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## SYMPLECTIC REDUCTION AND RIEMANN-ROCH FORMULAS FOR MULTIPLICITIES

REYER SJAMAAR

## Introduction

It is well-known that the presence of conserved quantities in a Hamiltonian dynamical system enables one to reduce the number of degrees of freedom of the system. This technique, which goes back to Lagrange and was treated in a modern spirit in papers of Marsden and Weinstein [17] and Meyer [21], is nowadays known as symplectic reduction.

In their paper [8] Guillemin and Sternberg considered the problem: what is the quantum analogue of symplectic reduction? In other words, when one quantizes both a mechanical system with symmetries and its reduced system, what is the relationship between the two quantum-mechanical systems that one obtains?

Recently a number of authors have made substantial progress in solving this problem, on which I shall report in this note. This development was brought about by work of Witten [27] and subsequent work of Jeffrey and Kirwan [11], Kalkman [13] and Wu [29] on cohomology rings of symplectic quotients. Another important idea turned out to be Lerman's technique of symplectic cutting or equivariant symplectic surgery [16], a generalization of the notions of blowing up and symplectic reduction.

## 1. Statement of the problem

The natural mathematical framework of Hamiltonian dynamics is symplectic geometry. A symplectic form on a smooth $\left(C^{\infty}\right)$ manifold $M$ is a smooth closed two-form $\omega$ on $M$ which is non-degenerate in the sense that at every point $m$ the alternating bilinear form $\omega_{m}$ on the tangent space $T_{m} M$ is non-degenerate. By elementary linear algebra, a symplectic form can exist on $M$ only if $M$ is evendimensional. A symplectic manifold is a pair $(M, \omega)$, where $M$ is a manifold and

[^0]$\omega$ a symplectic form on $M$. The simplest example of a symplectic manifold is the phase space $\mathbf{R}^{2 n}=\mathbf{R}^{n} \times \mathbf{R}^{n}$ of elementary mechanics, whose symplectic form is the constant symplectic form given in coordinates $q_{1}, q_{2}, \ldots, q_{n}, p_{1}, p_{2}, \ldots, p_{n}$ by
\[

$$
\begin{equation*}
\omega=\sum_{k} d q_{k} \wedge d p_{k} \tag{1}
\end{equation*}
$$

\]

A theorem of Darboux states that near every point on a symplectic manifold one can find a system of local coordinates $\left(q_{1}, q_{2}, \ldots, q_{n}, p_{1}, p_{2}, \ldots, p_{n}\right)$ (called Darboux coordinates) in which the symplectic form has the simple form (1). Thus the only local symplectic invariant is the dimension, and most symplectic phenomena of interest are global in nature.

Let $f$ be a smooth $\left(C^{\infty}\right)$ function on a symplectic manifold $(M, \omega)$. The Hamiltonian vector field (or symplectic gradient) of $f$ is the unique vector field $\Xi_{f}$ on $M$ satisfying $\omega\left(\Xi_{f}, v\right)=d f(v)$ for all vector fields $v$. This is well-defined because the form $\omega$ is non-degenerate. The flow of the vector field $\Xi_{f}$ is called the Hamiltonian flow of the function $f$. In a system of Darboux coordinates the Hamiltonian flow of $f$ is given by the well-known equations of Hamilton:

$$
\frac{d q_{k}}{d t}=\frac{\partial f}{\partial p_{k}} \quad \text { and } \quad \frac{d p_{k}}{d t}=-\frac{\partial f}{\partial q_{k}} \quad \text { for } k=1, \ldots, n
$$

For example, if $M=\mathbf{R}^{2 n}$ and $f(q, p)=p_{k}$, the Hamiltonian flow of $f$ is a uniform translation along the $q_{k}$-axis, and if $f(q, p)=\left(q_{k}^{2}+p_{k}^{2}\right) / 2$, the Hamiltonian flow of $f$ is a uniform rotation of period $2 \pi$ in the $\left(q_{k}, p_{k}\right)$-plane.

Quantization refers to a rule $Q$ assigning to every symplectic manifold $M$ a Hilbert space $Q(M)$ and to each observable quantity (that is, a function) on the manifold a selfadjoint operator $Q(f)$ on the Hilbert space in such a manner that the following conditions are satisfied. Firstly, the operator $Q(f)$ should depend linearly on the function $f$; secondly, to every constant function should be assigned the corresponding multiplication operator on the Hilbert space. The third condition is Dirac's commutator condition, which says that

$$
\begin{equation*}
[Q(f), Q(g)]=i Q(\{f, g\}) \tag{2}
\end{equation*}
$$

where $\{f, g\}=\omega\left(\Xi_{f}, \Xi_{g}\right)$ denotes the Poisson bracket of the functions $f$ and $g$.
The fourth and last condition is an irreducibility condition, which can be phrased as follows. Consider a finite-dimensional subspace $\mathfrak{g}$ of the space of smooth functions on a symplectic manifold $M$. Let us assume that $\mathfrak{g}$ is closed under the Poisson bracket, so that it forms a Lie algebra. Let us also assume that the Hamiltonian vector fields of the functions in $\mathfrak{g}$ are complete. Then the Hamiltonian flows of the functions in $\mathfrak{g}$ generate an action on $M$ of a connected finite-dimensional Lie group $G$ with Lie algebra $\mathfrak{g}$. The transpose of the inclusion map $i: \mathfrak{g} \rightarrow C^{\infty}(M)$, the $\operatorname{map} \Phi: M \rightarrow \mathfrak{g}^{*}$ defined by $\Phi(m)(\xi)=i(\xi)(m)$, is called a momentum map for the action of $G$. This notion generalizes the notions of linear momentum (where $M=\mathbf{R}^{n} \times \mathbf{R}^{n}$ and $G=\mathbf{R}^{n}$ is the group of translations acting on the first factor) and angular momentum (where $M=\mathbf{R}^{3} \times \mathbf{R}^{3}$ and $G=\mathrm{SO}(3)$ is the group of rotations acting diagonally). Because the action is generated by Hamiltonian vector fields, we say that $M$ is a Hamiltonian $G$-manifold. Now note that (2) says that $-i Q$ is a homomorphism from the Lie algebra of functions (with the Poisson bracket) to the Lie algebra of anti-selfadjoint operators on the Hilbert space $Q(M)$. The operators $-i Q(\xi)$ for $\xi \in \mathfrak{g}$ therefore define a representation of the Lie algebra $\mathfrak{g}$ on $Q(M)$. Under some mild assumptions this representation can be integrated to a unitary
representation of $G$ on $Q(M)$. The irreducibility condition requires that if $G$ acts transitively on $M$, then the representation $Q(M)$ should be irreducible.

For instance, let $M$ be the phase space $\mathbf{R}^{2 n}$, on which the Poisson bracket is given by

$$
\{f, g\}=\sum_{k}\left(\frac{\partial f}{\partial p_{k}} \frac{\partial g}{\partial q_{k}}-\frac{\partial g}{\partial p_{k}} \frac{\partial f}{\partial q_{k}}\right)
$$

Let $\mathfrak{g}$ be the subspace spanned by the constant function 1 and the coordinate functions $q_{1}, q_{2}, \ldots, q_{n}, p_{1}, p_{2}, \ldots, p_{n}$. Clearly, $\mathfrak{g}$ is closed under the Poisson bracket, and the corresponding group $G$ (known as the Heisenberg group) contains the full group of translations $\mathbf{R}^{2 n}$ and therefore acts transitively. For the functions in $\mathfrak{g}$, Dirac's condition (2) specializes to $\left[Q\left(p_{k}\right), Q\left(q_{l}\right)\right]=i \delta_{k l}$, which is the mathematical expression of Heisenberg's uncertainty principle. This relation is satisfied if we take $Q\left(\mathbf{R}^{2 n}\right)$ to be Schrödinger's quantization, the space of square-integrable functions in the $q$-coordinates. Here to the functions $q_{k}$ are assigned the operators "multiplication by $q_{k}$ " and to the functions $p_{k}$ the differential operators $-i \partial / \partial q_{k}$. The representation of $\mathfrak{g}$ defined in this manner can be integrated to a unitary representation of $G$ on $Q\left(\mathbf{R}^{2 n}\right)$. The Stone-von Neumann theorem asserts that this is indeed an irreducible representation and that it is, up to equivalence, the only one satisfying Heisenberg's uncertainty principle.

The trouble with the foregoing is that it has been known for a long time that a quantization scheme $Q$ with the four properties discussed above does not exist. In the example of $\mathbf{R}^{2 n}$ it was shown by Van Hove that with the operators $Q\left(q_{k}\right)$ and $Q\left(p_{k}\right)$ being defined as above a consistent assignment $f \mapsto Q(f)$ cannot be made whenever $f$ contains terms of order greater than two in the $p \mathrm{~s}$ and $q \mathrm{~s}$. To resolve this problem, over the past decades several amendments to the rules of quantization have been proposed. Most of these aim at finding a suitable subclass of the class of all functions that can be quantized in a consistent fashion and depend upon a choice of some auxiliary structure on the manifold and therefore work well for special types of symplectic manifolds only. One of the most successful is the Kirillov-Kostant-Souriau theory of geometric quantization, a good reference for which is Woodhouse's book [28]. Within the framework of these theories some rather limited uniqueness results are known, which assert that in special situations the result of the quantization process is to some extent independent of the choice of the auxiliary data. Despite these difficulties, it has turned out to be fruitful from a heuristic point of view to pretend that a quantization scheme exists that is more or less unique and grosso modo obeys the four rules set forth above. Typically, one expects the quantization of a non-compact manifold to be infinite-dimensional, whereas the quantization of a compact manifold ought to be finite-dimensional.

With this caveat in mind, let us now consider an arbitrary Hamiltonian $G$ manifold $M$ with momentum map $\Phi$. The technique of symplectic reduction referred to in the introduction consists of two steps: first one imposes constraints by setting each of the functions in $\mathfrak{g}$ equal to zero, and then one factors the constraint set by the action of the group $G$. The resulting topological space, $\Phi^{-1}(0) / G$, is called the reduced phase space or symplectic quotient of $M$ and is denoted by $M_{0}$. We shall assume that $M_{0}$ is a smooth manifold; it then carries a natural symplectic form $\omega_{0}$ induced by the symplectic form on $M$. (A sufficient condition for $M_{0}$ to be a manifold is that the action of $G$ on the constraint set $\Phi^{-1}(0)$ be proper and free.)

This situation arises in classical mechanics when one considers a Hamiltonian vector field $\Xi_{f}$ on $M$ possessing a number of integrals of motion, that is, a Lie algebra $\mathfrak{g}$ of functions that Poisson-commute with $f$. The point of the construction is that the Hamiltonian flow of $f$ is invariant under the group $G$ and descends to a Hamiltonian flow on the symplectic quotient $M_{0}$.

A surprising link between this set-up and geometric invariant theory was discovered by Guillemin and Sternberg [8], Kirwan [14] and others, who found that if $M$ is a nonsingular complex projective variety, $M_{0}$ can be identified with a geometric quotient in the sense of Mumford. We shall have nothing further to say about Hamiltonian dynamics or geometric invariant theory, but instead focus on the relationship between reduction and quantization.

After these preliminaries we can now state the reduction-versus-quantization problem. It was formulated by Guillemin and Sternberg in approximately the following terms:
"Theorem". Let $M$ be a Hamiltonian $G$-manifold with momentum map $\Phi$, and let $M_{0}=\Phi^{-1}(0) / G$ denote the symplectic quotient of $M$. Then there is an isomorphism of Hilbert spaces,

$$
\begin{equation*}
Q\left(M_{0}\right) \cong Q(M)^{G} \tag{3}
\end{equation*}
$$

where $Q(M)^{G}$ denotes the subspace of $G$-fixed vectors in the representation $Q(M)$.
This assertion is sometimes loosely paraphrased by saying that quantization commutes with reduction. It is possible to reduce $M$ at values $\lambda \in \mathfrak{g}^{*}$ different from 0 , and for these values there is an analogous statement involving the isotypical subspaces of $Q(M)$, but in this note I shall only consider the case $\lambda=0$, which is the most important one.

In view of the non-existence of the quantization functor $Q$, Guillemin and Sternberg's "theorem" is to be thought of as a guiding principle, which one can only hope to make sense of by judiciously bending the rules or by imposing sufficiently strong hypotheses. (This is the reason why they called it a "theorem", not a conjecture.) Consider for example $M=\mathbf{R}^{2 n}$ and the Lie algebra $\mathfrak{g}$ generated by the single function $p_{1}$. Then $G$ is the group of translations along the $q_{1}$-axis, so that $M_{0}=\mathbf{R}^{2(n-1)}$ and $Q\left(M_{0}\right)$ is the space of square-integrable functions in $q_{2}, q_{3}, \ldots$, $q_{n}$. On the other hand, the space $Q(M)^{G}$, being the space of square-integrable functions in $q_{1}, q_{2}, \ldots, q_{n}$ that are annihilated by $-i \partial / \partial q_{1}$, is $\{0\}$. Although the statement of the "theorem" is not strictly correct, we can simply force it to be correct by slightly enlarging $Q(M)$ before taking the $G$-invariants.

In the context of geometric quantization, rigorous versions of the "theorem" have been stated and proved by Guillemin and Sternberg [8] for Kähler manifolds, by Gotay [5] for cotangent bundles, and in an infinite-dimensional example of interest in gauge theory by Axelrod et al. [1]. A general procedure for quantizing a reduced system is the BRST method, which was put on a mathematical footing by Kostant and Sternberg [15] and by means of which the "theorem" was verified in a number of examples by Duval et al. [4]. (They also found a physically interesting example where it is false.) For a version of the "theorem" in the setting of microlocal analysis, see [9].

## 2. Quantization as an equivariant index

A definition of quantization that is apparently due to Bott has recently turned out to lead to particularly elegant versions of the "theorem" in the case where both the symplectic manifold $M$ and the group $G$ are compact. The remainder of this note will be devoted to these new results. An underlying assumption, which is common to most quantization schemes, is that the symplectic form $\omega$ is integral in the sense that for every integral cycle $c$ on $M$ the period $\int_{c} \omega$ is an integer. By the Chern-Weil theory of connections and curvature this implies that there exists a complex line bundle $L$ on $M$ with a Hermitian fibre metric such that the curvature form of $L$ is equal to $\omega$. Bott's idea was to define $Q(M)$ as a push-forward in $K$-theory:

$$
Q(M)=\pi_{*}([L])
$$

where $[L]$ denotes the class of $L$ in the $K$-theory of $M$ and $\pi$ is the map sending $M$ to a space consisting of one point. In this approach $Q(M)$ is an element of the $K$-theory of a point, that is, a virtual vector space. This has the disadvantage that the dimension of $Q(M)$ may be negative and also that there is no natural choice of an inner product. But this is more than compensated for by the fact that the dimension of $Q(M)$ is equal to the index of a rolled-up Dolbeault complex, which by the Atiyah-Singer index formula is given by an integral of explicit differential forms on $M$. An additional advantage over some earlier approaches is that $M$ is not required to possess a complex structure or any other type of polarization. Apart from the symplectic form, the only thing needed is a compatible almost complex structure, which always exists, and the dimension of $Q(M)$ is independent of it. We shall call the space $Q(M)$ the almost complex quantization of $M$.

How far does almost complex quantization comply with the rules discussed in the previous section? Consider a compact connected group $G$ acting on $M$ in a Hamiltonian fashion. In favourable circumstances the action of $G$ can be lifted to an action on $L$ by bundle transformations. (For Hamiltonian actions, there always exists a natural lift of the action of the Lie algebra $\mathfrak{g}$ to an action of $\mathfrak{g}$ on $L$; see e. g. [8]. Under a mild topological condition this action can then be integrated to an action of $G$.) In other words, $L$ defines a class in the equivariant $K$-theory of $M$, and $Q(M)$ is therefore an element of the equivariant $K$-theory of a point, that is, a virtual representation of $G$. Infinitesimally, we have a virtual representation of the Lie algebra $\mathfrak{g}$; the functions in $\mathfrak{g}$ are thus quantized as operators $Q(f)$ on the virtual vector space $Q(M)$. Moreover, if the action on $M$ is transitive, it can be deduced from the Borel-Weil-Bott theorem that $Q(M)$ is an irreducible representation.

Furthermore, by the equivariant index theorem the character of the virtual representation $Q(M)$ associated to a Hamiltonian $G$-manifold $M$ can be calculated as an integral of explicit equivariant differential forms on $M$. This gives some hope that a detailed comparison between the equivariant index formula on the manifold $M$ and the ordinary index formula on the symplectic quotient $M_{0}$ might lead to a proof of the "theorem".

This hope was fulfilled by the work of Vergne [23, 24, 26], Guillemin [7], Jeffrey and Kirwan [10, 12], Duistermaat et al. [3], and Meinrenken [18, 19, 20].

The source of this development was a formula proposed by Witten in [27] known as the nonabelian (or more correctly "not necessarily abelian") localization formula. The formula expresses integrals over $M$ of $G$-equivariant differential forms in terms
of data localized near the critical points of the norm square of the momentum map. One of its purposes is to determine cup products of cohomology classes on the symplectic quotient $M_{0}$ in terms of data on $M$.

In [11] Jeffrey and Kirwan proved a version of Witten's localization formula expressing integrals of equivariant forms over $M$ in terms of integrals not over the critical points of the norm square of the momentum map, but over the components of the fixed point set of the maximal torus $T$ of $G$. As a corollary they obtained a formula, the residue formula, which calculates integrals of forms over the quotient $M_{0}$ in terms of integrals of certain equivariant forms over the $T$-fixed points in $M$. In the special case of a circle action Witten's localization formula was proved by Kalkman [13] and Wu [29].

The index theorem says that the dimension of $Q\left(M_{0}\right)$ can be computed by evaluating the cohomology class $C h\left(L_{0}\right) T d\left(M_{0}\right)$ on the fundamental class of $M_{0}$. Here $C h\left(L_{0}\right)$ denotes the Chern character of $L_{0}$, the Hermitian line bundle on the quotient $M_{0}$ whose curvature form is equal to the reduced symplectic form $\omega_{0}$, and $T d\left(M_{0}\right)$ denotes the Todd class of $M_{0}$. The Chern character $C h\left(L_{0}\right)$ is represented by the differential form $\exp \omega_{0}$, which appears naturally in the Witten-Jeffrey-Kirwan localization formula. For this reason, a promising way to tackle the quantization problem is to apply the localization formula to the Todd class of $M_{0}$.

This is essentially the approach taken in the papers [7, 23, 26, 18] and [12], leading to several proofs of the "theorem" valid under various hypotheses, which I shall not state in detail here. In brief, in [7] Guillemin used the localization formula to reduce the proof of (3) to the verification of a combinatorial identity involving lattice points in polytopes, which he observed to be true in the case of a quasi-free torus action. In [18] Meinrenken extended these ideas to the case of a general torus action. His paper also contains a proof of an asymptotic version of (3) valid for arbitrary compact groups. In $[23,26]$ Vergne independently gave two proofs of (3) for torus actions, one based on localization formulas in equivariant cohomology, the other on the index theorem for transversally elliptic operators. (Note however that her definition of quantization differs from ours in that she twists the quantizing line bundle by the square root of the canonical line bundle.) In [12] Jeffrey and Kirwan used their residue formula to give a proof of (3) for groups of rank one under a mild hypothesis on the critical values of the momentum map.

A seemingly very different method was used in the papers [3] and [19] based on Lerman's technique of symplectic cutting. The two approaches are however not unrelated, since it is possible to give a proof of Jeffrey and Kirwan's residue formula using symplectic cutting; see [16]. Duistermaat et al. [3] gave a simple proof of (3) for circle actions, whereas Meinrenken's paper [19] contains the first proof valid for general compact groups.

More recently, Jeffrey and Kirwan [10] have also obtained a proof valid in this generality based on the residue formula. However, their result neither follows from nor implies Meinrenken's: on one hand they relax the condition of positivity of the quantizing line bundle imposed by Meinrenken; on the other hand they need to impose a condition stating roughly that 0 should be not too close to the singular values of $\Phi$.

In all of the above papers the assumption is made that 0 is a regular value of the momentum map. Under this assumption the reduced space is usually not a manifold, but an orbifold, a space with finite-quotient singularities. It is possible
to extend the definition of quantization given above to the category of symplectic orbifolds.

In the sequel I shall present a sketch of a proof of (3) following the ideas of [3] and [19], where I shall for simplicity assume that $G$ is a torus.

## 3. The equivariant index and symplectic cutting

3.1. Reduction and quantization. Let $M$ be a Hamiltonian $G$-manifold, where $G$ is a torus, and let $\Phi: M \rightarrow \mathfrak{g}^{*}$ be the momentum map. We assume that $M$ is compact and that the symplectic form on $M$ is integral. We also assume that the symplectic quotient $M_{0}$ is a smooth manifold. Instead of the virtual representation $Q(M)$, it is more convenient to consider its character, which we shall denote by $R R(M)$. We denote by $R R(M)^{G}$ the integral of $R R(M)$ over $G$, that is, the dimension of the space of $G$-invariants $Q(M)^{G}$. Similarly, we let $R R\left(M_{0}\right)$ be the dimension of the virtual vector space $Q\left(M_{0}\right)$. Our goal is to outline a proof of the following version of Guillemin and Sternberg's "theorem":

Theorem 3.1. $R R(M)^{G}=R R\left(M_{0}\right)$.
The first step is to prove a special case of this statement. By a theorem of Atiyah, Guillemin and Sternberg, the image of $\Phi$ is a convex polytope, which we shall denote by $\Delta$. Furthermore, the preimage of a vertex of $\Delta$ consists of a single component of the fixed-point set $M^{G}$. Consequently, if 0 is a vertex of $\Delta$, the symplectic quotient $M_{0}$ is identical to the fibre $\Phi^{-1}(0)$ and is a smooth symplectic submanifold of $M$.

Theorem 3.2 ([20]). If 0 is not contained in $\Delta$, then $R R(M)^{G}=0$. If 0 is a vertex of $\Delta$, then $R R(M)^{G}=R R\left(M_{0}\right)$.

See Section 3.2 for a discussion of the proof of this result.
The next step is to reduce the general case to the case where 0 is a vertex. The idea is to subdivide $\Delta$ into smaller polytopes, each of which corresponds roughly speaking to a $G$-invariant symplectic submanifold of $M$, and to prove a gluing formula for the equivariant index akin to the gluing formula for the topological Euler characteristic.

More explicitly, let $\mathcal{C}$ be a fan in $\mathfrak{g}^{*}$, that is, a finite collection of strictly convex polyhedral cones in $\mathfrak{g}^{*}$ satisfying the following conditions:
$\{0\} \in \mathcal{C} ;$
if $C_{1}$ and $C_{2}$ are in $\mathcal{C}$, then $C_{1} \cap C_{2}$ is in $\mathcal{C}$;
if $C_{1} \in \mathcal{C}$ and $C_{2}$ is a face of $C_{1}$, then $C_{2} \in \mathcal{C}$.
In addition, we require that $\mathcal{C}$ be complete (the union of all cones in $\mathcal{C}$ is the whole of $\mathfrak{g}^{*}$ ), rational (every one-dimensional cone in $\mathcal{C}$ contains a nonzero integral weight), and simplicial (the set of one-dimensional faces of every cone in $\mathcal{C}$ is linearly independent).

For a generic choice of such a fan $\mathcal{C}$ it is possible to define a collection of compact Hamiltonian $G$-manifolds $M_{C}$ with momentum maps $\Phi_{C}$ having the following properties:

1. $\Phi_{C}\left(M_{C}\right)=C \cap \Delta$, so if $0 \in \Delta$, then 0 is a vertex of the polytope $\Phi_{C}\left(M_{C}\right)$;
2. $\Phi_{C}^{-1}(0)$ is symplectically isomorphic to $M_{0}$;
and to prove the following formula.

Theorem 3.3 (gluing formula, [19]).

$$
(-1)^{\operatorname{dim} \Delta} R R(M)=\sum_{C \in \mathcal{C}}(-1)^{\operatorname{dim} C \cap \Delta} R R\left(M_{C}\right)
$$

See Section 3.3 for the definition of the spaces $M_{C}$ and some remarks on the proof of the gluing formula.

By Theorem 3.2 and properties (1) and (2) above, $R R\left(M_{C}\right)^{G}=R R\left(M_{0}\right)$ for every cone $C$. Consequently, by Theorem 3.3 ,

$$
(-1)^{\operatorname{dim} \Delta} R R(M)^{G}=\sum_{C \in \mathcal{C}}(-1)^{\operatorname{dim} C \cap \Delta} R R\left(M_{0}\right)
$$

Moreover, $\sum_{C \in \mathcal{C}}(-1)^{\operatorname{dim} C \cap \Delta}=(-1)^{\operatorname{dim} \Delta}$, because the fan $\mathcal{C}$ is complete. This proves Theorem 3.1.
3.2. The index formula. By the equivariant index theorem of Atiyah, Segal and Singer the character $R R(M): G \rightarrow \mathbf{C}$ can be written as a sum

$$
\begin{equation*}
R R(M)=\sum_{F \in \mathcal{F}} \chi_{F} \tag{4}
\end{equation*}
$$

where $\mathcal{F}$ denotes the collection of components of the fixed point set $M^{G}$, and where each of the contributions $\chi_{F}$ is a rational function on $G$ given by an integral of equivariant differential forms on $F$ as follows. Choose a $G$-invariant almost complex structure on $M$ compatible with the symplectic form. Then the tangent bundles of $M$ and of $F$, and hence the normal bundle $N_{F}$ of $F$ in $M$, are complex vector bundles. If $\xi \in \mathfrak{g}$ is generic in the sense that it generates a dense one-parameter subgroup of $G$, then

$$
\begin{equation*}
\chi_{F}(\exp \xi)=e^{i \Phi^{\xi}(F)} \int_{F} \frac{e^{\omega} T d(F)}{D_{F}(\xi)} \tag{5}
\end{equation*}
$$

Here $\Phi^{\xi}(F)$ is the constant value of the function $\Phi^{\xi}$ on the submanifold $F, T d(F)$ is a form representing the Todd class of $F$, and $D_{F}(\xi)$ is the equivariant form

$$
\begin{equation*}
D_{F}(\xi)=\prod_{j}\left(1-\exp \left(2 \pi i \alpha_{j F}(\xi)+c_{j F}\right)\right) \tag{6}
\end{equation*}
$$

The meaning of $c_{j F}$ and $\alpha_{j F}$ is as follows. One formally splits the normal bundle of $F$ in $M$ into a sum of complex $G$-equivariant line bundles: $N_{F}=E_{1 F} \oplus E_{2 F} \oplus$ $\cdots \oplus E_{r F}$, where $r=\operatorname{rank} N_{F}$, and defines $c_{j F} \in \Omega^{2}(F)$ to be the Chern form of $E_{j F}$ and $\alpha_{j F} \in \mathfrak{g}^{*}$ the weight of the $G$-action on the fibre of $E_{j F}$ at any point of $F$. (The $c_{j F}$ and the $\alpha_{j F}$ may not be well-defined, but any symmetric function of them is, such as the form $D_{F}(\xi)$. )

On the other hand we can write the character as a finite sum

$$
\begin{equation*}
R R(M)(\exp \xi)=\sum_{\lambda \in \Lambda} N(\lambda) e^{2 \pi i \lambda(\xi)} \tag{7}
\end{equation*}
$$

where $\Lambda$ is the lattice of integral weights in $\mathfrak{g}^{*}$ and the $N(\lambda)$ are integers.
Now assume that 0 is not contained in the polytope $\Delta$, or that it is a vertex of $\Delta$. Let $C$ be the smallest convex cone with vertex 0 containing $\Delta$. Choose the vector $\xi$ such that $\mu(\xi) \geq 0$ for all $\mu \in C$. In (4) and (7) we can substitute $i t \xi$ for $\xi$, where $t$ is real, and study the asymptotics as $t \rightarrow \infty$.

If the fixed-point component $F$ is different from $\Phi^{-1}(0)=M_{0}$, then $\Phi^{\xi}(F)>0$, so (5) implies that $\chi_{F}(\exp i t \xi)=O\left(t^{-\infty}\right)$ for $t \rightarrow \infty$.

If 0 is not in $\Delta$, we infer from this that $R R(M)(\exp i t \xi)=O\left(t^{-\infty}\right)$ as $t \rightarrow \infty$. This means that the constant term $N(0)=R R(M)^{G}$ in the Fourier expansion (7) vanishes. This proves the first assertion of Theorem (3.2).

If 0 is a vertex of $\Delta$, then 0 is the minimum value of $\Phi^{\xi}$, and therefore $\alpha_{j F}(\xi)>0$ for $F=M_{0}$, and hence $\chi_{M_{0}}=\int_{M_{0}} e^{\omega} \operatorname{Td}\left(M_{0}\right)+O\left(t^{-\infty}\right)$. Applying the RiemannRoch theorem to $M_{0}$ and adding all the terms $\chi_{F}$, we obtain

$$
\begin{equation*}
R R(M)(\exp i t \xi)=R R\left(M_{0}\right)+O\left(t^{-\infty}\right) \tag{8}
\end{equation*}
$$

as $t \rightarrow \infty$. In particular, $R R(M)(\exp i t \xi)$ is bounded for large $t$. Comparing this with (7), we see that $N(\lambda)=0$ if $\lambda$ is not in $C$ (because for generic $\xi$ the exponents $2 \pi i \lambda(\xi)$ are all distinct), and therefore

$$
\begin{equation*}
R R(M)(\exp i t \xi)=N(0)+O\left(t^{-\infty}\right) \tag{9}
\end{equation*}
$$

as $t \rightarrow \infty$. The estimates (8) and (9) immediately imply the second part of Theorem 3.2.
3.3. The gluing formula. The notation and conventions are as in Section 3.1. For every cone $C$ in the fan $\mathcal{C}$ let $\langle C\rangle$ denote the linear span of $C$. Let $\mathfrak{g}_{C}$ be the annihilator in $\mathfrak{g}$ of $\langle C\rangle$, and let $G_{C}$ be the analytic subgroup $\exp \mathfrak{g}_{C}$ of $G$. The symplectic cut of $M$ with respect to $C$ is defined by induction on the dimension of $C$ :

$$
M_{C}=\Phi^{-1}(\stackrel{\circ}{C}) / G_{C} \cup \bigcup_{C^{\prime}<C} M_{C^{\prime}}
$$

where $\dot{C}$ denotes the relative interior of $C$ and the union is over all faces $C^{\prime}$ of $C$. The map $\Phi_{C}: M_{C} \rightarrow \mathfrak{g}^{*}$ is obtained by restricting $\Phi$ to $\Phi^{-1}(C)$ and pushing it down to $M_{C}$. The image of $\Phi_{C}$ is equal to $C \cap \Delta$. Also, $G$ acts in a natural way on $M_{C}$ with the subgroup $G_{C}$ acting trivially.

Clearly, $M_{C^{\prime}}$ is a subset of $M_{C}$ if $C^{\prime}<C$. The subset $\Phi^{-1}(\stackrel{\circ}{C}) / G_{C}$ is open and dense in $M_{C}$. If $C=\{0\}, M_{C}$ is simply the symplectic quotient $M_{0}$. If $C$ is top-dimensional $(\operatorname{dim} C=\operatorname{dim} G)$, then the open dense piece is $\Phi^{-1}(\dot{C})$ and is equivariantly diffeomorphic to an open piece of $M$.

It can be shown that if the fan $\mathcal{C}$ is generic with respect to the polytope $\Delta$, then for every cone $C$ the space $M_{C}$ is a symplectic manifold and the $G$-action is Hamiltonian with momentum map $\Phi_{C}$. Here "generic" means, among other things, that for every fixed-point component $F$ of $M$ the image $\Phi(F)$ has to be in a top-dimensional cone of $\mathcal{C}$.
(Here I have slurred over an interesting but somewhat technical complication: even for generic fans the $M_{C}$ are usually not manifolds, but orbifolds. For the sake of simplicity I shall ignore this problem and pretend they are manifolds. This gap in our proof of Theorems 3.1-3.3 can be filled by invoking index theorems for orbifolds due to Atiyah, Kawasaki and Vergne. See e. g. [25].)

Finally, a word on the proof of the gluing formula, Theorem 3.3. Let us denote the set of fixed-point components in $M_{C}$ by $\mathcal{F}_{C}$ and write the equivariant index of $M_{C}$ as a sum of fixed-point contributions as in (4): $R R\left(M_{C}\right)=\sum_{F \in \mathcal{F}_{C}} \chi_{C, F}$. The gluing formula then amounts to the identity

$$
\begin{equation*}
(-1)^{\operatorname{dim} \Delta} \sum_{F \in \mathcal{F}} \chi_{F}=\sum_{C \in \mathcal{C}}(-1)^{\operatorname{dim} C \cap \Delta} \sum_{F \in \mathcal{F}_{C}} \chi_{C, F} . \tag{10}
\end{equation*}
$$

To evaluate the right-hand side, we note that the fixed-point components of the cut manifolds $M_{C}$ fall into two classes:

1. $F \subset M_{C}$ where $\operatorname{dim} C=\operatorname{dim} G$ and $\Phi_{C}(F) \in \dot{C}$. These are the fixed-point components coming from fixed-point components in $M$. Indeed, since $\Phi_{C}^{-1}(\dot{C})$ is isomorphic to $\Phi^{-1}(\dot{C}), F$ is an isomorphic copy of a fixed-point component in $M$, which we shall also denote by $F$, and $\chi_{C, F}=\chi_{F}$. Consequently the sum $\sum_{F} \chi_{C, F}$ over all these $F$ s is equal to $\sum_{F \in \mathcal{F}} \chi_{F}=R R(M)$.
2. $F \subset M_{C}$ for some $C$ with $\operatorname{dim} C<\operatorname{dim} G$. These are fixed-point components that do not correspond to fixed-point components in $M$, but are introduced by the cutting process. An isomorphic copy of $F$, which we shall also denote by $F$, occurs in every $M_{C^{\prime}}$ such that $\Phi(F) \in C^{\prime}$. To finish the proof of (10) it suffices to show that for all such $F$

$$
\begin{equation*}
\sum_{\substack{C \\ \Phi(F) \in C}}(-1)^{\operatorname{dim} C \cap \Delta} \chi_{C, F}=0 . \tag{11}
\end{equation*}
$$

Let me briefly indicate the proof of this identity in the important special case $F=M_{0}$, the fixed-point component that is contained in $M_{C}$ for every $C \in \mathcal{C}$. Using (5) we see that in this case (11) amounts to

$$
\begin{equation*}
\sum_{C \in \mathcal{C}}(-1)^{\operatorname{dim} C \cap \Delta} \frac{1}{D_{C, M_{0}}(\xi)}=0 \tag{12}
\end{equation*}
$$

where $D_{C, M_{0}}$ is the equivariant form defined as in (6) associated to the normal bundle $N_{C}$ to $M_{0}$ in $M_{C}$. In fact,

$$
\begin{equation*}
D_{C, M_{0}}(\xi)=\prod_{j}\left(1-\exp \left(2 \pi i \lambda_{j}(\xi)+\lambda_{j}(c)\right)\right) \tag{13}
\end{equation*}
$$

where $\lambda_{j} \in \Lambda$ are the generators of the one-dimensional faces of $C$, and $c \in$ $\Omega^{2}\left(M_{0}, \mathfrak{g}\right)$ is the curvature form of the principal $G$-bundle $\Phi^{-1}(0) \rightarrow M_{0}$.

The identity (12) boils down to a purely combinatorial identity involving the fan $\mathcal{C}$. Take an element $\xi$ of the complexified Lie algebra $\mathfrak{g}^{\mathbf{C}}$ and an integral weight $\lambda$. Then $z=\exp \xi$ is an element of the complexified torus $G^{\mathbf{C}}$ and it makes sense to define $z^{\lambda}=\exp 2 \pi i \lambda(\xi)$. To each cone $C$ we associate the rational function $f_{C}$ on $G^{\mathbf{C}}$ defined by $f_{C}(z)=\prod_{j}\left(1-z^{\lambda_{j}}\right)^{-1}$. A multi-variable generalization of the identity $(1-z)^{-1}+\left(1-z^{-1}\right)^{-1}=1$ says that

$$
\sum_{C \in \mathcal{C}}(-1)^{\operatorname{dim} C} f_{C}=0
$$

Substituting the equivariant form $\exp \left(2 \pi i \lambda_{j}(\xi)+\lambda_{j}(c)\right)$ for $z^{\lambda_{j}}$ and using (13), we obtain (12).
3.4. Further developments. If 0 is not a regular value of the momentum map $\Phi$, the quotient $M_{0}$ is usually not a manifold or even an orbifold, but contains serious singularities. It is of interest to extend Theorem 3.1 to this situation. Recent joint work of Meinrenken and the author based on the methods outlined in this note shows that this is indeed possible. (Previously, a version of Theorem 3.1 for singular quotients was obtained in [22] under the hypothesis that $M$ admits a compatible complex structure.)

The statement of Theorem 3.1 can be made sense of in the setting of presymplectic and Spin ${ }^{\text {C }}$-manifolds and was proved for completely integrable torus actions on such manifolds by Grossberg and Karshon [6]. Further results in this direction have been announced by Canas da Silva et al. [2].

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Cornell University, Ithaca, New York 14853-7901
E-mail address: sjamaar@math. cornell.edu


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