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# SYMPLECTIC REAL BOTT MANIFOLDS

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ABSTRACT. A real Bott manifold is the total space of an iterated $\mathbb{R}P^1$-bundle over a point, where each $\mathbb{R}P^1$-bundle is the projectivization of a Whitney sum of two real line bundles. In this paper, we characterize real Bott manifolds which admit a symplectic form. In particular, it turns out that a real Bott manifold admits a symplectic form if and only if it is cohomologically symplectic. In this case, it admits even a Kähler structure. We also prove that any symplectic cohomology class of a real Bott manifold can be represented by a symplectic form. Finally, we study the flux of a symplectic real Bott manifold.

## 1. Introduction

A *real Bott tower* (of height $n$) is a sequence of $\mathbb{R}\mathrm{P}^1$-bundles:

$$M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 = \{\text{a point}\},$$

where each $\mathbb{R}\mathrm{P}^1$-bundle $M_i \to M_{i-1}$ is the projectivization of a Whitney sum of two real line bundles on $M_{i-1}$. Each $M_i$ is called a *real Bott manifold*. Clearly $M_1 = \mathbb{R}\mathrm{P}^1$ and $M_2 = (\mathbb{R}\mathrm{P}^1)^2$ or a Klein bottle. If every bundle in the tower is trivial, then $M_n = (\mathbb{R}\mathrm{P}^1)^n$. However, there are many choices of non-trivial bundles at each stage in the tower, and it is known that there are many different diffeomorphism classes in real Bott manifolds ([7], [5]). A real Bott manifold is also an example of a real toric manifold which admits a flat Riemannian metric ([7]).

Although orientable real Bott manifolds occupy a small portion in all real Bott manifolds ([4]), the number of orientable ones of dimension $n$ approaches infinity as $n$ approaches infinity. Among those orientable ones, some are *symplectic*, i.e., admit a symplectic form. In this paper we give a complete characterization of symplectic real Bott manifolds (Theorem 3.1). In particular, we prove that among real Bott manifolds $M$ the following are equivalent:

(1) $M$ is cohomologically symplectic,
(2) $M$ is symplectic,
(3) $M$ admits a Kähler structure.

We call a $2n$-dimensional closed manifold $M$ *cohomologically symplectic* if there is a cohomology class $\alpha \in \mathrm{H}^2(M;\mathbb{R})$ such that $\alpha^n \neq 0$. We remark that the implication (3) $\Rightarrow$ (2) $\Rightarrow$ (1) always holds, but the reverse implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) do not hold in general, as is well-known. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is

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cohomologically symplectic but not symplectic because it does not admit an almost complex structure, and a certain $T^2$-bundle over $T^2$ constructed in [9] is symplectic but does not admit a Kähler structure.

This paper is organized as follows. In Section 2 we recall the quotient description of real Bott manifolds. In Section 3 we state and prove our main theorem. In Section 4 we study the flux group of a symplectic real Bott manifold.

Throughout this paper, all cohomology will be de Rham cohomology over $\mathbb{R}$.

## 2. Quotient description of real Bott manifolds

In this section, we recall the quotient description of real Bott manifolds (see [7] and [5] for details) and determine the cohomology ring of a real Bott manifold.

Let $\mathfrak{B}(n)$ be the the set of $n \times n$ upper triangular matrices whose entries are 0's and 1's with zero diagonal entries. For a matrix $A \in \mathfrak{B}(n)$, $A^i_j$ denotes the $(i, j)$ entry of $A$, and $A^i$ (respectively, $A_j$) denotes the $i$-th row (respectively, the $j$-th column) of $A$. Let $S^1$ be the unit circle in $\mathbb{C}$. We then define the involution $a_i$ on $T^n := (S^1)^n$ by

$$a_i(z_1, \dots, z_n) := (z_1, \dots, z_{i-1}, -z_i, z_{i+1}^{(-1)^{A^i_{i+1}}}, \dots, z_n^{(-1)^{A^i_n}}) \tag{2.1}$$

for $(z_1, \dots, z_n) \in T^n$ and $i = 1, \dots, n$. Let $G(A)$ denote the transformation group on $T^n$ generated by the $a_i$'s. Then the quotient space $M(A) := T^n/G(A)$ is known to be a real Bott manifold, and every real Bott manifold can be obtained as $M(A)$ for some $A \in \mathfrak{B}(n)$. The real Bott manifold $M(A)$ does not determine uniquely the matrix $A$; however, all geometrical information on $M(A)$ can be read from $A$. For example,

$$M(A) \text{ is orientable } \iff \sum_{j=1}^{n} A^i_j = 0 \text{ in } \mathbb{Z}/2 \text{ for any } i \tag{2.2}$$

(see [7]).

It is also helpful to describe $M(A)$ as the quotient of $\mathbb{R}^n$ by affine transformations. In fact, let $\Gamma(A)$ denote the affine transformation group on $\mathbb{R}^n$ generated by $s_i$'s defined by

$$s_i(u_1, \dots, u_n) := (u_1, \dots, u_{i-1}, u_i + \frac{1}{2}, (-1)^{A^i_{i+1}} u_{i+1}, \dots, (-1)^{A^i_n} u_n) \tag{2.3}$$

for $i = 1, \dots, n$. Then, an exponential map from $\mathbb{R}$ to $S^1$ sending $u$ to $\exp(2\pi\sqrt{-1}u)$ induces a diffeomorphism from $\mathbb{R}^n/\Gamma(A)$ onto $T^n/G(A) = M(A)$.

Let $du_1, \dots, du_n$ denote the standard 1-forms on $\mathbb{R}^n$. Since each $du_j$ is invariant under parallel translations on $\mathbb{R}^n$, it descends to a closed 1-form on $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$, which we also denote by $du_j$. The (de Rham) cohomology ring $\mathrm{H}^*(T^n)$ of $T^n$ is the exterior algebra in $n$ variables $[du_1], \dots, [du_n]$ over $\mathbb{R}$, where $[du_j]$ denotes the cohomology class represented by the 1-form $du_j$. It follows from (2.1) or (2.3) that the endomorphism $a_i^*$ of $\mathrm{H}^*(T^n)$ induced by $a_i \in G(A)$ is given by

$$a_i^*([du_j]) = \begin{cases} [du_j] & \text{if } A^i_j = 0, \\ -[du_j] & \text{if } A^i_j = 1. \end{cases} \tag{2.4}$$

We note that since $M(A) = T^n/G(A)$ and $G(A)$ is a finite group, we have

$$\mathrm{H}^*(M(A)) = \mathrm{H}^*(T^n)^{G(A)} \tag{2.5}$$

(see [3, Theorem 2.4 on p. 120], for example), where the right hand side denotes the $G(A)$-invariants in $\mathrm{H}^*(T^n)$.

**Lemma 2.1.** *Let $J$ be a subset of $\{1,\dots,n\}$. Then $\prod_{j\in J}[du_j]\in \mathrm{H}^*(T^n)$ is $G(A)$-invariant if and only if $\sum_{j\in J} A_j = 0$ in $\mathbb{Z}/2$.*

*Proof.* By (2.4), we have

$$a_i^*(\prod_{j\in J}[du_j]) = (-1)^{\sum_{j\in J} A_j^i}\prod_{j\in J}[du_j].$$

Thus, $\prod_{j\in J}[du_j]$ is fixed by $a_i^*$ if and only if $\sum_{j\in J} A_j^i = 0$ in $\mathbb{Z}/2$. This implies the lemma since $G(A)$ is generated by the $a_i$'s. □

## 3. Main theorem

The following is our main theorem in this paper.

**Theorem 3.1.** *Let $A\in\mathfrak{B}(2n)$. The following conditions are equivalent:*

(1) *$M(A)$ is cohomologically symplectic; that is, there exists an $\alpha\in \mathrm{H}^2(M(A))$ such that $\alpha^n$ is non-zero.*
(2) *There exist $n$ subsets $\{j_1, j_{n+1}\},\dots,\{j_n, j_{2n}\}$ of $\{1,2,\dots,2n\}$ such that*
    - *$\coprod_k^n\{j_k, j_{k+n}\} = \{1,2,\dots,2n\}$ and*
    - *$A_{j_1} = A_{j_{n+1}},\dots,A_{j_n} = A_{j_{2n}}$.*
(3) *There exists a symplectic form on $M(A)$.*
(4) *There exists a Kähler structure on $M(A)$.*

*Moreover, any $\alpha\in \mathrm{H}^2(M(A))$ in (1) can be represented by a symplectic form on $M(A)$.*

*Proof.* Because any closed symplectic manifold is cohomologically symplectic and any Kähler manifold is a symplectic manifold, it suffices to prove implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4).

Proof of (1) $\Rightarrow$ (2). Assume that there exists a de Rham cohomology class $\alpha\in \mathrm{H}^2(M(A))$ such that $\alpha^n\neq 0$. We identify $\mathrm{H}^*(M(A))$ with $\mathrm{H}^*(T^n)^{G(A)}$ by (2.5). Then it follows from Lemma 2.1 that we can write $\alpha$ uniquely as

$$\alpha = \sum_{j<k, A_j=A_k} c_{j,k}[du_j\wedge du_k] \quad \text{with some } c_{j,k}\in\mathbb{R}. \tag{3.1}$$

Thus $\alpha^n\neq 0$ implies condition (2).

Proof of (2) $\Rightarrow$ (4). Assume that $A\in\mathfrak{B}(2n)$ satisfies condition (2), namely $A_{j_k} = A_{j_{k+n}}$ for $k=1,\dots,n$. Then we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by

$$z_k := u_{j_k} + \sqrt{-1}u_{j_{k+n}}$$

for $k=1,\dots,n$. Consider the standard Hermitian metric on $\mathbb{C}^n$. Then, $\Gamma(A)$ acts on $\mathbb{C}^n$ as biholomorphisms and isometries. In fact, through the above identification, it follows from (2.3) that the action of $s_i\in\Gamma(A)$ on $\mathbb{C}^n$ is given by

$$s_i(z_1,\dots,z_n)_k = \begin{cases} z_k+\frac{1}{2} & \text{if } i = j_k,\\ z_k+\frac{\sqrt{-1}}{2} & \text{if } i = j_{k+n},\\ z_k & \text{if } A^i_{j_k} = A^i_{j_{k+n}} = 0 \text{ and } i\neq j_k, j_{k+n},\\ -z_k & \text{if } A^i_{j_k} = A^i_{j_{k+n}} = 1,\end{cases}$$

where the left hand side denotes the $k$-th component of $s_i(z_1, \dots, z_n)$. Thus the quotient $M(A) = \mathbb{C}^n / \Gamma(A)$ inherits the standard Kähler structure on $\mathbb{C}^n$.

Finally, we shall prove the last statement in the theorem. As observed above, $\alpha \in \mathrm{H}^2(M(A))$ is of the form (3.1). We then define the differential closed 2-form $\omega$ on $\mathbb{R}^{2n}$ by

$$\omega := \sum_{j<k, A_j = A_k} c_{j,k} du_j \wedge du_k. \tag{3.2}$$

Comparing (3.1) with (3.2), one sees that the condition $\alpha^n \neq 0$ implies that $\omega^n$ is nowhere zero. Thus $\omega$ is a symplectic form on $\mathbb{R}^{2n}$. Since $\omega$ is invariant under the $\Gamma(A)$-action on $\mathbb{R}^{2n}$, $\omega$ descends to a symplectic form on the quotient $M(A) = \mathbb{R}^{2n} / \Gamma(A)$, and this represents the given class $\alpha$. □

**Example 3.2.** Let $A \in \mathfrak{B}(4)$. If $A$ is the zero matrix, then $M(A)$ is the 4-dimensional torus and symplectic. Suppose that $A$ is non-zero and $M(A)$ is symplectic. Then it follows from Theorem 3.1 (2) that $A$ is one of the following:

$$\begin{pmatrix} 0&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}.$$

Real Bott manifolds $M(A)$ for $A$ above are diffeomorphic to each other but not diffeomorphic to the 4-dimensional torus ([7], [5]). One sees that $M(A)$ is the total space of a non-trivial orientable $T^2$-bundle over $T^2$. Note that total spaces of orientable $T^2$-bundles over $T^2$ are symplectic and classified in [6].

Now we determine the type of $M(A)$. Suppose

$$A = \begin{pmatrix} 0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}.$$

Then, $A_1 = A_2$ and $A_3 = A_4$. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by

$$z_1 := \frac{1}{2}u_1 + \frac{\sqrt{-1}}{2}u_2,$$
$$z_2 := \frac{1}{2}u_3 + \frac{\sqrt{-1}}{2}u_4.$$

Then, the action of $s_1$, $s_2$, $s_3$ and $s_4$ on $\mathbb{C}^2$ are given by

$$s_1(z_1, z_2) = (z_1 + 1, -z_2),$$
$$s_2(z_1, z_2) = (z_1 + \sqrt{-1}, z_2),$$
$$s_3(z_1, z_2) = (z_1, z_2 + 1)$$

and

$$s_4(z_1, z_2) = (z_1, z_2 + \sqrt{-1}).$$

Let $E$ denote the elliptic curve $\mathbb{C} / \mathbb{Z} \oplus \sqrt{-1}\,\mathbb{Z}$. Then,

$$M(A) = \mathbb{C}^2 / \Gamma(A) \cong \mathbb{C} \times E / \langle s_1, s_2 \rangle.$$

Hence our $M(A)$ is of type $\{-I, I, (0,0)\}$ in [6, Table 1] and is a hyper-elliptic surface of type (a1) in [2, p. 148].

Finally we note that if

$$A = \begin{pmatrix} 0&1&1&0\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

then $M(A)$ is orientable by (2.2), but not symplectic. Therefore the class of symplectic real Bott manifolds is strictly smaller than that of orientable real Bott manifolds.

## 4. The flux group

In this section, we will study the flux group of a symplectic real Bott manifold. To do that, we recall the definition of a flux group for a general symplectic manifold.

Let $(M, \omega)$ be a closed symplectic manifold. A diffeomorphism $\phi : M \to M$ is called a *symplectomorphism* if $\phi^*\omega = \omega$ and the group of symplectomorphisms of $(M, \omega)$ is denoted by $\mathrm{Symp}(M, \omega)$. Associated to a smooth function $f : M \to \mathbb{R}$, the Hamiltonian vector field $X_f$ is defined by $i_{X_f}\omega = df$. For a one-parameter family $\{f_t\}_{0\leq t\leq 1}$ of functions, we obtain a one-parameter family $\{X_{f_t}\}_{0\leq t\leq 1}$ of Hamiltonian vector fields, and integrating $\{X_{f_t}\}$, we obtain a one-parameter family $\{\phi_t\}_{0\leq t\leq 1}$ of diffeomorphisms defined by

$$\frac{d}{dt}\phi_t = X_{f_t} \circ \phi_t \quad \text{and} \quad \phi_0 = \mathrm{id}\,.$$

The time-one map $\phi_1$ is a symplectomorphism and is called a *Hamiltonian diffeomorphism.* It is known that all Hamiltonian diffeomorphisms of $(M, \omega)$ form a subgroup, denoted $\mathrm{Ham}(M, \omega)$, of the identity component $\mathrm{Symp}_0(M, \omega)$ of $\mathrm{Symp}(M, \omega)$. For a symplectic isotopy $\{\phi_t\}$, that is, an isotopy through symplectomorphisms, we obtain a one-parameter family $\{X_t\}$ of vector fields defined by

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

The *flux* of $\{\phi_t\}$ is then defined to be

$$\int_0^1 [i_{X_t}\omega]dt \in \mathrm{H}^1(M). \tag{4.1}$$

It is known that the flux depends only on the homotopy class of symplectic isotopies with fixed end points $\phi_0 = \mathrm{id}$ and $\phi_1$, so that it defines a homomorphism

$$\mathrm{Flux} : \mathrm{Symp}_0(M, \omega) \to \mathrm{H}^1(M)/\Gamma_\omega,$$

where $\Gamma_\omega$ is the image of the fundamental group $\pi_1(\mathrm{Symp}_0(M, \omega))$ by the flux and is called the *flux group* of $(M, \omega)$. The solution of the flux conjecture ([8]) says that the subgroup $\Gamma_\omega$ of $\mathrm{H}^1(M)$ is closed and discrete. According to [1], the kernel of Flux is exactly equal to $\mathrm{Ham}(M, \omega)$; in other words, we have an exact sequence

$$\{1\} \to \mathrm{Ham}(M, \omega) \to \mathrm{Symp}_0(M, \omega) \xrightarrow{\mathrm{Flux}} \mathrm{H}^1(M)/\Gamma_\omega.$$

Now, we consider the flux of a symplectic real Bott manifold.

**Theorem 4.1.** *Let $M(A)$ be a real Bott manifold with a symplectic form $\omega$ given by (3.2). Then, the flux group $\Gamma_\omega$ is a lattice group of $\mathrm{H}^1(M(A))$ of full rank.*

*Proof.* It follows from Lemma 2.1 that $\mathrm{H}^1(M(A))$ is generated by $[du_j]$ with $A_j = 0$, and since $M(A)$ is symplectic, the number of zero columns in $A$ is even by Theorem 3.1, so that $\mathrm{H}^1(M(A))$ is even dimensional. Let $2r$ be the dimension of $\mathrm{H}^1(M(A))$. We may assume that $\mathrm{H}^1(M(A))$ is generated by $du_1, \ldots, du_{2r}$ by changing the suffices of the coordinates. Moreover, through a linear coordinate change of the first $2r$ coordinates $u_1, \dots, u_{2r}$, we may assume that the symplectic form $\omega$ on $M(A)$ is of the form

$$\omega = \sum_{i=1}^{r} du_i \wedge du_{i+r} + \sum_{j<k, A_j = A_k \neq 0} c_{j,k} du_j \wedge du_k. \tag{4.2}$$

Since $M(A) = T^{2n}/G(A)$ and $A_p = 0$ for $p = 1, \dots, 2r$, the multiplication of $S^1$ on the $p$-th coordinate on $T^{2n}$ for $1 \leq p \leq 2r$ descends to an $S^1$-action on $M(A)$ and defines a symplectic isotopy $\{\phi_t^p\}$. The one-parameter family $\{X_t^p\}$ of vector fields associated with $\{\phi_t^p\}$ is then $\partial/\partial u_p$ (possibly up to a non-zero constant), so that it follows from (4.1) and (4.2) that

$$\text{the flux of } \{\phi_t^p\} = \int_0^1 [i_{X_t^p}\omega] dt = \int_0^1 [du_q] dt = [du_q],$$

where $q = p + r$ if $1 \leq p \leq r$ and $q = p - r$ if $r + 1 \leq p \leq 2r$. This shows that $\Gamma_\omega$ spans $\mathrm{H}^1(M(A))$ over $\mathbb{R}$. Since $\Gamma_\omega$ is closed and discrete in $\mathrm{H}^1(M(A))$ as remarked before, it must be a lattice group of $\mathrm{H}^1(M(A))$ of full rank. □

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