model for the rational homology disk.

It is not hard to list the combinatorial constraints a plumbing tree must satisfy in order for the proofs of Fintushel-Stern and Park to be applied. In [29] these constraints were explicitly spelled out and the family of plumbing trees satisfying these properties has been determined. The combinatorial conditions are, however, sometimes too weak to ensure the existence of the rational homology disk needed to perform the geometric operation. In [29] a number of examples were shown to admit such rational homology disks. (Some of these examples were already applied by Michalogiorgaki [16] for constructing exotic 4-manifolds.) Whether these new examples for rational blow-down can be executed in the symplectic category, however, remained an open question.

In this paper we show that some of the plumbing trees found in [29] can be, in fact, blown down symplectically. (For the definition of *symplectic* rational blow-down see also [30, Definition 1.1].) Our main result refers to a class of plumbing trees which we describe in the following definition. However, it is worth pointing out here that we are able to prove the existence of  $\omega$ -convex neighborhoods (which amounts to half of our main result) for a much larger class of plumbing trees, namely all star-shaped negative definite trees.

Definition 1.1.

- The plumbing tree given by Figure 1 will be denoted by  $\Gamma_{p,q,r}$  ( $p, q, r \geq 0$ ). We denote the set of these plumbing trees with  $\mathcal{W}$ .

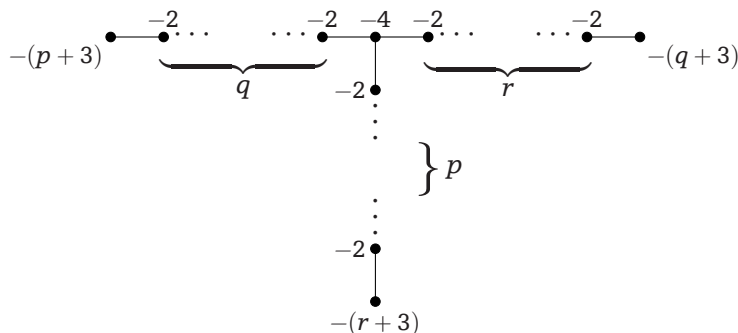
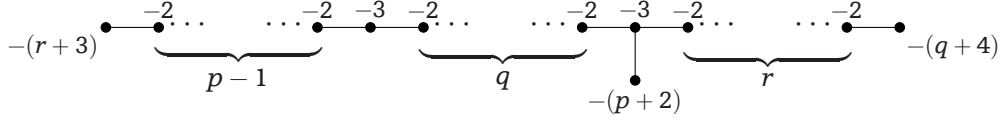
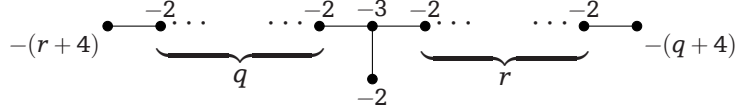


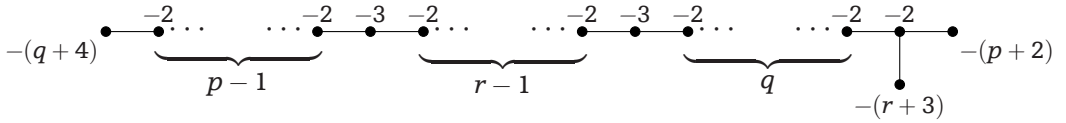
Figure 1 The graph  $\Gamma_{p,q,r}$  in  $\mathcal{W}$ .



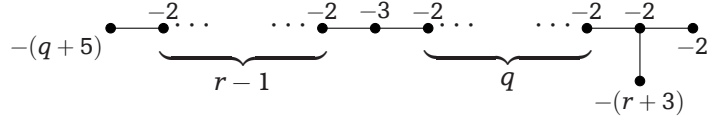
**Figure 2** The graph  $\Delta_{p,q,r}$  for  $p \geq 1$  and  $q, r \geq 0$ .



**Figure 3** The graph  $\Delta_{0,q,r}$ .



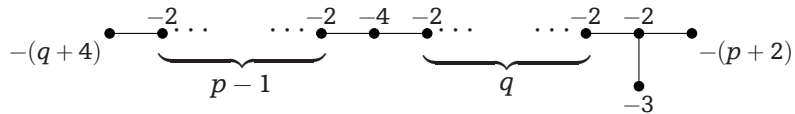
**Figure 4** The graph  $\Lambda_{p,q,r}$  for  $p, r \geq 1$  and  $q \geq 0$ .



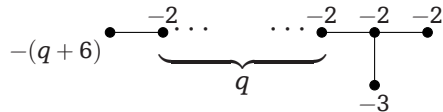
**Figure 5** The graph  $\Lambda_{0,q,r}$  for  $r \geq 1$  and  $q \geq 0$ .

- The plumbing tree of Figure 2 will be denoted by  $\Delta_{p,q,r}$  ( $p \geq 1$  and  $q, r \geq 0$ ). The slight modification of the graph  $\Delta_{p,q,r}$  when  $p = 0$  is shown in Figure 3. The set of graphs  $\Delta_{p,q,r}$  with  $p, q, r \geq 0$  will be denoted by  $\mathcal{N}$ .
- The plumbing graph of Figure 4 is  $\Lambda_{p,q,r}$  with  $p, r \geq 1$  and  $q \geq 0$ . Modifications of these graphs for  $p = 0, r \geq 1$  and  $p \geq 1, r = 0$  and finally for  $p = r = 0$  are shown by Figures 5, 6 and 7. The set of graphs  $\Lambda_{p,q,r}$  with  $p, q, r \geq 0$  will be denoted by  $\mathcal{M}$ .
- Finally let us denote the union  $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M}$  simply by  $\mathcal{G}$ .

**Theorem 1.2.** Suppose that  $(X, \omega)$  is a given symplectic 4-manifold and the symplectic spheres  $S_i \subset (X, \omega)$  intersect each other  $\omega$ -orthogonally and according to the plumbing tree  $\Gamma$ , which is an element of  $\mathcal{G}$ . Then there is a rational homology disk  $B_\Gamma$  and a tubular neighbourhood  $S_\Gamma$  of  $\cup S_i \subset X$  such that the symplectic form  $\omega$  extends from  $X - \text{int} S_\Gamma$  to  $X_\Gamma = (X - \text{int} S_\Gamma) \cup B_\Gamma$ . In short, graphs in  $\mathcal{G}$  can be symplectically blown down.  $\square$



**Figure 6** The graph  $\Lambda_{p,q,0}$  for  $p \geq 1$  and  $q \geq 0$ .



**Figure 7** The graph  $\Lambda_{0,q,0}$  for  $q \geq 0$ .

The proof relies on the gluing theorem of symplectic manifolds along  $\omega$ -convex (and  $\omega$ -concave) boundaries, as explained in [3]. (Recall that “ $\omega$ -convex”, resp. “ $\omega$ -concave”, is synonymous with “strongly convex”, resp. “strongly concave”.) According to this scheme, a strong convex symplectic filling of the contact 3-manifold  $(M_1, \xi_1)$  can be symplectically glued to a strong concave symplectic filling of  $(M_2, \xi_2)$  provided there is a contactomorphism between  $(M_1, \xi_1)$  and  $(M_2, \xi_2)$ . Therefore one way of proving Theorem 1.2 is to show that

- (1) the symplectic spheres admit a neighbourhood such that the complement of it is a strong concave filling of the boundary contact 3-manifold,
- (2) the rational homology disk  $B_\Gamma$  admits a symplectic structure which is a strong convex filling of its boundary (equipped with some contact structure), and finally
- (3) the desired gluing contactomorphism exists.

There are many possibilities for verifying (1) above. Since the plumbing graph under examination is negative definite, a classical theorem of Grauert [12] shows the existence of a  $J$ -convex neighbourhood of the spheres, which in this dimension shows that the complement is a weak concave filling of its boundary. The concave analogue of Eliashberg’s deformation argument [2] (stating that a weak concave filling of a rational homology 3-sphere can be perturbed to be a strong filling, cf. [19]) would complete the argument—but no concave analogue of the “convex” theorem of Eliashberg mentioned above is available at the moment. We will rather use an explicit way of constructing symplectic structures on the plumbing 4-manifold  $M_\Gamma$  (associated to the plumbing graph  $\Gamma$ ) and invoke standard neighbourhood theorems to find the right  $\omega$ -convex neighborhoods.

For (2) we only need to note that the rational homology disks for the plumbing graphs of  $\mathcal{G}$  are given as deformations of appropriate surface singularities (see [29]), therefore are equipped with Stein structures. (Alternatively,  $B_\Gamma$  can be given by Weinstein handle attachments, according to the Kirby diagram described in [29].)

Finally, the existence of the desired contactomorphism (listed under (3) above) will be shown by using the classification of tight contact structures on the boundary 3-manifolds.

## 2 Convex neighborhoods of certain plumbings

We prove a slightly stronger result than is needed in the proof of our main result. More precisely, we will consider a more general class of graphs than  $\mathcal{G}$ .

**Theorem 2.1.** Suppose that  $(X, \omega)$  is a given symplectic 4-manifold, and the symplectic spheres  $S_i$  intersect each other  $\omega$ -orthogonally and according to a star-shaped graph  $\Gamma$  with negative definite intersection form. Then  $\cup S_i$  admits an  $\omega$ -convex neighbourhood  $S_\Gamma$ .  $\square$

The proof of Theorem 2.1 will easily follow from the main result of this section below. (As customary, we will denote the positive half line  $(0, \infty)$  by  $\mathbb{R}^+$ .)

**Theorem 2.2.** Suppose that  $\Gamma$  is a star-shaped negative definite graph and  $a_i \in \mathbb{R}^+$  ( $i = 1, \dots, |\Gamma|$ ) are given. Then there is a symplectic structure  $\omega_a$  on the plumbing 4-manifold  $M_\Gamma$  associated to the graph  $\Gamma \in \mathcal{G}$  such that

- the spheres  $P_i$  corresponding to the vertices of the plumbing graph are symplectic and  $\omega$ -orthogonal,
- $\int_{P_i} \omega_a = a_i$  and
- any neighbourhood of  $\cup P_i$  contains an  $\omega_a$ -convex neighbourhood.  $\square$

Proof of Theorem 2.1 from Theorem 2.2. Suppose that  $(X, \omega)$  is given with  $S_i \subset X$  intersecting each other according to  $\Gamma$  and take  $a_i = \int_{S_i} \omega$ . Consider the plumbing 4-manifold  $M_\Gamma$  corresponding to  $\Gamma$  and equip  $M_\Gamma$  with the symplectic structure  $\omega_a$  provided by Theorem 2.2. By Moser's Theorem  $\cup S_i \subset (X, \omega)$  and  $\cup P_i \subset (M_\Gamma, \omega_a)$  admit symplectomorphic neighborhoods (cf. also [30, Proposition 3.5]), which, by the third conclusion of Theorem 2.2 concludes the proof.  $\blacksquare$

The idea of the proof of Theorem 2.2 is to construct the desired neighbourhood for each of the legs independently, using toric techniques as in [30], then to construct

the desired neighbourhood for the central vertex, and then to glue the pieces together carefully. These techniques will certainly generalize to larger classes of graphs, but here we focus on star-shaped graphs. Also, there should be a way to state these results in terms of embedded graphs in  $\mathbb{R}^3$  with edges of rational slope, inspired by the “tropical” ideas of Mikhalkin, but in these simple cases it is easier to avoid this perspective.

It is convenient to work with 5-tuples of the form  $(X, \omega, C, f, V)$  where:

- (1)  $X$  is a 4-manifold,
- (2)  $\omega$  is a symplectic form on  $X$ ,
- (3)  $C$  is a collection of symplectically embedded surfaces in  $X$  intersecting each other  $\omega$ -orthogonally,
- (4)  $f$  is a smooth function  $f: X \rightarrow [0, \infty)$  such that  $f^{-1}(0) = C$  and  $f$  has no critical values in  $(0, \infty)$ , and
- (5)  $V$  is a Liouville vector field defined on  $X \setminus C$  such that  $df(V) > 0$ .

If we can produce such a 5-tuple with  $C$  being a collection of spheres intersecting according to the given graph  $\Gamma$  with symplectic areas  $a_i$ , then we will have proved Theorem 2.2. In our proof we will construct one such 5-tuple for each leg of  $\Gamma$  and one for the central vertex, and then glue them together.

Let us call such a 5-tuple a “neighbourhood 5-tuple”. Theorem 2.2 follows from a much more technical statement:

**Proposition 2.3.** Suppose that, for some  $m \in \mathbb{N}$  and some  $n_1, \dots, n_m \in \mathbb{N}$ , we are given points in the upper half plane  $\{P_{i,j} = (x_{i,j}, y_{i,j}) \in \mathbb{R}^2, y_{i,j} > 0 \mid 1 \leq i \leq m, 0 \leq j \leq n_i + 2\}$ , with  $P_{i,j} \neq P_{i,j+1}$ . For each  $i, j$  with  $1 \leq i \leq m$  and  $0 \leq j \leq n_i + 1$ , let  $L_{i,j}$  be the line containing  $P_{i,j}$  and  $P_{i,j+1}$  and let  $E_{i,j}$  be the compact oriented line segment from  $P_{i,j}$  to  $P_{i,j+1}$ . Now require that each  $L_{i,j}$  has rational or infinite slope, so that there is a well-defined primitive integral tangent vector  $\tau_{i,j} = (u_{i,j}, v_{i,j})^T \in \mathbb{Z}^2$  parallel to  $E_{i,j}$  pointing from  $P_{i,j}$  to  $P_{i,j+1}$ . (Here “primitive” means that  $\gcd(u_{i,j}, v_{i,j}) = 1$  with  $\gcd(1, 0)$  defined to be 1.) Record the positive real numbers  $\lambda_{i,j}$  such that the vector from  $P_{i,j}$  to  $P_{i,j+1}$  is equal to  $\lambda_{i,j}\tau_{i,j}$ . Suppose there is a fixed constant  $y_0 > 0$  such that  $y_{i,0} = y_0$  for all  $i = 1, \dots, m$  and suppose that  $x_{1,0} + x_{2,0} + \dots + x_{m,0} > 0$ . Suppose furthermore that the following conditions are satisfied for each  $i = 1, \dots, m$ :

- (1)  $\tau_{i,0} = (1, 0)^T$ ; i.e.  $x_{i,0} < x_{i,1}$  and  $y_{i,0} = y_{i,1}$ .
- (2)  $L_{i,n_i+1}$  passes through the origin  $O = (0, 0)$ ; i.e.  $(x_{i,n_i+2}, y_{i,n_i+2}) = \rho(x_{i,n_i+1}, y_{i,n_i+1})$  for some  $\rho \in \mathbb{R}^+$ .

(3) For each  $j = 0, \dots, n_i$ ,  $\det(\tau_{i,j}, \tau_{i,j+1}) = +1$ .

(4) For each  $j = 0, \dots, n_i$ ,  $\det(\tau_{i,j}, P_{i,j}^T) > 0$ .

Then there exists a neighbourhood 5-tuple  $(X, \omega, C, f, V)$  such that  $C$  is a collection of symplectically embedded 2-spheres  $S_0, S_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n_i$  intersecting  $\omega$ -orthogonally according to an  $m$ -legged star-shaped graph with  $S_0$  the central vertex intersecting each  $S_{i,1}$  and with  $S_{i,1}, \dots, S_{i,n_i}$  the  $i$ 'th leg. Furthermore, the areas and self-intersections are given as follows:

(1) For each  $i = 1, \dots, m$  and each  $j = 1, \dots, n_i$ ,  $S_{i,j} \cdot S_{i,j} = -\det(\tau_{i,j-1}, \tau_{i,j+1})$ , and the area of  $S_{i,j}$  is  $2\pi\lambda_{i,j}$ .

(2) Noting that, for each  $i = 1, \dots, m$ ,  $\tau_{i,1} = (u_{i,1}, 1)^T$ , we have that  $S_0 \cdot S_0 = u_{1,1} + u_{2,1} + \dots + u_{m,1}$ .

(3) The area of  $S_0$  is  $2\pi(x_{1,1} + x_{2,1} + \dots + x_{m,1})$ . □

Proof. Throughout this proof, identify  $S^1$  with  $\mathbb{R}/2\pi\mathbb{Z}$  through the  $\mathbb{R}/2\pi\mathbb{Z}$ -valued coordinate function  $q$  (possibly decorated with subscripts); i.e.  $\int_{S^1} dq = 2\pi$ .

To build the neighbourhood 5-tuples for the legs, we give a detailed description of the toric setup as we need it and the basic results that come from the techniques in [30, 31]. (We are essentially ignoring the global group action point of view that is fundamental to traditional toric geometry, and taking the more local topological point of view developed by Symington. Also, here we make essential use of the correspondence between radial vector fields on images of moment maps and Liouville vector fields on the domains of moment maps; this is not standard in the toric geometry literature but is critical to Symington's work and ours.)

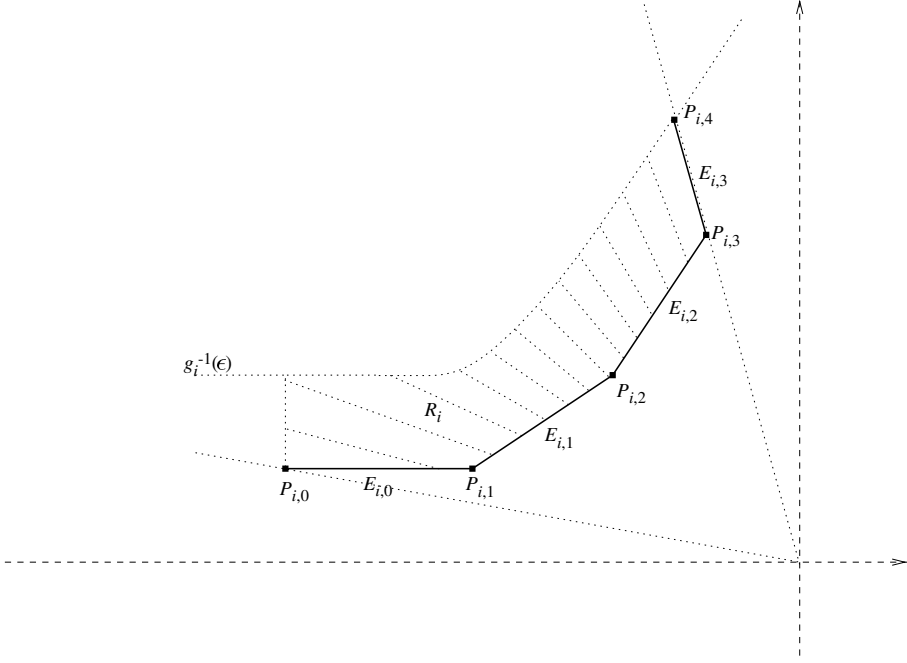
For each  $i = 1, \dots, m$ , the following is the construction of a neighbourhood 5-tuple  $(X_i, \omega_i, C_i, f_i, V_i)$  for the  $i$ 'th leg. Here  $C_i$  will be  $D_{i,0} \cup S_{i,1} \cup S_{i,2} \cup \dots \cup S_{i,n_i}$  where  $D_{i,0}$  is a small open disk neighbourhood in  $S_0$  of  $S_0 \cap S_{i,1}$ . An example is illustrated in Figure 8.

For each  $j = 0, \dots, n_i + 1$ , let  $H_{i,j}$  be the closed half-plane with  $\partial H_{i,j} = L_{i,j}$  and not containing the origin  $O$ , and consider the convex region  $H_i = H_{i,0} \cap \dots \cap H_{i,n_i+1}$ . Choose a smooth function  $g_i: H_i \rightarrow [0, \infty)$  with no critical values in  $(0, \infty)$  such that:

(1)  $g_i^{-1}(0) = (L_{i,0} \cup L_{i,1} \cup \dots \cup L_{i,n_i}) \cap H_i$ .

(2) For some small  $\epsilon > 0$  (smaller than  $x_{i,1} - x_{i,0}$ ),  $g_i$  restricted to  $(-\infty, x_{i,0} + \epsilon] \times [y_{i,0}, y_{i,0} + \epsilon]$  is given by  $g_i(x, y) = y - y_{i,0}$ .

(3) On a small neighbourhood of  $L_{i,n_i+1} \cap H_i$ ,  $g_i(x, y)$  is equal to the Euclidean distance from  $(x, y)$  to  $L_{i,n_i}$ .



**Figure 8** An example of a two-dimensional “moment-map image” of the neighbourhood for one leg of a star-shaped plumbing graph. Here  $n_i = 2$ .

- (4) The level sets of  $g_i$  are transverse to the radial vector field radiating out from the origin; i.e.  $dg_i(x\partial_x + y\partial_y) > 0$  whenever  $dg_i \neq 0$ .

Finally consider the region  $R_i = g_i^{-1}[0, \epsilon) \setminus ((-\infty, x_{i,0}] \times \mathbb{R})$ , and relabel things so that  $E_{i,0} \cap R_i$  is now called  $E_{i,0}$  and  $E_{i,n_i+1} \cap R_i$  is now called  $E_{i,n_i+1}$ . (These are the initial and final edges of the polygonal boundary of  $R_i$ , and they are bounded but noncompact, while the other edges  $E_{i,1}, \dots, E_{i,n}$  are compact.) Also let  $P_{i,n_i+2}$  now denote the new terminal point of  $E_{i,n_i+1}$ , so that  $E_{i,n_i+1}$  is the half-open interval from  $P_{i,n_i+1}$  to  $P_{i,n_i+2}$ , containing  $P_{i,n_i+1}$  and not  $P_{i,n_i+2}$ . This 2-dimensional data is the “template” for our neighbourhood 5-tuple, which we now describe in some detail.

Let  $Y_i = \{(p_1, q_1, p_2, q_2) \in \mathbb{R} \times S^1 \times \mathbb{R} \times S^1 \mid (p_1, p_2) \in R_i\}$  with symplectic form  $\eta_i = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$  and let  $\pi_i: Y_i \rightarrow R_i$  be the projection  $(p_1, q_1, p_2, q_2) \mapsto (p_1, p_2)$ . For each edge  $E_{i,j}$  let  $Y_{i,j} = [0, \lambda_{i,j}] \times S^1$  with coordinates  $(p, q)$  and symplectic form  $\eta_{i,j} = dp \wedge dq$  and let  $\pi_{i,j}: Y_{i,j} \rightarrow E_{i,j}$  be the projection defined by  $(p, q) \mapsto (x_{i,j} + pu_{i,j}, y_{i,j} + pv_{i,j})$ . Also, for each  $j$ , consider the map  $\psi_{i,j}: \pi_i^{-1}(E_{i,j}) \rightarrow Y_{i,j}$  defined by  $(p_1, q_1, p_2, q_2) \mapsto (p, q)$  where  $p$  is such that  $(p_1, p_2) = (x_{i,j} + pu_{i,j}, y_{i,j} + pv_{i,j})$  and  $q = u_{i,j}q_1 + v_{i,j}q_2$ . In other words, the  $S^1 \times S^1$  in  $Y_i$  above a point  $(p_1, p_2) \in E_{i,j}$  collapses to the  $S^1$  in  $Y_{i,j}$  above  $(p_1, p_2)$ , with



the  $(-v_{i,j}, u_{i,j})$ -curves in  $S^1 \times S^1$  being the curves that collapse to points. Note that on  $\pi_i^{-1}(E_{i,j})$  we have  $\pi_i = \pi_{i,j} \circ \psi_{i,j}$ . Then there exists a unique symplectic 4-manifold  $(X_i, \omega_i)$  equipped with the following smooth maps:

- (1) A map  $\mu_i: X_i \rightarrow \mathbb{R}^2$  with  $\mu_i(X_i) = R_i$ . (This is the “moment map” for a Hamiltonian torus action on  $(X_i, \omega_i)$ , after identifying  $\mathbb{R}^2$  with the dual of the Lie algebra of  $S^1 \times S^1$ .)
- (2) A map  $\phi_i: Y_i \rightarrow X_i$  such that  $\pi_i = \mu_i \circ \phi_i$  and such that  $\phi_i$  is a symplectomorphism from  $(\pi_i^{-1}(R_i \setminus \partial R_i), \eta_i)$  to  $(\mu_i^{-1}(R_i \setminus \partial R_i), \omega_i)$ . We think of this as giving us coordinates  $(p_1, q_1, p_2, q_2)$  on  $X_i$  with respect to which  $\omega_i = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ , except that the  $S^1$ -valued coordinates  $(q_1, q_2)$  degenerate along  $\mu_i^{-1}(\partial R_i)$  in various ways determined by the next set of maps.
- (3) A map  $\phi_{i,j}: Y_{i,j} \rightarrow \mu_i^{-1}(E_{i,j})$ , for each  $j = 0, \dots, n_i + 1$ , such that  $\pi_{i,j} = \mu_i \circ \phi_{i,j}$  and such that  $\phi_{i,j}$  restricts to a symplectomorphism from  $(\pi_{i,j}^{-1}(E_{i,j} \setminus \partial E_{i,j}), \eta_{i,j})$  to  $(\mu_i^{-1}(E_{i,j} \setminus \partial E_{i,j}), \omega_i)$ .

In addition, for each point  $P_{i,j} \in R_i$ ,  $\mu_i^{-1}(P_{i,j})$  is a single point. The following are consequences of this fact and the relationships amongst these maps:

- (1) For each point  $P \in R_i \setminus \partial R_i$ ,  $\mu_i^{-1}(P)$  is a torus.
- (2) For each point  $P \in E_{i,j} \setminus \partial E_{i,j}$ ,  $\mu_i^{-1}(P)$  is a circle.
- (3) For each compact edge  $E_{i,j}$ ,  $j = 1, \dots, n_i$ ,  $\mu_i^{-1}(E_{i,j})$  is a symplectically embedded 2-sphere  $S_{i,j}$  with symplectic area  $2\pi\lambda_{i,j}$ .
- (4) For each of the two noncompact edges  $E_{i,0}$  and  $E_{i,n_i+1}$ ,  $D_{i,j} = \mu_i^{-1}(E_{i,j})$  is a symplectically embedded open disk with symplectic area  $2\pi\lambda_{i,j}$  ( $j = 0$  or  $n_i + 1$ ).
- (5) Each point  $\mu_i^{-1}(P_{i,j})$ , for  $j = 1, \dots, n_i + 1$ , is a point of  $\omega$ -orthogonal intersection between  $\mu_i^{-1}(E_{i,j-1})$  and  $\mu_i^{-1}(E_{i,j})$ .
- (6) Given an embedded arc  $\gamma: [0, 1] \rightarrow R_i$  meeting  $\partial R_i$  transversely at one point  $\gamma(1)$  in the interior of an edge  $E_{i,j}$ , the submanifold  $\mu_i^{-1}(\gamma([0, 1]))$  is a solid torus naturally parameterized using the coordinates  $(t, q_1, q_2)$  as  $[0, 1] \times S^1 \times S^1$ , with  $\{1\} \times S^1 \times S^1$  collapsed to  $S^1$  so that the  $(-v_{i,j}, u_{i,j})$  curves collapse to points.
- (7) Each 2-sphere  $S_{i,j} = \mu_i^{-1}(E_{i,j})$ , for  $j = 1, \dots, n_i$ , has self-intersection  $S_{i,j} \cdot S_{i,j} = -\det(\tau_{i,j-1}, \tau_{i,j+1})$ . This can be seen by using the preceding point to understand the topology of the  $S^1$ -bundle over  $S^2$  which is the boundary of a tubular neighborhood of  $S_{i,j}$ . This bundle is the inverse image

under  $\mu_i$  of a parallel copy of  $E_{i,j}$ , translated into the interior of  $R_i$  and extended until it intersects the two adjacent edges  $E_{i,j-1}$  and  $E_{i,j+1}$ . Thus the bundle is the result of Dehn filling the two boundary components of  $[0, 1] \times S^1 \times S^1$  according to the rule in the preceding point, which gives an Euler class of  $-\det(\tau_{i,j-1}, \tau_{i,j+1})$ .

Furthermore, focusing attention on the end  $\mu_i^{-1}((x_{i,0}, x_{i,0} + \epsilon) \times [y_{i,0}, y_{i,0} + \epsilon])$ , which is diffeomorphic to  $(x_{i,0}, x_{i,0} + \epsilon) \times S^1 \times D$ , where  $D$  is an open disk in  $\mathbb{R}^2$  of an appropriate radius, everything can be written explicitly as follows, using coordinates  $t \in (x_{i,0}, x_{i,0} + \epsilon)$ ,  $\alpha \in S^1$  and polar coordinates  $(r, \theta) \in D$ :

- (1)  $p_1 = t, q_1 = \alpha, p_2 = y_{i,0} + \frac{1}{2}r^2, q_2 = \theta$ , so that  $\mu_i(t, \alpha, r, \theta) = (t, y_{i,0} + \frac{1}{2}r^2)$ .
- (2)  $\omega_i = dt \wedge d\alpha + r dr \wedge d\theta$ .
- (3)  $f_i(t, \alpha, r, \theta) = \frac{1}{2}r^2$ .

Finally, to get the desired convexity, the radial vector field on  $\mathbb{R}^2$  radiating out from the origin lifts to a Liouville vector field  $V_i$  defined on  $X_i \setminus \pi_i^{-1}(E_{i,0} \cup \dots \cup E_{i,n})$  which is, in  $(p_1, q_1, p_2, q_2)$  coordinates, given by  $V_i = p_1 \partial_{p_1} + p_2 \partial_{p_2}$ . (The fact that  $V_i$  is defined on  $\pi_i^{-1}(E_{i,n_i+1})$  comes from the fact that the line  $L_{i,n_i+1}$  passes through  $O$ , so that  $E_{i,n_i+1}$  is tangent to the radial vector field.) On the end  $\mu_i^{-1}((x_{i,0}, x_{i,0} + \epsilon) \times [y_{i,0}, y_{i,0} + \epsilon]) \cong (x_{i,0}, x_{i,0} + \epsilon) \times S^1 \times D$ , with coordinates  $(t, \alpha, r, \theta)$  as above,  $V_i$  is given by  $V = t \partial_t + (\frac{1}{2}r + \frac{y_{i,0}}{r}) \partial_r$ . ■

Thus we see that  $(X_i, \omega_i, C_i = \mu_i^{-1}(E_{i,0} \cup \dots \cup E_{i,n_i}), f_i = g_i \circ \mu_i, V_i)$  is the desired neighbourhood 5-tuple for the legs and that, furthermore, we have everything written down explicitly in local coordinates on the end of  $X_i$  which is a neighbourhood of the disk  $D_{i,0}$ .

Now we construct the neighbourhood 5-tuple for the central vertex using the following lemma (which we have stated in a form which allows also for positive genus, in case it should ever be useful):

**Lemma 2.4.** Suppose that we are given a compact connected surface  $\Sigma$  with  $m > 0$  boundary components  $\partial_1 \Sigma, \dots, \partial_m \Sigma$ ,  $k \geq 0$  negative real numbers  $c_1, \dots, c_k$ , and  $m - k > 0$  positive real numbers  $c_{k+1}, \dots, c_m$ , with  $c_1 + \dots + c_k + c_{k+1} + \dots + c_m > 0$ . Then there exists a symplectic form  $\beta$  on  $\Sigma$  and a Liouville vector field  $W$  defined on all of  $\Sigma$ , pointing in along  $\partial_i \Sigma$  for  $1 \leq i \leq k$  and pointing out along  $\partial_i \Sigma$  for  $k + 1 \leq i \leq m$ , such that, for each  $i = 1, \dots, m$ , a collar neighbourhood of  $\partial_i \Sigma$  can be parameterized as  $(c_i - \epsilon, c_i] \times S^1$  with coordinates  $(t, \alpha)$  with respect to which  $\beta = dt \wedge d\alpha$  and  $W = t \partial_t$ . □

**Proof.** Note that the condition that  $c_1 + \dots + c_m > 0$  is a necessary condition because  $2\pi(c_1 + \dots + c_m) = \int_{\partial \Sigma} t d\alpha = \int_{\partial \Sigma} \iota_W \beta = \int_{\Sigma} \beta > 0$ . There are probably numerous ways

to see that this lemma is true; in fact we need only extend the 1-forms  $td\alpha$  on each end to a 1-form  $\gamma$  on all of  $\Sigma$  such that  $d\gamma > 0$ . Our idea here is to apply Weinstein's techniques [33] for attaching symplectic handles to symplectic manifolds with convex boundary in the relatively trivial case of dimension 2; in this case the "contact forms" on the 1-dimensional boundary components are nowhere zero 1-forms. Once we get the 1-forms on the boundary correct, then flowing in along the Liouville vector fields produces the correct parameterization of the ends.

First we claim that the lemma is true if  $m - k = 1$ , in which case the vector field should point out along only one boundary component. Start with the disjoint union  $\coprod_{i=1}^k ([c_i - \epsilon, c_i] \times S^1, dt \wedge d\alpha)$  of  $k$  symplectic cylinders with the Liouville vector field  $t\partial_t$  (or a very small disk if  $k = 0$  with the radial vector field) and attach Weinstein's 1-handles to the boundaries along which the Liouville vector fields point out, to make a symplectic surface diffeomorphic to  $\Sigma$ , with a Liouville vector field with the correct behavior along  $\partial_1\Sigma, \dots, \partial_k\Sigma$  and pointing out along  $\partial_m\Sigma$ . Now simply attach a symplectization collar to  $\partial_m\Sigma$  if  $\int_{\partial_m\Sigma} \iota_W\beta$  is too small, or remove a collar (by flowing backwards along  $W$ ) if it is too big, and thus achieve the correct behavior along  $\partial_m\Sigma$ .

Now if  $m - k > 1$ , we can first build a symplectic surface  $(\Sigma', \beta')$  with the same genus as  $\Sigma$  and  $k+1$  boundary components, with a Liouville vector field  $W$ , with  $k$  boundary components along which  $W$  points in and the behavior is correct, and one boundary component  $\partial_+$  along which the vector field points out, arranging that  $\int_{\partial_+} \iota_{W'}\beta' < 2\pi(c_{k+1} + \dots + c_m)$ . Then use the same ideas as in the preceding paragraph to build a  $(m - k + 1)$ -punctured sphere with one boundary component  $\partial_-$  along which the Liouville vector field points in, so as to be able to glue to  $\partial_+$ , and with  $m - k$  components along which the vector field points out with the correct behavior. Then glue the two pieces together.

The neighbourhood 5-tuple needed for the central vertex is now  $(X_0 = \Sigma \times D, \omega_0 = \beta + \frac{1}{2\pi}rdr \wedge d\theta, C_0 = \Sigma \times \{(0, 0)\}, f_0 = \frac{1}{2}r^2, V_0 = W + (\frac{1}{2}r + \frac{Y_0}{r})\partial_r)$ . Because  $y_0 = Y_{i,0}$  for each  $i = 1, \dots, m$ , if we implement the above lemma using  $c_i = x_{i,0} + \delta$  for some small  $\delta > 0$ , and for  $\Sigma$  an  $m$ -punctured sphere, we will be able to glue the 5-tuples together. The area of the central sphere  $S_0$  will then be  $2\pi(x_{1,1} + x_{2,1} + \dots + x_{m,1})$  by Stokes' theorem.

It remains to compute  $S_0 \cdot S_0$ . To do this note that, for some small  $\delta > 0$ , the submanifold  $N$  of  $X$  defined by

$$N = (\Sigma \times D_{\sqrt{2\delta}}) \cup \bigcup_{i=1}^m \mu_i^{-1}(R_i \cap ((x_{i,0}, \infty) \times [y_{i,0}, y_{i,0} + \delta)))$$

(where  $D_{\sqrt{2\delta}}$  is an open disk of radius  $\sqrt{2\delta}$ ) is a tubular neighbourhood of  $S_0$ . Then we can see  $\partial N$  as diffeomorphic to  $\Sigma \times S^1$  with the  $i$ 'th component  $(\partial_i \Sigma) \times S^1$  of  $\partial(\Sigma \times S^1)$  filled in with a solid torus so that the  $(-v_{i,1}, u_{i,1})$  curves in  $(\partial_i \Sigma) \times S^1 = S^1 \times S^1$  bound disks. But since  $v_{i,1} = 1$  this means that the  $(1, -u_{i,1})$  curves are filled, so this 3-manifold is the  $S^1$ -bundle over  $S^2$  of Euler class  $-u_{1,1} - u_{2,1} - \dots - u_{m,1}$ . Since the base  $S^2$  in this case is precisely the central vertex sphere  $S_0$ , this tells us that  $S_0 \cdot S_0 = -u_{1,1} - \dots - u_{m,1}$ . ■

Now we need to translate Proposition 2.3 into a proof of Theorem 2.2.

**Proof of Theorem 2.2.** Given a star-shaped plumbing graph  $\Gamma$ , label the spheres  $S_0, S_{i,j}$ ,  $1 \leq i \leq m, 1 \leq j \leq n_i$ , so that  $S_0$  corresponds to the central  $m$ -valent vertex of  $\Gamma$  and  $S_{i,1}, \dots, S_{i,n_i}$  correspond in order to the vertices on the  $i$ 'th leg, with  $S_0 \cdot S_{i,1} = 1$ . Let the corresponding self-intersections be  $s_0, s_{i,j}$  and let the areas be  $a_0, a_{i,j}$ . We need to produce the points  $P_{i,j}$ ,  $1 \leq i \leq m, 0 \leq j \leq n_i + 2$  satisfying the conditions of Proposition 2.3, giving the desired self-intersections and areas.

We first consider the vectors  $\tau_{i,j} \in \mathbb{Z}^2$ ,  $1 \leq i \leq m, 0 \leq j \leq n_i + 1$ . We are forced to have  $\tau_{i,0} = (1, 0)^T$  and  $\tau_{i,1} = (u_{i,1}, 1)^T$ . We can make a choice for the values of  $u_{i,1}$  so long as  $u_{1,1} + \dots + u_{m,1} = -s_0$ , so that  $S_0 \cdot S_0 = -u_{1,1} - \dots - u_{m,1} = s_0$ . Now note that each  $\tau_{i,j+1}$  for  $j = 1, \dots, n_i$  is completely determined by  $\tau_{i,j-1}$  and  $\tau_{i,j}$  and the constraints that  $\det(\tau_{i,j-1}, \tau_{i,j+1}) = -s_{i,j}$  and  $\det(\tau_{i,j}, \tau_{i,j+1}) = +1$ , so that our choices for  $\tau_{i,0}$  and  $\tau_{i,1}$  determine  $\tau_{i,2}, \dots, \tau_{i,n_i+1}$ . In fact,  $\tau_{i,j+1} = -\tau_{i,j-1} - s_{i,j}\tau_{i,j}$ .

For each  $i = 1, \dots, m$ , let  $\sigma_i = u_{i,n_i+1}/v_{i,n_i+1}$ , the reciprocal of the slope of the terminal vector  $\tau_{i,n_i+1}$ . We claim that if  $\Gamma$  is negative definite then, for any choice of  $u_{1,1}, u_{2,1}, \dots, u_{m,1}$  such that  $u_{1,1} + u_{2,1} + \dots + u_{m,1} = -s_0$ , we will automatically have  $\sigma_1 + \dots + \sigma_m > 0$ . Given this claim, the existence of suitable points  $P_{i,j}$  can be seen as follows:

Having determined the vectors  $\tau_{i,j}$ , the points  $P_{i,j}$  for  $i = 1, \dots, m, j = 2, \dots, n_i + 1$  are completely determined by the  $P_{i,1}$ 's and the areas  $a_{i,j}$ . Furthermore, the fact that  $P_{i,n_i+1}$  must lie on the line  $L_{i,n_i+1}$  passing through the origin tangent to the vector  $\tau_{i,n_i+1}$  constrains  $P_{i,1}$  to lie on a particular line parallel to  $\tau_{i,n_i+1}$ ; i.e. there is some constant  $K_i$ , determined by the  $\tau_{i,j}$ 's and the  $a_{i,j}$ 's, such that  $x_{i,1} = \sigma_i y_{i,1} + K_i$ . Because the points  $P_{i,j}$  must all lie to the left of the line  $L_{i,n_i+1}$ , which passes through the origin, we know that  $K_i < 0$ . The precise location of  $P_{i,n_i+2}$  on  $L_{i,n_i+1}$  is not important, so we ignore it. The precise location of  $P_{i,0}$  is also not important, provided it is close enough to  $P_{i,1}$  and provided  $x_{1,1} + x_{2,1} + \dots + x_{m,1} > 0$ . Recall that we have the additional constraint that  $y_{i,1} = y_{i,0} = y_0$  for some  $y_0$ ; thus we can think of each  $x_{i,1}$  as determined by a linear function of  $y_0$ :  $x_{i,1} = \sigma_i y_0 + K_i$ . We must now choose  $y_0$  appropriately so that

$2\pi(x_{1,1} + x_{2,1} + \cdots + x_{m,1}) = a_0$ , the given area of  $S_0$ . We have  $x_{1,1} + x_{2,1} + \cdots + x_{m,1} = (\sigma_1 + \cdots + \sigma_m)y_0 + (K_1 + \cdots + K_m)$ . But since  $\sigma_1 + \cdots + \sigma_m > 0$  and  $K_1 + \cdots + K_m < 0$ , we can realize any given positive value for  $x_{1,1} + \cdots + x_{m,1}$  by choosing an appropriate  $y_0 > 0$ .

Now to prove the claim, we first appeal to [18, Theorem 5.2] stating that a star-shaped graph  $G$  is negative definite if and only if  $s_0 + r_1 + \cdots + r_m < 0$ , where  $r_i$  is the negative reciprocal of the continued fraction corresponding to the  $i$ 'th leg, i.e.

$$r_i = -\frac{1}{s_{i,1} - \frac{1}{s_{i,2} - \cdots - \frac{1}{s_{i,n_i}}}} \quad (2.1)$$

In the proof of [32, Theorem 9.20] Symington shows that if  $\tau_0, \dots, \tau_{n+1}$ ,  $\tau_i = (u_i, v_i)^T$ , is a list of vectors in  $\mathbb{Z}^2$  with  $\tau_0 = (0, -1)^T$ ,  $\tau_1 = (1, 0)^T$ ,  $\det(\tau_i, \tau_{i+1}) = 1$  and  $\det(\tau_{i-1}, \tau_{i+1}) = b_i$ , then the reciprocal slope  $\sigma$  of the last vector is given by

$$\sigma = \frac{u_{n+1}}{v_{n+1}} = b_1 - \frac{1}{b_2 - \cdots - \frac{1}{b_n}}$$

We can change such a list to one where  $\tau_0 = (1, 0)^T$  and  $\tau_1 = (u, 1)^T$  for some given  $u \in \mathbb{Z}$  by applying the linear transformation

$$\tau_i \mapsto \begin{pmatrix} u & -1 \\ 1 & 0 \end{pmatrix} \tau_i$$

This transforms  $\sigma$  to  $u - 1/\sigma$ . In other words, after applying this transformation,  $\sigma$  is now given by:

$$\sigma = u - \frac{1}{b_1 - \frac{1}{b_2 - \cdots - \frac{1}{b_n}}}$$

Applying this to each leg of our star-shaped graph, with the initial condition  $\tau_{i,1} = (u_{i,1}, 1)^T$  and  $\det(\tau_{i,j-1}, \tau_{i,j+1}) = -s_{i,j}$  we get that

$$\sigma_i = u_{i,1} + \frac{1}{s_{i,1} - \frac{1}{s_{i,2} - \cdots - \frac{1}{s_{i,n_i}}}} = u_{i,1} - r_i$$

Thus  $\sigma_1 + \cdots + \sigma_m = -(s_0 + r_1 + \cdots + r_m)$  and the claim is proved.  $\blacksquare$

**Remark 2.5.** Since Lemma 2.4 allows for any genus, Theorem 2.1 is actually true even when the surface corresponding to the central vertex has positive genus. It seems that

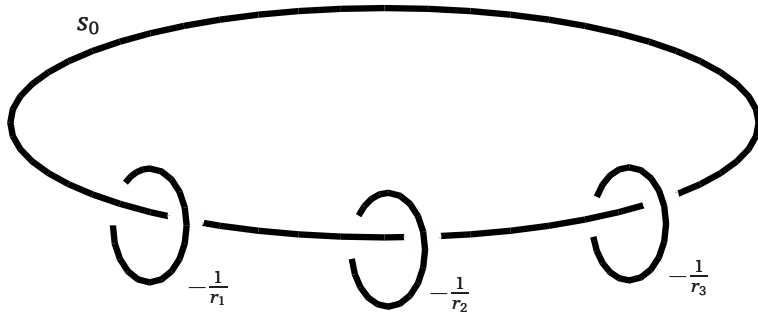
a similar picture provides  $\omega$ -convex neighbourhood for any collection of symplectic spheres intersecting each other according to a *negative definite* plumbing tree, i.e. the assumption of Theorem 2.2 on  $\Gamma$  being star-shaped can be removed. We hope to return to this question in a future project.

### 3 Tight contact structures on certain Seifert fibered 3-manifolds

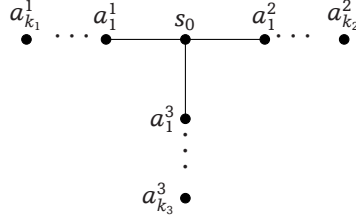
In order to find the right gluing map for performing the rational blow-down process, we will invoke a classification result of tight contact structures on certain small Seifert fibered 3-manifolds. Let us start with some generalities.

As is customary, we say that a 3-manifold  $Y$  is a *small Seifert fibered space* if it can be given by the surgery diagram of Figure 9. Here we assume that  $s_0 \in \mathbb{Z}$  and  $r_i \in (0, 1) \cap \mathbb{Q}$  (then  $(s_0; r_1, r_2, r_3)$  are the *normalized Seifert invariants* of  $M = M(s_0; r_1, r_2, r_3)$ ). By applying the inverse slam dunk operation (cf. [11, Figure 5.30]), the diagram of Figure 9 can be easily transformed into a star-shaped plumbing as it is given by Figure 10. The plumbing coefficients  $[a_1^i, \dots, a_{k_i}^i]$  on the  $i^{\text{th}}$  leg (satisfying  $a_j^i \in \mathbb{Z}$  and  $a_j^i \leq -2$ ) are specified by the continued fraction coefficients of the rational number  $-\frac{1}{r_i} < -1$  (cf. Equation (2.1)). Notice that all plumbing graphs in the Introduction (Figures 1–7) give rise to 4-manifolds with small Seifert fibered 3-manifold boundary.

In a remarkable series of papers [20, 21] Ozsváth and Szabó introduced an invariant for  $\text{spin}^c$  3-manifolds: the (mod 2) *Ozsváth-Szabó homology group*  $\widehat{HF}(Y, \mathfrak{t})$  of a closed  $\text{spin}^c$  3-manifold  $(Y, \mathfrak{t})$  is a finite dimensional vector space over the field  $\mathbb{Z}_2$  of two elements. For a rational homology sphere the dimension of this vector space is odd for every  $\text{spin}^c$  structure. We say that a rational homology sphere  $Y$  is an *L-space* if



**Figure 9** Surgery diagram for the Seifert fibered 3-manifold  $M(s_0; r_1, r_2, r_3)$ .



**Figure 10** Plumbing diagram of a 4-manifold with boundary  $M(s_0; r_1, r_2, r_3)$ .

$\widehat{HF}(Y, \mathbf{t}) = \mathbb{Z}_2$  for every  $\text{spin}^c$  structure  $\mathbf{t} \in \text{Spin}^c(Y)$ , that is,  $Y$  has the simplest possible Ozsváth-Szabó homologies.

Recall that a contact structure  $\xi$  on any 3-manifold naturally induces a  $\text{spin}^c$  structure, which we will denote by  $\mathbf{t}_\xi$ . In addition,  $(Y, \xi)$  also gives rise to an element  $c(Y, \xi) \in \widehat{HF}(-Y, \mathbf{t}_\xi)$ , the *contact invariant* of  $(Y, \xi)$ . (For the definition and basic properties of  $c(Y, \xi)$  see [23, 14].)

As the next result (a compilation of theorems of Wu, Ghiggini and Plamenevskaya) shows, tight contact structures on small Seifert fibered 3-manifolds with simple Ozsváth-Szabó homologies (and  $s_0 \leq -2$ ) admit a simple classification scheme. More precisely

**Theorem 3.1.** Suppose that the small Seifert fibered 3-manifold  $M = M(s_0; r_1, r_2, r_3)$  satisfies  $s_0 \leq -2$  and  $M$  is an  $L$ -space. Then two tight contact structures  $\xi_1, \xi_2$  on  $M$  are isotopic if and only if  $\mathbf{t}_{\xi_1} = \mathbf{t}_{\xi_2}$ .  $\square$

*Proof.* One direction of the statement is obvious: isotopic contact structures are homotopic as 2-plane fields, hence induce the same  $\text{spin}^c$  structure. The converse, however, is more subtle, since it states that in these 3-manifolds the  $\text{spin}^c$  structure (which is a homotopic invariant of the contact structure) determines the tight contact structure up to isotopy.

Let  $\Gamma$  denote the star-shaped plumbing graph we get from the surgery diagram of  $M = M(s_0; r_1, r_2, r_3)$  by inverse slam dunks. The corresponding 4-manifold  $Z_\Gamma$  is then given by a sequence of 2-handle attachments along the unknots corresponding to the vertices of  $\Gamma$ . Working with a projection now it is easy to see that each unknot can be isotoped until it becomes the Legendrian unknot, i.e., a Legendrian knot isotopic to the unknot with Thurston-Bennequin invariant  $-1$ . The assumption  $s_0 \leq -2$  ensures that by adding sufficiently many zig-zags to these Legendrian unknots we end up with a Legendrian link on which contact  $(-1)$ -surgery yields  $Z_\Gamma$  together with a Stein structure on it [1, 10]. Notice, however, that there is a certain freedom in adding the zig-zags to the unknots:

moving a zig-zag from left to right (or from right to left) will change the rotation number of the particular knot, which in turn will change the first Chern class of the resulting Stein structure on  $Z_\Gamma$ . According to a result of Lisca and Matić [13] Stein structures on a fixed 4-manifold with different first Chern classes induce nonisotopic tight contact structures on the boundary of the 4-manifold. The number of tight contact structures one can distinguish in this way can be easily computed from the framing coefficients of the graph  $\Gamma$ . Furthermore, it was shown by Plamenevskaya [24] that if two Stein structures on a 4-manifold induce two contact structures  $\xi_1, \xi_2$ , and for the two Stein structures we have  $c_1(J_1) \neq c_1(J_2)$  then for the (mod 2 reduced) contact Ozsváth-Szabó invariants we have  $c(M, \xi_1) \neq c(M, \xi_2)$ . In summary, the Legendrian surgery construction described above allowed us to construct a finite set of Stein fillable (hence tight) contact structures on  $M$  (corresponding to the different choices of left and right zig-zags), and all these structures have different (and nonzero) contact Ozsváth-Szabó invariants.

Now we are ready to prove the theorem. Using convex surface techniques, it was shown by Wu [34] that for a small Seifert 3-manifold with  $s_0 \leq -3$  the above set of Stein fillable contact structures, in fact, contains *all* tight contact structures on the 3-manifold at hand. (For these 3-manifolds the  $L$ -space condition is automatically satisfied, cf. Lemma 4.1.) For the remaining  $s_0 = -2$  case we argue as follows: According to [15] the assumption for  $M$  being an  $L$ -space implies that  $-M$  admits no transverse contact structure. In [7] this property is translated to a numerical condition for  $r_1, r_2, r_3$ , which in turn, implies that the number of tight contact structures on  $M$  is bounded above by the number of Stein fillable contact structures constructed by Legendrian surgery along the Legendrian links described above. In short, under the assumption of the theorem, the Legendrian surgery construction described at the beginning of the proof produces all tight contact structures on  $M$ .

Appealing to the  $L$ -space property again, now we can easily conclude the proof: if  $\xi_1, \xi_2$  are nonisotopic tight contact structures on  $M$  then according to the above said we have  $c(M, \xi_1) \neq c(M, \xi_2)$ . Assuming that for the induced  $\text{spin}^c$  structures  $\mathbf{t}_{\xi_1} = \mathbf{t}_{\xi_2} = \mathbf{t}$  holds we easily reach a contradiction: in this case the contact invariants  $c(M, \xi_1), c(M, \xi_2) \in \widehat{HF}(-M, \mathbf{t}) = \mathbb{Z}_2$  are distinct, nonzero elements, which is clearly impossible. ■

**Remark 3.2.** In fact, a very similar proof (together with the classification of tight contact structures on small Seifert fibered 3-manifolds with  $s_0 \geq 0$  [8]) applies for  $s_0 \geq 0$  as well. (It is worth mentioning that if  $s_0(M) \geq 0$  then  $s_0(-M) \leq -3$ , therefore any  $M$



with  $s_0(M) \geq 0$  is still an  $L$ -space.) The only difference is in the construction of the appropriate 4-manifold (carrying the Stein structures): for  $s_0 \geq 0$  we need to introduce an appropriate Stein 1-handle as well, cf. the argument of [8]. We will not use this fact in the present paper. A similar statement (that is, that tight contact structures with isomorphic  $\text{spin}^c$  structures are isotopic) is expected in the case when  $s_0 = -1$  and the 3-manifold is an  $L$ -space; for related discussion and partial results see [9]. Most probably the same statement also holds for *strongly fillable* contact structures on boundaries of negative definite plumbings which are  $L$ -spaces. Considering all tight contact structures, such a statement cannot be true, since many of these 3-manifolds are toroidal, hence contain infinite families of homotopic, nonisomorphic tight contact structures. (These infinite families are constructed by inserting Giroux torsions, resulting in contact structures which are not strongly fillable [6].) For strongly fillable structures, however, one can expect that the strong filling can be chosen to be diffeomorphic to the plumbing 4-manifold, and then an adaptation of the proof above would provide the same result.

#### 4 The proof of Theorem 1.2

We start this section by proving that the result of Theorem 3.1 applies for the plumbing manifolds given by graphs in  $\mathcal{G}$ .

**Lemma 4.1.** Suppose that  $\Gamma \in \mathcal{G}$ . Then the star-shaped graph  $\Gamma$  is negative definite and the 3-manifold  $\partial M_\Gamma$  is an  $L$ -space. □

Proof. The plumbing graph  $\Gamma \in \mathcal{G}$  is negative definite since it embeds in a negative definite lattice [29]. (Alternatively, a simple computation and the application of [18, Theorem 5.2] shows the same.) There are many ways to verify that  $\partial M_\Gamma$  is an  $L$ -space. The direct application of the algorithm of [22] (which applies for negative definite plumbing graphs with at most one “bad” vertex) implies the result at once, although this proof involves some computations. (In fact, for graphs in  $\mathcal{W} \cup \mathcal{N}$  the result is stated in [22], since these graphs involve no “bad” vertices.) Alternatively, we can argue as follows: as [29, Examples 8.3 and 8.4] show, the normal surface singularity defined by  $\Gamma \in \mathcal{G}$  admits a rational homology disk smoothing and this property implies that the surface singularity is rational, cf. [29, Proposition 2.3]. (In yet another way, Laufer’s algorithm can be easily applied to verify rationality of the surface singularities defined by graphs in  $\mathcal{G}$ .) The computation presented in [17] shows that the link of a rational singularity is an  $L$ -space, concluding our argument. ■

After all these preparations, now we are ready to turn to the proof of Theorem 1.2:

**Proof of Theorem 1.2.** Suppose that the  $S_i$  are symplectic spheres in the symplectic 4-manifold  $(X, \omega)$ , intersecting each other  $\omega$ -orthogonally, and according to the plumbing tree  $\Gamma \in \mathcal{G}$ . By Theorem 2.1 there is a neighbourhood  $S_\Gamma$  of  $\cup S_i$  such that its complement is a strong concave filling of its contact boundary. The contact structure on the boundary  $\partial S_\Gamma$  induces the  $\text{spin}^c$  structure which extends to  $S_\Gamma$  as  $\mathbf{s}$  with the property that  $c_1(\mathbf{s})$  satisfies the adjunction equality for all  $S_i$  (since the  $S_i$  are all smooth symplectic submanifolds).

Now consider the normal surface singularity given by  $\Gamma$ . As such, it defines a contact structure on its link, which induces the  $\text{spin}^c$  structure  $\mathbf{t}$ . Resolving the singularity doesn't change this contact structure (and hence the  $\text{spin}^c$  structure  $\mathbf{t}$ ), but provides a 4-manifold diffeomorphic to  $S_\Gamma$ , with a  $\text{spin}^c$  structure  $\mathbf{s}_{can}$ . Since the vertices of  $\Gamma$  in the resolution correspond to complex curves, the first Chern class of  $\mathbf{s}_{can}$  also satisfies the adjunction equality. This shows that  $c_1(\mathbf{s}) = c_1(\mathbf{s}_{can})$ , and since  $S_\Gamma$  is a simply connected 4-manifold, we conclude that  $\mathbf{s} = \mathbf{s}_{can}$ . This fact, however, identifies the  $\text{spin}^c$  structures of the contact structures on  $\partial S_\Gamma$ , which in the light of Theorem 3.1 implies that the contact structure on  $\partial S_\Gamma$  induced by the Liouville vector field is isotopic to the one induced on the link of the singularity. This last observation verifies the existence of the desired gluing contactomorphism, hence the symplectic gluing can be performed as in [3], concluding the proof. ■

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