# SYMPLECTIC REPRESENTATIONS OF ALGEBRAIC GROUPS SATISFYING A CERTAIN ANALYTICITY CONDITION

BY

ICHIRO SATAKE

*The University of Chicago, Chicago, Ill., U.S.A.*(1)

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## Introduction

1. The starting point of this study was a problem of Kuga on the determination of all symplectic representations of a semi-simple (algebraic) group of hermitian type satisfying a certain analyticity condition ([8]; for a more precise formulation, see 6.1). In my previous paper [9] ([9 a]), I have solved this problem from the geometrical point of view, or in other words, over the field of real numbers **R**(2). But, the aim of the problem lying

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(1) Partly supported by NSF grant GP 3903.

(2) Mumford and Tate have considered a similar problem from a somewhat different point of view and obtained a similar classification independently (at least, for absolutely irreducible representations satisfying the condition ($H_2$)), see Mumford, Families of abelian varieties, *Proceedings of the Symposia in Pure Mathematics*, Vol. 9, 1966, 347–351. Meanwhile, in some special cases, holomorphic imbeddings of a symmetric domain into another symmetric domain have been studied by several mathematicians in connection with the theory of automorphic functions. Cf. [4], [7]; and also Eichler's Nancy note.

primarily in the construction of families of (polarized) abelian varieties, the problem should be considered over the field of rational numbers **Q**, rather than **R**. This requires naturally a more systematic treatment of symplectic representations of an algebraic group, on the one hand, and considerations of certain "$k$-forms" (e.g. $k$-forms of a Clifford algebra and of a spin group(1)), on the other. The main purpose of this paper is to achieve these points.

In Part I, we shall develop a generality on symplectic representations of a (reductive) algebraic group $G$ defined over a field $k_0$ of characteristic zero. We shall see that the determination of all symplectic representations of $G$ can be reduced to that of all bilinear or hermitian invariants for the absolutely irreducible representations of $G$ in a division algebra with involution (3.6, Th. 1). Part II is devoted to the determination of this division algebra (denoted by $\mathfrak{K}_{\varrho_1}$), and this will be done by determining a certain invariant of $G$ (denoted by $\gamma(G) \in H^2(k_0, Z)$, $Z$: the center of $G$) which generalizes the Hasse-Minkowski invariant(2). Combining these results with our earlier ones, we shall be able to obtain, in Part III, an almost complete classification over **Q** of the symplectic representations $\varrho$ of the described type. Namely, let $G$ be a Zariski-connected semi-simple algebraic group defined over **Q** of hermitian type and $\varrho$ a **Q**-rational symplectic representation of $G$ satisfying the condition $(H_1)$. Then, by the general theory, the problem can first be reduced to the case where the representation $\varrho$ is **Q**-primary (i.e. a direct sum of mutually equivalent **Q**-irreducible representations). In this paper, we shall make an additional assumption on $\varrho$ that $\varrho$ comes essentially from an absolutely irreducible representation of just one absolutely simple factor of $G$ (7.1, (9)); then we may assume without any loss of generality that $G$ is **Q**-simple (and so $\varrho$ to be almost faithful). Under these assumptions, it turns out (§ 8) that, besides the "standard solutions" (coming from the identical representation) for the groups of type (I), (II), (III.1), (III.2), investigated already by Shimura [12], [13] from the other direction(3), we have also non-standard solutions for the groups of type (I′), (IV.1), (IV.2) (and for the groups of the mixed type (II–IV.2) if the number of the quaternion variables is four). These solutions give rise to analytic families of polarized abelian varieties over symmetric domains of type (I), (IV), which are contained in Shimura's families as "subfamilies" (in the sense specified in Appendix). One notes that the above-

(1) The $k$-forms of an (even) Clifford algebra have been studied recently by Jacobson [6] and others.

(2) In his first manuscript, the author treated only those cases which are needed in Part III by a more direct method. The generalization as presented here, especially the introduction of $\gamma(G)$, was suggested to him by the Referee, to whom the author is very grateful.

(3) In the notation of [12], [13], these correspond to the Types IV, III, I, II, respectively. In Shimura's theory, the group $G_{\mathbf{R}}$ has no compact factor except for Type (I) (which is an essential consequence of his construction) and the representation $\varrho$ is **Q**-irreducible (which is merely a conventional assumption).

mentioned additional condition on $\varrho$ is automatically satisfied, if the group $G_{\mathbf{R}}$ has no compact factor ([9]). Without this assumption, the conclusion would become much more complicated, for one would then have to consider various "mixed types", including the type (II–IV.2) in the $\mathbf{Q}$-simple case (9.2).

2. *Notation.* Following the general convention in algebraic geometry, we shall fix a universal domain $\Omega$ (of characteristic zero) and consider all algebraic groups, vector-spaces, ... etc. as having coordinates in $\Omega$. (For our purpose, it is enough to assume $\Omega$ to be just an algebraically closed field, which contains the field of complex numbers $\mathbf{C}$, in Part III.) Thus, for instance, a vector-space $V$ defined over $k$ means a vector-space over $\Omega$ containing a distinguished vector-space $V_k$ over $k$ such that $V = V_k \otimes_k \Omega$; then, for any $K \supset k$, we can speak of the set of $K$-rational points $V_K$ which is a vector-space over $K$ and is identified with $V_k \otimes_k K$. A (linear) algebraic group $G$ defined over $k$ acting on $V$ is a subgroup of $GL(V)$ defined by polynomial equations in the matrix entries (with respect to any basis of $V_k$) with coefficients in $k$. For any $K \supset k$, one puts $G_K = G \cap GL(V_K)$.

For the convenience of the reader, we recall here briefly the notion of $k$-form of an algebraic group. Let $K \supset k$, and let $G_0$ be a (linear) algebraic group defined over $K$. A *$k$-form* (or more precisely a *$K/k$-form*) of $G_0$ is a pair $(G, f)$ formed of an algebraic group $G$ defined over $k$ and an isomorphism $f$ defined over $K$ (or, as we shall call more briefly, a $K$-isomorphism) of $G$ onto $G_0$. Now suppose that $k$ is perfect and $K/k$ is finite; we denote by $\mathcal{G}(k)$ the Galois group of $\bar{k}/k$, $\bar{k}$ denoting the algebraic closure of $k$. If $(G, f)$ is a $k$-form of $G_0$, then, for every $\sigma \in \mathcal{G}(k)$, $\varphi_\sigma = f^\sigma \circ f^{-1}$ is a $\bar{k}$-isomorphism of $G_0$ onto $G_0^\sigma$ (depending only on $\sigma|K$) satisfying the condition $\varphi_\sigma^\tau \circ \varphi_\tau = \varphi_{\sigma\tau}$ for all $\sigma, \tau \in \mathcal{G}(k)$; this last condition is equivalent to saying that, if one puts $g^{[\sigma]} = \varphi_\sigma^{-1}(g^\sigma)$ for $g \in (G_0)_{\bar{k}}$, $\sigma \in \mathcal{G}(k)$, then one has $(g^{[\sigma]})^{[\tau]} = g^{[\sigma\tau]}$. Conversely, it is known ([11], [15]) that, given a collection of isomorphisms $\{\varphi_\sigma\}$ satisfying this condition, one can construct a $k$-form $(G, f)$ such that $\varphi_\sigma = f^\sigma \circ f^{-1}$. Moreover, let $(G', f')$ be another $K/k$-form of $G_0$ with $\varphi'_\sigma = f'^\sigma \circ f'^{-1}$. If there is a $k$-isomorphism $\varphi$ of $G$ onto $G'$, then $\psi = f' \circ \varphi \circ f^{-1}$ is a $K$-automorphism of $G_0$ satisfying $\varphi'_\sigma = \psi^\sigma \circ \varphi_\sigma \circ \psi^{-1}$ for all $\sigma \in \mathcal{G}(k)$, and *vice versa*. In particular, if $K/k$ is a Galois extension with the Galois group $\mathcal{G}(K/k)$ and if $G_0$ is defined over $k$, then the $k$-isomorphism-classes of $K/k$-forms of $G_0$ are in a one-to-one correspondence with the elements of first cohomology set $H^1(\mathcal{G}(K/k), \mathrm{Aut}_K(G_0))$, where $\mathrm{Aut}_K(G_0)$ is the group of all $K$-automorphisms of $G_0$.

These considerations apply, of course, to other kinds of algebraic systems, too. For instance, any central simple algebra $\mathfrak{A}$ of dimension $m^2$ defined over $k$ (and splitting over $K$), together with its unique absolutely irreducible representation $\theta_1$ (defined over $K$), can be considered as a $k$-form ($K/k$-form) of a total matric algebra $\mathfrak{M}_m$ (defined over the prime field). Since all automorphisms of $\mathfrak{M}_m$ are inner, one may identify $\mathrm{Aut}(\mathfrak{M}_m)$ with $PL(m)=GL(m)/\mathbf{G}_{\mathrm{m}}$, $\mathbf{G}_{\mathrm{m}}$ denoting the multiplicative group of the universal domain. If $\{\Phi_\sigma\}$ is a system of representatives in $GL(m, \bar{k})$ of the (continuous) 1-cocycle corresponding to $\mathfrak{A}$, one has $\Phi_\sigma^\tau \Phi_\tau = \lambda_{\sigma,\tau}\Phi_{\sigma\tau}$ with $\lambda_{\sigma,\tau}\in(\bar{k})^*$, where $\{\lambda_{\sigma,\tau}\}$ is a (continuous) 2-cocycle of $\mathcal{G}(k)$ in $(\mathbf{G}_{\mathrm{m}})_{\bar{k}}=(\bar{k})^*$. The mapping $\mathfrak{A}\to(\lambda_{\sigma,\tau})\in H^2(k, \mathbf{G}_{\mathrm{m}})$ $(=H^2(\mathcal{G}(k), (\bar{k})^*))$ is clearly multiplicative and gives rise to (the inverse of) the well-known isomorphism of the Brauer group $\mathcal{B}(k)$ and $H^2(k, \mathbf{G}_{\mathrm{m}})$ ([11]). Replacing $GL(m)$ by $SL(m)$, one may also obtain a canonical isomorphism $\mathcal{B}(k)\cong H^2(k, \mathsf{E})$, where $\mathsf{E}$ denotes the multiplicative group of all roots of unity in $\bar{k}$. We shall denote by $c(\mathfrak{A})$ the inverse of the 2-cohomology class (or, by abuse of notation, a 2-cocycle representing it) in $H^2(k, \mathsf{E})$ corresponding to the algebra-class of $\mathfrak{A}$.

Returning to an algebraic group $G_1$ defined over a finite extension $K$ of $k$, one defines the group $R_{K/k}(G_1)$ defined over $k$ as follows ([15]). Let $\{\sigma_1(=\mathrm{id.}), ..., \sigma_d\}$ $(d=[K:k])$ be a complete set of representatives of $\mathcal{G}(K)\backslash\mathcal{G}(k)$. Then $R_{K/k}(G_1)$ is defined as a $k$-form $(G, f)$ of $G_0=\prod_{i=1}^d G_1^{\sigma_i}$ such that, denoting by $p_i$ the projection of $G_0$ onto the $i$th factor $G_1^{\sigma_i}$, one has $p_i^\sigma\circ f^\sigma=p_j\circ f$ if $\mathcal{G}(K)\sigma_i\sigma=\mathcal{G}(K)\sigma_j$; this last condition is equivalent to saying that putting $p=p_1\circ f$, one has $f(g)=(p^{\sigma_i}(g))$ for $g\in G$. The pair $(G, p)$ is uniquely characterized by the following universality: Whenever one has an algebraic group $G'$ defined over $k$ and a $K$-homorphism $\varphi_1$ of $G'$ into $G_1$, there exists (uniquely) a $k$-homomorphism $\varphi$ of $G'$ into $G$ such that $\varphi_1=p\circ\varphi$. For the groups of rational points one has the canonical isomorphism $(R_{K/k}(G_1))_k\cong G_{1K}$ (induced by $p$). If $G_1$ has any additional algebraic structure (e.g. vector-space, associative algebra, etc.), then so does also $R_{K/k}(G_1)$.

## Part I. Symplectic representations of algebraic groups

Throughout Part I, we fix once and for all a field $k_0$ of characteristic zero and a reductive algebraic group $G$ defined over $k_0$. As is well-known, for any field $K$ containing $k_0$, a representation of $G$ defined over $K$ is completely reducible in $K$, namely, it is $K$-equivalent to a direct sum of a certain number of $K$-irreducible representations, which are uniquely determined up to the order and $K$-equivalence. It is only this property of $G$ that will be used essentially in the following considerations.

### § 1. Primary representations

**1.1.** Let $(V, \varrho)$ be a representation defined over $k_0$ of $G$, where $V$ is a (finite-dimensional) vector-space defined over $k_0$ and $\varrho$ is a $k_0$-homomorphism of $G$ into $GL(V)$. (Sometimes $\varrho$ alone is called a representation, while $V$ is referred to as a representation-space.) Let $(V_1, \varrho_1)$ be an absolutely irreducible representation defined over $\bar{k}_0$ (=the algebraic closure of $k_0$) contained in $(V, \varrho)$, i.e., $V_1$ is a $\varrho(G)$-invariant subspace of $V$, defined over $\bar{k}_0$, such that the restriction $\varrho_1 = \varrho \mid V_1$ is $\bar{k}_0$-irreducible. Denote by $\mathcal{G}$ the Galois group of $\bar{k}_0/k_0$. Then, for every $\sigma \in \mathcal{G}$, we have a "conjugate" representation $(V_1^\sigma, \varrho_1^\sigma)$, also contained in $(V, \varrho)$; namely, $V_1^\sigma$ is again a $\varrho(G)$-invariant subspace of $V$, defined over $\bar{k}_0$, and we have $\varrho_1^\sigma \sim \varrho \mid V_1^\sigma$. Now we define two subgroups of $\mathcal{G}$ as follows:

$$\begin{aligned} \mathcal{G}_{\varrho_1} &= \{\sigma \in \mathcal{G} \mid \varrho_1^\sigma \sim \varrho_1\}, \\ \mathcal{G}' &= \{\sigma \in \mathcal{G} \mid V_1^\sigma = V_1\}. \end{aligned} \tag{1}$$

Then it is clear that $\mathcal{G}' \subset \mathcal{G}_{\varrho_1}$. Denote further by $K_{\varrho_1}$ and $K'$ the subfields of $\bar{k}_0$ corresponding to $G_{\varrho_1}$ and $G'$, respectively; $K'$ is then the smallest field containing $k_0$ over which the subspace $V_1$ (and hence $\varrho_1$) is defined. Therefore, $K'$ is a finite extension of $k_0$, and we have $K' \supset K_{\varrho_1} \supset k_0$. Put $[K_{\varrho_1} : k_0] = d$ and fix once and for all a system of representatives $\{\tau_1, \ldots, \tau_d\}$ of $\mathcal{G}_{\varrho_1} \backslash \mathcal{G}$, i.e., one puts

$$\mathcal{G} = \bigcup_{i=1}^{d} \mathcal{G}_{\varrho_1} \tau_i. \tag{2}$$

It should be noted that the definition of $\mathcal{G}_{\varrho_1}$ (and hence $K_{\varrho_1}$) depends only on the ($\bar{k}_0$-) equivalence-class of $\varrho_1$, while that of $G'$ (and hence $K'$) does depend essentially on the imbedding of $(V_1, \varrho_1)$ in $(V, \varrho)$.

Now we denote by $\hat{V}_1$ (resp. $\tilde{V}_1$) the sum of all $\varrho(G)$-invariant subspaces $W$ of $V$, defined over $\bar{k}_0$, such that $\varrho \mid W \sim \varrho_1$ (resp. $\varrho \mid W \sim \varrho_1^\tau$ with some $\tau \in \mathcal{G}$). Then, $\hat{V}_1$ (resp. $\tilde{V}_1$), being invariant under all $\sigma \in \mathcal{G}_{\varrho_1}$ (resp. $\mathcal{G}$), is a subspace of $V$ defined over $K_{\varrho_1}$ (resp. $k_0$). (Actually, $K_{\varrho_1}$ is the smallest field containing $k_0$ over which $\hat{V}_1$ is defined.) Clearly one has an isomorphism (of respresentation-spaces)

$$\hat{V}_1 \cong m V_1 \quad (\text{over } \bar{k}_0), \tag{3}$$

or, in other notation, $\hat{\varrho}_1 = \varrho \mid \hat{V}_1 \sim m\varrho_1$, where $m$ is a positive integer, called the "multiplicity" of $\varrho_1$ in $\varrho$. Since, for every $\tau \in \mathcal{G}$, $\hat{V}_1^\tau$ is the sum of all $\varrho(G)$-invariant subspaces $W$ of $V$, defined over $\bar{k}_0$, such that $\varrho \mid W \sim \varrho_1^\tau$, one sees at once that $\tilde{V}_1$ is decomposed into the direct sum of the following form:

$$\tilde{V}_1 = \sum_{i=1}^{d} \hat{V}_1^{\tau_i}. \tag{4}$$

Thus one has

$$\varrho \mid \tilde{V}_1 \sim m \sum_{i=1}^{d} \varrho_1^{\tau_i} \quad (\text{over } \bar{k}_0). \tag{5}$$

According to a general notation in algebraic geometry (see Introduction, 2), one may also write (4) in the form

$$\tilde{V}_1 = R_{K_{\varrho_1}/k_0}(\hat{V}_1). \tag{4'}$$

*If one starts from another absolutely irreducible $\varrho(G)$-invariant subspace $V_2$ of $V$* defined over $\bar{k}_0$, which is not contained in $\tilde{V}_1$, and forms $\hat{V}_2$, $\tilde{V}_2$ in a similar way, then one has $\tilde{V}_1 \cap \tilde{V}_2 = \{0\}$. Repeating this process, one finally obtains, by virtue of the complete reducibility of $V$ (over $\bar{k}_0$), a direct decomposition of $V$:

$$V = \sum \tilde{V}_i. \tag{6}$$

*Definition.* A (completely reducible) representation $(V, \varrho)$ defined over $k_0$ of $G$ is called *$k_0$-primary,* if for any two absolutely irreducible $\varrho(G)$-invariant subspaces $W$ and $W'$ of $V$, both defined over $\bar{k}_0$, one has $(\varrho | W)^\tau \sim \varrho | W'$ with some $\tau \in \mathcal{G} = \mathcal{G}(k_0)$.

From the above construction, $\varrho | \tilde{V}_i$'s are all $k_0$-primary and $\tilde{V}_i$'s are maximal with respect to this property; and clearly the decomposition (6) is uniquely characterized by these properties. $(\tilde{V}_i, \varrho | \tilde{V}_i)$'s are called *$k_0$-primary components* of $(V, \varrho)$. $((\hat{V}_1, \hat{\varrho}_1)$ may be called an absolutely primary component of $(V, \varrho)$.)

1.2. It is obvious that, for a given absolutely irreducible representation $(V_1, \varrho_1)$ defined over $\bar{k}_0$, there exists always a $k_0$-primary representation $(V, \varrho)$ containing $(V_1, \varrho_1)$. (For instance, let $K'$ be a finite extension of $k_0$ over which $(V_1, \varrho_1)$ is defined, and take $R_{K'/k_0}(V_1, \varrho_1)$.) As we have seen above, such a representation $(V, \varrho)$ can be written uniquely in the following form:

$$V = R_{K_{\varrho_1}/k_0}(\hat{V}_1), \qquad \hat{V}_1 \cong m V_1 \ \text{(over } \bar{k}_0\text{)},$$

Now the $k_0$-equivalence-class of $\varrho$ depends only on the multiplicity $m$. This will follow immediately from the following (well-known) lemma:

LEMMA 1. *Let $(V, \varrho)$ and $(V', \varrho')$ be two representations of $G$ defined over $k_0$. If there exists a monomorphism of $V$ into $V'$ (viewed as representation-spaces), then there exists a $k_0$-monomorphism of $V$ into $V'$.*

*Proof* (after Ono). Let $\mathcal{L}$ be the vector-space formed of all linear mappings of $V$ into $V'$, and let $\mathcal{H}$ be the subspace of $\mathcal{L}$ formed of all homomorphisms of $V$ into $V'$ viewed as representation-spaces, i.e., all $\varphi \in \mathcal{L}$ such that $\varrho'(g) \circ \varphi = \varphi \circ \varrho(g)$ for all $g \in G$. Then, $\mathcal{L}$ is a vector-space defined over $k_0$ and $\mathcal{H}$ is a linear subspace of $\mathcal{L}$ also defined over $k_0$, so that $\mathcal{H}_{k_0}$, the set of all $k_0$-rational points in $\mathcal{H}$, is everywhere dense in $\mathcal{H}$ in the sense of Zariski topology. On the other hand, from the assumption, the subset $\mathcal{H}^{(0)}$ of $\mathcal{H}$ formed of all *monomorphisms* (=*injective homomorphisms*) of $V$ into $V'$ is clearly a non-empty Zariski ($k_0$-)open set in $\mathcal{H}$. Therefore, one has $\mathcal{H}^{(0)} \cap \mathcal{H}_{k_0} \neq \emptyset$, q.e.d.

We denote by $m(\varrho_1, k_0)$ the smallest possible multiplicity of $\varrho_1$ in the representations defined over $k_0$. It follows from the above Lemma that a $k_0$-primary representation $\varrho$ containing $\varrho_1$ is $k_0$-irreducible if and only if $m=m(\varrho_1, k_0)$, and the ($k_0$-)equivalence-class of such $\varrho$ is unique. In general, for a $k_0$-primary representation $\varrho$ containing $\varrho_1$, the multiplicity $m$ is a multiple of $m(\varrho_1, k_0)$ and $\varrho$ is ($k_0$-)equivalent to the direct sum of $m/m(\varrho_1, k_0)$ copies of the (unique) $k_0$-irreducible representation containing $\varrho_1$. Thus a $k_0$-primary representation can also be defined as a representation defined over $k_0$ of $G$ which is a direct sum of a certain number of mutually ($k_0$-)equivalent $k_0$-irreducible representations.

**1.3.** We shall now explain how the structure of a $k_0$-primary representation can be described in terms of Galois cohomology. To start with, let $(V_1, \varrho_1)$ be an absolutely irreducible representation of $G$ defined over $\bar{k}_0$. For every $\sigma \in \mathcal{G}_{\varrho_1}$, denote by $\varphi_\sigma$ a $\bar{k}_0$-isomorphism of $V_1$ onto $V_1^\sigma$ (viewed as representation-spaces), i.e., a linear isomorphism defined over $\bar{k}_0$ such that

$$\varrho_1^\sigma(g) = \varphi_\sigma \circ \varrho_1(g) \circ \varphi_\sigma^{-1} \quad \text{for all } g \in G. \tag{7}$$

By Schur's lemma, such a $\varphi_\sigma$ is uniquely determined up to a scalar multiple. It follows, in particular, that, for every $\sigma, \tau \in \mathcal{G}_{\varrho_1}$, one has

$$\varphi_\sigma^\tau \circ \varphi_\tau = \lambda_{\sigma,\tau} \varphi_{\sigma\tau} \tag{8}$$

with $\lambda_{\sigma,\tau} \in \bar{k}_0$. $\{\lambda_{\sigma,\tau}\}$ then becomes a 2-cocycle of $\mathcal{G}_{\varrho_1}$ in $(\bar{k}_0)^*$ (=the multiplicative group of non-zero elements in $\bar{k}_0$), whose cohomology class is uniquely determined. If one takes a finite Galois extension $K''$ of $K_{\varrho_1}$ over which $(V_1, \varrho_1)$ is defined, then the system $\{\varphi_\sigma\}$ can be chosen in such a way that all the $\varphi_\sigma$'s are defined over $K''$ (Lemma 1) and that $\varphi_\sigma$ depends only on the restriction of $\sigma$ on $K''$; then one has $\lambda_{\sigma,\tau} \in K''$. Thus we may assume, whenever necessary, that $(\lambda_{\sigma,\tau})$ is actually a 2-cocycle of the Galois group $\mathcal{G}(K''/K_{\varrho_1})$ in $K''^*$. (Without specifying $K''$, one sometimes says that $(\lambda_{\sigma,\tau})$ is a "continuous" 2-cocycle.)

Now, as we have seen in 1.1, the structure of a $k_0$-primary representation $(V, \varrho)$ containing $(V_1, \varrho_1)$ is uniquely determined by that of the absolutely primary component $\hat{V}_1$, which can be considered as a "$K_{\varrho_1}$-form" of $mV_1$. For our purpose, it will be more convenient to regard it as a $K_{\varrho_1}$-form of $V_1 \otimes V_2$, where $V_2$ is an $m$-dimensional row-vector space (defined over the prime field) on which $G$ operates trivially. Let $\psi$ be a $\bar{k}_0$-isomorphism of $V_1 \otimes V_2$ onto $\hat{V}_1$ (viewed as representation-spaces) which we write in the following form:

$$\psi(x \times (u_i)) = \sum_{i=1}^m \psi_i(x)\, u_i, \tag{9}$$

where $\psi_i$ $(1 \leqslant i \leqslant m)$ is a $\bar{k}_0$-monomorphism of $V_1$ into $\hat{V}_1$. Then one has $\varrho(g) \circ \psi = \psi \circ (\varrho_1(g) \otimes 1)$, or what is the same,

$$\varrho(g)\circ\psi_i=\psi_i\circ\varrho_1(g) \quad (1\leqslant i\leqslant m).$$

Applying $\sigma\in\mathcal{G}_{\varrho_1}$ on the both sides of this equality, one obtains

$$\varrho(g)\circ\psi_i^\sigma=\psi_i^\sigma\circ\varrho_1^\sigma(g)=\psi_i^\sigma\circ\varphi_\sigma\circ\varrho_1(g)\circ\varphi_\sigma^{-1}.$$

It then follows by Schur's lemma that $\psi_i^\sigma\circ\varphi_\sigma$ is a linear combination of $\psi_i$ $(1\leqslant i\leqslant m)$, so that one may write

$$\begin{pmatrix}\psi_1^\sigma\circ\varphi_\sigma\\ \vdots\\ \psi_m^\sigma\circ\varphi_\sigma\end{pmatrix}=\Phi_\sigma\begin{pmatrix}\psi_1\\ \vdots\\ \psi_m\end{pmatrix} \tag{10}$$

with a non-singular $m\times m$ matrix $\Phi_\sigma$ with entries in $\bar{k}_0$. The system $\{\Phi_\sigma\}$ satisfies clearly the relation

$$\Phi_\sigma^\tau\,\Phi_\tau=\lambda_{\sigma,\tau}\,\Phi_{\sigma\tau}. \tag{8'}$$

It follows that one has

$$\psi^{-\sigma}\circ\psi(x\otimes u)=\varphi_\sigma(x)\otimes u\Phi_\sigma^{-1} \quad \text{for } x\in V_1,\ u\in V_2,$$

or in other words (cf. Introduction, 3), $(\hat{V}_1, \psi^{-1})$ is a $K_{\varrho_1}$-form of $V_1\otimes V_2$ corresponding to the operation of the Galois group defined by

$$(x\otimes u)^{[\sigma]}=\varphi_\sigma^{-1}(x^\sigma)\otimes(u^\sigma\Phi_\sigma). \tag{11}$$

Conversely, it is easy to see that, given a system $\{\Phi_\sigma\}$ in $GL(m, \bar{k}_0)$ satisfying (8'), one can define a $K_{\varrho_1}$-form $(\hat{V}_1, \psi^{-1})$ of $V_1\otimes V_2$ (viewed as a representation-space) by the operation of the Galois group given by (11). It is also trivial that if $\{\Phi'_\sigma\}$ is another system satisfying the same conditions as $\{\Phi_\sigma\}$, two $K_{\varrho_1}$-forms corresponding to $\{\Phi_\sigma\}$ and $\{\Phi'_\sigma\}$ are $K_{\varrho_1}$-isomorphic, if and only if the two systems are "cohomologous" in the sense that one has $\Phi'_\sigma=\Psi^\sigma\Phi_\sigma\Psi^{-1}$ with $\Psi\in GL(m, \bar{k}_0)$. But, as we have already seen in 1.2, there exists only one $K_{\varrho_1}$-isomorphism class of $\hat{V}_1$ of the given dimension, so that there is also only one cohomology class of such $\{\Phi_\sigma\}$. (In this form, our result is a special case of the well-known theorem in Galois cohomology: Theorem 900 of Hilbert. Cf., e.g., [11].)

*Example* 1. Let $K'$ be a finite extension of $K_{\varrho_1}$ over which $(V_1, \varrho_1)$ is defined, and put $\hat{V}_1=R_{K'/K_{\varrho_1}}(V_1)$. Let $m=[K':K_{\varrho_1}]$, $\mathcal{G}'=\mathcal{G}(\bar{k}_0/K')$, and let $\{\sigma_1, \ldots, \sigma_m\}$ be a system of representatives of $\mathcal{G}'\backslash\mathcal{G}_{\varrho_1}$. Then $\hat{V}_1\cong\sum_{i=1}^m V_1^{\sigma_i}\cong mV_1$ (over $\bar{k}_0$), so that $\hat{V}_1$ is an absolutely primary representation-space with multiplicity $m$. In the above notation, the monomorphism $\psi_i$: $V_1\to\hat{V}_1$ can be taken to be equal to $\psi_1^{\sigma_i}\circ\varphi_{\sigma_i}$ $(1\leqslant i\leqslant m)$. Then, if $\mathcal{G}'\sigma_i\sigma=\mathcal{G}'\sigma_j$, one has $\psi_i^\sigma\circ\varphi_\sigma=\lambda_{\sigma_i,\sigma}\psi_j$, i.e., $\Phi_\sigma$ is an $m\times m$ matrix whose $(i, j)$-th entry is equal to $\lambda_{\sigma_i,\sigma}$, if $\sigma_i\sigma\sigma_j^{-1}\in\mathcal{G}'$, and zero otherwise. We shall see later (3.3, Ex. 2) that, in case all central division algebras over $K_{\varrho_1}$ are represented by a crossed product (e.g., in the case $k_0=\mathbf{Q}$), all absolutely primary, $K_{\varrho_1}$-irreducible representation-spaces $\hat{V}_1$ can be obtained in this manner.

## § 2. Invariant alternating forms

**2.1.** By a "symplectic" representation defined over $k_0$ of $G$, we understand a triple $(V, A, \varrho)$, where $(V, \varrho)$ is a representation of $G$ defined over $k_0$ and $A$ is a non-degenerate $\varrho(G)$-invariant alternating form defined over $k_0$ on $V\times V$ or, as we shall call more briefly, an invariant alternating form on $V$ defined over $k_0$. The notions of equivalence (or isomorphism), direct sum, etc., of symplectic representations are defined in the obvious way.

Let $(V, A, \varrho)$ be a symplectic representation of $G$ defined over $k_0$, and let

$$V=\sum \tilde{V}_i$$

be the decomposition of $V$ into the direct sum of $k_0$-primary components. Let $(V_1, \varrho_1)$ be, as before, an absolutely irreducible representation contained in $\tilde{V}_1$. Then, since ${}^t\varrho^{-1}\sim\varrho$, an absolutely irreducible representation equivalent to the "contagredient" representation $(V_1^*, {}^t\varrho_1^{-1})$ ($V_1^*$ denoting the dual space of $V_1$) is also contained in $(V, \varrho)$. Now we shall distinguish the following three cases:

(a) ${}^t\varrho_1^{-1}\sim\varrho_1$,
(b) ${}^t\varrho_1^{-1}\nsim\varrho_1$, but $\sim\varrho_1^{\sigma_0}$ with some $\sigma_0\in\mathcal{G}$,
(c) ${}^t\varrho_1^{-1}\nsim\varrho_1^{\tau}$ for all $\tau\in\mathcal{G}$.

It should be noted that these conditions, being invariant under the operation of the Galois group, depend only on the $k_0$-primary component $\tilde{V}_1$ containing $(V_1, \varrho_1)$. The absolutely irreducible representation $\varrho_1$ (or the $k_0$-primary representation containing $\varrho_1$) will be called *of type* (a), (b), (c) (over $k_0$) according to the cases.

In the case (c), a representation equivalent to ${}^t\varrho_1^{-1}$ is contained in a $k_0$-primary component different from $\tilde{V}_1$, say $\tilde{V}_2$. Then, it is clear that the restrictions $A\,|\,\tilde{V}_1\times\tilde{V}_i$ for $i\neq 2$ are all identically zero, so that $A\,|\,\tilde{V}_1\times\tilde{V}_2$ must be non-degenerate, and $\tilde{V}_1+\tilde{V}_2$ is a direct summand of the symplectic representation-space $V$. Moreover, $\tilde{V}_2$ may be identified with the dual representation-space $\tilde{V}_1^*$ of $\tilde{V}_1$ by the bilinear form $A\,|\,\tilde{V}_1\times\tilde{V}_2$, so that one has

$$\varrho\,|\,\tilde{V}_2\sim{}^t(\varrho\,|\,\tilde{V}_1)^{-1}\sim m\sum_{i=1}^{d}({}^t\varrho_1^{-1})^{\tau_i}.$$

Conversely, if $(\tilde{V}_2, \varrho\,|\,\tilde{V}_2)$ is equivalent to $(\tilde{V}_1^*, {}^t(\varrho\,|\,\tilde{V}_1)^{-1})$, then one can define an invariant alternating form defined over $k_0$ on $\tilde{V}_1+\tilde{V}_2$ in a natural way. Thus, in this case, to have a symplectic representation-space $\tilde{V}_1+\tilde{V}_2$ defined over $k_0$ is essentially the same as to have a ($k_0$-primary) representation-space $\tilde{V}_1$ (without alternating form) defined over $k_0$. Incidentally, in our later considerations (Part III), this case will not occur at all.

In the cases (a), (b), a representation equivalent to ${}^t\varrho_1^{-1}$ is again contained in $\tilde{V}_1$. It is then clear that $A\,|\,\tilde{V}_1\times\tilde{V}_i(i\neq 1)$ are all identically zero, while $A\,|\,\tilde{V}_1\times\tilde{V}_1$ is non-degenerate; in other words, $\tilde{V}_1$ becomes a direct summand of the symplectic representation-space $V$. From these, one concludes that a symplectic representation $(V, A, \varrho)$ can be decomposed into a direct sum of the symplectic representations defined over $k_0$, each one of which is either of the form $(\tilde{V}_i, A\,|\,\tilde{V}_i, \varrho\,|\,\tilde{V}_i)$ (cases (a), (b)) or of the form $(\tilde{V}_i+\tilde{V}_{i'}, A\,|\,(\tilde{V}_i+\tilde{V}_{i'}), \varrho\,|\,\tilde{V}_i+\varrho\,|\,\tilde{V}_{i'})$ (case (c)), and that this decomposition is unique in an obvious sense. Since the case (c) is of no further interest to us, we shall restrict ourselves to the case where $V$ is $k_0$-primary; such a symplectic representation $(V, A, \varrho)$ will also be called $k_0$-primary.

**2.2.** Let $(V, A, \varrho)$ be a $k_0$-primary symplectic representation of type (a). Then it is clear that the restrictions $A\,|\,(\hat{V}_1^{\tau_i}\times\hat{V}_1^{\tau_j})$ are all identically zero except for $i=j$ and that, if one puts $\hat{A}_1=A\,|\,\hat{V}_1\times\hat{V}_1$, $\hat{A}_1$ is an invariant alternating form defined over $K_{\varrho_1}$ on $\hat{V}_1$ and one has $A\,|\,(\hat{V}_1^{\tau_i}\times\hat{V}_1^{\tau_i})=\hat{A}_1^{\tau_i}$. Thus one obtains

$$A(\sum x_i, \sum y_i)=\sum_{i=1}^{d}\hat{A}_1^{\tau_i}(x_i, y_i) \tag{12}$$

for all $x_i, y_i\in\hat{V}_1^{\tau_i}$. We shall express this simply by writing

$$A=\operatorname{tr}_{K_{\varrho_1}/k_0}(\hat{A}_1). \tag{12$'$}$$

Now, from the assumption, there is a non-degenerate $\varrho_1(G)$-invariant bilinear form $B_1$ on $V_1\times V_1$, determined uniquely up to a scalar multiple (Schur's lemma). If $(V_1, \varrho_1)$ is defined over $K'$, $B_1$ may also be taken to be defined over the same field $K'$ (Lemma 1). In the notation in 1.3, it follows that one has

$$\hat{A}_1(\psi_i(x), \psi_j(y))=B_1(x, y)\beta_{ij} \quad \text{for all } x, y\in V_1$$

with $\beta_{ij}\in\bar{k}_0$. Putting $B_2=(\beta_{ij})$ and $B_2(u, v)=(u_i)\,B_2{}^t(v_i)$ for all $u=(u_i)$, $v=(v_i)\in V_2$, one has from (9)

$$\hat{A}_1(\psi(x\otimes u), \psi(y\otimes v))=B_1(x, y)\,B_2(u, v). \tag{13}$$

Clearly this relation, in turn, determines $B_1$ and $B_2$ uniquely up to scalar multiples. Since $\hat{A}_1$ is alternating, it follows first that one of the bilinear forms $B_1$ and $B_2$ is symmetric and the other is alternating. Secondly, applying $\sigma \in \mathcal{G}_{\varrho_1}$ on the both sides of (13) and in view of (11), one obtains the relations:

$$B_1^\sigma(\varphi_\sigma(x), \varphi_\sigma(y)) = \lambda_\sigma B_1(x, y), \tag{14}$$

$$B_2^\sigma(u\Phi_\sigma^{-1}, v\Phi_\sigma^{-1}) = \lambda_\sigma^{-1} B_2(u, v) \quad \text{or} \quad \Phi_\sigma B_2 {}^t\Phi_\sigma = \lambda_\sigma B_2^\sigma. \tag{14'}$$

Conversely, it is easy to see that, if one has non-degenerate bilinear forms $B_1$ on $V_1$ and $B_2$ on $V_2$ satisfying these conditions and if $B_1$ is $\varrho_1(G)$-invariant, then by (13) one can define an invariant alternating form $\hat{A}_1$ on $\hat{V}_1$ defined over $K_{\varrho_1}$, and then by (12) an invariant alternating form $A$ on $V$ defined over $k_0$. It should be noted that from (14) or (14′) one obtains the relation

$$\lambda_{\sigma,\tau}^2 = \lambda_\sigma^\tau \lambda_\tau \lambda_{\sigma\tau}^{-1} \sim 1. \tag{15}$$

**2.3.** Let us now consider the case (b), where one has ${}^t\varrho_1^{-1} \sim \varrho_1^{\sigma_0}$ with some $\sigma_0 \in \mathcal{G}, \notin \mathcal{G}_{\varrho_1}$. Since $\varrho_1^{\sigma_0^2} \sim {}^t(\varrho_1^{\sigma_0})^{-1} \sim \varrho_1$, one has $\sigma_0^2 \in \mathcal{G}_{\varrho_1}$. Next, for every $\sigma \in \mathcal{G}_{\varrho_1}$, one has

$$\varrho_1^{\sigma_0\sigma\sigma_0^{-1}} \sim ({}^t\varrho_1^{-1})^{\sigma\sigma_0^{-1}} \sim ({}^t\varrho_1^{-1})^{\sigma_0^{-1}} \sim \varrho_1,$$

so that, putting $\bar{\sigma} = \sigma_0\sigma\sigma_0^{-1}$, one has $\bar{\sigma} \in \mathcal{G}_{\varrho_1}$. This means that $\mathcal{G}_{\varrho_1} \cup \mathcal{G}_{\varrho_1}\sigma_0$ is a subgroup of $\mathcal{G}$ containing $\mathcal{G}_{\varrho_1}$ as a normal subgroup of index 2. We denote by $K_0$ the subfield of $\bar{k}_0$ corresponding to $\mathcal{G}_{\varrho_1} \cup \mathcal{G}_{\varrho_1}\sigma_0$. Then, it follows that $K_{\varrho_1}^{\sigma_0} = K_{\varrho_1}$ and $K_{\varrho_1}$ is a quadratic extension of $K_0$ with the Galois group $\mathcal{G}(K_{\varrho_1}/K_0) = \{1, \sigma_0 | K_{\varrho_1}\}$. In the following, we shall fix once and for all an element $\alpha$ in $K_0$ such that $K = K_0(\sqrt{\alpha})$. Also, we shall take a system of representatives $\{\tau_1, \ldots, \tau_d\}$ of $\mathcal{G}_{\varrho_1}\backslash\mathcal{G}$ in such a way that $\tau_{d/2+i} = \sigma_0\tau_i$ $(1 \leqslant i \leqslant d/2)$; then one has

$$\mathcal{G} = \bigcup_{i=1}^{d/2} (\mathcal{G}_{\varrho_1} \cup \mathcal{G}_{\varrho_1}\sigma_0)\,\tau_i.$$

Under these assumptions, it is clear that $A\,|\,(\hat{V}_1^{\tau_i} \times \hat{V}_1^{\tau_j})$ are all identically zero except for $j \equiv d/2 + i \pmod d$, and, if one puts

$$F(x, y) = \sqrt{\alpha}\, A(x^{\sigma_0}, y) \quad \text{for } x, y, \in \hat{V}_1, \tag{16}$$

$F$ is a non-degenerate $\varrho(G)$-invariant hermitian form on $\hat{V}_1$ with respect to $\sigma_0$, i.e., it is linear in the variable $y$ and satisfies the relation

$$F(x, y)^{\sigma_0} = F(y, x^{\sigma_0^2}) \quad \text{for all} \quad x, y \in \hat{V}_1. \tag{17}$$

(For simplicity, we shall henceforth suppose that $\sigma_0$ is extended to an automorphism of the "universal domain".) Therefore, putting

$$\hat{\hat{V}}_1 = R_{K_{\varrho_1}/K_0}(\hat{V}_1) = \hat{V}_1 + \hat{V}_1^{\sigma_0},\ \hat{\hat{A}}_1 = A \mid \hat{\hat{V}}_1,$$

one has

$$\hat{\hat{A}}_1(x+x', y+y') = \sqrt{\alpha}^{-1}\{F(x'^{\sigma_0^{-1}}, y) - F(y'^{\sigma_0^{-1}}, x)\}$$

$$= \sqrt{\alpha}^{-1}\{F(x'^{\sigma_0^{-1}}, y) - F^{\sigma_0}(x^{\sigma_0^{-1}}, y')\} \quad \text{for all } x, y \in \hat{V}_1, x', y' \in \hat{V}_1^{\sigma_0}. \quad (16')$$

On the other hand, as $A \mid \hat{\hat{V}}_1^{\tau_i} = \hat{\hat{A}}_1^{\tau_i}$ $(1 \leqslant i \leqslant d/2)$, one may write

$$A = \mathrm{tr}_{K_0/k_0}(\hat{\hat{A}}_1). \quad (18)$$

*Remark.* It might have been more natural to consider a hermitian form $F'$ defined by

$$F'(x+x', y+y') = (\sqrt{\alpha}\, A(x', y),\ -\sqrt{\alpha}\, A(x, y')) \quad \text{for } x, y \in \hat{V}_1, x', y' \in \hat{V}_1^{\sigma_0};$$

which is a hermitian form on $\hat{\hat{V}}_1 = R_{K_{\varrho_1}/K_0}(\hat{V}_1)$ taking values in $R_{K_{\varrho_1}/K_0}(\text{field})$ with respect to the involution $(\xi, \eta) \to (\eta, \xi)$. But, if we consider $F$ and $F'$ restricted on $(\hat{V}_1)_{K_{\varrho_1}} = (\hat{\hat{V}}_1)_{K_0}$ which take values in $K_{\varrho_1} = (R_{K_{\varrho_1}/K_0}(\text{field}))_{K_0}$, then they reduce essentially to the same thing, as is seen from the relation

$$F'(x + x^{\sigma_0}, y + y^{\sigma_0}) = (F(x, y), F(x, y)^{\sigma_0}) \quad \text{for } x, y \in (\hat{V}_1)_{K_{\varrho_1}}.$$

Thus, one may write (16′) symbolically as $\hat{\hat{A}}_1 = \mathrm{tr}_{K_{\varrho_1}/K_0}(\sqrt{\alpha}^{-1}F)$. For instance, in the case $K_{\varrho_1} \subset \mathbf{C}$, $\alpha = -1$, one has $\hat{\hat{A}}_1 = 2\,\mathrm{Im}\, F$.

Now, from the assumption, there exists a non-degenerate $\varrho_1(G)$-invariant sesquilinear form (with respect to $\sigma_0$) $F_1$ on $V_1 \times V_1$, determined uniquely up to a scalar multiple. If $(V_1, \varrho_1)$ is defined over $K'$, $F_1$ may be taken to be defined over $K' \cup K'^{\sigma_0}$. It follows that one has $F(\psi_i(x), \psi_j(y)) = F_1(x, y)\beta_{ij}$ with $\beta_{ij} \in \bar{k}_0$. Putting $F_2 = (\beta_{ij})$ and $F_2(u, v) = (u_i^{\sigma_0}) F_2{}^t(v_i)$ for $u = (u_i)$, $v = (v_i) \in V_2$, one has from (9)

$$F(\psi(x \otimes u), \psi(y \otimes v)) = F_1(x, y)\, F_2(u, v). \quad (19)$$

Again this relation determines $F_1$ and $F_2$ up to scalar multiples. It follows from (17) and (11) that

$$F_1(x, y)^{\sigma_0} = \lambda F_1(y, \varphi_{\sigma_0^2}^{-1}(x^{\sigma_0^2})) \quad \text{for all } x, y \in V_1, \quad (20)$$

$${}^tF_2^{\sigma_0} = \lambda^{-1} F_2\, {}^t\Phi_{\sigma_0^2}, \quad (20')$$

with $\lambda \in \bar{k}_0$; and, for every $\sigma \in \mathcal{G}_{\varrho_1}$, one has

$$F_1^{\sigma}(\varphi_{\bar{\sigma}}(x), \varphi_\sigma(y)) = \lambda_\sigma F_1(x, y) \quad \text{for all } x, y \in V_1, \quad (21)$$

$$\Phi_{\bar{\sigma}}^{\sigma_0} F_2 {}^t\Phi_\sigma = \lambda_\sigma F_2^\sigma. \tag{21'}$$

with $\lambda_\sigma \in \bar{k}_0$. Conversely, it is also trivial that, if one has non-degenerate sesqui-linear forms $F_1$ on $V_1$ and $F_2$ on $V_2$ satisfying these conditions and if $F_1$ is $\varrho_1(G)$-invariant, then (19), (16′), (18) define an invariant alternating form $A$ on $V$ defined over $k_0$. One notes that from (20), (21), or (20′), (21′) one obtains the relations

$$\lambda_{\bar{\sigma},\bar{\tau}}^{\sigma_0} \lambda_{\sigma,\tau} = \lambda_\sigma^\tau \lambda_\tau \lambda_{\sigma\tau}^{-1} \sim 1. \tag{22 a}$$

$$\lambda_{\sigma_0^2} = \lambda^{\sigma_0+1}. \tag{22 b}$$

$$\lambda_{\bar{\sigma}}^{\sigma_0} \lambda_\sigma^{-1} = \lambda^{\sigma-1} \lambda_{\bar{\bar{\sigma}},\sigma_0^2} \lambda_{\sigma_0^2,\sigma}^{-1}. \tag{22 c}$$

## § 3. Formulations in terms of division algebras

**3.1.** We shall now translate the results so far obtained into the terminology of the theory of algebras. The notation being as before, let $(\mathfrak{A}_1, \theta_1)$ and $(\mathfrak{A}_2, \theta_2)$ be $K_{\varrho_1}$-forms of $\mathcal{E}(V_1)$ (= the algebra of all linear endomorphisms of $V_1$) and of $\mathcal{E}(V_2)$ ($=\mathcal{M}_m$) defined, respectively, by the operations of the Galois group given as follows:

$$\varphi^{[\sigma]} = \varphi_\sigma^{-1} \circ \varphi^\sigma \circ \varphi_\sigma \quad \text{for } \varphi \in \mathcal{E}(V_1)_{\bar{k}_0}, \tag{23}$$

$$\Phi^{[\sigma]} = \Phi_\sigma^{-1} \Phi^\sigma \Phi_\sigma \quad \text{for } \Phi \in \mathcal{M}_m(\bar{k}_0). \tag{23'}$$

$\mathfrak{A}_1$ and $\mathfrak{A}_2$ are then central simple algebras defined over $K_{\varrho_1}$ such that $c(\mathfrak{A}_1) \sim (\lambda_{\sigma,\tau}^{-1})$ and $c(\mathfrak{A}_2) \sim (\lambda_{\sigma,\tau})$ (see Introduction, 2). Since $(\hat{V}_1, \psi^{-1})$ is a $K_{\varrho_1}$-form of $V_1 \otimes V_2$ with respect to the operation of the Galois group given by (11), it follows that $\mathcal{E}(\hat{V}_1)$ may be identified with $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ as a $K_{\varrho_1}$-form of $\mathcal{E}(V_1 \otimes V_2) = \mathcal{E}(V_1) \otimes \mathcal{E}(V_2)$. More precisely, the identification is made in such a way that one has

$$\psi \circ (\varphi_1 \otimes \varphi_2) \circ \psi^{-1} = \theta_1^{-1}(\varphi_1) \otimes \theta_2^{-1}(\varphi_2) \quad \text{for all } \varphi_i \in \mathcal{E}(V_i). \tag{24}$$

Now, from (7) and (23), one has

$$\varrho_1(g)^{[\sigma]} = \varrho_1(g^\sigma) \quad \text{for all } g \in G_{\bar{k}_0}, \sigma \in \mathcal{G}_{\varrho_1};$$

this means that the rational mapping: $G \ni g \to \mathsf{P}_1(g) = \theta_1^{-1} \circ \varrho_1(g) \in \mathfrak{A}_1$ is defined over $K_{\varrho_1}$. It follows that $\varrho_1(G_{K_{\varrho_1}})$ is contained in $\theta_1((\mathfrak{A}_1)_{K_{\varrho_1}})$; but, since $\varrho_1$ is absolutely irreducible, the latter is the $K_{\varrho_1}$-linear closure of the former. Since $\hat{\varrho}_1 = \psi \circ (\varrho_1 \otimes 1) \circ \psi^{-1} = \mathsf{P}_1 \otimes 1$, it follows that $(\mathfrak{A}_1)_{K_{\varrho_1}} \otimes 1$ is the $K_{\varrho_1}$-linear closure of $\hat{\varrho}_1(G_{K_{\varrho_1}})$, or what amounts to the same, $1 \otimes (\mathfrak{A}_2)_{K_{\varrho_1}}$ is the centralizer of $\hat{\varrho}_1(G_{K_{\varrho_1}})$ in $\mathcal{E}(\hat{V}_1)_{K_{\varrho_1}}$.

Now, let $\mathfrak{K}_{\varrho_1}$ be a division algebra defined over $K_{\varrho_1}$ (i.e., an algebra defined over $K_{\varrho_1}$ such that $(\mathfrak{K}_{\varrho_1})_{K_{\varrho_1}}$ is division) representing the Brauer class of $\mathfrak{A}_1$. Then, one has $\mathfrak{A}_2 \sim \mathfrak{K}_{\varrho_1}^{-1}$. It follows, in particular, that, if $\mathfrak{K}_{\varrho_1}$ is of dimension $r^2$, then $r$ divides both dim $V_1$ and dim $V_2 = m$; hence we shall put

$$\dim V_1 = nr, \quad \dim V_2 = m = n'r. \tag{25}$$

As is well-known, $\hat{\varrho}_1$ is $K_{\varrho_1}$-irreducible, if and only if the centralizer of $\hat{\varrho}_1(G_{K_{\varrho_1}})$ in $\mathcal{E}(\hat{V}_1)_{K_{\varrho_1}}$ is division, i.e., $n'=1$. Thus one concludes that

$$m(\varrho_1, k_0) = r = (\dim \mathfrak{K}_{\varrho_1})^{1/2} \tag{26}$$

**3.2.** (In this paragraph, we shall abbreviate $K_{\varrho_1}$, $\mathfrak{K}_{\varrho_1}$ as $K$, $\mathfrak{K}$. We shall need only the relations $\mathfrak{A}_1 \cong \mathfrak{M}_n(\mathfrak{K})$, $\mathfrak{A}_2 \cong \mathfrak{M}_{n'}(\mathfrak{K}^{-1})$, where $\mathfrak{K}$ is not necessarily assumed to be division.) Let $\mathbf{V}_1$ be an $n$-dimensional right vector-space over $\mathfrak{K}$, or as we shall say more briefly, a *right* $\mathfrak{K}$*-space* defined over $K$. By definition, $\mathbf{V}_1$ is an $nr^2$-dimensional vector-space defined over $K$, provided with the right multiplication: $\mathbf{V}_1 \times \mathfrak{K} \ni (x, \xi) \to x\xi \in \mathbf{V}_1$, also defined over $K$, such that $\mathbf{V}_1$ has a basis over $\mathfrak{K}$ consisting of $n$ vectors. Then $\mathfrak{A}_1$ may be identified with $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$ (=the algebra of all $\mathfrak{K}$-linear endomorphisms of $\mathbf{V}_1$) in the following way.

Let us fix once and for all a $\bar{k}_0$-isomorphism $M$ of $\mathfrak{K}$ onto the total matric algebra $\mathfrak{M}_r$, or, what amounts to the same, a system of matrix-units $\varepsilon_{ij}$ $(1 \leqslant i, j \leqslant r)$ in $\mathfrak{K}_{\bar{k}_0}$. Then for each $\sigma \in \mathcal{G}_{\varrho_1}$, there exists an element $\eta_\sigma \in \mathfrak{K}_{\bar{k}_0}$, determined uniquely up to a scalar multiple, such that

$$M^\sigma(\xi) = M(\eta_\sigma^{-1} \xi \eta_\sigma) \quad \text{for all } \xi \in \mathfrak{K}, \tag{27}$$

or, what is the same,

$$\varepsilon_{ij}^\sigma = \eta_\sigma \varepsilon_{ij} \eta_\sigma^{-1} \quad (1 \leqslant i, j \leqslant r). \tag{27'}$$

For $\sigma, \tau \in \mathcal{G}_{\varrho_1}$, one has clearly

$$\eta_\sigma^\tau \eta_\tau = \mu_{\sigma,\tau} \eta_{\sigma\tau} \tag{28}$$

with $\mu_{\sigma,\tau} \in \bar{k}_0$, and $(\mu_{\sigma,\tau}) \in c(\mathfrak{K})$. On the other hand, one obtains a direct decomposition:

$$\mathbf{V}_1 = \sum_{i=1}^{r} \mathbf{V}_1 \varepsilon_{ii}, \tag{29}$$

where every $\mathbf{V}_1 \varepsilon_{ii}$ is an $nr$-dimensional vector-subspace defined over $\bar{k}_0$, invariant under $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$. Therefore, any one of $\mathbf{V}_1\varepsilon_{ii}$'s, say $V_1' = \mathbf{V}_1 \varepsilon_{11}$, gives a (unique) absolutely irreducible representation of $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$. In view of (27′), for each $\sigma \in \mathcal{G}_{\varrho_1}$, the right multiplication

$$R_{\eta_\sigma^{-1}} : x \longrightarrow x\eta_\sigma^{-1} \tag{30}$$

induces an isomorphism $\varphi_\sigma'$ of $V_1'$ onto $V_1'^\sigma = \mathbf{V}\varepsilon_{11}^\sigma$ (viewed as $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$-spaces). It follows that $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$ (together with the restriction map on $V_1'$) is a $K$-form of $\mathcal{E}(V_1')$ defined by the operation of the Galois group: $\varphi'^{[\sigma]}=\varphi_\sigma'^{-1}\varphi'^{\sigma}\varphi_\sigma'$. Therefore, identifying $\mathfrak{A}_1$ with $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$ is equivalent to fixing an isomorphism between $\mathcal{E}(V_1)$ and $\mathcal{E}(V_1')$ which gives rise to a $K$-isomorphism between the corresponding $K$-forms $\mathfrak{A}_1$ and $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$. By Skolem–Noether's theorem, such an isomorphism of $\mathcal{E}(V_1)$ onto $\mathcal{E}(V_1')$ comes from a $\bar{k}_0$-isomorphism $f_1$ of $V_1$ onto $V_1'$ (determined uniquely up to a scalar multiple) satisfying the relation

$$\varphi_\sigma' = \varkappa_\sigma f_1^\sigma \circ \varphi_\sigma \circ f_1^{-1} \quad \text{for all } \sigma \in \mathcal{G}_{\varrho_1} \tag{31}$$

with $\varkappa_\sigma \in \bar{k}_0$, and the identification of $\mathfrak{A}_1$ and $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$ is made in such a way that one has

$$X \mid V_1' = f_1 \circ \theta_1(X) \circ f_1^{-1} \quad \text{for } X \in \mathfrak{A}_1 = \mathcal{E}(\mathbf{V}_1/\mathfrak{K}). \tag{32}$$

It follows from (31) that one has

$$\mu_{\sigma,\tau}^{-1} = \frac{\varkappa_\sigma^\tau \varkappa_\tau}{\varkappa_{\sigma\tau}} \lambda_{\sigma,\tau} \sim \lambda_{\sigma,\tau}. \tag{33}$$

Quite similarly, let $\mathbf{V}_2$ be an $n'$-dimensional left $\mathfrak{K}$-space defined over $K$ and put $V_2' = \varepsilon_{11}\mathbf{V}_2$. For each $\sigma \in \mathcal{G}_{\varrho_1}$, the left multiplication $L_{\eta_\sigma}: x \to \eta_\sigma x$ induces an isomorphism of $V_2'$ onto $V_2'^\sigma = \varepsilon_{11}^\sigma \mathbf{V}_2$ (viewed as $\mathcal{E}(\mathfrak{K}\backslash\mathbf{V}_2)$-spaces). One identifies $\mathfrak{A}_2$ with $\mathcal{E}(\mathfrak{K}\backslash\mathbf{V}_2)$ (=the algebra of all $\mathfrak{K}$-linear endomorphisms of $\mathbf{V}_2$) in such a way that one has

$$X \mid V_2' = f_2 \circ (\theta_2(X)) \circ f_2^{-1} \quad \text{for all } X \in \mathfrak{A}_2 = \mathcal{E}(\mathfrak{K}\backslash\mathbf{V}_2), \tag{32'}$$

where $f_2$ is a $\bar{k}_0$-isomorphism of $V_2$ onto $V_2'$ satisfying the relation

$$f_2^{-\sigma} \circ L_{\eta_\sigma} \circ f_2 = \varkappa_\sigma' \Phi_\sigma^{-1} \quad \text{for every } \sigma \in \mathcal{G}_{\varrho_1} \tag{31'}$$

with $\varkappa_\sigma' \in \bar{k}_0$. It follows that $\quad \mu_{\sigma,\tau} = \dfrac{\varkappa_\sigma'^\tau \varkappa_\tau'}{\varkappa_{\sigma\tau}'} \lambda_{\sigma,\tau}^{-1}.$

Comparing this with (33), one sees that $\{\varkappa_\sigma \varkappa_\sigma'\}$ is a (continuous) 1-cocycle of $\mathcal{G}_{\varrho_1}$ in $(\bar{k}_0)^*$, so that by Hilbert's lemma (Th. 90) there exists an element $\theta \in \bar{k}_0$ such that one has $\varkappa_\sigma \varkappa_\sigma' = \theta^{\sigma-1}$. Therefore, replacing $f_2$ by $\theta f_2$, one may assume that $\varkappa_\sigma' = \varkappa_\sigma^{-1}$.

We can now form a tensor-product $\mathbf{V}_1 \otimes_{\mathfrak{K}} \mathbf{V}_2$ of $\mathbf{V}_1$ and $\mathbf{V}_2$ over $\mathfrak{K}$, which is an $nn'r^2$-dimensional vector-space defined over $K$, obtained from the ordinary tensor-product $\mathbf{V}_1 \otimes \mathbf{V}_2$ by identifying $(x_1\alpha)\otimes x_2$ with $x_1 \otimes (\alpha x_2)$, for $x_i \in \mathbf{V}_i$, $\alpha \in \mathfrak{K}$. We shall show that $\hat{V}_1$ is actually $K$-isomorphic to $\mathbf{V}_1 \otimes_{\mathfrak{K}} \mathbf{V}_2$ (as representation-spaces) by the mapping $(f_1 \otimes f_2)\circ\psi^{-1}$. First of all, one has

$$\mathbf{V}_1 \otimes_{\mathfrak{K}} \mathbf{V}_2 = (\sum V_1' \varepsilon_{1i}) \otimes_{\mathfrak{K}} (\sum \varepsilon_{i1} V_2') = V_1' \otimes_{\mathfrak{K}} V_2',$$

which can be identified with the ordinary tensor-product $V_1' \otimes V_2'$. Hence it is enough to show that one has

$$(f_1(x)\otimes_{\mathfrak{K}} f_2(u))^{\sigma}=(f_1\otimes f_2)((x\otimes u)^{[\sigma]})\quad \text{for all } x\in (V_1)_{\bar{k}_0},\ u\in (V_2)_{\bar{k}_0},\ \sigma\in \mathcal{G}_{\varrho_1}.$$

From (11), (31), (31′), the right-hand side is equal to

$$f_1(\varphi_\sigma^{-1}(x^\sigma))\otimes_{\mathfrak{K}} f_2(u^\sigma\Phi_\sigma)=\varkappa_\sigma f_1^\sigma(x^\sigma)\eta_\sigma\otimes_{\mathfrak{K}}\varkappa'_\sigma\eta_\sigma^{-1}f_2^\sigma(u^\sigma)=(f_1(x)\otimes_{\mathfrak{K}} f_2(u))^\sigma,$$

which proves our assertion. From (24), (32), (32′), it is easy to see that this isomorphism of $\hat{V}_1$ and $\mathbf{V}_1\otimes_{\mathfrak{K}}\mathbf{V}_2$ is compatible with the identification of $\mathcal{E}(\hat{V}_1)$ and $\mathfrak{A}_1\otimes\mathfrak{A}_2$ mentioned in 3.1. In particular, one has $\hat{\varrho}_1(g)=\mathsf{P}_1(g)\otimes 1$ for $g\in G$.

**3.3.** As the results of the above considerations, we obtain the following propositions.

PROPOSITION 1. *Let $G$ be an algebraic group defined over a field $k_0$ (of characteristic zero) and let $(V_1, \varrho_1)$ be an absolutely irreducible representation defined over $\bar{k}_0$ of $G$. Let $K_{\varrho_1}$ be a finite extension of $k_0$ defined in* 1.1. *Then there exists a uniquely determined central division algebra $\mathfrak{K}_{\varrho_1}$, a (finite-dimensional) right $\mathfrak{K}_{\varrho_1}$-space $\mathbf{V}_1$, both defined over $K_{\varrho_1}$, and a $K_{\varrho_1}$-homomorphism $\mathsf{P}_1$ of $G$ into $GL(\mathbf{V}_1/\mathfrak{K}_{\varrho_1})$ (=the group of all non-singular $\mathfrak{K}_{\varrho_1}$-linear automorphisms of $\mathbf{V}_1$) such that $\varrho_1$ is factorized in the following form:*

$$\varrho_1(g)=\theta_1(\mathsf{P}_1(g))\quad \textit{for all } g\in G, \tag{34}$$

*where $(V_1, \theta_1)$ is a (unique) absolutely irreducible representation defined over $\bar{k}_0$ of $\mathcal{E}(\mathbf{V}_1/\mathfrak{K}_{\varrho_1})$ (=the algebra of all $\mathfrak{K}_{\varrho_1}$-linear endomorphisms of $\mathbf{V}_1$).*

Here $K_{\varrho_1}$ is also uniquely characterized as the smallest extension of $k_0$ over which such a (non-commutative) representation $(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathsf{P}_1)$ can be constructed. In fact, if one has (34) over $K$, it follows that $\varrho_1^\sigma=\theta_1^\sigma\circ\mathsf{P}_1\sim\theta_1\circ\mathsf{P}_1=\varrho_1$ for all $\sigma\in\mathcal{G}(K)$, which shows that $K$ should contain the field defined in 1.1. As for the uniqueness of $\mathfrak{K}_{\varrho_1}$, it is enough to note that, if one has (34) with $\mathfrak{K}$, then one has

$$\theta_1^\sigma(X)=\varphi_\sigma\circ\theta_1(X)\circ\varphi_\sigma^{-1}\quad \text{for all } X\in\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$$

and for all $\sigma\in\mathcal{G}(K_{\varrho_1})$, which shows that the Brauer class of $\mathcal{E}(\mathbf{V}_1/\mathfrak{K})$, i.e., that of $\mathfrak{K}$, is just the one corresponding to the cohomology class of $(\lambda_{\sigma,\tau}^{-1})$. A (non-commutative) representation $(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathsf{P}_1)$ given above will sometimes be called an "absolutely irreducible representation" of $G$ in $\mathfrak{K}_{\varrho_1}$.

PROPOSITION 2. *The notation being as in Proposition* 1, *let $\mathbf{V}_2$ be a finite-dimensional left $\mathfrak{K}_{\varrho_1}$-space defined over $K_{\varrho_1}$ and put*

$$\left.\begin{aligned}&\hat{V}_1=\mathbf{V}_1\otimes_{\mathfrak{K}_{\varrho_1}}\mathbf{V}_2,\ \hat{\varrho}_1=\mathsf{P}_1\otimes\text{(triv.)},\\ &(V,\varrho)=R_{K_{\varrho_1}/k_0}(\hat{V}_1,\hat{\varrho}_1).\end{aligned}\right\} \tag{35}$$

*Then $(V, \varrho)$ is a $k_0$-primary representation of $G$ containing $(V_1, \varrho_1)$; and conversely all such representations of $G$ are obtained in this manner. The $k_0$-equivalence-class of $(V, \varrho)$ depends only on* $\dim \mathbf{V}_2$*; in particular, $(V_1, \varrho)$ is $k_0$-irreducible if and only if* $\dim \mathbf{V}_2 = 1$.

Denote by $R_{\mathfrak{K}_{\varrho_1}/K_{\varrho_1}}(\mathbf{V}_1, \mathsf{P}_1)$ the representation defined over $K_{\varrho_1}$ (of degree $nr^2$) obtained from $(\mathbf{V}_1, \mathsf{P}_1)$. Then the representation $(\hat{V}_1, \hat{\varrho}_1)$ defined by (35) is equivalent to the direct sum of $n'$ ($=\dim \mathbf{V}_2$) copies of $R_{\mathfrak{K}_{\varrho_1}/K_{\varrho_1}}(\mathbf{V}_1, \mathsf{P}_1)$, which in turn is equivalent to the direct sum of $r$ copies of $(V_1, \varrho_1)$; thus $(\hat{V}_1, \hat{\varrho}_1)$ is absolutely primary and so $(V, \varrho)$ is $k_0$-primary. This proves the first assertion. The rest is clear from what we have seen already. (One may note that, to obtain Proposition 2, we do not need the results stated at the end of 1.2 and in (26), so that these results can also be considered as consequences of Proposition 2.)

*Example* 2. Consider the case of a "crossed product" $\mathfrak{K} = (K''/K, \mu_{\sigma,\tau})$ (not necessarily division), where $K''$ is a finite Galois extension of $K$ with Galois group

$$\mathcal{G}(K''/K) = \{\sigma_1(=1), \dots, \sigma_r\}$$

and where $(\mu_{\sigma,\tau})$ is a 2-cocycle of $\mathcal{G}(K''/K)$ in $K''^*$. By definition, there exist, for every $\sigma \in \mathcal{G}(K''/K)$, an element $u_\sigma \in \mathfrak{K}_K$ and a monomorphism (of fields) $i$ of $K''$ into $\mathfrak{K}_K$ such that one has

$$\begin{cases} \mathfrak{K}_K = \sum_{j=1}^{r} u_{\sigma_j} i(K''), \\ u_\sigma u_\tau = u_{\sigma\tau} i(\mu_{\sigma,\tau}), \\ u_\sigma^{-1} i(\xi) u_\sigma = i(\xi^\sigma) \quad \text{for all } \sigma, \tau \in \mathcal{G}(K''/K), \xi \in K''. \end{cases} \tag{36}$$

An isomorphism $M$ of $\mathfrak{K}$ onto $\mathfrak{M}_r$, defined over $K''$, can be given by the relation

$$x(u_{\sigma_1}, \dots, u_{\sigma_r}) = (u_{\sigma_1}, \dots, u_{\sigma_r})\, i(M(x)) \quad \text{for all } x \in \mathfrak{K}_K.$$

Then, for every $\sigma \in \mathcal{G}(K''/K)$, one has $M^\sigma(x) = M(\eta_\sigma^{-1} x \eta_\sigma)$ with $\eta_\sigma$ given by

$$\eta_\sigma = \sum_{i=1}^{r} \mu_{\sigma_i, \sigma}\, \varepsilon_{\sigma_i \sigma, \sigma_i}, \tag{37}$$

where (and in the following) $\varepsilon_{\sigma_i, \sigma_j}$ stands for $\varepsilon_{i,j}$. It follows that

$$\varepsilon_{\sigma_i, \sigma_j}^{\sigma} = \frac{\mu_{\sigma_i, \sigma}}{\mu_{\sigma_j, \sigma}}\, \varepsilon_{\sigma_i \sigma, \sigma_j \sigma}, \tag{37'}$$

and $\eta_\sigma^\tau \eta_\tau = \mu_{\sigma,\tau} \eta_{\sigma\tau}$. Conversely, it is easy to see that, if $\eta_\sigma$ can be written in the form (37), or equivalently, if one has (37′), then $\mathfrak{K}$ becomes a crossed product.

Let us also remark that, in case $\mathfrak{K}_{\varrho_1}$ is a crossed product, the matrices $\Phi_\sigma$ (defined in 1.3) can be written in the form as given in Example 1. To see this, take a basis $(\mathfrak{e}_1, \ldots, \mathfrak{e}_{n'})$ of $\mathbf{V}_2$ over $\mathfrak{K}_{\varrho_1}$; then $\varepsilon_{1i}\mathfrak{e}_j$ $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n')$ form a basis of $V_2'$. Define a linear isomorphism $f_2$ of $V_2$ onto $V_2'$ by

$$V_2 \ni u = (u_{ij}) \longrightarrow f_2(u) = \sum u_{ij}\varepsilon_{1i}\mathfrak{e}_j \in V_2'.$$

Then, from (31)′ and (37) one obtains at once that

$$\Phi_\sigma = \varkappa_\sigma^{-1}(\mu_{\sigma_i,\sigma}^{-1}\delta_{\sigma_i\sigma,\sigma_j}) \otimes 1_{n'} \sim (\lambda_{\sigma_i,\sigma}\delta_{\sigma_i\sigma,\sigma_j}) \otimes 1_{n'}.$$

Thus one has

$$\hat{V}_1 \cong n' R_{K''/K_{\varrho_1}}(V_1'), \quad \text{i.e.,} \quad V \cong n' R_{K''/k_0}(V_1').$$

**3.4.** We shall now consider a $k_0$-primary symplectic representation $(V, A, \varrho)$ of type (a). We first contend that $\mathfrak{A}_1$ is then a simple algebra with an involution of the first kind defined over $K_{\varrho_1}$. In fact, since $B_1$ is symmetric or alternating, one can define an "involution" (i.e., involutorial anti-automorphism) $\iota$ of $\mathcal{E}(V_1)$ by

$$B_1(x, \varphi^\iota y) = B_1(\varphi x, y) \quad \text{for all } x, y \in V_1, \varphi \in \mathcal{E}(V_1). \tag{38}$$

Then, applying $\sigma \in \mathcal{G}_{\varrho_1}$ on the both sides of (38) and in view of (14), (23), one sees immediately that

$$(\varphi^\iota)^{[\sigma]} = (\varphi^{[\sigma]})^\iota,$$

which shows that the involution $\theta_1^{-1} \circ \iota \circ \theta_1$ of $\mathfrak{A}_1$, denoted again by $\iota$, is defined over $K_{\varrho_1}$.

Now, since $\mathfrak{A}_1 \sim \mathfrak{K}_{\varrho_1}$, $\mathfrak{K} = \mathfrak{K}_{\varrho_1}$ has also an involution of the first kind, denoted by $\iota_0$, defined over $K = K_{\varrho_1}$, by a theorem of Albert ([1], [2]). (In this and the next paragraphs, we shall again omit the subscript $\varrho_1$.) Then one has

$$M(\xi^{\iota_0}) = J^{-1\,t}M(\xi)J \quad \text{for } \xi \in \mathfrak{K} \tag{39}$$

with a matrix $J \in GL(r, \bar{k}_0)$, uniquely determined up to a scalar multiple, satisfying the relation

$${}^tJ = \varepsilon_0 J, \quad \varepsilon_0 = \pm 1. \tag{40}$$

Applying $\sigma \in \mathcal{G}_{\varrho_1}$ on (39), and in view of (27), one gets

$${}^tM(\eta_\sigma)JM(\eta_\sigma) = \mu_\sigma J^\sigma \tag{41}$$

with $\mu_\sigma \in \bar{k}_0$, whence it follows that

$$\mu_{\sigma,\tau}^2 = \mu_\sigma^\tau \mu_\tau \mu_{\sigma\tau}^{-1}. \tag{42}$$

Comparing this with (15), (33), one sees that the system $\{\varkappa_\sigma^2\lambda_\sigma\mu_\sigma\}$ becomes a (continuous)

1-cocycle of $\mathcal{G}_{\varrho_1}$ in $(\bar{k}_0)^*$, so that by Hilbert's lemma one can find $\theta \in (\bar{k}_0)^*$ such that one has $\varkappa_\sigma^2 \lambda_\sigma \mu_\sigma = \theta^{\sigma-1}$. Therefore, replacing $J$ by $\theta J$ (or $B_1$ by $\theta^{-1} B_1$), one may assume that

$$\varkappa_\sigma^2 \lambda_\sigma \mu_\sigma = 1. \tag{43}$$

Next, we shall establish a one-to-one correspondence between the symmetric or alternating forms $B_1$ on $V_1$ satisfying (14) and the $\varepsilon$-hermitian forms $\mathbf{F}_1$ on $\mathbf{V}_1$ with respect to $(\mathfrak{K}, \iota_0)$ defined over $K$. By the latter, we mean bilinear mappings $\mathbf{F}_1$ from $\mathbf{V}_1 \times \mathbf{V}_1$ into $\mathfrak{K}$ defined over $K$, ($\mathbf{V}_1$ and $\mathfrak{K}$ being considered as vector-spaces defined over $K$), satisfying the conditions

$$\begin{cases} \mathbf{F}_1(x\alpha, y\beta) = \alpha^{\iota_0} \mathbf{F}_1(x, y)\,\beta, \\ \mathbf{F}_1(y, x) = \varepsilon\, \mathbf{F}_1(x, y)^{\iota_0}, \quad \varepsilon = \pm 1, \end{cases} \tag{44}$$

for all $x, y \in \mathbf{V}_1$, $\alpha, \beta \in \mathfrak{K}$. To begin with, suppose such an $\mathbf{F}_1$ is given, and put

$$J \cdot M(\mathbf{F}_1(x, y)) = (B_{ij}(x, y)),$$

where $B_{ij}$'s are bilinear forms on $\mathbf{V}_1 \times \mathbf{V}_1$. From (44), (39), one sees at once that

$$B_{11} \,|\, \mathbf{V}_1 \varepsilon_{kk} \times \mathbf{V}_1 \varepsilon_{ll}$$

is identically zero except for $k = l = 1$, and one has

$$B_{ij}(x, y) = B_{11}(x\varepsilon_{i1}, y\varepsilon_{j1}) \quad \text{for all } x, y \in \mathbf{V}_1.$$

Thus, putting $B_1' = B_{11} \,|\, V_1' \times V_1'$ one has

$$J \cdot M(\mathbf{F}_1(x, y)) = (B_1'(x\varepsilon_{i1}, y\varepsilon_{j1})). \tag{45}$$

In the second place, from (40), (44), one sees that $B_1'$ is $\varepsilon_0\varepsilon$-symmetric (i.e., symmetric or alternating according as $\varepsilon_0\varepsilon = +1$ or $-1$). Finally, from the fact that $\mathbf{F}_1$ is defined over $K$ and from (41), (27), one obtains the relation

$$B_1'^{\sigma}(x\eta_\sigma^{-1}, y\eta_\sigma^{-1}) = \mu_\sigma^{-1} B_1'(x, y) \quad \text{for all } x, y \in V_1'.$$

Therefore, putting

$$B_1(x, y) = B_1'(f_1(x), f_1(y)) \quad \text{for } x, y \in V_1, \tag{46}$$

one gets, in view of (31) and (43), an $\varepsilon_0\varepsilon$-symmetric bilinear form $B_1$ on $V_1$ satisfying the relation (14). Conversely, it is easy to see that, given such a form $B_1$ on $V_1$, one can define an $\varepsilon$-hermitian form $\mathbf{F}_1$ on $\mathbf{V}_1$ with respect to $(\mathfrak{K}, \iota_0)$ defined over $K$ by (45), (46).

One notes that from (32) one has

$$J \cdot M(\mathbf{F}_1(Xx, Yy)) = (B_1(\theta_1(X) f_1^{-1}(x\varepsilon_{i1}), \theta_1(Y) f_1^{-1}(y\varepsilon_{j1})))$$

for all $x, y \in \mathbf{V}_1$, $X, Y \in \mathfrak{A}_1 = \mathcal{E}(\mathbf{V}_1/\mathfrak{K})$. It follows that $B_1$ is $\varrho_1(G)$-invariant if and only if $\mathbf{F}_1$ is $\mathsf{P}_1(G)$-invariant. It follows also that, for an involution of the first kind $\iota$ of $\mathfrak{A}_1$, $B_1$ satisfies (38), if and only if the corresponding $\mathbf{F}_1$ satisfies

$$\mathbf{F}_1(Xx, y) = \mathbf{F}_1(x, X^{\iota}y) \quad \text{for all } x, y \in \mathbf{V}_1,\ X \in \mathfrak{A}_1. \tag{38'}$$

Quite similarly, one can establish a one-to-one correspondence between the $\varepsilon$-hermitian forms $\mathbf{F}_2$ on $\mathbf{V}_2$ with respect to $(\mathfrak{K}, \iota_0)$ defined over $K$ and the $\varepsilon_0\varepsilon$-symmetric bilinear forms (or matrices) $B_2$ on $V_2$ satisfying (14') by the relations:

$$M(\mathbf{F}_2(x, y)) \cdot J^{-1} = (B_2'(\varepsilon_{1j}y, \varepsilon_{1i}x)) \quad \text{for } x, y \in \mathbf{V}_2, \tag{45'}$$

$$B_2(x, y) = B_2'(f_2(x), f_2(y)) \quad \text{for } x, y \in V_2. \tag{46'}$$

(Of course, in the definition of an $\varepsilon$-hermitian form $\mathbf{F}_2$ on $\mathbf{V}_2$ with respect to $(\mathfrak{K}, \iota_0)$, one should replace the first condition in (44) by

$$\mathbf{F}_2(\alpha x, \beta y) = \alpha \mathbf{F}_2(x, y) \beta^{\iota_0}$$

for all $x, y \in \mathbf{V}_2$, $\alpha, \beta \in \mathfrak{K}$.)

Combining these with the results obtained in 2.2, we get the following:

PROPOSITION 3. *The notation being as in Propositions* 1, 2, *suppose that* $(V_1, \varrho_1)$ *is of type* (a). *Then* $\mathfrak{K}_{\varrho_1}$ *has an involution* $\iota_0$ *of the first kind defined over* $K_{\varrho_1}$, *and there exists a non-degenerate* $\mathsf{P}_1(G)$-*invariant* $\varepsilon$-*hermitian form* $\mathbf{F}_1$ *on* $\mathbf{V}_1$ *with respect to* $(\mathfrak{K}_{\varrho_1}, \iota_0)$ *defined over* $K_{\varrho_1}$, *determined uniquely up to a scalar multiple. Let* $\mathbf{F}_2$ *be any non-degenerate* $(-\varepsilon)$-*hermitian form on* $\mathbf{V}_2$ *with respect to* $(\mathfrak{K}_{\varrho_1}, \iota_0)$ *defined over* $K_{\varrho_1}$, *and put*

$$\begin{cases} \hat{A}_1(x_1 \otimes_{\mathfrak{K}_{\varrho_1}} x_2, y_1 \otimes_{\mathfrak{K}_{\varrho_1}} y_2) = \mathrm{tr}_{\mathfrak{K}_{\varrho_1}}(\mathbf{F}_1(x_1, y_1)\, \mathbf{F}_2(y_2, x_2)) \quad \text{for all } x_1, y_1 \in \mathbf{V}_1,\ x_2, y_2 \in \mathbf{V}_2, \\ A = \mathrm{tr}_{K_{\varrho_1}/k_0}(\hat{A}_1), \end{cases} \tag{47}$$

$\mathrm{tr}_{\mathfrak{K}_{\varrho_1}}$ *denoting the reduced trace of* $\mathfrak{K}_{\varrho_1}$. *Then* $A$ *is a non-degenerate* $\varrho(G)$-*invariant alternating form on* $V$ *defined over* $k_0$; *and conversely all such forms* $A$ *are obtained in this manner.*

Rests to prove the last assertion. It suffices to show that, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are corresponding to $B_1$ and $B_2$, respectively, in the above sense, then the first relation in (47) is equivalent to (13). In fact, one has

$$\begin{aligned} \mathrm{tr}_{\mathfrak{K}}(\mathbf{F}_1(x_1, y_1)\, \mathbf{F}_2(y_2, x_2)) &= \mathrm{tr}\,(J \cdot M(\mathbf{F}_1(x_1, y_1)) \cdot M(\mathbf{F}_2(y_2, x_2)) \cdot J^{-1}) \\ &= \sum_{i,j=1}^{r} B_1'(x_1 \varepsilon_{i1}, y_1 \varepsilon_{j1})\, B_2'(\varepsilon_{1i} x_2, \varepsilon_{1j} y_2). \end{aligned}$$

But, since $x_1 \otimes_{\mathfrak{K}} x_2 = \sum_{i=1}^{r} x_1 \varepsilon_{1i} \otimes_{\mathfrak{K}} \varepsilon_{1i} x_2$, the element in $\hat{V}_1$ corresponding to $x_1 \otimes_{\mathfrak{K}} x_2$ under the isomorphism $\hat{V}_1 \cong \mathbf{V}_1 \otimes_{\mathfrak{K}} \mathbf{V}_2$ given in 3.2 is $\psi\left(\sum_{i=1}^{r}(f_1^{-1}(x_1 \varepsilon_{i1}) \otimes f_2^{-1}(\varepsilon_{1i} x_2)\right)$ and similarly for $y_1 \otimes_{\mathfrak{K}} y_2$. This proves our assertion.

*Example* 3. As is well-known, in the case $k_0 = \mathbf{Q}$, the quaternion algebras are the only division algebras with an involution of the first kind. A quaternion algebra $\mathfrak{K} = (\beta, \gamma)$ is a crossed product $(K''/K, \mu_{\sigma,\tau})$, where $K'' = K(\sqrt{\beta})$ is a quadratic extension of $K$ with Galois group $\{1, \sigma_0\}$ and where $\mu_{\sigma_0, \sigma_0} = \gamma$ and all the other $\mu_{\sigma,\tau}$'s are $=1$. Putting $\varepsilon_1 = i(\sqrt{\beta})$, $\varepsilon_2 = \mu_{\sigma_0}$ in the notation of Example 2, one has a basis $(1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2)$ of $\mathfrak{K}$ satisfying the relation

$$\varepsilon_1^2 = \beta, \quad \varepsilon_2^2 = \gamma, \quad \varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1.$$

If one takes the representation $M$ as given in Example 2, i.e.

$$M(\xi_0 + \xi_1\varepsilon_1 + \xi_2\varepsilon_2 + \xi_3\varepsilon_1\varepsilon_2) = \begin{pmatrix} \xi_0 + \xi_1\sqrt{\beta} & \gamma(\xi_2 + \xi_3\sqrt{\beta}) \\ \xi_2 - \xi_3\sqrt{\beta} & \xi_0 - \xi_1\sqrt{\beta} \end{pmatrix}, \tag{48}$$

one has $\eta_{\sigma_0} = \varepsilon_2$. On the other hand, for the canonical involution:

$$\iota_0 : \xi_0 + \xi_1\varepsilon_1 + \xi_2\varepsilon_2 + \xi_3\varepsilon_1\varepsilon_2 \longrightarrow \xi_0 - \xi_1\varepsilon_1 - \xi_2\varepsilon_2 - \xi_3\varepsilon_1\varepsilon_2$$

one has in (39)

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{49}$$

so that one has $\varepsilon_0 = -1$ and $\mu_{\sigma_0} = -\gamma$.

**3.5.** For a $k_0$-primary symplectic representation $(V, A, \varrho)$ of type (b), one can prove quite similarly as above, that $\mathfrak{A}_1$ has an "involution of the second kind" $\iota$ (with respect to $\sigma_0$) defined over $K$. By this, we mean a semi-linear anti-automorphism $\iota$ of $\mathfrak{A}_1$ (with respect to $\sigma_0$) defined over $K$ satisfying the relation

$$X^{\iota^2} = X^{\sigma_0^2} \quad \text{for all } X \in \mathfrak{A}_1. \tag{50}$$

In fact, defining a semi-linear anti-automorphism $\iota$ of $\mathcal{E}(V_1)$ by

$$F_1(x, \varphi^\iota y) = F_1(\varphi x, y) \quad \text{for all } x, y \in V_1,\ \varphi \in \mathcal{E}(V_1), \tag{51}$$

one verifies at once by (20), (21), (23) that

$$\varphi^{\iota^2} = \varphi^{[\sigma_0^2]}, \quad (\varphi^\iota)^{[\sigma]} = (\varphi^{[\sigma]})^\iota \quad \text{for all } \sigma \in \mathcal{G}_{\varrho_1},$$

which proves our assertion. (We denote $\theta_1^{-1}\iota\theta_1$ again by $\iota$.)

Now, since $\mathfrak{A}_1 \sim \mathfrak{K}$, $\mathfrak{K}$ has also an involution of the second kind (with respect to $\sigma_0$) defined over $K$, denoted by $\iota_0$, by a theorem of Albert (loc. cit.). Then, one has

$$M(\xi^{\iota_0}) = J^{-1}\,{}^tM(\xi)^{\sigma_0}J \tag{52}$$

with $J \in GL(r, \bar{k}_0)$, uniquely determined up to a scalar multiple. Applying this equality twice on $M(\xi^{\iota_0^2})$ and in view of the relation $\iota_0^2 = \sigma_0^2$ and (27), one has

$${}^tJ^{\sigma_0} = \mu^{-1}JM(\eta_{\sigma_0^2}) \tag{53}$$

with $\mu \in \bar{k}_0$. On the other hand, applying $\sigma \in \mathcal{G}_{\varrho_1}$ on (52) and in view of the relation $\iota_0\sigma = \bar{\sigma}\iota_0$ and (27), one has

$${}^tM(\eta_{\bar{\sigma}})^{\sigma_0}JM(\eta_\sigma) = \mu_\sigma J^\sigma \quad \text{for } \sigma \in \mathcal{G}_{\varrho_1} \tag{54}$$

with $\mu_\sigma \in \bar{k}_0$.

From (54), it follows that

$$\mu^{\sigma_0}_{\bar{\sigma},\bar{\tau}}\mu_{\sigma,\tau} = \mu^\tau_\sigma \mu_\tau \mu^{-1}_{\sigma\tau}. \tag{55 a}$$

Furthermore, easy calculations from (53), (54), combined with (27), (28), give the relations

$$\mu_{\sigma_0^2} = \mu^{\sigma_0+1}, \tag{55 b}$$

$$\mu^{\sigma_0}_{\bar{\sigma}}\mu^{-1}_\sigma = \mu^{\sigma-1}\mu_{\bar{\bar{\sigma}},\sigma_0^2}\mu^{-1}_{\sigma_0^2,\sigma}. \tag{55 c}$$

First, comparing (55a) with (22a) and (33), one sees that $\{\varkappa^{\sigma_0}_{\bar{\sigma}}\varkappa_\sigma\lambda_\sigma\mu_\sigma\}$ becomes a (continuous) 1-cocycle of $\mathcal{G}_{\varrho_1}$ in $(\bar{k}_0)^*$, so that by Hilbert's lemma one has $\varkappa^{\sigma_0}_{\bar{\sigma}}\varkappa_\sigma\lambda_\sigma\mu_\sigma = \theta_1^{\sigma-1}$ for all $\sigma \in \mathcal{G}_{\varrho_1}$ with a $\theta_1 \in \bar{k}_0$. Hence, replacing $J$ by $\theta_1 J$ (or $F_1$ by $\theta_1^{-1}F_1$), one may assume

$$\varkappa^{\sigma_0}_{\bar{\sigma}}\varkappa_\sigma\lambda_\sigma\mu_\sigma = 1. \tag{56 a}$$

Next, from (55b) and (22b), one has

$$(\varkappa_{\sigma_0^2}\lambda\mu)^{\sigma_0+1} = \varkappa^{\sigma_0}_{\sigma_0^2}\varkappa_{\sigma_0^2}\lambda_{\sigma_0^2}\mu_{\sigma_0^2} = 1.$$

On the other hand, from (55c) and (33), one has

$$(\varkappa_{\sigma_0^2}\lambda\mu)^{\sigma-1} = \varkappa^{\sigma-1}_{\sigma_0^2}\cdot\frac{\lambda^{\sigma_0}_{\bar{\sigma}}\mu^{\sigma_0}_{\bar{\sigma}}}{\lambda_\sigma\mu_\sigma}\cdot\frac{\varkappa^{\sigma_0^2}_{\bar{\bar{\sigma}}}\varkappa_{\sigma_0^2}\varkappa^{-1}_{\bar{\bar{\sigma}}\sigma_0^2}}{\varkappa^{\sigma}_{\sigma_0^2}\varkappa_\sigma\varkappa^{-1}_{\sigma_0^2\sigma}} = \frac{(\varkappa^{\sigma_0}_{\bar{\bar{\sigma}}}\varkappa_{\bar{\sigma}}\lambda_{\bar{\sigma}}\mu_{\bar{\sigma}})^{\sigma_0}}{\varkappa^{\sigma_0}_{\bar{\sigma}}\varkappa_\sigma\lambda_\sigma\mu_\sigma} = 1 \quad \text{for all } \sigma \in \mathcal{G}_{\varrho_1}.$$

which implies that $\varkappa_{\sigma_0^2}\lambda\mu \in K$. Therefore, again by Hilbert's lemma one may write $\varkappa_{\sigma_0^2}\lambda\mu = \theta_2^{\sigma_0-1}$ with $\theta_2 \in K$. Thus, replacing $J$ by $\theta_2 J$, (or $F_1$ by $\theta_2^{-1}F_1$) one may assume (without changing $\mu_\sigma$ and hence the relation (56 a)) that

$$\varkappa_{\sigma_0^2}\lambda\mu = 1. \tag{56 b}$$

*Remark.* It is known (Albert [1]) that, in case the Hilbert's irreducibility theorem holds for $k_0$, (which is surely the case for $k_0 = \mathbf{Q}$), $K$ has a finite Galois splitting field $K''$ over $K$ such that $K''^{\sigma_0} = K''$, $\sigma_0^2 | K'' = 1$ and $\bar{\sigma} | K'' = \sigma | K''$ for all $\sigma \in \mathcal{G}_{\varrho_1}$; in other words, $K''/K_0$ is a Galois extension whose Galois group $\mathcal{G}(K''/K_0)$ is the direct product of $\mathcal{G}(K''/K)$ and $\{1, \sigma_0 | K''\}$. Under this assumption, in taking $M$, $\eta_0$, $J$, etc., to be $K''$-rational, one may assume that $\eta_{\sigma_0^2} = 1$, $\mu = 1$, which reduces (53) to a simpler form: ${}^tJ^{\sigma_0} = J$. But, in this paper, we shall never need this simplification.

We can again establish a one-to-one correspondence between the sesqui-linear forms $F_1$ on $V_1$ (with respect to $\sigma_0$) satisfying (20), (21) and the hermitian forms $\mathbf{F}_1$ on $\mathbf{V}_1$ with respect to $(\mathfrak{K}, \iota_0)$ defined over $K$. By the latter, we mean sesqui-linear mapings $\mathbf{F}_1$ from $\mathbf{V}_1 \times \mathbf{V}_1$ into $\mathfrak{K}$ defined over $K$ satisfying the conditions

$$\begin{cases} \mathbf{F}_1(x\alpha, y\beta) = \alpha^{\iota_0}\mathbf{F}_1(x, y)\,\beta, \\ \mathbf{F}_1(x, y)^{\iota_0} = \mathbf{F}_1(y, x^{\sigma_0^2}) \quad \text{for all } x, y \in \mathbf{V}_1,\ \alpha, \beta \in \mathfrak{K}. \end{cases} \tag{57}$$

In fact, given such a hermitian form $\mathbf{F}_1$ on $\mathbf{V}_1$, one can prove, quite similarly as in 3.4, that there exists a sesqui-linear form $F_1'$ on $V_1'$ such that one has

$$J \cdot M(\mathbf{F}_1(x, y)) = (F_1'(x\varepsilon_{i1}, y\varepsilon_{j1})) \quad \text{for all } x, y \in \mathbf{V}_1, \tag{58}$$

and satisfying the relations

$$F_1'(x, y)^{\sigma_0} = \mu^{-1} F_1'(y, x^{\sigma_0^2}\eta_{\sigma_0^2}),$$

$$F_1'^{\sigma}(x\eta_{\bar{\sigma}}^{-1}, y\eta_{\sigma}^{-1}) = \mu_{\sigma}^{-1} F_1'(x, y) \quad \text{for all } x, y \in V_1'.$$

Then putting
$$F_1(x, y) = F_1'(f_1(x), f_1(y)) \quad \text{for } x, y \in V_1, \tag{59}$$

one concludes from (31), (56a), (56b) that $F_1$ becomes a sesqui-linear form on $V_1$ satisfying (20), (21). The converse is also immediate. Moreover, it is clear that $F_1$ is $\varrho_1(G)$-invariant, if and only if the corresponding $\mathbf{F}_1$ is $\mathsf{P}_1(G)$-invariant.

Quite similarly, one sees that there is a one-to-one correspondence between the hermitian forms $\mathbf{F}_2$ on $\mathbf{V}_2$ with respect to $(\mathfrak{K}, \iota_0)$ defined over $K$ and the sesqui-linear forms (or matrices) $F_2$ on $V_2$ satisfying (20'), (21') by the relations analogous to (45'), (46').

Combining these with the results obtained in 2.3, we get the following:

PROPOSITION 4. *The notation being as in Propositions 1, 2, suppose that $(V_1, \varrho_1)$ is of type* (b) *with respect to $\sigma_0$. Then $\mathfrak{K}_{\varrho_1}$ has an involution $\iota_0$ of the second kind (with respect to $\sigma_0$) defined over $K_{\varrho_1}$, and there exists a non-degenerate $\mathsf{P}_1(G)$-invariant hermitian form $\mathbf{F}_1$*

*on* $\mathbf{V}_1$ *with respect to* $(\mathfrak{K}_{\varrho_1}, \iota_0)$ *defined over* $K_{\varrho_1}$. *Let* $\mathbf{F}_2$ *be any non-degenerate hermitian form on* $\mathbf{V}_2$ *with respect to* $(\mathfrak{K}_{\varrho_1}, \iota_0)$ *defined over* $K_{\varrho_1}$, *and put*

$$\begin{cases} F(x, y) = \mathrm{tr}_{\mathfrak{K}_{\varrho_1}}(\mathbf{F}_1(x_1, y_1)\,\mathbf{F}_2(y_2, x_2)) \quad \textit{for} \quad x = x_1 \otimes_{\mathfrak{K}_{\varrho_1}} x_2,\ y = y_1 \otimes_{\mathfrak{K}_{\varrho_1}} y_2 \in \hat{V}_1 = \mathbf{V}_1 \otimes_{\mathfrak{K}_{\varrho_1}} \mathbf{V}_2, \\ \hat{\hat{A}}_1(x+x', y+y') = \sqrt{\alpha}^{-1}\{F(x'^{\sigma_0^{-1}}, y) - F(y'^{\sigma_0^{-1}}, x)\} \quad \textit{for} \quad x, y \in \hat{V}_1,\ x', y' \in \hat{V}_1^{\sigma_0}, \\ A = \mathrm{tr}_{K_0/k_0}(\hat{\hat{A}}_1), \end{cases} \tag{60}$$

*where* $K_0$ *is the fixed subfields of* $\sigma_0 \,|\, K_{\varrho_1}$ *in* $K_{\varrho_1}$ *and* $\alpha$ *is an element in* $K_0$ *such that* $K_{\varrho_1} = K_0(\sqrt{\alpha})$. *Then* $A$ *is a non-degenerate* $\varrho(G)$*-invariant alternating form on* $V$ *defined over* $k_0$; *and conversely all such forms* $A$ *are obtained in this manner.*

**3.6.** Let $(V, A, \varrho)$ be a $k_0$-primary symplectic representation of $G$ defined over $k_0$. Then $\varrho$ is a $k_0$-homomorphism of $G$ into $Sp(V, A)$, the symplectic group of $(V, A)$, viewed as an algebraic group defined over $k_0$. The notation being as before, we denote by $G_1' = U(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$ the "unitary group" of $(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$, i.e., the group of all $\mathfrak{K}_{\varrho_1}$-linear transformations of $\mathbf{V}_1$ leaving $\mathbf{F}_1$ invariant. In the case (a), $G_1'$ is a linear algebraic group defined over $K_{\varrho_1}$, operating on the underlying vector-space of $\mathbf{V}_1$, and $\mathsf{P}_1$ is a $K_{\varrho_1}$-homomorphism of $G$ into $G_1'$. In the case (b), $G_1'$ can be viewed as a linear algebraic group defined over $K_0$, operating on the underlying vector-space of $R_{K_{\varrho_1}/K_0}(\mathbf{V}_1)$, and $R_{K_{\varrho_1}/K_0}(\mathsf{P}_1)$ is a $K_0$-homomorphism of $G$ into $G_1'$. Quite similarly, the unitary group $G_2' = U(\mathfrak{K}_{\varrho_1} \backslash \mathbf{V}_2, \mathbf{F}_2)$ can be viewed as a linear algebraic group defined over $K_{\varrho_1}$ or $K_0$. Our results may then be summarized as follows:

THEOREM 1. *Let* $G$ *be an algebraic group defined over a field* $k_0$ *(of characteristic* 0*) and let* $\varrho : G \to G' = Sp(V, A)$ *be a (completely reducible)* $k_0$*-primary symplectic representation. Let* $K_{\varrho_1}$ *be a finite extension of* $k_0$ *defined in* 1.1, *i.e., the smallest field over which an absolutely primary component* $(\hat{V}_1, \hat{\varrho}_1)$ *of* $(V, \varrho)$ *is defined. Then:*

(i) *In case* (a), *there exist a central division algebra* $\mathfrak{K}_{\varrho_1}$ *with an involution of the first kind* $\iota_0$, *a right* $\mathfrak{K}_{\varrho_1}$*-space* $\mathbf{V}_1$ *with a non-degenerate* $\varepsilon$*-hermitian form* $\mathbf{F}_1$ *with respect to* $(\mathfrak{K}_{\varrho_1}, \iota_0)$, *a left* $\mathfrak{K}_{\varrho_1}$*-space* $\mathbf{V}_2$ *with a non-degenerate* $(-\varepsilon)$*-hermitian form* $\mathbf{F}_2$ *with respect to* $(\mathfrak{K}_{\varrho_1}, \iota_0)$, *all defined over* $K_{\varrho_1}$, *and a* $K_{\varrho_1}$*-homomorphism* $\mathsf{P}_1$ *of* $G$ *into* $U(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$, *which is absolutely irreducible as a representation of* $G$ *in* $\mathfrak{K}_{\varrho_1}$ *(in the sense of* 3.3*), such that* $\varrho$ *is factorized in the following manner:*

$$G \xrightarrow{\mathsf{P}_1} \left.\begin{matrix} U(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1) \\ \times \\ U(\mathfrak{K}_{\varrho_1}\backslash \mathbf{V}_2, \mathbf{F}_2) \end{matrix}\right\} \xrightarrow{\ \otimes\ } Sp(\hat{V}_1, \hat{A}_1) \xrightarrow{R_{K_{\varrho_1}/k_0}} Sp(V, A),$$

*where*
$$\begin{cases}\hat{V}_1=\mathbf{V}_1\otimes_{\mathfrak{K}_{\varrho_1}}\mathbf{V}_2,\\ \hat{A}_1=\mathrm{tr}_{\mathfrak{K}_{\varrho_1}}(\mathbf{F}_1\otimes{}^t\mathbf{F}_2),\end{cases}\qquad \begin{cases}V=R_{K_{\varrho_1}/k_0}(\hat{V}_1),\\ A=\mathrm{tr}_{K_{\varrho_1}/k_0}(\hat{A}_1).\end{cases}$$

(ii) *In case* (b) (*with respect to* $\sigma_0$) *let* $K_0$ *be the fixed subfield of* $\sigma_0|K_{\varrho_1}$ *in* $K_{\varrho_1}$ *and let* $\alpha$ *be an element in* $K_0$ *such that* $K_{\varrho_1}=K_0(\sqrt{\alpha})$. *Then there exist a central division algebra* $\mathfrak{K}_{\varrho_1}$ *with an involution of the second kind* $\iota_0$ *inducing* $\sigma_0$ *on the center, a right* (*resp. left*) $\mathfrak{K}_{\varrho_1}$*-space* $\mathbf{V}_1$ (*resp.* $\mathbf{V}_2$) *with a non-degenerate hermitian form* $\mathbf{F}_1$ (*resp.* $\mathbf{F}_2$) *with respect to* $(\mathfrak{K}_{\varrho_1},\iota_0)$, *all defined over* $K_{\varrho_1}$, *and a* $K_0$*-homomorphism* $R_{K_{\varrho_1}/K_0}(\mathsf{P}_1)$ *of* $G$ *into* $U(\mathbf{V}_1/\mathfrak{K}_{\varrho_1},\mathbf{F}_1)$, $\mathsf{P}_1$ *being an absolutely irreducible representation of* $G$ *into* $GL(\mathbf{V}_1/\mathfrak{K}_{\varrho_1})$, *such that* $\varrho$ *is factorized in the following manner:*

$$G\xrightarrow{R_{K_{\varrho_1}/K_0}(\mathsf{P}_1)}\left.\begin{matrix}U(\mathbf{V}_1/\mathfrak{K}_{\varrho_1},\mathbf{F}_1)\\ \times\\ U(\mathfrak{K}_{\varrho_1}\backslash\mathbf{V}_2,\mathbf{F}_2)\end{matrix}\right\}\xrightarrow{\ \otimes\ }U(\hat{V}_1,F)\subset Sp(\hat{\hat{V}}_1,\hat{\hat{A}}_1)\xrightarrow{R_{K_0/k_0}}Sp(V,A),$$

*where*
$$\begin{cases}\hat{V}_1=\mathbf{V}_1\otimes_{\mathfrak{K}_{\varrho_1}}\mathbf{V}_2,\\ F=\mathrm{tr}_{\mathfrak{K}_{\varrho_1}}(\mathbf{F}_1\otimes{}^t\mathbf{F}_2),\end{cases}\quad \begin{cases}\hat{\hat{V}}_1=R_{K_{\varrho_1}/K_0}(\hat{V}_1),\\ \hat{\hat{A}}_1=\mathrm{tr}_{K_{\varrho_1}/K_0}(\sqrt{\alpha}^{-1}F),\end{cases}\quad \begin{cases}V=R_{K_0/k_0}(\hat{\hat{V}}_1),\\ A=\mathrm{tr}_{K_0/k_0}(\hat{\hat{A}}_1).\end{cases}$$

*In either case* $\mathfrak{K}_{\varrho_1}$, $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathsf{P}_1$ *and the multiplicative equivalence-classes of* $\mathbf{F}_1$, $\mathbf{F}_2$ *are determined uniquely. Conversely, any symplectic representation* $\varrho$ *constructed in these manners is* $k_0$*-primary and of type* (a), (b), *respectively.*

In case $G$ is a connected semi-simple algebraic group, one may replace each unitary group in Theorem 1 by the corresponding special unitary group, i.e., the subgroup of the unitary group consisting of all elements with the reduced norm 1.

**3.7.** Finally, we add some remarks about how these data describing a symplectic representation behave under the extension of the ground field $k_0$. Let $\varrho_1$ be, as before, an absolutely irreducible representation of $G$ defined over $\bar{k}_0$. Let $k_0'$ be any extension of $k_0$ (contained in the same universal domain), and we denote the data relative to $k_0'$ by the same symbols with a prime (e.g., $K_{\varrho_1}'$, $\mathfrak{K}_{\varrho_1}'$, $\mathbf{V}_1'$, ...) as those denoting the corresponding data relative to $k_0$ (e.g., $K_{\varrho_1}$, $\mathfrak{K}_{\varrho_1}$, $\mathbf{V}_1$, ...). Then, from the definitions, it is clear that $K_{\varrho_1}'=k_0'K_{\varrho_1}$ and $\mathfrak{K}_{\varrho_1}'\sim\mathfrak{K}_{\varrho_1}$ over $K_{\varrho_1}'$; therefore, putting $r=r't$ ($\dim\mathfrak{K}_{\varrho_1}'=r'^2$), one has $\mathfrak{K}_{\varrho_1}\cong\mathfrak{M}_t(\mathfrak{K}_{\varrho_1}')$ over $K_{\varrho_1}'$. By means of this matrix expression, one obtains, as in 3.2, a $K_{\varrho_1}'$-isomorphism

$$\Theta:\mathcal{E}(\mathbf{V}_1/\mathfrak{K}_{\varrho_1})\to\mathcal{E}(\mathbf{V}_1'/\mathfrak{K}_{\varrho_1}'),\tag{61}$$

where $\mathbf{V}_1'$ is an $nt$-dimensional right $\mathfrak{K}_{\varrho_1}'$-space defined over $K_{\varrho_1}'$. Then one has a factorization $\varrho_1=\theta_1'\circ\mathsf{P}_1'$ of $\varrho_1$ relative to $k_0'$ (as given in Proposition 1) with

$$\theta_1' = \theta_1 \circ \Theta^{-1}, \quad \mathsf{P}_1' = \Theta \circ \mathsf{P}_1, \tag{62}$$

A $k_0$-primary representation $(V, \varrho)$ (containing $\varrho_1$) decomposes into a direct sum of a certain number of $k_0'$-primary components; let $(V', \varrho')$ be the one containing $\varrho_1$. Then, since the absolutely primary component $\hat{V}_1$ containing $\varrho_1$ is determined independently of the choice of the ground field, one has

$$V' = R_{K'_{\varrho_1}/k_0'}(\hat{V}_1), \quad \hat{V}_1 \cong \mathbf{V}_1' \otimes_{\mathfrak{R}'_{\varrho_1}} \mathbf{V}_2' \ (\text{over } K'_{\varrho_1}),$$

where $\mathbf{V}_2'$ is an $n't$-dimensional left $\mathfrak{R}'_{\varrho_1}$-space defined over $K'_{\varrho_1}$ on which $G$ is supposed to operate trivially. (We have to excuse the exceptional use of the notation $n'$ against the general convention settled above.)

Now, it is clear that, if $\varrho_1$ is of type (a) (resp. (c)) over $k_0$, so is it also over $k_0'$; if $\varrho_1$ is of type (b) over $k_0$, then $\varrho_1$ is of type (b) or (c) over $k_0'$. Here we shall be interested only in those cases where $\varrho_1$ is of type (a) or (b) over both $k_0$ and $k_0'$, for only such cases will occur in Part III, where we shall apply our theory with $k_0 = \mathbf{Q}$, $k_0' = \mathbf{R}$.

In the case (a) (over both $k_0$ and $k_0'$), one obtains an $\varepsilon'$-hermitian form $\mathbf{F}_1'$ on $\mathbf{V}_1'$ with respect to $(\mathfrak{R}'_{\varrho_1}, \iota_0')$ defined over $K'_{\varrho_1}$ (where $\varepsilon_0\varepsilon = \varepsilon_0'\varepsilon'$), either from $\mathbf{F}_1$ on $\mathbf{V}_1$ or from $B_1$ on $V_1$ as explained in 3.4, and $\Theta$ induces a $K'_{\varrho_1}$-isomorphism of the unitary groups:

$$U(\mathbf{V}_1/\mathfrak{R}_{\varrho_1}, \mathbf{F}_1) \cong U(\mathbf{V}_1'/\mathfrak{R}'_{\varrho_1}, \mathbf{F}_1'); \tag{63}$$

and similarly for $\mathbf{F}_2'$ on $\mathbf{V}_2'$. In the case (b) (over both $k_0$ and $k_0'$), the Galois automorphism $\sigma_0$ can be taken in common for $k_0$ and $k_0'$, so that one has $K_0' = k_0'K_0$, $K_0 = K_{\varrho_1} \cap K_0'$. The relation between $\mathbf{F}_i$ and $\mathbf{F}_i'$ $(i = 1, 2)$ is the same as above except that this time one has in place of (63) a $K_0'$-isomorphism induced by $R_{K'_{\varrho_1}/K_0'}(\Theta)$.

## Part II. Determination of $K_{\varrho_1}$ and $\mathfrak{R}_{\varrho_1}$

### § 4. Comparison with the quasi-split group

**4.1.** In the following, let $G$ be a connected semi-simple algebraic group defined over $k_0$. Let $T$ be a maximal torus in $G$ defined over $k_0$ and let $X$ be the character module of $T$. We shall fix once and for all a linear order in $X$ and let $\Delta$ be the corresponding fundamental system of roots. For a given absolutely irreducible representation $\varrho_1$ of $G$ we denote by $\lambda_{\varrho_1}$ the corresponding highest weight relative to $T$.

The Galois group $\mathcal{G}$ operates on $X$ in a natural manner and permutes the fundamental systems among themselves. Hence, for every $\sigma \in \mathcal{G}$, there exists a uniquely determined element $w_\sigma$ in the Weyl group $W$ (relative to $T$) such that one has $\Delta^\sigma = w_\sigma \Delta$. For $\chi \in X$, we put

$$\chi^{[\sigma]} = w_\sigma^{-1} \chi^\sigma. \tag{1}$$

Then clearly one has $\Delta^{[\sigma]} = \Delta$. It follows that

$$(\lambda_{\varrho_1})^{[\sigma]} = \lambda_{\varrho_1 \sigma}. \tag{2}$$

This implies the following

PROPOSITION 5. *In the above notation, one has*

$$\mathcal{G}_{\varrho_1} = \{\sigma \in \mathcal{G} \mid \lambda_{\varrho_1}^{[\sigma]} = \lambda_{\varrho_1}\}. \tag{3}$$

COROLLARY 1. *If $G$ is of Chevalley type over $k_0$, then one has $K_{\varrho_1} = k_0$ for all absolutely irreducible representations $\varrho_1$.*

COROLLARY 2. *If $G$ is absolutely simple and not of type $A_l (l \geq 2)$, $D_l$ or $E_6$, then one has $K_{\varrho_1} = k_0$ for all absolutely irreducible representations $\varrho_1$.*

In fact, under the assumption of Corollary 1 or 2, the operation $[\sigma]$ of the Galois group on $X$ is trivial, so that one has $\mathcal{G}_{\varrho_1} = \mathcal{G}$.

**4.2.** A connected semi-simple algebraic group $G$ is called $k_0$-quasi-split (or "of Steinberg type"), if there exists a Borel subgroup $B$ of $G$ defined over $k_0$. If the maximal torus $T$ and the fundamental system $\Delta$ are so chosen as to determine such a $B$, then one has $\Delta^\sigma = \Delta$ for all $\sigma \in \mathcal{G}$. (As is well-known, to get such $T$, $\Delta$, it suffices to take $T$ containing a maximal $k_0$-trivial torus $A$ and define $\Delta$ with respect to a linear order compatible with $X_0 \subset X$, $X_0$ denoting the annihilator of $A$ in $X$.) Hence in this case one has

$$\lambda_{\varrho_1^\sigma} = (\lambda_{\varrho_1})^\sigma \quad \text{for } \sigma \in \mathcal{G}. \tag{2'}$$

PROPOSITION 6. *If $G$ is $k_0$-quasi-split, then $\mathfrak{K}_{\varrho_1} \sim 1$ for all absolutely irreducible representation $\varrho_1$.*

*Proof.* Take $T$ and $\Delta$ as above. Let $V_1$ be the representation-space of $\varrho_1$ and let $x_1$ be an eigen-vector corresponding to the highest weight $\lambda_{\varrho_1}$ which is $\bar{k}_0$-rational; $x_1$ is then uniquely determined up to a scalar multiple. Then it is clear that, for each $\sigma \in \mathcal{G}_{\varrho_1}$, $x_1^\sigma$ is an eigen-vector in $V_1^\sigma$ corresponding to the weight $\lambda_{\varrho_1}^\sigma$, which is the highest one by (2′). Therefore one can normalize the isomorphism $\varphi_\sigma : V_1 \to V_1^\sigma$ in such a way that one has $\varphi_\sigma(x_1) = x_1^\sigma$ for all $\sigma \in \mathcal{G}_{\varrho_1}$. Then it follows that $\lambda_{\sigma,\tau} = 1$, so that one has $\mathfrak{K}_{\varrho_1} \sim 1$.

**4.3.** Let $G$ be a connected semi-simple algebraic group defined over $k_0$. It is known that there exists a $k_0$-quasi-split group $G^1$ from which $G$ is obtained by twisting with respect to inner automorphisms (see e.g. [17]); this means that there exists a $\bar{k}_0$-isomorphism $f$ of $G^1$ onto $G$ such that, for every $\sigma \in \mathcal{G}$, $f^\sigma \circ f^{-1}$ is an inner automorphism of $G$. Such $G^1$ is unique up to a $k_0$-isomorphism (see e.g. [19]). Put $f^\sigma \circ f^{-1} = I_{g_\sigma}$ with $g_\sigma \in G$, $I_{g_\sigma}$ denoting the inner automorphism: $g \to g_\sigma g g_\sigma^{-1}$ of $G$. Since $g_\sigma$ is uniquely determined modulo the center $Z$ of $G$, $c_{\sigma,\tau} = g_\sigma^\tau g_\tau g_{\sigma\tau}^{-1}$ is a 2-cocycle of $\mathcal{G}$ in $Z$, whose cohomology class is uniquely determined. We shall now show that this cohomology class does not depend on the choice of $f$ either. Let $f'$ be another $\bar{k}_0$-isomorphism of $G^1$ onto $G$ such that $f'^\sigma \circ f'^{-1} = I_{g'_\sigma}$ with $g'_\sigma \in G$ for every $\sigma \in \mathcal{G}$. It is known (see e.g. [17]) that $\mathrm{Aut}(G^1)$ splits into a semi-direct product of $\mathrm{Inn}(G^1)$ and a finite subgroup $U^1$ which is invariant under $\mathcal{G}$. Hence we can put $f^{-1} \circ f' = I_{g_1} \circ u$ with $g_1 \in G^1$ and $u \in U^1$. Then one has

$$I_{g'_\sigma} = f^\sigma \circ I_{g_1^\sigma} \circ u^\sigma \circ u^{-1} \circ I_{g_1^{-1}} \circ f^{-1} = I_{f(g_1)^\sigma} I_{g_\sigma} \circ (f \circ u^\sigma \circ u^{-1} \circ f^{-1}) \circ I_{f(g_1)^{-1}}.$$

This shows that $u^\sigma \circ u^{-1}$ is inner and so $u^\sigma = u$. It follows that $g'_\sigma = f(g_1)^\sigma g_\sigma f(g_1)^{-1}$ (mod. $Z$), whence our assertion. We shall denote the cohomology class of $(c_{\sigma,\tau})$ by $\gamma_{k_0}(G)$, or simply by $\gamma(G)$.(1) For any $k \supset k_0$, $\gamma_k(G)$ is the element in $H^2(k, Z)$ obtained from $\gamma(G)$ by restricting the Galois group.

Now let $\varrho_1$ be an absolutely irreducible representation of $G$ defined over $\bar{k}_0$. For every $\sigma \in \mathcal{G}$, one has $(\varrho_1 \circ f)^\sigma = \varrho_1^\sigma \circ (f^\sigma \circ f^{-1}) \circ f \sim \varrho_1^\sigma \circ f$, so that one has

$$\varrho_1^\sigma \sim \varrho_1 \Leftrightarrow (\varrho_1 \circ f)^\sigma \sim \varrho_1 \circ f.$$

Thus one concludes that $\mathcal{G}_{\varrho_1} = \mathcal{G}_{\varrho_1 \circ f}$, $K_{\varrho_1} = K_{\varrho_1 \circ f}$. Since $\mathfrak{R}_{\varrho_1 \circ f} \sim 1$ (Proposition 6), one may assume that $\varrho_1 \circ f$ is defined over $K_{\varrho_1}$. Then one has

$$\varrho_1^\sigma(g) = (\varrho_1 \circ f) \circ f^{-\sigma}(g) = \varrho_1(g_\sigma^{-1} g g_\sigma), \tag{4}$$

which shows that one can take $\varrho_1(g_\sigma)^{-1}$ as $\varphi_\sigma$. It then follows that, for $\sigma, \tau \in \mathcal{G}_{\varrho_1}$, one has

$$\lambda_{\sigma,\tau} = \varphi_\sigma^\tau \varphi_\tau \varphi_{\sigma\tau}^{-1} = \varrho_1^\tau(g_\sigma^{-\tau})\, \varrho_1(g_\tau^{-1})\, \varrho_1(g_{\sigma\tau}) = \varrho_1(c_{\sigma,\tau}^{-1}).$$

Since $Z$ is of finite order, $\lambda_{\sigma,\tau}$'s are all in $\mathsf{E}$ (=the group of all roots of unity in $\bar{k}_0$). In view of (4), the restriction of $\varrho_1$ on $Z$ is a $\mathcal{G}_{\varrho_1}$-homomorphism of $Z$ into $\mathsf{E}$, so that it induces a canonical homomorphism:

$$H^2(K_{\varrho_1}, Z) \to H^2(K_{\varrho_1}, \mathsf{E}),$$

which we shall denote by $\lambda_{\varrho_1}^*$. We have thus obtained the following

THEOREM 2. *Let $G$ be a connected semi-simple algebraic group defined over $k_0$, and let $\gamma(G)$ be the element of $H^2(k_0, Z)$ defined above, $Z$ denoting the center of $G$. Then, for every absolutely irreducible representation $\varrho_1$ of $G$, one has*

$$c(\mathfrak{R}_{\varrho_1}) = \lambda_{\varrho_1}^*(\gamma_{K_{\varrho_1}}(G)). \tag{5}$$

Thus, for the determination of $\mathfrak{R}_{\varrho_1}$, it is enough to determine $\gamma(G)$ for simply connected $G$.

(1) For a $\mathfrak{p}$-adic field $k_0$, the canonical map $H^1(G^1/Z^1) \to H^2(Z^1)$ is bijective, so that the $k_0$-isomorphism class of $G$ is uniquely determined by $f^{*-1}(\gamma(G)) \in H^2(k_0, Z^1)$. Cf. M. Kneser [17].

## § 5. Determination of $\gamma(G)$

**5.1.** Suppose first $G$ is decomposed into the direct product of $k_0$-simple factors: $G=\prod G^{(i)}$. Then it is clear that one has

$$\gamma(G)=\prod \gamma(G^{(i)}).$$

Therefore one may assume $G$ to be $k_0$-simple. Suppose further that $G$ is decomposable into the direct product of absolutely simple factors. Then one may put

$$G=\prod_{i=1}^{s} G_1^{\tau_i'}=R_{k/k_0}(G_1), \tag{6}$$

where $k$ is a finite extension of $k_0$, $G_1$ is an absolutely simple group defined over $k$, and $\{\tau_1', \dots, \tau_s'\}$ is a system of representatives of $\mathcal{G}(k)\backslash\mathcal{G}$. If $(G_1^1, f_1)$ is a quasi-split $k$-form of $G_1$ such that $f_1^\sigma \circ f_1^{-1}$ is an inner automorphism of $G_1$ for every $\sigma\in\mathcal{G}(k)$, then it is clear that $(R_{k/k_0}(G_1^1), \prod f_1^{\tau_i'})$ is a quasi-split $k_0$-form of $G$ satisfying the similar condition over $k_0$. Let $Z_1$ be the center of $G_1$. Then $Z=R_{k/k_0}(Z_1)$ is the center of $G$ and one has the canonical isomorphism (see [11]):

$$H^2(k, Z_1)\cong H^2(k_0, R_{k/k_0}(Z_1)), \tag{7}$$

which we shall denote, by abuse of notation, by $R_{k/k_0}{}^*$. (More precisely, for each $\sigma\in\mathcal{G}$, put $\tau_i'\sigma=\sigma(i)\tau_{i^\sigma}'$ with $\sigma(i)\in\mathcal{G}(k)$.) Then one has $R_{k/k_0}{}^*(c'_{\sigma',\tau'})=(c_{\sigma,\tau})$ with

$$c_{\sigma,\tau}=(c'_{\sigma(i^{\tau^{-1}\sigma^{-1}}),\,\tau(i^{\tau^{-1}})}) \quad \text{for all } \sigma, \tau\in\mathcal{G}.$$

In these notations, it is easy to see that

$$\gamma_{k_0}(G)=R_{k/k_0}{}^*(\gamma_k(G_1)). \tag{8}$$

Thus the determination of $\gamma(G)$ is reduced to the case where $G$ is absolutely simple.

**5.2.** The case $G=SL(n, \mathfrak{K})$, $\mathfrak{K}$ being a central simple division algebra of dimension $r^2$ defined over $k$. In this case $G^1=SL(nr)$, and the center $Z$ of $G$ can be identified canonically with $\mathsf{E}_{nr}$ (=the group of $nr$th roots of unity in $\bar{k}_0$) as a group with operators $\mathcal{G}(k)$.

In the notation of 3.2, one may put $f=M^{-1}$, $g_\sigma=\eta_\sigma 1_n$, and $c_{\sigma,\tau}=\mu_{\sigma,\tau}1_n$. Therefore, through the natural injection $H^2(k, Z)\to H^2(k, \mathsf{E})$, one has

$$\gamma(G)=c(\mathfrak{K}). \tag{9}$$

**5.3.** The case $G=SU(V/\mathfrak{K}, \mathrm{F})$, where $\mathfrak{K}$ is a central division algebra of dimension $r^2$ defined over a quadratic extension $k'$ of $k$ with a non-trivial Galois automorphism $\sigma_0$, having an involution $\iota_0$ of the second kind (with respect to $\sigma_0$), $\mathrm{V}$ is an $n$-dimensional right $\mathfrak{K}$-space, and $\mathbf{F}$ is a hermitian form on $\mathbf{V}$ with respect to $(\mathfrak{K}, \iota_0)$, both defined over $k'$. Regarding $\mathbf{F}$ as a linear mapping from $\mathbf{V}$ onto the dual of $\mathbf{V}^{\sigma_0}$ (which is $\iota_0^{-1}\sigma_0$-semi-linear if the dual of $V^{\sigma_0}$ is regarded as a left $\mathfrak{K}^{\sigma_0}$-space), we regard $G$ as a $k$-closed subgroup of $G' = R_{k'/k}(GL(\mathbf{V}/\mathfrak{K})) = GL(\mathbf{V}/\mathfrak{K}) \times GL(\mathbf{V}^{\sigma_0}/\mathfrak{K}^{\sigma_0})$ defined as follows:

$$G = \{(g_1, g_2) \in G' \mid n(g_1) = 1, {}^t g_2 \mathbf{F} g_1 = \mathbf{F}\},$$

$n$ denoting the reduced norm of $\mathcal{E}(\mathbf{V}/\mathfrak{K})$. We put $G^1 = SU(V, F_0)$, where $V$ is an $nr$-dimensional vector-space and $F_0$ is a hermitian form on $V$ (with respect to $\sigma_0$) with the maximal index $[nr/2]$, both defined over $k'$; in the similar way as above, $G^1$ is regarded as a $k$-closed subgroup of $G'^1 = R_{k'/k}(GL(V))$. The notation $(\varepsilon_{ij})$, $\eta_n$, ... being as in 3.2, 3.5, put

$$J \cdot M(\mathbf{F}(x, y)) = (F(x\varepsilon_{i1}, y\varepsilon_{j1}))$$

with a sesqui-linear form $F$ on $\mathbf{V}\varepsilon_{11} \times \mathbf{V}\varepsilon_{11}$ (with respect to $\sigma_0$), which will also be regarded as a linear mapping from $\mathbf{V}\varepsilon_{11}$ onto the dual of $\mathbf{V}^{\sigma_0}\varepsilon_{11}^{\sigma_0}$. Let $h_1$ be any linear isomorphism of $V$ onto $\mathbf{V}\varepsilon_{11}$ and put

$$h_2 = {}^tF^{-1} \circ {}^th_1^{-1} \circ {}^tF_0. \tag{10}$$

Then we have an isomorphism $f = (f_1, f_2) : G'^1 \to G'$ defined by

$$\begin{cases} f_1(g_1) \,|\, \mathbf{V}\ \varepsilon_{11} = h_1 g_1 h_1^{-1} & \text{for } g_1 \in GL(V), \\ f_2(g_2) \,|\, \mathbf{V}^{\sigma_0} \varepsilon_{11}^{\sigma_0} = h_2 g_2 h_2^{-1} & \text{for } g_2 \in GL(V^{\sigma_0}). \end{cases} \tag{11}$$

It is then clear that one has $f(G^1) = G$.

Now to calculate $f^\sigma \circ f^{-1}$, we first observe that $f_1^\sigma \circ f_1^{-1} = I_{g_\sigma}$ with $g_\sigma = f_1(h_1^{-1} \circ R_{\eta_\sigma} \circ h_1^\sigma)$, $R_{\eta_\sigma}$ denoting the right multiplication by $\eta_\sigma$. It follows that, if one puts $f' = (f_1, f_1^{\sigma_0})$, one has $f'^\sigma \circ f'^{-1} = I_{g'_\sigma}$ with

$$g'_\sigma = \begin{cases} (g_\sigma, g_{\bar{\sigma}}^{\sigma_0}) & \text{for } \sigma \in \mathcal{G}(k'), \\ (g_{\sigma_0\sigma}, g_{\sigma\sigma_0^{-1}}^{\sigma_0}) & \text{for } \sigma \notin \mathcal{G}(k'). \end{cases}$$

On the other hand, one has $f' \circ f^{-1} = I_{(1, h_0)}$ with $h_0 = f_2(h_2^{-1} h_1^{\sigma_0})$. It follows that $f^\sigma \circ f^{-1} = (f' \circ f^{-1})^{-\sigma} \circ (f'^\sigma \circ f'^{-1}) \circ (f' \circ f^{-1}) = I_{g''_\sigma}$ with

$$g''_\sigma = \begin{cases} (g_\sigma, h_0^{-\sigma} g_{\bar{\sigma}}^{\sigma_0} h_0) & \text{for } \sigma \in \mathcal{G}(k'), \\ (h_0^{-\sigma} g_{\sigma_0\sigma}, g_{\sigma\sigma_0^{-1}}^{\sigma_0} h_0) & \text{for } \sigma \notin \mathcal{G}(k'). \end{cases}$$

It is easy to see that $g''_\sigma$ is a similitude of the hermitian form $\mathbf{F}$ with the multiplicator:

$$\mu(g''_\sigma) = \begin{cases} \mu_\sigma & \text{for } \sigma \in \mathcal{G}(k'), \\ \mu^{-\sigma} \mu_{\sigma_0\sigma} \mu_{\sigma_0^2, \sigma\sigma_0^{-1}}^{\sigma_0} & \text{for } \sigma \notin \mathcal{G}(k'). \end{cases}$$

Now normalizing $\eta_\sigma$ and $J$ in such a way that $n(\eta_\sigma)=1$, $\det(J)=1$, we may assume that $\mu_{\sigma,\tau}$, $\mu_\sigma$, $\mu$ are all $r$th roots of unity. Moreover, taking $h_1$ suitably, one has $g_\sigma \in SL(\mathbf{V}/\mathfrak{K})$ for all $\sigma \in G(k')$. To have this, it suffices to take a $k'$-rational $\mathfrak{K}$-basis $(\mathrm{e}_1, \dots, \mathrm{e}_n)$ of $\mathbf{V}$ and a $k'$-rational basis $(e_1', \dots, e_{nr}')$ of $V$ and to take $h_1$ in such a way that the matrix of $h_1$ with respect to the basis $(e_i')$ and $(\mathrm{e}_i \varepsilon_{j1})_{1 \leq i \leq n,\, 1 \leq j \leq r}$ has the determinant one. Then it follows also that one has

$$n(h_0) = n(\mathbf{F}) \cdot \det(F_0)^{-1},$$

where $n(\mathbf{F}) = \det(M(\mathbf{F}(\mathrm{e}_i, \mathrm{e}_j)))$ and $\det(F_0) = \det(F_0(e_i', e_j'))$. Taking $F_0$ in the usual normal form, one has $\det(F_0) = (-1)^{[nr/2]} = (-1)^{nr(nr-1)/2}$.

Now we put

$$\beta = n(h_0)^{1/nr} = (-1)^{(nr-1)/2} n(\mathbf{F})^{1/nr}, \tag{12}$$

and

$$c_\sigma = \begin{cases} (1, \mu_\sigma) & \text{for } \sigma \in \mathcal{G}(k'), \\ (\beta^{-1} \mu^{-\sigma} \mu_{\sigma_0 \sigma} \mu^{\sigma_0}_{\sigma_0^2, \sigma\sigma_0^{-1}}, \beta) & \text{for } \sigma \notin \mathcal{G}(k'). \end{cases}$$

Then one has $c_\sigma^{-1} g_\sigma'' \in G$, and $f^\sigma \circ f^{-1} = I_{c_\sigma^{-1} g_\sigma''}$. Therefore calculating the coboundary of $(c_\sigma^{-1} g_\sigma'')$ (which is the same as that of $(c_\sigma^{-1} g_\sigma'))$, one obtains the following result

$$c_{\sigma,\tau} = \begin{cases} \mu_{\sigma,\tau} & \text{for } \sigma, \tau \in \mathcal{G}(k'), \\ \mu^{-\sigma_0}_{\sigma, \tau\sigma_0^{-1}} & \text{for } \sigma \in \mathcal{G}(k'), \tau \notin \mathcal{G}(k'), \\ \beta^{\tau-1} \mu_\tau \mu^{-\sigma_0}_{\sigma\sigma_0^{-1}, \bar{\tau}} & \text{for } \sigma \notin \mathcal{G}(k'), \tau \in \mathcal{G}(k'), \\ \beta^{1-\tau} \mu^\tau \mu^{-1}_{\sigma_0 \tau} \mu^{-\sigma_0}_{\sigma_0^2, \tau\sigma_0^{-1}} & \text{for } \sigma, \tau \notin \mathcal{G}(k'), \end{cases} \tag{13}$$

where we identify the center $Z$ of $G$ with $\mathsf{E}_{nr}$ (by the projection to the first factor). Note that the Galois group $\mathcal{G}(k)$ then operates on $Z$ by the following rule: $z^{[\sigma]} = z^\sigma$ for $\sigma \in \mathcal{G}(k')$, $= z^{-\sigma}$ for $\sigma \notin \mathcal{G}(k')$. If we put $SU(\mathfrak{K}, \iota_0) = \{\xi \in \mathfrak{K} \mid n(\xi) = 1, \xi^{\iota_0} \xi = 1\}$, one may write

$$\gamma(G) = \gamma'(\mathbf{F}) \cdot \gamma(SU(\mathfrak{K}, \iota_0)), \tag{14}$$

where $\gamma'(\mathbf{F})$ denotes the class of 2-cocycle defined as follows:

$$c'_{\sigma,\tau} = \begin{cases} 1 & \text{for } \sigma \in \mathcal{G}(k'), \\ \beta'^{\tau-1} & \text{for } \sigma \notin \mathcal{G}(k'), \tau \in \mathcal{G}(k'), \\ \beta'^{1-\tau} & \text{for } \sigma, \tau \notin \mathcal{G}(k'), \end{cases}$$

where $\beta' = (-1)^{(n-1)r/2} n(\mathbf{F})^{1/nr}$.

*Remark 1.* In case $r=2$, $\mathfrak{K}$ has a $k$-form $\mathfrak{K}_{\iota_0}$ defined by the operation of the Galois group: $\xi^{[\sigma_0]} = \xi^{\iota_0 \iota_1}$, $\iota_1$ denoting the canonical involution of the quaternion algebra $\mathfrak{K}$ (cf. [2], p. 161, Th. 21). It is not hard to see that one has

$$\gamma(SU(\mathfrak{K}, \iota_0)) = c(\mathfrak{K}_{\iota_0}),$$

where both sides are considered as elements of $H^2(k, \mathsf{E}_2)$.

*Remark 2.* In the similar sense as above, if $nr$ is even, $(\gamma'(\mathbf{F}))^{nr/2}$ can be identified with $c(\mathfrak{K}_1)$, where $\mathfrak{K}_1$ is a quaternion algebra defined by the crossed product $(k'/k, (-1)^{nr^2/2}n(\mathbf{F}))$.

Let $\{\omega_1, \ldots, \omega_{nr-1}\}$ be the fundamental weights of $G$ (relative to $T$ and $\Delta$) arranged in the usual order, and put $\lambda_{\varrho_1} = \sum m_i \omega_i$. Since $\omega_i^{[\sigma_0]} = \omega_{nr-i}$, one has by Proposition 5

$$K_{\varrho_1} = \begin{cases} k & \text{if } m_{nr-i} = m_i \text{ for all } i, \\ k' & \text{otherwise} \end{cases} \tag{15}$$

In case $K_{\varrho_1} = k'$, one has by Theorem 2

$$\mathfrak{K}_{\varrho_1} \sim \mathfrak{K}^{\Sigma m}. \tag{16}$$

In case $K_{\varrho_1} = k$, $\mathfrak{K}_{\varrho_1}$ is (at most) a quaternion algebra determined by

$$c(\mathfrak{K}_{\varrho_1}) = \begin{cases} \gamma(G)^{\frac{nr}{2} m_{nr/2}} & \text{if } nr \equiv 0 \text{ (mod. 2)}, \\ 1 & \text{otherwise}, \end{cases} \tag{16'}$$

where both sides are considered as elements of $H^2(k', \mathsf{E}_2)$.

**5.4.** The case $G = SU(\mathbf{V}/\mathfrak{K}, \mathbf{F})$, $\mathfrak{K}$ being a central division algebra of dimension $r^2$ with an involution $\iota_0$ of the first kind, $\mathbf{V}$ is an $n$-dimensional right $\mathfrak{K}$-space, and $\mathbf{F}$ is a non-degenerate $\varepsilon$-hermitian form on $\mathbf{V}$ with respect to $(\mathfrak{K}, \iota_0)$, all defined over $k$. Let $V$ be an $nr$-dimensional vector-space defined over $k$ and let $B = S$ or $A$ be a suitable $\varepsilon_0\varepsilon$-symmetric bilinear form on $V$ defined over $k$ such that $G^1$ is given by $SO(V, S)$ or $Sp(V, A)$, where we write $B = S$ or $A$ according as $\varepsilon_0\varepsilon = 1$ or $-1$.(1) Let $h$ be a linear isomorphism of $V$ onto $\mathbf{V}\varepsilon_{11}$ defined over $\bar{k}$ such that one has

(1) As is well known and also as will be seen from the following arguments, it suffices to take $B$ of the maximal index except for the case $\varepsilon_0\varepsilon = 1$ and $nr \equiv 0$ (mod. 2), in which case $B = S$ is given as follows: If $r = 1$, put $\delta = 1$. If $r > 1$, then $r$ is even; taking $J$ in such a way that det $(J) = 1$, one can find $\delta \in k$ such that $n(\eta_\sigma) = \sqrt{\delta}^{\sigma - 1} \mu_\sigma^{r/2}$. Then $B = S$ will be any symmetric bilinear form of index $nr/2-1$ with the determinant det $(S) \sim \delta^n n(\mathbf{F})$ (mod. $(k^*)^2$), for instance,

$$S = \begin{pmatrix} 0 & 1_{nr/2-1} \\ 1_{nr/2-1} & 0 \end{pmatrix} \dot{+} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{nr/2-1}\delta^n n(\mathbf{F}) \end{pmatrix}.$$

Note that, in case $\mathfrak{K}$ is a quaternion algebra, one can put $\delta = 1$.

$$J \cdot M(\mathbf{F}(x, y)) = B(h^{-1}(x\varepsilon_{i1}), h^{-1}(y\varepsilon_{j1})),$$

and define an isomorphism $f$ of $GL(V)$ onto $GL(\mathbf{V}/\mathfrak{K})$ by the relation

$$f(g) \,|\, \mathbf{V}\varepsilon_{11} = h \circ g \circ h^{-1}.$$

Then it is clear that $f(G^1) = G$. For each $\sigma \in \mathcal{G}(k)$, one has $f^\sigma \circ f^{-1} = I_{g_\sigma}$ with

$$g_\sigma = f(h^{-1} \circ R_{\eta_\sigma} \circ h^\sigma) \in GL(\mathbf{V}/\mathfrak{K}).$$

By what we have seen in 3.4, $g_\sigma$ is a similitude of $\mathbf{F}$ with the multiplicator $\mu_\sigma$, which is "proper" in the case where $\varepsilon_0\varepsilon = 1$ and $nr$ is even by our choice of $S$. So putting $g'_\sigma = \pm\mu_\sigma^{-\frac{1}{2}} g_\sigma$, one has $g'_\sigma \in G$ and $f^\sigma \circ f^{-1} = I_{g'_\sigma}$. Therefore, through the natural injection $Z \to \mathsf{E}_2$, one has

$$\gamma(G) = c(\mathfrak{K}). \tag{17}$$

**5.5.** In the case $\varepsilon_0\varepsilon = 1$ (i.e. $B = S$ : symmetric), the group $G$ in the preceding paragraph is not simply connected, so that one has to consider the universal covering group $(\tilde{G}, \varphi)$ of $G$. The corresponding quasi-split $k$-form is given by the universal covering group $(\tilde{G}^1, \varphi^1)$ of $G^1$. Let us first recall briefly the construction of the "spin group" $\tilde{G}^1$ and its twisted form $\tilde{G}$ after Jacobson [6].

Let $C = C(V, S)$ be the Clifford algebra of $(V, S)$, i.e. an associative algebra (over the universal domain) with the unit element 1 generated by all $x \in V$ with the defining relations $x^2 = S(x, x)$; and let $C^+$ denote its even part, i.e. the subalgebra of $C$ spanned by all products of an even number of vectors in $V$. Let further $\iota$ be the canonical involution of $C$, i.e. the involution of $C$ defined by $(x_1 \ldots x_\nu)^\iota = x_\nu \ldots x_1$ $(x_i \in V)$. Then $(\tilde{G}^1, \varphi^1)$ is given as follows (cf. [3]):

$$\begin{cases} \tilde{G}^1 = \{g \in C^+ \,|\, g^\iota g = 1,\ gVg^{-1} = V\}, \\ \varphi^1(g)\,(x) = gxg^{-1} \qquad \text{for } g \in G, x \in V. \end{cases} \tag{18}$$

Next, we define a $k$-form $(\mathfrak{C}, \bar{f}^{-1})$ of $C^+$ by the following (well-defined) operation of the Galois group:

$$(xy)^{[\sigma]} = \mu_\sigma^{-1} f^{-1}(g_\sigma)\, x^\sigma \cdot f^{-1}(g_\sigma)\, y^\sigma = f^{-1}(g'_\sigma)\, x^\sigma \cdot f^{-1}(g'_\sigma)\, y^\sigma \quad \text{for } x, y \in V, \tag{19}$$

where $g_\sigma$, $g'_\sigma$ are as defined in 5.4. Then it is immediate that the $k$-isomorphism class of $\mathfrak{C}$ is uniquely determined only by $(\mathbf{V}, \mathbf{F})$ (independently of the choice of $\varepsilon_{ij}$, $h$, etc.) and that, when $\mathfrak{K} \sim 1$, this $\mathfrak{C}$ can be identified with the ordinary even Clifford algebra of $(\mathbf{V}, \mathbf{F})$. Moreover it is clear that one has

$$(x^{\iota})^{[\sigma]} = (x^{[\sigma]})^{\iota} \quad \text{for all } x \in C^{+},$$

that means that the corresponding involution of $\mathfrak{C}$, denoted again by $\iota$, is defined over $k$.

Now, as is well known (cf. [3], II, 3.4), the group $\{g \in C^{+} | gVg^{-1} = V\}$ is generated by the products of two vectors $x$, $y$ in $V$ with $S(x, x) \neq 0$, $S(y, y) \neq 0$. It follows that $\tilde{G}^1$ is invariant under all $[\sigma]$ $(\sigma \in \mathcal{G}(k))$, and therefore $\tilde{G} = \tilde{f}(\tilde{G}^1)$ is defined over $k$. To prove that the covering homomorphism of $\tilde{G}$ onto $G$ is also defined over $k$, we need the following characterization of $\varphi^1$:

LEMMA 2. *Let $\psi$ be a homomorphism of $\tilde{G}^1$ into $G^1$ satisfying the relation*

$$g(xy)g^{-1} = \psi(g)x \cdot \psi(g)y \tag{20}$$

*for all $g \in \tilde{G}^1$, $x, y \in V$. Then one has $\psi = \varphi^1$.*

*Proof.* Put $\chi(g) = \psi(g)\varphi^1(g)^{-1}$. Then one has $\chi(g)x \cdot \chi(g)y = xy$ for all $x, y \in V$, so that the orthogonal transformation $\chi(g)$ can be extended to an automorphism of $C$ which is trivial on $C^{+}$. Applying this automorphism on the second formula of (18), one obtains $g \cdot \chi(g)(x) \cdot g^{-1} = \chi(g) \circ \varphi(g)(x)$, i.e. $\varphi(g) \circ \chi(g) = \chi(g) \circ \varphi(g)$ for all $g \in \tilde{G}^1$, whence follows that $\chi(g)$ is in the center of $G^1$ and so $= \pm 1$. Since $G^1$ is connected, $\chi$ must be trivial, q.e.d.

Now applying $[\sigma]$ on (20) with $\psi = \varphi^1$, one has

$$g^{[\sigma]}(x \cdot y)g^{-[\sigma]} = \varphi^1(g)^{[\sigma]}(x) \cdot \varphi^1(g)^{[\sigma]}(y)$$

where $\varphi^1(g)^{[\sigma]} = f^{-1} \circ f^{\sigma}(\varphi^1(g)^{\sigma})$. It follows from the Lemma that $\varphi^1(g)^{[\sigma]} = \varphi^1(g^{[\sigma]}) = \varphi^1(\tilde{f}^{-1} \circ \tilde{f}^{\sigma}(g^{\sigma}))$, whence follows that $\varphi = f \circ \varphi^1 \circ (\tilde{f}^{-1} | \tilde{G})$ is defined over $k$.(1) Thus one obtains the following commutative diagram:

$$\begin{array}{ccccc} C^{+} & \supset & \tilde{G}^1 & \xrightarrow{\varphi^1} & G^1 \\ \tilde{f}\downarrow & & \downarrow & & \downarrow f \\ \mathfrak{C} & \supset & \tilde{G} & \xrightarrow{\varphi} & G \end{array} \tag{21}$$

**5.6.** (1) *The case* $nr \equiv 1$ (mod. 2). The center of $G$ being trivial, one has $\gamma(G) = c(\mathfrak{K}) = 1$ (i.e. $r = 1$). On the other hand, one has $C^{+} \simeq \mathfrak{M}_{2^{\frac{1}{2}(n-1)}}$ over $k$ and so $\mathfrak{C}$ is a central simple algebra (with involution of the first kind). Taking $\tilde{g}_{\sigma} \in \tilde{G}$ in such a way that $\varphi(\tilde{g}_{\sigma}) = g'_{\sigma}$, one has from (19) $\tilde{f}^{\sigma} \circ \tilde{f}^{-1} = I_{\tilde{g}_{\sigma}}$. Therefore, identifying the center $\tilde{Z}$ of $\tilde{G}$ with $\mathsf{E}_2$, one has

(1) This can be proven more directly as follows: Denote by $\tilde{\varphi}_{\sigma}$ the (inner) automorphism of $C$ extending the proper orthogonal transformation $f^{-1}(g'_{\sigma})^{-1}$; then one has $\tilde{f}^{-\sigma} \circ \tilde{f} = \tilde{\varphi}_{\sigma} | C^{+}$. Put further $\varphi_{\sigma} = f^{-\sigma} \circ f = I_{f^{-1}(g'_{\sigma})^{-1}}$. It suffices to show that $\varphi^1 \circ (\tilde{\varphi}_{\sigma} | \tilde{G}^1) = \varphi_{\sigma} \circ \varphi^1$. For $\tilde{g} \in \tilde{G}$ one has $\varphi^1 \circ \tilde{\varphi}_{\sigma}(\tilde{g}) = I_{\tilde{\varphi}_{\sigma}(\tilde{g})} | V = (\tilde{\varphi}_{\sigma} \circ I_{\tilde{g}} \circ \tilde{\varphi}_{\sigma}^{-1}) | V = f^{-1}(g'_{\sigma})^{-1} \circ \varphi^1(\tilde{g}) \circ f^{-1}(g'_{\sigma}) = (\varphi_{\sigma} \circ \varphi^1)(\tilde{g})$, which proves our assertion. The fact that $\tilde{G}$ is defind over $k$ can also be given a similar proof.

$$\gamma(\tilde{G}) = c(\mathfrak{C}). \tag{22}$$

Let $\{\omega_1, ..., \omega_{\frac{1}{2}(n-1)}\}$ be the fundamental weights of $\tilde{G}$ ordered in the usual way (i.e. $\omega_{\frac{1}{2}(n-1)}$ corresponding to the spin representation) and put $\lambda_{\varrho_1} = \sum m_i \omega_i$. Then $\varrho_1$ is a representation of $G$ if and only if $m_{\frac{1}{2}(n-1)}$ is even. One has $K_{\varrho_1} = k$ for all $\varrho_1$ and

$$\mathfrak{R}_{\varrho_1} \sim \mathfrak{C}^{m_{\frac{1}{2}(n-1)}} \text{ (over } k), \tag{23}$$

for the spin representation is, by definition, the restriction on $\tilde{G}$ of the (unique) absolutely irreducible representation of $\mathfrak{C}$.

(2) *The case* $nr \equiv 0$ (mod. 2). In this case, one has

$$C^+ \cong \mathcal{M}_{2^{\frac{1}{2}nr-1}} \oplus \mathcal{M}_{2^{\frac{1}{2}nr-1}} \quad \text{over} \quad k' = k(\sqrt{(-1)^{\frac{1}{2}nr}} \det(S)).$$

Let
$$\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2 \tag{24}$$

be the corresponding direct decomposition of $\mathfrak{C}$. Again taking $\tilde{g}_\sigma \in \tilde{G}$ such that $\varphi(\tilde{g}_\sigma) = g'_\sigma$, one has $\tilde{f}^\sigma \circ \tilde{f}^{-1} = I_{\tilde{g}_\sigma}$. This implies in the first place that the decomposition (24) is also defined over $k'$. Now we have to distinguish the following two cases:

(2.1) *The case* $nr \equiv 0$ (mod. 4). In this case, $\iota$ leaves $\mathfrak{C}_1$ and $\mathfrak{C}_2$ invariant and induces an involution of the first kind in each of them. It follows that, if one denotes by $1'$, $1''$ the unit elements of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, respectively, the center $\tilde{Z}$ of $\tilde{G}$ is given by $\{\pm 1' \pm 1''\}$. If $k' = k$, the Galois group operates trivially on $\tilde{Z}$ and so, through the identification $\tilde{Z} = \mathsf{E}_2 \times \mathsf{E}_2$, one has

$$\gamma(\tilde{G}) = (c(\mathfrak{C}_1), c(\mathfrak{C}_2)). \tag{25}$$

If $k' \supsetneqq k$, One may write $\mathfrak{C} = R_{k'/k}(\mathfrak{C}_1)$ and identify $\tilde{Z}$ with $R_{k'/k}(\mathsf{E}_2)$. Then one has

$$\gamma_k(\tilde{G}) = R_{k'/k}{}^*(c_{k'}(\mathfrak{C}_1)), \tag{25'}$$

where $R_{k'/k}{}^*$ denotes the canonical isomorphism $H^2(k', \mathsf{E}_2) \to H^2(k, R_{k'/k}(\mathsf{E}_2))$.

(2.2) *The case* $nr \equiv 2$ (mod. 4). In this case, $\iota$ interchanges $\mathfrak{C}_1$ and $\mathfrak{C}_2$ each other. It follows that, in the same notation as above, $\tilde{Z}$ is a cyclic group of order 4 generated by $\sqrt{-1}(1' - 1'')$. If $k' = k$, the projection on the $i$th factor ($i = 1, 2$) gives a $\mathcal{G}(k)$-isomorphism of $\tilde{Z}$ onto $\mathsf{E}_4$, and through this one has

$$\text{proj}_i^* (\gamma(\tilde{G})) = c(\mathfrak{C}_i). \tag{26}$$

It follows that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are of exponent $2^\nu (\nu \leqslant 2)$. If $k' \supsetneqq k$, let $\sigma_0$ be an element of $\mathcal{G}(k)$ which induces a non-trivial automorphism of $k'/k$. Then $\iota\sigma_0$ induces on each $\mathfrak{C}_i$ an involution of the second kind (in the sense of 3.5), which we denote by $\iota_i$. Put $G_1' = SU(\mathfrak{C}_1, \iota_1)$. Then the projection on the first factor gives an injection of $\tilde{Z}$ into the center of $G_1'$ (with the operation of the Galois group $\mathcal{G}(k)$), and through this one has

$$\mathrm{proj}_1^* \, (\gamma_k(\tilde{G})) = \gamma_k(SU(\mathfrak{C}_1, \iota_1)). \tag{26'}$$

It follows, in particular,

$$\mathrm{proj}_1^* \, (\gamma_{k'}(\tilde{G})) = c_{k'}(\mathfrak{C}_1),$$

so that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are again of exponent $2^\nu$ ($\nu \leqslant 2$).

Returning to the general case $nr \equiv 0$ (mod. 2), let $\{\omega_1, \ldots, \omega_{\frac{1}{2}nr}\}$ be the fundamental weights of $\tilde{G}$, where $\omega_{\frac{1}{2}nr-1}$ and $\omega_{\frac{1}{2}nr}$ correspond to the spin representations $\varrho^{(1)}$ and $\varrho^{(2)}$ given by $\mathfrak{C}_1$ and $\mathfrak{C}_2$, respectively. Put $\lambda_{\varrho_1} = \sum m_i \omega_i$. Then one has

$$K_{\varrho_1} = \begin{cases} k & \text{if } m_{\frac{1}{2}nr-1} = m_{\frac{1}{2}nr}, \\ k' & \text{otherwise}, \end{cases} \tag{27}$$

$$\mathfrak{K}_{\varrho_1} \sim \mathfrak{K}^{\sum_i^{\frac{1}{2}nr-2} m_i} \otimes \mathfrak{C}_1^{m_{\frac{1}{2}nr-1}} \otimes \mathfrak{C}_1^{m_{\frac{1}{2}nr}} \quad \text{over } k'. \tag{28}$$

On the other hand, $\varrho_1$ is a representation of $G$ if and only if $m_{\frac{1}{2}nr-1} \equiv m_{\frac{1}{2}nr}$ (mod. 2), and for such a representation one has

$$\mathfrak{K}_{\varrho_1} \sim \mathfrak{K}^{\sum_1^{\frac{1}{2}nr-2} m_i + (\frac{1}{4}nr) m_{\frac{1}{2}nr-1} + (\frac{1}{4}nr-1) m_{\frac{1}{2}nr}} \quad \text{over } K_{\varrho_1}. \tag{28'}$$

Comparing these two expressions, one obtains the following relations due to Jacobson ([6]): [1]

When $nr \equiv 0$ (mod. 4), $\quad \mathfrak{C}_1 \otimes \mathfrak{C}_2 \sim \mathfrak{K}$ over $k'$; (29)

When $nr \equiv 2$ (mod. 4), $\quad \mathfrak{C}_1 \otimes \mathfrak{C}_2 \sim 1$, $\mathfrak{C}_1^2 \sim \mathfrak{C}_2^2 \sim \mathfrak{K}$ over $k'$. (30)

(These relations can also be obtained by comparing the restrictions of $\varphi$, $\varrho^{(1)}$, $\varrho^{(2)}$ on $\tilde{Z}$.) Note that, in case $\mathfrak{K} \sim 1$, one has $\mathfrak{C}_1 \sim \mathfrak{C}_2 \sim C'$, $C'$ denoting the (full) Clifford algebra of $(\mathbf{V}, \mathbf{F})$ in the ordinary sense.

**5.7.** For the exceptional groups, our result being still incomplete, we shall restrict ourselves to the case where $k$ is a local field or a number-field. Let $G$ be a simply connected absolutely simple group of exceptional type defined over $k$ and let $Z$ be the center of $G$;

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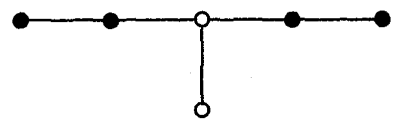
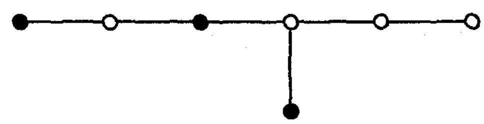
(1) In [16], p. 173, Cartan writes "Il en est de même pour le groupe $g_2$, qui admet donc, ainsi que $g_1$, d'indice 1." But this clearly contradicts Jacobson's result. In Cartan's notation, one should say that one of $g_1$ and $g_2$ is of index 1, while the other is of index $-1$.





we may assume $G$ to be of type $D_4$, $E_6$ or $E_7$, for otherwise $Z$ is trivial. In case $k$ is a local field, one obtains the following results (cf. [16], [17]). If $G^1$ is not of Chevalley type over $k$ (i.e. the cases ${}^3D_4$, ${}^6D_4$, ${}^2E_6$), one has $H^2(k, Z)=1$ and so *a fortiori* $\gamma(G)=1$. (The same is also true over number-fields.) Hence suppose $\mathcal{G}^1$ to be of Chevalley type. In case $G$ is of type $(E_6)$, $Z$ is $\mathcal{G}(k)$-isomorphic to $\mathsf{E}_3$, so that $H^2(k, Z)\cong H^2(k, \mathsf{E}_3)$ is cyclic of order 3. When $k$ is a $\mathfrak{p}$-adic field and $G$ has the diagram as indicated, $G$ contains a $k$-closed subgroup $k$-isomorphic to $SL(2, \mathfrak{K}_3)$ where $\mathfrak{K}_3$ is a central division algebra of dimension 9 defined over $k$; in this case, one has $\gamma(G)=c(\mathfrak{K}_3)$. Otherwise one has $\gamma(G)=1$. In case $G$ is of type $(E_7)$, $Z$ is isomorphic to $\mathsf{E}_2$ and so $H^2(k, Z)$ is of order 2. When $G$ has the diagram as indicated, or when $G_k$ is compact ($k=\mathbf{R}$), one has $\gamma(G)=c(\mathfrak{K}_2)$, where $\mathfrak{K}_2$ is the (unique) quaternion algebra defined over $k$. In all other cases, one has $\gamma(G)=1$. For the group $G$ defined over an algebraic number-field $k$, $\gamma_k(G)$ can be determined in virtue of Hasse principle for $H^2(k, Z)$. (See T. Ono, On the relative theory of Tamagawa numbers, *Ann. of Math.*, 82 (1965), 88–111, especially p. 107.)

**5.8.** We shall add here few remarks on the determination of $\mathbf{F}_1$, which is also indispensable for the description of symplectic representations. We use the notation introduced in § 4.

(i) Let $w_0$ be the (unique) element in the Weyl group $W$ such that $w_0\Delta=-\Delta$. Then an absolutely irreducible representation $\varrho_1$ of $G$ is of type (a) if and only if one has $-w_0(\lambda_{\varrho_1})=\lambda_{\varrho_1}$. (For instance, if $G$ has no simple factors of type $A_l$ $(l\geqslant 2)$, $D_l$ ($l$ : odd) or $E_6$, one has $w_0=-1$, so that $\varrho_1$ is always of type (a).) In that case, putting $\Delta=\{\alpha_1, \ldots, \alpha_l\}$, one may write $2\lambda_{\varrho_1}=\sum n_i\alpha_i$ with $n_i\in\mathbf{Z}$, $n_i\geqslant 0$. Then, in the notation of Part I, one has $\varepsilon_0\varepsilon=(-1)^{\sum n_i}$ ([18]).(1)

(ii) In the same notation, $\varrho_1$ is of type (b), if and only if

(1) D. N. Verma gave recently a simpler proof for this formula independent of the classification theory. Iwahori has also gotten another formula determining $\varepsilon_0\varepsilon$.

$$-w_0(\lambda_{\varrho_1}) = \lambda_{\varrho_1}^{[\sigma_0]} \neq \lambda_{\varrho_1} \quad \text{for some } \sigma_0 \in \mathcal{G}.$$

In both cases (a), (b), the determination of $\mathbf{F}_1$ can easily be reduced to the case where $G$ is $k_0$-simple (cf. 9.2).

## Part III. Symplectic representations of semi-simple algebraic groups of hermitian type satisfying the condition $(\mathrm{H}_1)$

### § 6. Observations over R

**6.1.** Let $G$ be a (Zariski-)connected semi-simple algebraic group defined over $\mathbf{Q}$. $G_{\mathbf{R}}$ is then a semi-simple Lie group with a finite number of connected components (in the usual topology). One denotes by $G_{\mathbf{R}}^0$ the identity component of $G_{\mathbf{R}}$. We assume $G$ to be "of hermitian type", i.e., denoting by $\mathcal{K}$ a maximal compact subgroup of $G_{\mathbf{R}}$, we assume that the associated symmetric space $\mathcal{D} = G_{\mathbf{R}}/\mathcal{K}$ has a $G_{\mathbf{R}}^0$-invariant complex structure and thus becomes a symmetric (bounded) domain. This implies, as is well-known, that all simple factors of $G_{\mathbf{R}}$ (viewed as a Lie group) are either non-compact and corresponding to irreducible symmetric domains, or compact; thus all absolutely simple factors of $G$ (viewed as an algebraic group) are defined over $\mathbf{R}$.

As explained in the Introduction, our main problem is the following: for a given maximal compact subgroups $\mathcal{K}$ of $G_{\mathbf{R}}$, determine all symplectic representations $(V, A, \varrho)$ of $G$ defined over $\mathbf{Q}$ (or $\mathbf{R}$) together with a maximal compact subgroup $\mathcal{K}'$ of $G'_{\mathbf{R}} = Sp(V, A)_{\mathbf{R}}$ containing $\varrho(\mathcal{K})$, such that the induced mapping from $\mathcal{D} = G_{\mathbf{R}}/\mathcal{K}$ into $\mathcal{D}' = G'_{\mathbf{R}}/\mathcal{K}'$ is holomorphic with respect to the given complex structures on $\mathcal{D}$ and $\mathcal{D}'$ (Condition $(\mathrm{H}_1)$). In terms of Lie algebras, this condition is expressed as follows ([9]). Let $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{k}$, $\mathfrak{k}'$ be the Lie algebras of $G_{\mathbf{R}}$, $G'_{\mathbf{R}}$, $\mathcal{K}$, $\mathcal{K}'$, respectively. Then there exists a (uniquely determined) element $H_0$ in the center of $\mathfrak{k}$ such that $\mathrm{ad}(H_0)$ induces on the factor space $\mathfrak{g}/\mathfrak{k}$, identified with the tangent vector-space to $\mathcal{D}$ at the origin, the given complex structure of it. For brevity, we shall call such an element $H_0$ in $\mathfrak{g}$ an *H-element* for $G_{\mathbf{R}}$ (or $G$). Let $H_0'$ be an $H$-element for $G'_{\mathbf{R}}$ determining the maximal compact subgroup $\mathcal{K}'$ and the given complex structure on $\mathcal{D}'$. Then, the condition $(\mathrm{H}_1)$ may be expressed as

$$d\varrho([H_0, X]) = [H_0', d\varrho(X)] \quad \text{for all } X \in \mathfrak{g}, \tag{$\mathrm{H}_1$}$$

$d\varrho$ denoting the homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ induced by $\varrho$. (This condition clearly implies that $d\varrho(\mathfrak{k}) \subset \mathfrak{k}'$ and so $\varrho(\mathcal{K}) \subset \mathcal{K}'$, for $\mathcal{K}$ and $\mathcal{K}'$ are Zariski-connected algebraic subgroups of $G_{\mathbf{R}}$ and $G'_{\mathbf{R}}$ corresponding to $\mathfrak{k}$ and $\mathfrak{k}'$, respectively.) We shall also consider the following stronger condition:

$$d\varrho(H_0) = H_0'. \tag{$\mathrm{H}_2$}$$

As is well known, the maximal compact subgroups $\mathcal{K}'$ of $G'_{\mathbf{R}} = Sp(V, A)_{\mathbf{R}}$ $(= G'^{0}_{\mathbf{R}})$ are corresponding (in a one-to-one way) to the complex structures $I$ on $V_{\mathbf{R}}$ such that

$$A(x, Iy) \quad (x, y \in V_{\mathbf{R}}) \text{ is symmetric and positive-definite,} \tag{1}$$

by the relation that $I$ is $\mathcal{K}'$-invariant; and, if $H'_0$ is an $H$-element determining $\mathcal{K}'$, one has $H'_0 = \pm \frac{1}{2} I$. In this paper, we shall always choose the complex structure on $\mathcal{D}'$ in such a way that we have the plus sign here. Then the condition $(\mathrm{H}_1)$ may also be written as

$$d\varrho([H_0, X]) = \tfrac{1}{2}[I, d\varrho(X)] \quad \text{for all } X \in \mathfrak{g}. \tag{2}$$

Thus, *given a maximal compact subgroup* $\mathcal{K}$ *of* $G_{\mathbf{R}}$ *and a* $G^0_{\mathbf{R}}$*-invariant complex structure on* $\mathcal{D}$ *(or, what amounts to the same, given an* $H$*-element* $H_0$ *for* $G_{\mathbf{R}}$*), our problem is to determine all symplectic representations* $(V, A, \varrho)$ *of* $G$ *defined over* $\mathbf{Q}$ *(or* $\mathbf{R}$*) together with a complex structure* $I$ *on* $V_{\mathbf{R}}$ *satisfying the conditions* (1), (2).

**6.2.** Considering the above problem over $\mathbf{R}$, we shall show in the first place that a solution $(V, A, \varrho; I)$ can be decomposed into a direct sum of $\mathbf{R}$-primary solutions. (Cf. [9], Th. 1.)

We first assert that all $\mathbf{R}$-primary components of $V$ are invariant under the complex structure $I$. In fact, let $W$ be any $\mathbf{R}$-irreducible $\varrho(G)$-invariant subspace of $V$. Since $G$ is Zariski-connected, the condition (2) implies that $I - 2d\varrho(H_0)$ commutes with all $\varrho(g)$ $(g \in G)$; or, in other words, the linear transformation $I - 2d\varrho(H_0)$ is an endomorphism of $V$ viewed as a representation-space. Hence the image $(I - 2d\varrho(H_0))(W)$ of $W$ is contained in the same $\mathbf{R}$-primary component of $V$ as $W$, and therefore so does also

$$I(W) \subset (I - 2d\varrho(H_0))(W) + W.$$

This proves our assertion.

Combining this with the condition (1), we see that, in the notation of 2.1, the case (c) does not occur for any $\mathbf{R}$-primary component of $V$. Therefore, denoting by $(V^{(i)}, \varrho^{(i)})$ the $\mathbf{R}$-primary components of $(V, \varrho)$ and putting $A^{(i)} = A \mid V^{(i)}$, $\varrho^{(i)} = \varrho \mid V^{(i)}$, $I^{(i)} = I \mid V^{(i)}$, we conclude that a solution $(V, A, \varrho; I)$ decomposes into the direct sum as follows:

$$(V, \varrho) = \sum (V^{(i)}, \varrho^{(i)}), \quad A = \sum A^{(i)}, \quad I = \sum I^{(i)},$$

where each $(V^{(i)}, A^{(i)}, \varrho^{(i)}; I^{(i)})$ satisfies again the conditions (1), (2). Thus, the problem over $\mathbf{Q}$ (resp. $\mathbf{R}$) is reduced to the case when $(V, \varrho)$ is $\mathbf{Q}$- (resp. $\mathbf{R}$-) primary.

By 3.7, the above consideration also implies that, when one has a $\mathbf{Q}$-primary solution $(V, A, \varrho; I)$, $\varrho$ is of type (a) or (b) over $\mathbf{Q}$ and in either case all the $\mathbf{R}$-primary components of $\varrho$ are of the same type over $\mathbf{R}$. Moreover, in the case (b), one can take as $\sigma_0$ (an extension of) the usual complex conjugation, so that one has $\sigma_0^2 = 1$ in $\mathbf{C}$.

There is a "trivial" solution of our problem over $\mathbf{Q}$ (resp. $\mathbf{R}$), i.e., the one obtained by taking $(V, \varrho)$ to be a trivial representation of $G$ (of any degree) defined over $\mathbf{Q}$ (resp. $\mathbf{R}$) and $A$ and $I$ arbitrarily under the only condition (1). The associated mapping of the symmetric domain $\mathcal{D}$ into a Siegel space is also trivial in the sense that the image of the mapping reduces to a point. Such a solution is, of course, of no interest at all from our point of view. Therefore, in the following, we shall assume that *the representation* $\varrho$ *is* $\mathbf{Q}$-*(resp.* $\mathbf{R}$-*) primary and non-trivial.*

**6.3.** Applying the main results of Part I to the case $k_0 = \mathbf{R}$, we shall now study non-trivial, $\mathbf{R}$-primary solutions $(V, A, \varrho; I)$ more closely. We first assume that $d\varrho(H_0) \neq 0$; as we shall see, this condition is equivalent to saying that the associated mapping of the symmetric domain $\mathcal{D}$ is non-trivial. Moreover, it will also imply that the complex structure $I$ is uniquely determined only by $(V, \varrho)$ and $H_0$ (under the condition $(\mathrm{H}_1)$). In this and the next paragraphs, to simplify the notation, we shall consider the real vector-space $V_{\mathbf{R}}$ instead of the vector-space $V$ over the universal domain.

In the notation of Part I, we have the following four possibilities (where $\mathfrak{K}_{\varrho_1}$ stands for $(\mathfrak{K}_{\varrho_1})_{K_{\varrho_1}}$):

$(\mathrm{a}_1)$ $K_{\varrho_1} = \mathfrak{K}_{\varrho_1} = \mathbf{R}$,

$(\mathrm{a}_2)$ $K_{\varrho_1} = \mathbf{R}$, $\mathfrak{K}_{\varrho_1} = \mathbf{K}$ ( = the real quaternion algebra),

$(\mathrm{a}_3)$ $K_{\varrho_1} = \mathfrak{K}_{\varrho_1} = \mathbf{C}$,

(b) $K_{\varrho_1} = \mathfrak{K}_{\varrho_1} = \mathbf{C}$, $K_0 = \mathbf{R}$.

As we shall see, the case $(\mathrm{a}_3)$ does not occur; in other words, in the case (a) $\varrho$ is always absolutely primary.

*The case* $(\mathrm{a}_1)$. One has (Proposition 2)

$$\begin{cases} V_{\mathbf{R}} = V_1 \otimes_{\mathbf{R}} V_2, \\ \varrho = \varrho_1 \otimes \text{triv.}, \end{cases} \tag{3 a$_1$}$$

where $V_1$ and $V_2$ are vector-spaces over $\mathbf{R}$ and $\varrho_1$ is an absolutely irreducible representation of $G_{\mathbf{R}}$ in $\mathbf{R}$.

Moreover (under the conditions $(\mathrm{H}_1)$ and $d\varrho(H_0) \neq 0$) one has

$$A = A_1 \otimes S_2, \tag{4 a$_1$}$$

$$I = I_1 \otimes 1_{V_2}, \tag{5 a$_1$}$$

where $A_1$ (resp. $S_2$) is a non-degenerate, real, $\varrho_1(G_{\mathbf{R}})$-invariant alternating (resp. symmetric) bilinear form on $V_1$ (resp. $V_2$) and $I_1$ is a $(\varrho_1(\mathcal{K})$-invariant) complex structure on $V_1$, satisfying the following conditions:

$$\begin{cases} A_1(x, I_1 y)\ (x, y \in V_1) \text{ is symmetric and (positive) definite,} \\ S_2 \text{ is (positive) definite,} \end{cases} \tag{1 a$_1$}$$

$$d\varrho_1(H_0) = \tfrac{1}{2} I_1. \tag{2 a$_1$}$$

$1_{V_2}$ denotes the identity transformation of the vector-space $V_2$.

In fact, since the centralizer of $\varrho(G) = \varrho_1(G) \otimes 1_{V_2}$ in $\mathcal{E}(V_{\mathbf{R}})$ is equal to $1_{V_1} \otimes \mathcal{E}(V_2)$, the condition (2) implies that

$$I = 2d\varrho_1(H_0) \otimes 1_{V_2} + 1_{V_1} \otimes \varphi_2$$

$\varphi_2$ being an endomorphism of $V_2$. As $I^2 = -1$, one has

$$4(d\varrho_1(H_0))^2 \otimes 1 + 4\, d\varrho_1(H_0) \otimes \varphi_2 + 1 \otimes (\varphi_2{}^2 + 1) = 0.$$

Since $d\varrho_1(H_0)\ (\neq 0)$ and $1_{V_1}$ are linearly independent, (for $\operatorname{tr}(d\varrho_1(H_0)) = 0$), one obtains the relations

$$(d\varrho_1(H_0))^2 = \lambda 1_{V_1} + \mu\, d\varrho_1(H_0),$$
$$\varphi_2 = -\mu 1_{V_2}, \quad \lambda = -\tfrac{1}{4}(\mu^2 + 1).$$

But, the trace of $I$ being also equal to zero, one must have $\mu = 0$, and so $\varphi_2 = 0$, which proves (5a$_1$) and (2a$_1$). Then, by Proposition 3, (4a$_1$) and (1a$_1$) follow from (1) immediately. (Replacing $A_1$, $S_2$ by $-A_1$, $-S_2$, if necessary, one may assume that $A_1(x, Iy)$ and $S_2$ are positive-definite.)

*The case* (a$_2$). One has

$$\begin{cases} V_{\mathbf{R}} = \mathbf{V}_1 \otimes_{\mathbf{K}} \mathbf{V}_2, \\ \varrho = \mathsf{P}_1 \otimes \text{triv.}, \end{cases} \tag{3 a$_2$}$$

where $\mathbf{V}_1$ (resp. $\mathbf{V}_2$) is a right (resp. left) $\mathbf{K}$-space and $\mathsf{P}_1$ is an absolutely irreducible representation of $G_{\mathbf{R}}$ in $\mathbf{K}$. In this case, by a similar argument as above, one has

$$A = \operatorname{tr}_{\mathbf{K}/\mathbf{R}}(\mathbf{F}_1 \otimes {}^t\mathbf{F}_2), \tag{4 a$_2$}$$

$$I = \mathbf{I}_1 \otimes 1_{\mathbf{V}_2}, \tag{5 a$_2$}$$

where $\mathbf{F}_1$ (resp. $\mathbf{F}_2$) is a non-degenerate, quaternionic, $\mathsf{P}_1(G_{\mathbf{R}})$-invariant skew-hermitian (resp. hermitian) form on $\mathbf{V}_1$ (resp. $\mathbf{V}_2$) and $\mathbf{I}_1$ is a $\mathbf{K}$-linear complex structure on $\mathbf{V}_1$, satisfying the following conditions:

$$\begin{cases} \mathbf{F}_1(x, \mathbf{I}_1 y)\ (x, y \in \mathbf{V}_1) \text{ is quaternionic hermitian and (posititive) definite,} \\ \mathbf{F}_2 \text{ is (positive) definite,} \end{cases} \tag{1 a$_2$}$$

$$d\mathsf{P}_1(H_0) = \tfrac{1}{2} \mathbf{I}_1. \tag{2 a$_2$}$$

(Note that the multiplicative equivalence-class of the quaternionic skew-hermitian forms on $\mathbf{V}_1$ is unique.)

*The case* (b). One has

$$\begin{cases} V_{\mathbf{R}} = V_1 \otimes_{\mathbf{C}} V_2 \text{ (viewed as a vector-space over } \mathbf{R}), \\ \varrho = \varrho_1 \otimes \text{triv.}, \end{cases} \tag{3 b}$$

where $V_1$ and $V_2$ are vector-spaces over $\mathbf{C}$ and $\varrho_1$ is an absolutely irreducible representation of $G_{\mathbf{R}}$ in $\mathbf{C}$. Moreover one has

$$A = 2 \operatorname{Im}(F_1 \otimes F_2), \tag{4 b}$$

$$I = (\sqrt{-1}\, T_1) \otimes 1_{V_2}, \tag{5 b}$$

where $F_i$'s are non-degenerate, hermitian forms on $V_i$ $(i=1, 2)$, $F_1$ being $\varrho_1(G_{\mathbf{R}})$-invariant, and $T_1$ is a $\mathbf{C}$-linear transformation on $V_1$ with $T_1{}^2 = 1$, satisfying the following conditions:

$$\begin{cases} F_1(x, T_1 y)\ (x, y \in V_1) \text{ is hermitian and (positive) definite,} \\ F_2 \text{ is (positive) definite,} \end{cases} \tag{1 b}$$

$$d\varrho_1(H_0) = \frac{\sqrt{-1}}{2}(T_1 + \mu 1_{V_1}), \tag{2 b}$$

where $\mu$ is a rational number, uniquely determined by the property that $T_1 + \mu 1_{V_1}$ is of trace zero. (If $F_1$ is of signature $(p, q)$ and if $F_1(x, T_1, y)$ is positive-definite, one has $\mu = (q-p)/(p+q)$. Note also that the condition $d\varrho(H_0) \neq 0$ implies that $F_1$ is indefinite.)

In the case $(a_3)$, one would again have (3b). But, proceeding just as above, one has $A = B_1 \otimes B_2$, $I = (\sqrt{-1}\, T_1) \otimes 1_{V_2}$, where $B_i$'s are symmetric or alternating bilinear forms on $V_i$ $(i=1, 2)$ and $T_1$ is a $\mathbf{C}$-linear transformation on $V_1$ with $T_1{}^2 = 1$. Then, clearly, condition (1) can never be satisfied.

In the notation of Theorem 1, the corresponding special unitary groups $G_1'$ and $G_2'$ the are given as follows:

| | $(a_1)$ | $(a_2)$ | (b) |
|---|---|---|---|
| $G'_{1\mathbf{R}} =$ | $Sp(V_1, A_1)$ | $SU(\mathbf{V}_1/\mathbf{K}, \mathbf{F}_1)$ | $SU(V_1, F_1)$ |
| $G'_{2\mathbf{R}} =$ | $SO(V_2, S_2)$ | $SU(\mathbf{K}\backslash\mathbf{V}_2, \mathbf{F}_2)$ | $SU(V_2, F_2)$ |

Thus, one sees that, in every case, the group $G'_{1\mathbf{R}}$ is non-compact and of hermitian type (type (III), (II), (I), respectively) and the group $G'_{2\mathbf{R}}$ is compact. (Actually, by the classification-theory, the above table exhausts all the possibilities of such pairs $(G_1', G_2')$.) An $H$-element for $G_{1\mathbf{R}}'$ is given by

$$H_{01}' = \tfrac{1}{2} I_1, \quad \tfrac{1}{2}\mathbf{I}_1(^1), \quad \frac{\sqrt{-1}}{2}(T_1 + \mu\, 1_{V_1}), \tag{6}$$

respectively, and the condition (2*) means that the absolutely irreducible representation $\varrho_1$ or $\mathsf{P}_1$ of $G_\mathbf{R}$ into $G_{1\mathbf{R}}'$ satisfies the condition $(\mathrm{H}_2)$ with respect to the $H$-elements $H_0$ and $H_{01}'$. (In the cases $(\mathrm{a}_1)$, $(\mathrm{a}_2)$, $\varrho$ itself satisfies $(\mathrm{H}_2)$ with respect to $H_0$ and $H_0'$.) Thus, summing up, we obtain (the "only if" parts of) the following theorem. (The "if" part is trivial.)

THEOREM 3. *Let $G$ be a connected semi-simple algebraic group, defined over* $\mathbf{R}$, *of hermitian type with an $H$-element $H_0$ (see 4.1), and let $(V, A, \varrho)$ be a (non-trivial)* $\mathbf{R}$*-primary symplectic representation of $G$ defined over* $\mathbf{R}$. *Then, $\varrho$ satisfies the conditions* $(\mathrm{H}_1)$ *and* $d\varrho(H_0) \neq 0$, *if and only if, in the notation of Theorem 1, the special unitary group* $G_{1\mathbf{R}}' = SU(\mathbf{V}_1/\mathfrak{K}, \mathbf{F}_1)_\mathbf{R}$ *is non-compact and of hermitian type,* $G_{2\mathbf{R}}' = SU(\mathfrak{K}\backslash\mathbf{V}_2, \mathbf{F}_2)_\mathbf{R}$ *is compact and the representation* $\mathsf{P}_1(^2)$ *of* $G_\mathbf{R}$ *into* $G_{1\mathbf{R}}'$ *satisfies the condition* $(\mathrm{H}_2)$. *More precisely, $\varrho$ satisfies* $(\mathrm{H}_1)$ *with respect to the $H$-elements $H_0$ and $H_0' = \frac{1}{2}I$, $I$ being a complex structure on $V_\mathbf{R}$ satisfying (1), if and ony if there exists an $H$-element $H_{01}'$ for $G_{1\mathbf{R}}'$ such that one has*

$$d\mathsf{P}_1(H_0) = H_{01}' \quad (\textit{condition } (\mathrm{H}_2)), \tag{2*}$$

$$H_0' = \begin{cases} H_{01}' \otimes 1_{\mathbf{V}_2} & \textit{in case } (\mathrm{a}), \\ H_{01}' \otimes 1_{V_2} - \dfrac{\mu}{2}\sqrt{-1}\, 1_V & \textit{in case } (\mathrm{b}), \end{cases} \tag{5*}$$

*where $\mu$ is a (uniquely determined) rational number. Thus, in this case, $I$ is uniquely determined by $(V, \varrho)$ and $H_0$.*

(1) The interpretation given here for $H$-elements in the case $(\mathrm{a}_2)$ is slightly different from the one given in [9]. The relation between them is as follows. Fix an isomorphism $M: \mathbf{K} \otimes_\mathbf{R} \mathbf{C} \to \mathfrak{M}_2(\mathbf{C})$, as given in 3.4, Example 3, and put $V_1 = (\mathbf{V}_1 \otimes_\mathbf{R} \mathbf{C})\varepsilon_{11}$ ($2n$-dimensional complex vector-space). Then, as explained in 3.4, there corresponds, to $\mathbf{F}_1$, a complex symmetric bilinear form $S$ on $V_1$ satisfying the relation

$$\bar{S}(x\varepsilon_2, y\varepsilon_2) = -\gamma S(x, y).$$

Therefore, if one puts $F(x, y) = \sqrt{-1}\, S(\bar{x}\varepsilon_2, y)\ (x, y \in V_1)$, $F$ becomes a usual hermitian form on $V_1$ and one has by (45)

$$M(\mathbf{F}_1(x+\bar{x}, y+\bar{y})) = \begin{pmatrix} -\sqrt{-1}\, F(x, y) & -\gamma\, \overline{S(x, y)} \\ -S(x, y) & \sqrt{-1}\, \overline{F(x, y)} \end{pmatrix}$$

for $x, y \in V_1$. If one puts $\mathbf{I}_1 | V_1 = \sqrt{-1}\, T$, then the conditions for $\mathbf{I}_1$ stated in the text is equivalent to saying that $T^2 = 1$, $S(x, Ty)$ is alternating and $F(x, Ty)$ is hermitian and positive-definite. Thus $\frac{1}{2}\sqrt{-1}\, T$ is an $H$-element in the sense given in [9].

(2) When considered as a homomorphism of an algebraic group, this $\mathsf{P}_1$ should be replaced by $R_{\mathbf{C}/\mathbf{R}}(\mathsf{P}_1)$ in the case (b).

**6.4.** In the case $d\varrho(H_0)=0$, the roles of $\mathbf{V}_1$ and $\mathbf{V}_2$ (in the formulas (4), (5)) are interchanged. For instance, in the case ($\mathrm{a}_1$), one sees immediately that one has ($3\mathrm{a}_1$) with

$$A=S_1\otimes A_2, \tag{4$'$ a$_1$}$$

$$I=1_{V_1}\otimes I_2, \tag{5$'$ a$_1$}$$

where $S_1$ (resp. $A_2$) is a non-degenerate, real, $\varrho_1(G_{\mathbf{R}})$-invariant symmetric (resp. alternating) bilinear form on $V_1$ (resp. $V_2$) and $I_2$ is a complex structure on $V_2$, satisfying the following relations:

$$\begin{cases} S_1 \text{ is (positive) definite,} \\ A_2(x, I_2 y)\ (x, y\in V_2) \text{ is symmetric and (positive) definite.}\end{cases} \tag{1$'$ a$_1$}$$

Similar results are also obtained in the other cases ($\mathrm{a}_2$), (b), (the case ($\mathrm{a}_3$) again missing); in the case (b), one has (3b), (4b) with

$$I=(\sqrt{-1}\,1_{V_1})\otimes T_2, \tag{5$'$ b}$$

and the condition (1′b) says that $F_1$ and $F_2(x, T_2 y)$ are (positive) definite. In this last case the signature of $F_2$ can be arbitrary, and it may happen that $F_2$ itself is definite and $T_2=\pm 1_{V_2}$.

Thus one obtains the following supplement to Theorem 3.

THEOREM 3′. *The notation being as in Theorem* 3, $\varrho$ *satisfies the conditions* ($\mathrm{H}_1$) *with respect to the H-element* $H_0$ *and* $H_0'$ *and* $d\varrho(H_0)=0$, *if and only if* $G_{1\mathbf{R}}'$ *is compact. In that case,* $G_{2\mathbf{R}}'$ *is of hermitian type with an H-element* $H_{02}'$ *satisfying the relation:*

$$H_0'=\begin{cases} 1_{\mathbf{V}_1}\otimes H_{02}' & \textit{in case } (\mathrm{a}), \\ 1_{\mathbf{V}_1}\otimes H_{02}'-\dfrac{\mu}{2}\sqrt{-1}\,1_V & \textit{in case } (\mathrm{b}),\end{cases} \tag{5$'$*}$$

*where* $\mu$ *is a rational number (depending only on the signature of* $F_2$*); in particular, in case* (a), $G_{2\mathbf{R}}'$ *is non-compact.*

Contrary to the case $d\varrho(H_0)\neq 0$, the $H$-element $H_{02}'$ in (5′*) (as well as the signature of $F_2$ in case (b)) can be taken arbitrarily. One should also note that, in case $d\varrho$ is faithful, one has $d\varrho(H_0)=0$ if and only if $G_{\mathbf{R}}$ is compact.

## § 7. Observations over Q

**7.1.** Let $G$ be a connected semi-simple algebraic group, defined over $\mathbf{Q}$, of hermitian type, and let $(V, A, \varrho; I)$ be a non-trivial $\mathbf{Q}$-primary solution of our problem. First of all, it is clear that we may assume, without any loss of generality, that $G$ is simply connected (as an algebraic group); then $G$ is decomposed into the direct product of absolutely simple factors defined over $\bar{\mathbf{Q}}$ as follows:

$$G = G_1 \times \ldots \times G_s. \tag{7}$$

Let $(V_1, \varrho_1)$ be an absolutely irreducible representation of $G$ defined over $\bar{\mathbf{Q}}$, contained in $(V, \varrho)$. Then by a well-known theorem in the representation-theory $\varrho_1$ is decomposed in the following form:

$$\varrho_1 = \bigotimes_{i=1}^{s} \varrho_{1i} \circ p_i, \tag{8}$$

where $p_i$ denotes the projection of $G$ onto $G_i$ and $\varrho_{1i}$ is an absolutely irreducible representation of $G_i$ defined over $\bar{\mathbf{Q}}$. In the following, we shall restrict ourselves to the case where $\varrho_{1i}$'s are all trivial except one of them, say $\varrho_{11}$; i.e., *we shall assume that* $\varrho_1$ *is of the form*

$$\varrho_1 = \varrho_{11} \circ p_1. \tag{9}$$

By virtue of [9], Th. 2, this is surely the case, if $G_{\mathbf{R}}$ has no compact factor.

Now, let $\mathcal{G}$ be the Galois group of $\bar{\mathbf{Q}}/\mathbf{Q}$, put

$$\mathcal{G}_{G_1} = \{\sigma \in \mathcal{G} \mid G_1^{\sigma} = G_1\};$$

and denote by $k$ the subfield of $\bar{\mathbf{Q}}$ corresponding to $\mathcal{G}_{G_1}$; then $k$ is the smallest field over which the subgroup $G_1$ is defined. Since the conjugates of $G_1$ are all defined over $\mathbf{R}$, $k$ is a totally real number-field (of finite degree). If one puts

$$\mathcal{G} = \bigcup_{i=1}^{s_0} \mathcal{G}_{G_1} \tau_i', \tag{10}$$

then one has $s_0$ distinct conjugates $G_1^{\tau_i'}$ in the decomposition (7), and for each $\sigma \in \mathcal{G}_{G_1}\tau_i'$ one has $\varrho_1^{\sigma} = \varrho_{11}^{\sigma} \circ p_1^{\tau_i'}$ where $p_1^{\tau_i'}$ is the projection of $G$ onto $G_1^{\tau_i'}$ and $\varrho_{11}^{\sigma}$ is a representation of $G_1^{\tau_i'}$. Thus $\varrho$ is essentially a representation of $\prod_{i=1}^{s_0} G_1^{\tau_i'} = R_{k/\mathbf{Q}}(G_1)$, which is nothing else than a $\mathbf{Q}$-simple factor of $G$ containing $G_1$. Therefore, in the following, *we shall further assume that $G$ is* $\mathbf{Q}$*-simple*, i.e., *$G$ is of the form:*

$$G = \prod_{i=1}^{s} G_1^{\tau_i'} = R_{k/\mathbf{Q}}(G_1), \tag{11}$$

*where $G_1$ is a connected (simply connected) absolutely simple algebraic group defined over $k$.* It then follows that $\varrho$ is almost faithful (i.e., has a finite kernel) and so $d\varrho$ is faithful.

*Remark.* When we consider discrete subgroups in $G_{\mathbf{R}}$, the above argument should be supplemented by the following observation. Suppose one has a discrete subgroup $\Gamma$ in

$G_{\mathbf{R}}$ such that $\Gamma\backslash G_{\mathbf{R}}$ is of finite volume and that $\varrho(\Gamma)$ is contained in $\Gamma'=Sp(L, A)$, $L$ being a lattice in $V_{\mathbf{Q}}$ (see Introduction, 2). Put $G^{(1)}=R_{k/\mathbf{Q}}(G_1)$ and let $G^{(2)}$ be the complementary partial product in the decomposition (7), which is also defined over $\mathbf{Q}$. Then, by the assumption (9), $\varrho^{(1)}=\varrho|G^{(1)}$ is almost faithful and $\varrho|G^{(2)}$ is trivial. It follows that the projection of $\Gamma$ on $G_{\mathbf{R}}^{(1)}$ is also discrete, for it is contained in the discrete subgroup $\varrho^{(1)^{-1}}(\Gamma')$ of $G_{\mathbf{R}}^{(1)}$. Therefore, denoting by $\Gamma^{(i)}$ the projections of $\Gamma$ on $G_{\mathbf{R}}^{(i)}$ $(i=1, 2)$, one can conclude that $\Gamma$ is commensurable with $\Gamma^{(1)}\times\Gamma^{(2)}$. (See e.g. H. Shimizu, On Discontinuous groups operating on the product of the upper half planes, *Ann. of Math.*, 77 (1963), p. 40) Consequently, the considerations on the quotient space $\Gamma\backslash\mathcal{D}$ is essentially reduced to that on the direct product of the quotient spaces $\Gamma^{(1)}\backslash\mathcal{D}^{(1)}$ and $\Gamma^{(2)}\backslash\mathcal{D}^{(2)}$, $\mathcal{D}^{(i)}$ denoting the symmetric domains associated with $G_{\mathbf{R}}^{(i)}$ $(i=1, 2)$, and we have a family of abelian varieties only on the first factor $\Gamma^{(1)}\backslash\mathcal{D}^{(1)}$. Thus, for the study of families of abelian varieties, we may restrict ourselves to the case $G=G^{(1)}$.

One should also note that, under the assumption (11), the commensurability class of $\Gamma$ is uniquely determined, i.e., $\Gamma$ is automatically commensurable with $G_{\mathbf{Z}}$.

**7.2.** We shall note here that, under the above assumptions, one has $K_{\varrho_1}=K_{\varrho_{11}}$, $\mathfrak{K}_{\varrho_1}=\mathfrak{K}_{\varrho_{11}}$ and if

$$\varrho_{11}=\theta_1\circ \mathsf{P}_{11} \tag{12}$$

is a factorization of $\varrho_{11}$ as given in Proposition 1, then

$$\varrho_1=\theta_1\circ(\mathsf{P}_{11}\circ p_1) \tag{13}$$

is the corresponding factorization of $\varrho_1$. In fact, for $\sigma\in\mathcal{G}$, one has $\varrho_1^\sigma=\varrho_{11}^\sigma\circ p_1^\sigma$, so that one has $\varrho_1^\sigma\sim\varrho_1$, if and only if $p_1^\sigma=p_1$ and $\varrho_{11}^\sigma\sim\varrho_{11}$; and the condition $p_1^\sigma=p_1$ is equivalent to $\sigma\in\mathcal{G}_{G_1}$. Therefore one has $\mathcal{G}_{\varrho_1}=\mathcal{G}_{\varrho_{11}}\subset\mathcal{G}_{G_1}$, namely $K_{\varrho_1}=K_{\varrho_{11}}\supset k$. Moreover, if $\mathsf{P}_{11}$ is a representation of $G_1$ into $GL(\mathbf{V}_1/\mathfrak{K}_{\varrho_{11}})$ satisfying (12), then one has also (13). This implies, by the uniqueness of the factorization in Proposition 1, that $\mathfrak{K}_{\varrho_1}=\mathfrak{K}_{\varrho_{11}}$, and our last assertion follows.

**7.3.** The notation and the assumptions being as above, let us first consider the case (a). By Theorem 1 and 7.2, the representation $\varrho$ can be factorized as follows:

$$G_{/\mathbf{Q}}\xrightarrow{p_1}G_{1/k}\xrightarrow{\mathsf{P}_{11}}\left.\begin{matrix}G_1'=SU(\mathbf{V}_1/\mathfrak{K}_{\varrho_1},\mathbf{F}_1)_{/K_{\varrho_1}}\\ \times\\ G_2'=SU(\mathfrak{K}_{\varrho_1}\backslash\mathbf{V}_2,\mathbf{F}_2)_{/K_{\varrho_1}}\end{matrix}\right\}\xrightarrow{\otimes}Sp(\hat{V}_1,\hat{A}_1)_{/K_{\varrho_1}}\xrightarrow{R_{K_{\varrho_1}/\mathbf{Q}}}G'=Sp(V,A)_{/\mathbf{Q}}, \tag{14a}$$

or what amounts to the same,

$$G = R_{k/\mathbf{Q}}(G_1) \xrightarrow{R_{K_{\varrho_1}/\mathbf{Q}}(\mathsf{P}_{11}\circ p_1)} \left.\begin{matrix} R_{K_{\varrho_1}/\mathbf{Q}}(G_1') \\ \times \\ R_{K_{\varrho_1}/\mathbf{Q}}(G_2') \end{matrix}\right\} \xrightarrow{\otimes} G' = Sp(V, A). \tag{14' a}$$

Here $\mathsf{P}_{11}$ is almost faithful, since $G_1$ is absolutely simple and $\varrho$ is non-trivial.

By § 4, all $\mathbf{R}$-primary components of $V$ are absolutely primary, so that the decomposition

$$V = \sum_{i=1}^{d} \hat{V}_1^{\tau_i}, \quad \text{with } \hat{V}_1 = \mathbf{V}_1 \otimes_{\mathfrak{K}_{\varrho_1}} \mathbf{V}_2$$

is exactly the decomposition of $V$ into the $\mathbf{R}$-primary components. This implies in the first place that $K_{\varrho_1}$ is a totally real number-field. Next, applying what we have said in 3.7 and Theorem 2.2′ to each component $\hat{V}_1^{\tau_i}$, one sees that

$$R_{K_{\varrho_1}/\mathbf{Q}}(G_1') = \prod_{i=1}^{d} G_1'^{\tau_i} \quad \text{and} \quad R_{K_{\varrho_1}/\mathbf{Q}}(G_2') = \prod_{i=1}^{d} G_2'^{\tau_i}$$

are of hermitian type and that, for each $i$, one of $(G_1'^{\tau_i})_{\mathbf{R}}$ and $(G_2'^{\tau_i})_{\mathbf{R}}$ is compact, while the other is non-compact. Thus, in the classification of 8.1, $R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$ is of type (III.1), (III.2) or (II) (see the next paragraph). Moreover, one can take respective $H$-elements $H_{01}'$ and $H_{02}'$ for $R(G_1')_{\mathbf{R}}$ and $R(G_2')_{\mathbf{R}}$ in such a way that one has

$$d(R(\mathsf{P}_1))(H_0) = H_{01}' \quad (\mathsf{P}_1 = \mathsf{P}_{11}\circ p_1), \tag{15}$$

$$H_{01}' \otimes 1 + 1 \otimes H_{02}' = H_0'. \tag{16 a}$$

Thus the representation $R(\mathsf{P}_1)$ satisfies the condition $(\mathrm{H}_2)$ with respect to the $H$-elements $H_0$ and $H_{01}'$.

Conversely, it is clear that, if $R(G_1')$, $R(G_2')$ and $R(\mathsf{P}_1)$ are taken to satisfy all these conditions, then the representation $\varrho$ defined by the above diagram (14a) or (14′a) satisfies the condition $(\mathrm{H}_1)$. Thus one sees that the essential part of our problem lies in *the determination of the absolutely irreducible (almost faithful) representation* $\varrho_{11} = \theta_1 \circ \mathsf{P}_{11}$ *of* $G_1$ *such that* $G_1' = SU(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$ *is of hermitian type (as described above) and that* $R_{K_{\varrho_1}/\mathbf{Q}}(\mathsf{P}_{11}\circ p_1)$: $G = R_{k/\mathbf{Q}}(G_1) \to R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$ *satisfies the condition* $(\mathrm{H}_2)$. In fact, this will first determine uniquely $K_{\varrho_1}$, $\mathfrak{K}_{\varrho_1}$, $\mathbf{V}_1$ and $\mathbf{F}_1$ (up to a scalar multiple), and then (as we shall see it more explicitly in the next paragraph) settle the rest of the problem, i.e., the determinations of $\mathbf{V}_2$, $\mathbf{F}_2$ and $I$, almost automatically. A complete list of such absolutely irreducible representations $\varrho_{11}$ will be given in § 8.

Denoting by $\mathcal{D}_1'$ and $\mathcal{D}_2'$ the symmetric domains associated with $R(G_1')_{\mathbf{R}}$ and $R(G_2')_{\mathbf{R}}$ respectively, one obtains from (14′a) the following diagram:

$$\begin{array}{c} \mathcal{D} \longrightarrow \left.\begin{array}{c} \mathcal{D}_1' \\ \times \\ \mathcal{D}_2' \end{array}\right\} \longrightarrow \mathcal{D}'. \end{array} \tag{17}$$

If one takes invariant complex structures on $\mathcal{D}_1'$ and $\mathcal{D}_2'$ determined by $H_{01}'$ and $H_{02}'$, respectively, these associated mappings become holomorphic. As we have seen above, $\mathcal{D}_1' \times \mathcal{D}_2'$ has exactly $d$ irreducible components. When one fixes $(V, A, \varrho)$ and $H_0$ (together with the factorization (14′a)), the complex structure $I$ giving a solution of our problem, i.e., satisfying the conditions (1), (2), is parametrized by an $H$-element $H_{02}'$ for $R(G_2')_{\mathbf{R}}$. In other words, if $d_2$ is the number of irreducible components of $\mathcal{D}_2'$, the disjoint sum of $2^{d_2}$ copies of $\mathcal{D}_2'$ provided with various complex structures may be regarded as a "parameter-space" of the solutions.

*Remark.* From the almost faithfulness of $\varrho_{11}$, it follows that $(G_1'^{\tau_i})_{\mathbf{R}}$ is compact, if and only if $(G_1^{\tau_i})_{\mathbf{R}}$ is compact. Thus, in particular, if $G_{\mathbf{R}}$ has no compact factor, then $R(G_1')_{\mathbf{R}}$ has no compact factor either and so $R(G_2')_{\mathbf{R}}$ is compact. In this case, $I$ is uniquely determined by $(V, \varrho)$ and $H_0$, and the parameter-space $\mathcal{D}_2'$ reduces to a point.

**7.4.** To describe the solutions more explitly, let us now consider the cases separately, according as $\mathfrak{K}_{\varrho_1} \sim 1$ or $\nsim 1$. First in case $\mathfrak{K}_{\varrho_1} \sim 1$, all $\mathbf{R}$-primary components $(\hat{V}_1^{\tau_i}, \hat{A}_1^{\tau_i}, \varrho^{(i)}(=\hat{\varrho}_1^{\tau_i}))$ of $(V, A, \varrho)$ are of type $(\mathrm{a}_1)$. Since we have at least one index $i$ for which one has $d\varrho^{(i)}(H_0) \neq 0$, it follows that $\mathbf{F}_1 = A_1$ is alternating and so $\mathbf{F}_2 = S_2$ is symmetric. Thus one has

$$\begin{cases} V = R_{K_{\varrho_1}/\mathbf{Q}}(V_1 \otimes V_2), \\ A = \mathrm{tr}_{K_{\varrho_1}/\mathbf{Q}}(A_1 \otimes S_2), \end{cases} \tag{18 a$_1$}$$

and $G_1' = Sp(V_1, A_1)$, $G_2' = SO(V_2, S_2)$. ($R(G_1')$ is of type (III.1).) Then, one has $d\varrho^{(i)}(H_0) \neq 0$ for all $1 \leqslant i \leqslant d$, and therefore

$$I = \sum_{i=1}^{d} I_1^{(i)} \otimes 1, \tag{16 a$_1$}$$

where $I_1^{(i)} = 2d\varrho_1^{\tau_i}(H_0)$ is a complex structure on $(V_1^{\tau_i})_{\mathbf{R}}$. With suitable $\varepsilon_i = \pm 1$ $(1 \leqslant i \leqslant d)$, one has

$$\begin{cases} \varepsilon_i A_1^{\tau_i}(x, I_1^{(i)} y) \gg 0, \\ \varepsilon_i S_2^{\tau_i} \gg 0. \end{cases} \tag{19 a$_1$}$$

Now, we shall indicate the process of obtaining a solution $(V, A, p; I)$. First find an absolutely irreducible representation $\varrho_{11}$ of $G_1$ into $G_1' = Sp(V_1, A_1)$, defined over $K$, such that $R_{K/\mathbf{Q}}(\varrho_{11} \circ p_1)$ satisfies $(\mathrm{H}_2)$ with respect to the $H$-elements $H_0$ and $H_{01}' = \frac{1}{2}\sum I_1^{(i)}$, $I_1^{(i)}$ being a complex structure on $(V_1^{\tau_i})_{\mathbf{R}}$. Next, one determines the "distribution of signs"

$(\varepsilon_i)$ by the first inequalities in $(19a_1)$. Then, taking any vector-space $V_2$ defined over $K$ and any symmetric bilinear form $S_2$ on it also defined over $K$ and satisfying the second inequalities in $(19a_1)$ with this $(\varepsilon_i)$, one constructs a solution $(V, A, \varrho; I)$ by means of $(18a_1)$ $(K_{\varrho_1}=K)$ and $(16a_1)$.

One notes that, taking an element $\omega \in K$ with the given distribution of signs $(\varepsilon_i)$ and replacing $A_1$ and $S_2$ by $\omega A_1$ and $\omega^{-1}S_2$, respectively, one may always obtain the solution with $\varepsilon_i = +1$ for all $i$.

Next, let $\mathfrak{K}_{\varrho_1} \nsim 1$; then $\mathfrak{K}_{\varrho_1}$, having an involution of the first kind, is a quaternion division algebra defined over $K_{\varrho_1}$. We shall rearrange the indices in such a way that

$$\begin{aligned} d\varrho^{(i)}(H_0) &\neq 0 \quad \text{for } 1 \leqslant i \leqslant d_1, \\ d\varrho^{(i)}(H_0) &= 0 \quad \text{for } d_1+1 \leqslant i \leqslant d. \end{aligned} \tag{20}$$

First consider the case where $\mathbf{F}_1$ is hermitian and $\mathbf{F}_2$ is skew-hermitian (with respect to the canonical involution of $\mathfrak{K}_{\varrho_1}$). ($R(G_1')$, $R(G_2')$ are of type (III.2), (II), respectively.) Then, for $1 \leqslant i \leqslant d_1$, one has $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{R}} \sim 1$ and the corresponding $\mathbf{R}$-primary component $\varrho^{(i)}$ is of type $(a_1)$, while, for $d_1+1 \leqslant i \leqslant d$, $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{R}} \nsim 1$ and $\varrho^{(i)}$ is of type $(a_2)$. More precisely, for $1 \leqslant i \leqslant d_1$, taking an $\mathbf{R}$-isomorphism $M^{(i)} : \mathfrak{K}_{\varrho_1}^{\tau_i} \to \mathfrak{M}_2$ (as given in 3.4, Example 3), one can define $V_1^{(i)} = (\mathbf{V}_1^{\tau_i}\varepsilon_{11})_{\mathbf{R}}$, $V_2^{(i)} = (\varepsilon_{11}\mathbf{V}_2^{\tau_i})_{\mathbf{R}}$ and $A_1^{(i)}$, $S_2^{(i)}$ by the relations (I, 45), (I, 45′). This allows us to identify $(\hat{V}_1^{\tau_i})_{\mathbf{R}}$ with $V_1^{(i)} \otimes V_2^{(i)}$ and $\hat{A}_1^{\tau_i}$ with $A_1^{(i)} \otimes S_2^{(i)}$, and one has isomorphisms: $(G_1'^{\tau_i})_{\mathbf{R}} \cong Sp(V_1^{(i)}, A_1^{(i)})$, $(G_2'^{\tau_i})_{\mathbf{R}} \cong SO(V_2^{(i)}, S_2^{(i)})$. Then one has

$$I = \sum_{i=1}^{d_1} I_1^{(i)} \otimes 1 + \sum_{i=d_1+1}^{d} 1 \otimes \mathbf{I}_2^{(i)}, \tag{16′}$$

where $I_1^{(i)} = 2d\mathsf{P}_1^{\tau_i}(H_0) \mid V_1^{(i)}$, and $\mathbf{I}_2^{(i)}$ is a $\mathbf{K}$-linear complex structure on $(\mathbf{V}_2^{\tau_i})_{\mathbf{R}}$. Also, with suitable $\varepsilon_i$, one has

$$\begin{cases} \varepsilon_i A_1^{(i)}(x, I_1^{(i)}y) \gg 0, \\ \varepsilon_i S_2^{(i)} \gg 0, \ (1 \leqslant i \leqslant d_1) \end{cases} \tag{19′ $a_1$}$$

$$\begin{cases} \varepsilon_i \mathbf{F}_1^{\tau_i} \gg 0, \\ \varepsilon_i \mathbf{F}_2^{\tau_i}(\mathbf{I}_2^{(i)}x, y) \gg 0. \ (d_1+1 \leqslant i \leqslant d) \end{cases} \tag{19′ $a_2$}$$

(Remark that the $\varepsilon_i$'s for $1 \leqslant i \leqslant d_1$ depend not only on the choice of $\mathbf{F}_1$ and $\mathbf{F}_2$ but also on the matrix-representations $M^{(i)}$.)

In this case, when one has $\mathsf{P}_{11} : G_1 \to G_1'$ such that $R_{K/\mathbf{Q}}(\mathsf{P}_{11} \circ p_1)$ satisfies $(\mathrm{H}_2)$ with respect to the $H$-elements $H_0$ and $H_{01}' = \frac{1}{2}\sum_{i=1}^{d_1} I_1^{(i)}$, a solution can be constructed, first determining the "distribution of signs" $(\varepsilon_i)$ by the first inequalities in $(19'a_1)$ and $(19'a_2)$, and then taking any $\mathbf{F}_2$ satisfying the second inequalities in $(19'a_1)$ and $(19'a_2)$. This time, $\mathbf{I}_2^{(i)}$'s $(d_1+1 \leqslant i \leqslant d)$ can be taken quite arbitrarily and $I$ is then given by (16′).

The case when $\mathbf{F}_1$ is skew-hermitian and $\mathbf{F}_2$ is hermitian (i.e., $R(G_1')$, $R(G_2')$ are of type (II), (III.2), respectively) can be treated quite similarly, just in interchanging the roles of $\mathbf{F}_1$ and $\mathbf{F}_2$. Namely, for $1\leqslant i\leqslant d_1$, one has $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{R}}\nsim 1$ and $\varrho^{(i)}$ is of type $(\mathrm{a}_2)$, while, for $d_1+1\leqslant i\leqslant d$, $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{R}}\sim 1$ and $\varrho^{(i)}$ is of type $(\mathrm{a}_1)$; for $d_1+1\leqslant i\leqslant d$, one gets a symmetric (resp. alternating) bilinear form $S_1^{(i)}$ (resp. $A_2^{(i)}$) on a real vector-space $V_1^{(i)}$ (resp. $V_2^{(i)}$) such that one has $(\hat{V}_1^{\tau_i})_{\mathbf{R}}=V_1^{(i)}\otimes V_2^{(i)}$ and $\hat{A}_1^{\tau_i}=S_1^{(i)}\otimes A_2^{(i)}$. Then one has

$$I=\sum_{i=1}^{d_1}\mathbf{I}_1^{(i)}\otimes 1+\sum_{i=d_1+1}^{d}1\otimes I_2^{(i)}, \tag{16''}$$

where $\mathbf{I}_1^{(i)}=2d\mathsf{P}_1^{\tau_i}(H_0)$ is a $\mathbf{K}$-linear complex structure on $(\mathbf{V}_1^{\tau_i})_{\mathbf{R}}$ $(1\leqslant i\leqslant d_1)$ and $I_2^{(i)}$ is a complex structure on $V_2^{(i)}$ $(d_1+1\leqslant i\leqslant d)$. Moreover, with suitable $\varepsilon_i$, one has

$$\begin{cases}\varepsilon_i\mathbf{F}_1^{\tau_i}(\mathbf{I}_1^{(i)}x, y)\gg 0,\\ \varepsilon_i\mathbf{F}_2^{\tau_i}\gg 0, \quad (1\leqslant i\leqslant d_1)\end{cases} \tag{19$''$ a$_1$}$$

$$\begin{cases}\varepsilon_i S_1^{(i)}\gg 0,\\ \varepsilon_i A_2^{(i)}(x, I_2^{(i)}y)\gg 0. \quad (d_1+1\leqslant i\leqslant d)\end{cases} \tag{19$''$ a$_2$}$$

Thus it is again clear that the determination of $\mathsf{P}_{11}$ is sufficient.

7.5. In the case (b), the special unitary groups $G_1'=SU(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$ and $G_2'=SU(\mathfrak{K}_{\varrho_1}\backslash\mathbf{V}_2, \mathbf{F}_2)$ being algebraic groups defined over $K_0$, the diagram (14a) in 5.3 should be modified as follows:

$$G_{/\mathbf{Q}}\xrightarrow{p_1}G_{1/k}\xrightarrow{R_{K_{\varrho_1}/K_0}(\mathsf{P}_{11})}\left.\begin{matrix}G'_{1/K_0}\\ \times\\ G'_{2/K_0}\end{matrix}\right\}\xrightarrow{\otimes}Sp(\hat{\hat{V}}_1, \hat{\hat{A}}_1)_{/K_0}\xrightarrow{R_{K_0/\mathbf{Q}}}G'=Sp(V, A)_{/\mathbf{Q}}. \tag{14 b}$$

By a similar reason as in 5.3, one sees that $K_0$ is totally real, $K_{\varrho_1}$ is totally imaginary, and that

$$V=\sum_{i=1}^{d/2}\hat{\hat{V}}_1^{\tau_i}\quad(\text{with }\hat{\hat{V}}_1=R_{K_{\varrho_1}/K_0}(V_1\otimes_{K_{\varrho_1}}V_2))$$

is exactly the decomposition of $V$ into $\mathbf{R}$-primary components. Thus, replacing $K_{\varrho_1}$, $d$, $\mathsf{P}_{11}$, $\hat{V}_1$, $\hat{A}_1$, respectively by $K_0$, $d/2$, $R_{K_{\varrho_1}/K_0}(\mathsf{P}_{11})$, $\hat{\hat{V}}_1$, $\hat{\hat{A}}_1$, everywhere, one sees that all results stated in 7.3 remain true in the case (b), except for the following two points: (i) it can happen that both $(G_1'^{\tau_i})_{\mathbf{R}}$ and $(G_2'^{\tau_i})_{\mathbf{R}}$ are compact, so that the number of irreducible components of $\mathcal{D}_1'\times\mathcal{D}_2'$ is $\leqslant d/2$; (ii) the equality (16a) should be replaced by a congruence modulo scalar multiplications (in each $\mathbf{R}$-primary component) by certain purely imaginary numbers.

To be more explicit, for each $1 \leqslant i \leqslant d/2$, let $F_1^{(i)}$, $F_2^{(i)}$ be (usual) hermitian forms on complex vector-spaces $V_1^{(i)} = (\mathbf{V}_1^{\tau_i} \varepsilon_{11})_{\mathbf{C}}$, $V_2^{(i)} = (\varepsilon_{11} \mathbf{V}_2^{\tau_i})_{\mathbf{C}}$, obtained from $\mathbf{F}_1^{\tau_i}$, $\mathbf{F}_2^{\tau_i}$, respectively (as explained in 3.5; since $\sigma_0^2 = 1$, we may assume $\lambda = 1$ in (I.20, 20′)). Then one has

$$(\hat{\hat{V}}_1^{\tau_i})_{\mathbf{R}} = V_1^{(i)} \otimes_{\mathbf{C}} V_2^{(i)}, \quad \hat{\hat{A}}_1^{\tau_i} = 2 \operatorname{Im}(F_1^{(i)} \otimes F_2^{(i)}), \quad (G_1'^{\tau_i})_{\mathbf{R}} \cong SU(V_1^{(i)}, F_1^{(i)}), \quad (G_2'^{\tau_i})_{\mathbf{R}} \cong SU(V_2^{(i)}, F_2^{(i)}).$$

Suppose that $F_1^{(i)}$ $(1 \leqslant i \leqslant d_1/2)$ is indefinite with the signature $(p_i, q_i)$ and $F_2^{(i)}$ $(d_1/2 + 1 \leqslant i \leqslant d/2)$ is definite. Then one has

$$I = \sqrt{-1}\Big(\sum_{i=1}^{d_1/2} T_1^{(i)} \otimes_{\mathbf{C}} 1_{V_2^{(i)}} + \sum_{i=d_1/2+1}^{d/2} 1_{V_1^{(i)}} \otimes_{\mathbf{C}} T_2^{(i)}\Big), \tag{16 b}$$

where $T_1^{(i)}$ $(1 \leqslant i \leqslant d_1/2)$ is a $\mathbf{C}$-linear transformation on $V_1^{(i)}$ with $T_1^{(i)2} = 1$ such that one has

$$d\,\mathsf{P}_1^{\tau_i}(H_0) \mid V_1^{(i)} = \frac{\sqrt{-1}}{2}(T_1^{(i)} + \mu_i 1_{V_1^{(i)}}) \tag{15 b}$$

for some rational number $\mu_i$ and where $T_2^{(i)}$ $(d_1/2 + 1 \leqslant i \leqslant d/2)$ is a $\mathbf{C}$-linear transformation on $V_2^{(i)}$ with $T_1^{(i)2} = 1$. With suitable $\varepsilon_i$, one has

$$\begin{cases} \varepsilon_i F_1^{(i)}(x, T_1^{(i)} y) \gg 0, \\ \varepsilon_i F_2^{(i)} \gg 0, \quad (1 \leqslant i \leqslant d_1/2) \end{cases} \tag{19 b}$$

$$\begin{cases} \varepsilon_i F_1^{(i)} \gg 0, \\ \varepsilon_i F_2^{(i)}(x, T_2^{(i)} y) \gg 0. \quad (d_1/2 + 1 \leqslant i \leqslant d/2) \end{cases} \tag{19′ b}$$

(Note that $T_1^{(i)}$ and $\mu_i = \varepsilon_i(q_i - p_i)/(p_i + q_i)$ are determined uniquely by (15b).)

Thus, when one has $R_{K/K_0}(\mathsf{P}_{11})$: $G_1 \to G_1'$ such that $R_{K/\mathbf{Q}}(\mathsf{P}_{11} \circ p_1)$ satisfies $(\mathrm{H}_2)$ with respect to the $H$-elements $H_0$ and $H_{01}' = \frac{1}{2}\sqrt{-1}\sum(T_1^{(i)} + \mu_i 1)$, a solution can be constructed, first determining $T_1^{(i)}$ $(1 \leqslant i \leqslant d_1)$ by (15b) and the distribution of signs $(\varepsilon_i)$ by the first inequalities in (19b) (or by the relation $\mu_i = (q_i - p_i)/(p_i + q_i)$) and (19′b), and then taking any $\mathbf{F}_2$ satisfying the second inequalities in (19b) and (19′b); here $T_2^{(i)}$'s can be taken quite arbitrarily and $I$ is then given by (16b).

## § 8. List of solutions

**8.1.** We are now in position to give a list of all possible $\mathbf{Q}$-simple algebraic groups $G = R_{k/\mathbf{Q}}(G_1)$ of hermitian type having actually a (non-trivial) symplectic representation satisfying all the above requirements. It suffices to give a list of the corresponding absolutely simple algebraic groups $G_1$ defined over a totally real number-field $k$. In the following, we arrange the indices $i$ $(1 \leqslant i \leqslant s,\ s = [k : Q])$ in such a way that $(G_1^{\tau_i})_{\mathbf{R}}$ is non-compact for $1 \leqslant i \leqslant s_1$ $(s_1 > 1)$ and compact for $s_1+1 \leqslant i \leqslant s$. A "quaternion algebra" means always a central quaternion division algebra, and a "quaternionic" $\varepsilon$-hermitian form means an $\varepsilon$-hermitian form with respect to the canonical involution of the quaternion algebra.

(I) $G_1 = SU(\mathbf{V}_1/\mathfrak{K}, \mathbf{F}_1)$ (simply connected), where

$k'$ = totally imaginary quadratic extension of $k$,

$(\mathfrak{K}, \iota_0)$ = central division algebra of dimension $r^2$ with an involution of the second kind (with respect to the complex conjugation $\sigma_0$), both defined over $k'$,

$\mathbf{V}_1$ = $n$-dimensional right $\mathfrak{K}$-space defined over $k'$,

$\mathbf{F}_1$ = non-degenerate hermitian form on $\mathbf{V}_1$ with respect to $(\mathfrak{K}, \iota_0)$ defined over $k'$.

(One denotes by $(p_i, q_i)$ the signature of the (usual) hermitian form $F_1^{(i)}$ obtained from $\mathbf{F}_1^{\tau_i}$ as in 3.5. So one has $\mathrm{Min}\{p_i, q_i\} > 0$ for $1 \leqslant i \leqslant s_1$ and $=0$ for $s_1+1 \leqslant i \leqslant s$.)

(II) $G_1 = SU(\mathbf{V}_1/\mathfrak{K}, \mathbf{F}_1)$, where

$\mathfrak{K}$ = quaternion division algebra defined over $k$ such that $(\mathfrak{K}^{\tau_i})_{\mathbf{R}} = \mathbf{K}$ for $1 \leqslant i \leqslant s_1$ and $= \mathfrak{M}_2(\mathbf{R})$ for $s_1+1 \leqslant i \leqslant s$,

$\mathbf{V}_1$ = $n$-dimensional right $\mathfrak{K}$-space defined over $k$ $(n \geqslant 3)$,

$\mathbf{F}_1$ = non-degenerate, quaternionic skew-hermitian form on $\mathbf{V}_1$ defined over $k$ such that, for $s_1+1 \leqslant i \leqslant s$, the real symmetric bilinear form $S_1^{(i)}$, obtained from $\mathbf{F}_1^{\tau_i}$ as in 3.4, is definite.

(III.1) $G_1 = Sp(V_1, A_1)$ (simply connected), where

$V_1$ = $n$-dimensional vector-space defined over $k$,

$A_1$ = non-degenerate algernating form on $V_1$ defined over $k$.

(In this case, one has $s_1 = s$.)

(III.2) $G_1 = SU(\mathbf{V}_1/\mathfrak{K}, \mathbf{F}_1)$ (simply connected), where

$\mathfrak{K}$ = quaternion division algebra defined over $k$ such that one has $(\mathfrak{K}^{\tau_i})_{\mathbf{R}} = \mathfrak{M}_2(\mathbf{R})$ for $1 \leqslant i \leqslant s_1$ and $= \mathbf{K}$ for $s_1+1 \leqslant i \leqslant s$,

$\mathbf{V}_1$ = $n$-dimensional right $\mathfrak{K}$-space defined over $k$,

$\mathbf{F}_1$ = non-degenerate, quaternionic hermitian form on $\mathbf{V}_1$ defined over $k$ such that, for $s_1+1 \leqslant i \leqslant s$, the real quaternionic hermition form $\mathbf{F}_1^{\tau_i}$ is definite.

(IV.1) $G_1$ = simply connected covering group of $SO(W, S)$, where

$W$ = $l$-dimensional vector-space defined over $k$ $(l \geqslant 3, l \neq 4)$,

$S$ = non-degenerate symmetric bilinear form on $W$, defined over $k$, such that $S^{\tau_i}$ (considered over $\mathbf{R}$) is of the signature $(l-2, 2)$ or $(2, l-2)$ for $1 \leqslant i \leqslant s_1$ and is definite for $s_1+1 \leqslant i \leqslant s$.

(IV.2) $G_1$ = simply connected covering group of $SU(\mathbf{W}/\mathfrak{K}, \mathbf{H})$, where

$\mathfrak{K}$ = "totally indefinite" quaternion division algebra over $k$, (i.e., one has $(\mathfrak{K}^{\tau_i})_{\mathbf{R}} \sim 1$ for all $1 \leqslant i \leqslant s$),

$\mathbf{W}$ = $l$-dimensional right $\mathfrak{K}$-space defined over $k$ ($l' \geqslant 3$),

$\mathbf{H}$ = non-degenerate, quaternionic skew-hermitian form on $\mathbf{W}$, defined over $k$, such that the real symmetric bilinear form $S^{(i)}$, obtained from $\mathbf{H}^{\tau_i}$, has the same signature as in the case (IV.1).

*Remark.* Besides these, there are $\mathbf{Q}$-simple algebraic groups of hermitian type, of the mixed type (II–IV.2), of the exceptional type $(D_4)$ coming from the "triality", and of types $(E_6)$ and $(E_7)$. But by [9] we know already that for the last two types there is no solution of our problem. Also, as we shall see in 8.4, the mixed type (II–IV.2) can occur (under the assumption (9)) only for $l' = 4$, and the exceptional $(D_4)$ cannot occur at all.

**8.2.** We shall now determine, for a given $G = R_{k/\mathbf{Q}}(G_1)$, all non-trivial, $\mathbf{Q}$-primary solutions $(V, A, \varrho; I)$ (satisfying the condition (9)) of our problem. As we have seen in 7.3–7.5, it is enough to determine all absolutely irreducible representations $\varrho_{11} = \theta_1 \circ \mathsf{P}_{11}$ of $G_1$ of type (a) or (b) such that $R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$ or $R_{K_0/\mathbf{Q}}(G_1')$ with $G_1' = SU(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$ is of type (III.1), (III.2), (II) or (I) and that $R_{K_{\varrho_1}/\mathbf{Q}}(\mathsf{P}_{11} \circ p_1) : G = R_{k/\mathbf{Q}}(G_1) \to R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$ or $R_{K_0/\mathbf{Q}}(G_1')$ satisfies the condition $(\mathrm{H}_2)$. Considering $G_1$ and $\varrho_{11}$ over $\mathbf{R}$, one will then have a simple Lie group $G_{1\mathbf{R}}$ of hermitian type and an absolutely irreducible representation $\varrho_{11}$ of it into a certain (complex) unitary group satisfying the condition $(\mathrm{H}_2)$. But, we have already a complete list of such representations $\varrho_{11}$ ([9], 3.10), whence we can conclude that, except for the case $G_1 = (D_4)$, any one of these representations, taken to be defined over $\bar{\mathbf{Q}}$, satisfies actually all the above conditions. We shall prove this last point in the following Proposition:

PROPOSITION 7. *Let $G_1$ be an absolutely simple algebraic group defined over a totally real number-field $k$, such that $G_{1\mathbf{R}}$ is non-compact and $G = R_{k/\mathbf{Q}}(G_1)$ is of hermitian type, but not of type $(D_4)$. Then, for an absolutely irreducible representation $\varrho_{11}$ of $G_1$ defined over $\bar{\mathbf{Q}}$ of type* (a) (resp. (b)), *the following four conditions are equivalent:*

(i) *If $\varrho_{11} = \theta_1 \circ \mathsf{P}_{11}$ is the factorization of $\varrho_{11}$ relative to $k_0 = k$ and if $G_1' = SU(\mathbf{V}_1/\mathfrak{K}, \mathbf{F}_1)$ is the corresponding special unitary group, then $R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$* (resp. $R_{K_0/\mathbf{Q}}(G_1')$) *is of type* (III.1), (III.2), (II) (resp. (I)) *and $R_{K_{\varrho_1}/\mathbf{Q}}(\mathsf{P}_{11} \circ p_1) : G = R_{k/\mathbf{Q}}(G_1) \to R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$* (resp. $R_{K_0/\mathbf{Q}}(G_1')$) *satisfies the condition* $(\mathrm{H}_2)$.

(ii) *The notation being as in* (i), *$G_{1\mathbf{R}}'$ is of hermitian type* (III), (II) (resp. (I)) *and* $\mathsf{P}_{11}$ (resp. $R_{\mathbf{C}/\mathbf{R}}(\mathsf{P}_{11})$) : $G_{1\mathbf{R}} \to G_{1\mathbf{R}}'$ *satisfies* $(\mathrm{H}_2)$.

(iii) *If* $\varrho_{11}=\theta_1^{(1)}\circ \mathsf{P}_{11}^{(1)}$ *is the factorization of* $\varrho_{11}$ *relative to* $k_0=\mathbf{R}$ *and if* $G_1^{(1)}=SU(\mathbf{V}_1^{(1)}/\mathfrak{K}^{(1)}, \mathbf{F}_1^{(1)})$ *is* (*the group of* $\mathbf{R}$*-rational points of*) *the corresponding special unitary group, then* $G_1^{(1)}$ *is of hermitian type* (III), (II) (resp. (I)) *and* $\mathsf{P}_{11}^{(1)}: G_{1\mathbf{R}}\to G_{11}^{(1)}$ *satisfies* ($\mathrm{H}_2$).

(iv) $\varrho_{11}(G_{1\mathbf{R}})$ *has a hermitian invariant* $F_1$ *and* $\varrho_{11}: G_{1\mathbf{R}}\to SU(V_{1\mathbf{C}}, F_1)$ *satisfies* ($\mathrm{H}_2$).

In fact, (i)$\Rightarrow$(ii) is evident. The equivalence (ii)$\Leftrightarrow$(iii) follows immediately from the existence of an $\mathbf{R}$-(resp. $\mathbf{C}$-)isomorphism $\Theta: \mathcal{E}(\mathbf{V}_1/\mathfrak{K})_{\mathbf{R}}\to\mathcal{E}(\mathbf{V}_1^{(1)}/\mathfrak{K}^{(1)})$, inducing an isomorphism $G'_{1\mathbf{R}}\cong G_1^{(1)}$, such that one has $\theta_1=\theta_1^{(1)}\circ\Theta$, $\mathsf{P}_{11}^{(1)}=\Theta\circ\mathsf{P}_{11}$ (see 3.7). ($\theta_1^{(1)}$ may be taken to be id. except for the case where $G_1^{(1)}$ is of type (II).) The equivalence (iii)$\Leftrightarrow$(iv) is clear in the case (b) ($\mathfrak{K}^{(1)}=\mathbf{C}$). In the case (a), if $G_1^{(1)}$ is of type (III) or (II), there exists an invariant hermitian form $F_1$ on the representation-space $\mathbf{V}_{1\mathbf{C}}$ of $\theta_1^{(1)}$ and the monomorphism $\theta_1^{(1)}: G_1^{(1)}\to SU(\mathbf{V}_{1\mathbf{C}}, F_1)$ satisfies ($\mathrm{H}_2$). (In the notation of [9], $d\theta_1^{(1)}$ is nothing but the canonical injection: $(\mathrm{III})_p$ or $(\mathrm{II})_p\to(\mathrm{I})_{p,p}$.) Hence one has (iii)$\Rightarrow$(iv). The converse (iv)$\Rightarrow$(iii) follows either directly from the definitions or from the list given in [9].

Now, suppose $\varrho_{11}$ satisfies (iv). In view of the list in [9], one sees that every conjugate $\varrho_{11}^\tau$ of $\varrho_{11}$ is a representation of $G_1^\tau$ of the same kind as $\varrho_{11}$, excepting the case where $G_1$ is of type ($D_4$). Since $\varrho_{11}^\tau=\theta_1^\tau\circ\mathsf{P}_{11}^\tau$ is the factorization of $\varrho_{11}^\tau$ (relative to $k_0=k$), this implies, by virtue of (iv)$\Rightarrow$(ii), that (ii) holds for the corresponding conjugate $\mathsf{P}_{11}^\tau: (G_1^\tau)_{\mathbf{R}}\to(G_1'^\tau)_{\mathbf{R}}$ of $\mathsf{P}_{11}$ as long as $(G_1^\tau)_{\mathbf{R}}$ is non-compact. On the other hand, if $(G_1^\tau)_{\mathbf{R}}$ is compact, then $\varrho_{11}^\tau((G_1^\tau)_{\mathbf{R}})$ is contained in a compact unitary group, so that $(G_1'^\tau)_{\mathbf{R}}$ is also compact (see 4.4). Thus we conclude that (i) holds for $\varrho_{11}$. This completes the proof.

**8.3.** We finally obtain the following list of all possible $\varrho_{11}=\theta_1\circ\mathsf{P}_{11}$ excepting the case $G_1=(D_4)$, which we shall treat separately in 8.4. The first four solutions will be called "standard".

(I) ($nr\geqslant 3$): Case (b). $K_{\varrho_1}=k'$, $K_0=k$, $\mathfrak{K}_{\varrho_1}=\mathfrak{K}$, and $\mathbf{V}_1, \mathbf{F}_1$ are the same as given in the first list. Denoting by $\pi_1$ the canonical projection: $R_{K_{\varrho_1}/K_0}(GL(\mathbf{V}_1))\to GL(\mathbf{V}_1)$, one has $\mathsf{P}_{11}=\pi_1$ or $\pi_1^{\sigma_0}$ (restricted on $G_1$), so that $R_{K_{\varrho_1}/K_0}(\mathsf{P}_{11})=\mathrm{id}$.

(II) ($n\geqslant 5$): Case (a). $K_{\varrho_1}=k$, $\mathfrak{K}_{\varrho_1}=\mathfrak{K}$, and $\mathbf{V}_1, \mathbf{F}_1$ are the same as given in the first list; and $\mathsf{P}_{11}=\mathrm{id}$.

(III.1): Case (a). $K_{\varrho_1}=k$, $\mathfrak{K}_{\varrho_1}\sim 1$, $\mathbf{V}_1=V_1$, $\mathbf{F}_1=A_1$; and $\mathsf{P}_{11}=\varrho_{11}=\mathrm{id}$.

(III.2): Case (a). $K_{\varrho_1}=k$, $\mathfrak{K}_{\varrho_1}=\mathfrak{K}$, and $\mathbf{V}_1, \mathbf{F}_1$ are the same as in the first list; and $\mathsf{P}_{11}=\mathrm{id}$.

(I′) (Special case of (I) where one has $p_i$ or $q_i=1$ for all $1\leqslant i\leqslant s_1$): one has $\varrho_{11}=\Lambda_\nu\circ\theta_1\circ\pi_1$ ($1\leqslant\nu\leqslant[nr/2]$), where $\Lambda_\nu$ denotes a skew-symmetric tensor representation of degree $\nu$ of $GL(V_1)$. Case (a) and $K_{\varrho_1}=k$, if $\nu=nr/2$, and case (b) and $K_{\varrho_1}=k'$, $K_0=k$ otherwise.

For the determination of $\mathfrak{K}_{\varrho_1}$ and $\mathbf{F}_1$, see 5.3 and [9], 3.2; especially when $\nu = nr/2$, one has $\varepsilon_0\varepsilon = (-1)^{nr/2}$.

(IV.1): $\varrho_{11}$ is a spin representation. Let $C^+$ be the even Clifford algebra of $(W, S)$. When $l \equiv 1$ (2), one is in Case (a), $K_{\varrho_1} = k$, $\mathfrak{K}_{\varrho_1} \sim C^+$. One can identify $C^+$ with $\mathcal{E}(\mathbf{V}_1/\mathfrak{K}_{\varrho_1})$, so that $\mathsf{P}_{11}$ becomes the inclusion mapping. $\mathbf{F}_1$ is the $\varepsilon$-hermitian form on $\mathbf{V}_1$ corresponding to the canonical involution $\iota$ of $C^+$; one has

$$\varepsilon_0\varepsilon = \begin{cases} 1 & \text{if } l \equiv \pm 1 \ (8), \\ -1 & \text{if } l \equiv \pm 3 \ (8), \end{cases} \tag{21}$$

and accordingly $G_1' = SU(\mathbf{V}_1, \mathbf{F}_1)$ is of type (II) or (III) ([9], 3.7). When $l \equiv 0$ (2), put $k' = k(\sqrt{(-1)^{l/2} \det(S)})$. Let $\varrho^{(i)}$ be the spin representations of $G_1$ obtained by the simple component $C_i^+$ $(i = 1, 2)$ of $C^+$ and denote by $\mathbf{V}_1^{(i)}, \mathbf{F}_1^{(i)}, \mathsf{P}_{11}^{(i)}, \ldots$ the corresponding data. When $l \equiv 0$ (4), one is in Case (a), $K_{\varrho^{(i)}} = k'$, $\mathfrak{K}_{\varrho^{(i)}} \sim C_i^+$ $(\sim C)$. One can identify $C_i^+$ with $\mathcal{E}(\mathbf{V}_1^{(i)}/\mathfrak{K}_{\varrho^{(i)}})$, so that $\mathsf{P}_{11}^{(i)}$ is the restriction on $G_1$ of the projection of $C^+$ on the $i$th factor. The hermitian form $\mathbf{F}_1^{(i)}$ is the $\varepsilon$-hermitian form on $\mathbf{V}_1^{(i)}$ corresponding to the restriction on $C_i^+$ of the canonical involution $\iota$; one has

$$\varepsilon_0\varepsilon = \begin{cases} 1 & \text{if } l \equiv 0 \ (8), \\ -1 & \text{if } l \equiv 4 \ (8), \end{cases} \tag{21'}$$

and accordingly $G_1' = SU(\mathbf{V}_1^{(i)}, \mathbf{F}_1^{(i)})$ is of type (II) or (III) ([9], 3.6). If $k' \underset{\neq}{\supset} k$, $\varrho^{(1)}$ and $\varrho^{(2)}$ being conjugate over $k$, one gets only one solution. When $l \equiv 2$ (4), one is in Case (b), $K_{\varrho^{(i)}} = k'$, $K_0 = k$, and $\mathfrak{K}_{\varrho^{(i)}}$, $\mathsf{P}_{11}^{(i)}$ are the same as above. The hermitian form $\mathbf{F}_1^{(i)}$ is the one corresponding to the involution of the second kind $\iota_i = \iota\sigma_0 | C_i^+$. $\varrho^{(1)}$ and $\varrho^{(2)}$, being conjugate over $k$, give one and the same solution.

(IV.2): $\varrho_{11}$ is a spin representation (see 5.5, 5.6). Replacing $l$, $C^+$, $C_i^+$ by $2l'$, $\mathfrak{C}$, $\mathfrak{C}_i$, respectively, one obtains the similar result as in the case (IV.1), $l \equiv 0$ (2).

*Remark.* For the group of type (II) $(n = 3)$, the identical representation and two spin representations (which are mutually conjugate over $k$) are solutions of our problem. But, since this group is isomorphic to a group of type (I) $(n = 1, r = 4)$ and these solutions correspond to those given in (I′), we omitted them from the list. On the other hand, for the group of type (IV.1) $(l = 8)$, (IV.2) $(l' = 4)$, no modification is needed, so that they are included in the list.

8.4. In this paragraph, we shall consider the groups of the mixed type (II–IV.2) $(l' > 4)$ and the groups of type $(D_4)$ (which is not of type (IV.1), (IV.2)). Let $(G_1, \varphi)$ be the universal covering group of $SU(\mathbf{W}/\mathfrak{K}, \mathbf{H})$, where $\mathfrak{K}$ is a quaternion division algebra defined over $k$ such that $(\mathfrak{K}^{\tau_i})_{\mathbf{R}}=\mathfrak{K}$ for $1\leqslant i\leqslant s'$ and $=\mathfrak{M}_2(\mathbf{R})$ for $s'+1\leqslant i\leqslant s$, $\mathbf{W}$ an $l'$-dimensional right $\mathfrak{K}$-space, and $\mathbf{H}$ a non-degenerate quaternionic skew-hermitian form on $\mathbf{W}$, both defined over $k$, such that the real symmetric bilinear form $S^{(i)}$, obtained from $\mathbf{H}^{\tau_i}$ for $s'+1\leqslant i\leqslant s$, is of signature $(2l'-2, 2)$ or $(2, 2l'-2)$ for $s'+1\leqslant i\leqslant s_1$ and is definite for $s_1+1\leqslant i\leqslant s$. Let $\mathfrak{C}=\mathfrak{C}_1+\mathfrak{C}_2$ be the corresponding "twisted" Clifford algebra, as constructed in 5.5, and let $\rho^{(j)}$ be the spin representation of $G_1$ obtained by $\mathfrak{C}_j$ $(j=1, 2)$.

First let $l'>4$ and $0<s'<s_1$, and suppose we have a solution $\rho_{11}=\theta_1\circ \mathsf{P}_{11}$ of our problem. Then, one has $\mathsf{P}_{11}^{\tau_i'}=\varphi^{\tau_i'}$ for $1\leqslant i\leqslant s'$ and $\rho_{11}^{\tau_i'}=\rho^{(1)\tau_i'}$ or $\rho^{(2)\tau_i'}$ for $s'+1\leqslant i\leqslant s_1$, whence one would have $\theta_1\circ\varphi=\rho^{(1)}$ or $\rho^{(2)}$, which is a contradiction (cf. 5.6). Thus there is no solution for the group of the mixed type (II–IV.2) $(l'>4)$.

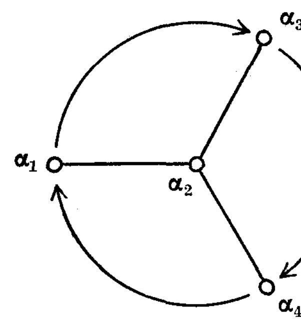
Next, let $l'=4$ and we shall examine the condition for the existence of solution. Let $\tau\in\mathcal{G}(k)\tau_i'$. In view of Jacobson's theorem (5.6), we have the following possibilities for the algebra-class of $(\mathfrak{C}_j^\tau)_{\mathbf{R}}$:

| | $1\leqslant i\leqslant s'$ | $s'+1\leqslant i\leqslant s_1$ | $s_1+1\leqslant i\leqslant s$ |
|---|---|---|---|
| $(\mathfrak{C}_1^\tau)_{\mathbf{R}}\sim$ | $\mathbf{K}$ or $\mathbf{R}$ | $\mathbf{K}$ | $\mathbf{R}$ |
| $(\mathfrak{C}_2^\tau)_{\mathbf{R}}\sim$ | $\mathbf{R}$ or $\mathbf{K}$ | $\mathbf{K}$ | $\mathbf{R}$ |

Now, if $\rho_{11}=\theta_1\circ\mathsf{P}_{11}$ is a solution, then, for $s'+1\leqslant i\leqslant s_1$, one has again $\rho_{11}^\tau=\rho^{(1)\tau}$ or $\rho^{(2)\tau}$, while, for $1\leqslant i\leqslant s'$, one has $\rho_{11}^\tau=\theta_1^\tau\circ\varphi^{\tau_i'}$ or $=\rho^{(j)\tau}$ where $(\mathfrak{C}_j^\tau)_{\mathbf{R}}\sim\mathbf{K}$ (cf. [9], 3.3). Thus, in case $s'=s_1$ (i.e. $G_1$ is of type (II)), $\mathsf{P}_{11}=\varphi$ is always a solution, and *$\rho_{11}=\rho^{(j)}$ is a solution if and only if one has $(\mathfrak{C}_j^\tau)_{\mathbf{R}}\sim\mathbf{K}$ for all $\tau\in\mathcal{G}(k)\tau_i'$, $1\leqslant i\leqslant s'$*. This condition implies that $k'=k(\sqrt{n(\mathbf{H})})=k$, for otherwise one would have $(\mathfrak{C}_1^{\tau_i'})_{\mathbf{R}}\sim(\mathfrak{C}_2^{\tau_i'})_{\mathbf{R}}\sim\mathbf{K}$ $(1\leqslant i\leqslant s')$, which is impossible. Therefore the above condition is equivalent to saying *that $k'=k$ and $(\mathfrak{C}_j^{\tau_i'})_{\mathbf{R}}\sim\mathbf{K}$ for all $1\leqslant i\leqslant s'$*. In case $0<s'<s_1$ (i.e. $G_1$ is of the mixed type (II–IV.2)), the only possibility is $\rho_{11}=\rho^{(j)}$ and this occurs under the same condition as above. In both cases, the data describing $\rho_{11}=\rho^{(j)}$ is as given in 8.3, (IV.1) $(l=8)$; especially the corresponding group $G_1'$ (which is in this case $k$-isogeneous to $G_1$) is of type (II).

Finally, let us consider the "exceptional" $(D_4)$ which comes from the triality. $k$ being, as before, a totally real number-field, let $G_1$ be a simply connected algebraic group of type $(D_4)$ such that $(G_1^{\tau_i'})_{\mathbf{R}}$ is non-compact and corresponds to an irreducible symmetric domain (of type $(\mathrm{II})_4\cong(\mathrm{IV})_6$) for $1\leqslant i\leqslant s_1$, and is compact for $s_1+1\leqslant i\leqslant s$. Take any quadratic form of 8 variables, defined over $\mathbf{Q}$, of the signature $(6, 2)$ (say, $\sum_{i=1}^6 X_i^2-X_7^2-X_8^2$), and let $G_0$ be the corresponding spin group. Then there exists an isomorphism $\mathrm{f}: G_0\to G_1$, defined over $\overline{\mathbf{Q}}\cap\mathbf{R}$.

Now, let $\Delta=\{\alpha_1, ..., \alpha_4\}$ be a $\mathcal{G}(k)$-fundamental system of $G_1$ in the sense of [19]. By definition, $G_1$ is an exceptional $(D_4)$, if and only if the operation of $\{[\sigma]|\sigma\in\mathcal{G}(k)\}$ (as defined in 4.1) is transitive on the set of three extreme vertices $\alpha_1$, $\alpha_3$, $\alpha_4$ of the diagram of $\Delta$ (see the figure). On the other hand, if $G_1$ has a solution $\varrho_{11}$ of our problem, then, for any $\sigma\in\mathcal{G}(k)$, $\varrho_{11}^\sigma\circ f$ is one of the two spin representations of $G_0$. Through the fixed isomorphism $f$, one can distinguish two fundamental weights of $G_1$ (relative to $\Delta$), say




$\omega_3$, $\omega_4$, corresponding to the two spin representations of $G_0$. Then one has $\lambda_{\varrho_{11}^\sigma}=\omega_3$ or $\omega_4$ for all $\sigma\in\mathcal{G}(k)$, $\lambda_{\varrho_{11}^\sigma}$ denoting the heighest weight of $\varrho_{11}^\sigma$ (relative to $\Delta$). In view of the relation $\lambda_{\varrho_{11}^\sigma}=\lambda_{\varrho_{11}}^{[\sigma]}$, this implies that $\{\omega_3, \omega_4\}$ and hence$\{\alpha_3, \alpha_4\}$ is left invariant under $[\sigma]$ $(\sigma\in\mathcal{G}(k))$. This contradiction proves the non-existence of solution for the exceptional $(D_4)$.

## § 9. Examples and generalizations

**9.1.** We shall give here some examples to indicate how the "distribution of signs" can be determined for the non-standard solutions. We keep the notation of the preceding section.

*Example* 1. (I') $(\nu<nr/2)$ Let $G_1=SU(\mathbf{V}_1/\mathfrak{K}, \mathbf{F}_1)$. For each $1\leqslant i\leqslant s=d/2$, let $V_1^{(i)}$ and $F_1^{(i)}$ be a complex vector-space and a (complex) hermitian form on it obtained from $\mathbf{V}_1^{\tau_i}$, $\mathbf{F}_1^{\tau_i}$, respectively. Let $\Lambda_\nu(V_1^{(i)})$ denote the $\nu$th exterior product of $V_1^{(i)}$, and $\Lambda_\nu(F_1^{(i)})$ the hermitian form on $\Lambda_\nu(V_1^{(i)})$ obtained from $F_1^{(i)}$ in the canonical way (see [9], (34)). Then one has

$$(G_1^{\tau_i})_{\mathbf{R}}\cong SU(V_1^{(i)}, F_1^{(i)}),\quad (G_1'^{\tau_i})_{\mathbf{R}}\cong SU(\Lambda_\nu(V_1^{(i)}), \Lambda_\nu(F_1^{(i)})).$$

Through these isomorphisms, the $H$-elements for $R(G_1)_{\mathbf{R}}$, $R(G_1')_{\mathbf{R}}$ are expressed as follows:

$$H_0=\frac{\sqrt{-1}}{2}\sum_{i=1}^{s_1}\left(T_1^{(i)}+\varepsilon_i\frac{q_i-p_i}{p_i+q_i}\right),$$

$$H_{01}'=\frac{\sqrt{-1}}{2}\sum_{i=1}^{s_1}\left(T_{1,\nu}^{(i)}+\varepsilon_i'\frac{q_i'-p_i'}{p_i'+q_i'}\right),$$

where $(p_i, q_i)$ is the signature of $F_1^{(i)}$, $\varepsilon_i = \pm 1$, and $T_1^{(i)}$ is a $\mathbf{C}$-linear transformation on $V_1^{(i)}$ with $T_1^{(i)2} = 1$ such that $\varepsilon_i F_1^{(i)}(x, T_1^{(i)}y)$ is positive-definite, and similarly for the second equation. Now suppose the condition $(\mathrm{H}_2): R(\mathsf{P}_1)\ (H_0) = H'_{01}$ is satisfied. If $p_i = nr-1$, $q_i = 1$, $\varepsilon_i = 1$, then one has

$$p'_i = \binom{nr-1}{\nu}, \quad q'_i = \binom{nr-1}{\nu-1}, \quad \varepsilon'_i = 1$$

(see [9], 3.2). In general, for $1 \leqslant i \leqslant s_1$, put $\eta_i = 1$ or $-1$ according as $q_i = 1$ or $p_i = 1$; then applying this result to $\eta_i F_1^{(i)}, \varepsilon_i \eta_i T_1^{(i)}$, one gets easily $\varepsilon'_i = \varepsilon_i \eta_i^{\nu+1}$. For $s_1 + 1 \leqslant i \leqslant s$, $\varepsilon_i$ and $\varepsilon'_i$ being determined by the conditions $\varepsilon_i F_1^{(i)} \gg 0$ and $\varepsilon'_i \Lambda_\nu(F_1^{(i)}) \gg 0$, one clearly obtains $\varepsilon'_i = \varepsilon_i^\nu$.

*Example* 2. (IV.1) We can take $\mathfrak{K}_{\varrho_1}$ and $\mathbf{V}_1$ in the following form:

$$\mathfrak{K}_{\varrho_1} = \varepsilon C^+ \varepsilon, \quad \mathbf{V}_1 = C^+ \varepsilon, \tag{22}$$

where $\varepsilon$ is a $k'$-rational idempotent in $C^+$, which is indecomposable in $k'$. (In the following, we put $k' = k$, if $l$ is odd.) In the case (a), one may further assume that $\varepsilon^\iota = \varepsilon$, so that $\mathfrak{K}_{\varrho_1}$ is invariant under $\iota$ ([2], p. 156, Th. 12); then $\iota$ induces in $\mathfrak{K}_{\varrho_1}$ an involution of the first kind.

*The case* $l \equiv 3, 4, 5$ (8) *and* $\mathfrak{K}_{\varrho_1}$ *is a quaternion algebra* (i.e., $R(G'_1)$ is of type (III.2)). In this case, one sees easily (e.g., by counting the number of linearly independent $\iota$-symmetric elements in $\mathfrak{K}_{\varrho_1}$) that $\iota | \mathfrak{K}_{\varrho_1}$ coincides with the canonical involution $\iota_0$ of $\mathfrak{K}_{\varrho_1}$. Therefore one may take $\mathbf{F}_1$ in the form

$$\mathbf{F}_1(x, y) = x^\iota y \quad \text{for } x, y \in \mathbf{V}_1. \tag{23}$$

For $1 \leqslant i \leqslant d_1$, let $\{\varepsilon_{jk}^{(i)}\}$ be a system of matrix-units in $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{R}}$ and put

$$\begin{cases} V_1^{(i)} = (\mathbf{V}_1^{\tau_i} \varepsilon_{11}^{(i)})_{\mathbf{R}}, \\ A_1^{(i)}(x, y) = -\operatorname{tr}(\varepsilon_{12}^{(i)} x^\iota y) \quad \text{for } x, y \in V_1^{(i)}. \end{cases} \tag{24}$$

Let further $(e_1^{(i)}, \ldots, e_l^{(i)})$ be an orthogonal basis of $(W^{\tau_i})_{\mathbf{R}}$ such that $S^{\tau_i}(e_j^{(i)}, e_j^{(i)}) = 1$ $(1 \leqslant j \leqslant l-2$ or $2)$ and $= -1$ $(l-1$ or $3 \leqslant j \leqslant l)$. Reordering these basis if necessary, one may assume that the $H$-element $H_0$ for $R(G_1)_{\mathbf{R}}$ is compatible with the usual bounded domain realization of $\mathcal{D} = R(G_1)_{\mathbf{R}}/\mathcal{K}$ obtained from these basis (see [9], 3.5). Then, if $dR(\mathsf{P}_1)(H_0) = H'_{01} = \frac{1}{2}\sum_{i=1}^{d_1} I_1^{(i)}$, one has ([9], 3.6, 7)

$$I_1^{(i)}(x) = \begin{cases} e_{l-1}^{(i)} e_l^{(i)} x & \text{if } S^{\tau_i} \text{ is of signature } (l-2, 2), \\ -e_1^{(i)} e_2^{(i)} x & \text{if } S^{\tau_i} \text{ is of signature } (2, l-2). \end{cases} \tag{25}$$

We put $e_{-}^{(i)} = e_{l-1}^{(i)} e_l^{(i)}$ or $-e_1^{(i)} e_2^{(i)}$ according to the cases, and call $\beta^{(i)} (\in \mathbf{R})$ the (1, 2)-component of $\varepsilon^{\tau_i} e_{-}^{(i)} \varepsilon^{\tau_i}$, i.e. one puts $\varepsilon_{11}^{(i)} e_{-}^{(i)} \varepsilon_{22}^{(i)} = \beta^{(i)} \varepsilon_{12}^{(i)}$. Then, one has

$$\beta^{(i)} A_1^{(i)}(x, I_1^{(i)} y) = -\beta^{(i)} \operatorname{tr}(\varepsilon_{12}^{(i)} x^t e_{-}^{(i)} y) = \operatorname{tr}(e_{-}^{(i)-1} x^t e_{-}^{(i)} y).$$

Since $x \to e_{-}^{(i)-1} x^t e_{-}^{(i)}$ is a positive involution of $(C^{+\tau_i})_{\mathbf{R}}$ (cf. [10]), this shows that the $i$th sign $\varepsilon_i$ is given by $\operatorname{sign}(\beta^{(i)})$ for $1 \leqslant i \leqslant d_1$. (In particular, one has $\beta^{(i)} \neq 0$.) For $d_1 + 1 \leqslant i \leqslant d$, $x \to x^t$ is a positive involution of $(C^{+\tau_i})_{\mathbf{R}}$, so that $F_1^{\tau_i}$ is positive-definite. Thus one has $\varepsilon_i = 1$.

In case $\mathfrak{K}_{\varrho_1} \sim 1$ (i.e., $R(G_1')$ is of type (III.1)), taking $k'$-rational matrix-units $\varepsilon_{jk}$ in the splitting quaternion algebra and putting $\varepsilon_{jk}^{(i)} = \varepsilon_{jk}^{\tau_i}$, $A_1^{(i)} = A_1^{\tau_i}$, etc., one obtains the same result.

*The case* $l \equiv -1, 0, 1$ (8) (i.e., $R(G_1')$ is of type (II)). In this case, one has

$$\xi^t = a_0^{-1} \xi^{\iota_0} a_0 \quad \text{for } \xi \in \mathfrak{K}_{\varrho_1} \tag{26}$$

with $a_0 \in (\mathfrak{K}_{\varrho_1})_{k'}$, $a_0^{\iota_0} = -a_0$, and so one may put

$$\mathbf{F}_1(x, y) = a_0 x^t y = x^{\iota_0} a_0 y. \tag{23'}$$

For $d_1 + 1 \leqslant i \leqslant d$, let $\{\varepsilon_{jk}^{(i)}\}$ be a system of matrix-units in $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{R}}$ and define $V_1^{(i)}$ and $S_1^{(i)}$ similarly as (24). Let $\alpha^{(i)} (\in \mathbf{R})$ be the (1, 2)-component of $a_0^{\tau_i}$. Then, quite similarly as above, one obtains $\varepsilon_i = \operatorname{sign}(\alpha^{(i)} n(a_0^{\tau_i}))$. For $1 \leqslant i \leqslant d_1$, let $e_{-}^{(i)}$ and $H_0$ be as before. Then, one has $a_0^{\tau_i} e_{-}^{(i)} \varepsilon^{\tau_i} = \delta^{(i)} \varepsilon^{\tau_i}$ with $\delta^{(i)} \in \mathbf{R}$, and $\varepsilon_i$ is given by $\varepsilon_i = \operatorname{sign}(\delta^{(i)})$.

*The case* $l \equiv 2$ (4) (i.e., $R(G_1')$ is of type (I)). In this case, one may assume in (22) that $\varepsilon \in C_1^+$ and $\varepsilon^{\iota_1} = \varepsilon$; then $\iota_1$ leaves $\mathfrak{K}_{\varrho_1}$ invariant and induces in it an involution of the second kind. One also assumes that $\mathfrak{K}_{\varrho_1} \nsim 1$. (The other case is easier.) For each $i$, one can take a matrix representation $M^{(i)}$ of $(\mathfrak{K}_{\varrho_1}^{\tau_i})_{\mathbf{C}}$ such that

$$\text{for } M^{(i)}(\xi) = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}, \quad \text{one has } M^{(i)}(\xi^{\iota_1}) = \begin{pmatrix} \bar{\xi}_{11} & \eta \bar{\xi}_{21} \\ \eta \bar{\xi}_{12} & \bar{\xi}_{22} \end{pmatrix},$$

where $\eta = -1$ $(1 \leqslant i \leqslant s_1 = d_1/2)$, $\eta = 1$ $(s_1 + 1 \leqslant i \leqslant s = d/2)$. One puts

$$\mathbf{F}_1(x, y) = x^{\iota_1} y \quad \text{for } x, y \in \mathbf{V}_1, \tag{23''}$$

and for each $i$

$$F_1^{(i)}(x, y) = \operatorname{tr}(\bar{x}^t y) \quad \text{for } x, y \in V_1^{(i)} = (\mathbf{V}_1^{\tau_i} \varepsilon_{11}^{(i)})_{\mathbf{C}}. \tag{24''}$$

Then $F_1^{(i)}$ is of signature $(2^{l/2-2}, 2^{l/2-2})$ for $1 \leqslant i \leqslant s_1$ and positive-definite for $s_1 + 1 \leqslant i \leqslant s$. $e_{-}^{(i)}$ and $H_0$ being as before, one can show that the (1, 1)-component of $\varepsilon^{\tau_i} e_{-}^{(i)} \varepsilon^{\tau_i}$ is $= \sqrt{-1}\, \beta^{(i)}$ with $\beta^{(i)} \in \mathbf{R}$, and one has $\varepsilon_i = \operatorname{sign}(\beta^{(i)})$ for $1 \leqslant i \leqslant s_1$. For $s_1 + 1 \leqslant i \leqslant s$, one has $\varepsilon_i = 1$.

**9.2.** We shall conclude this paper by a brief indication about the most general solutions not necessarily satisfying the condition (9). For simplicity, we shall treat only the case (a).

To begin with, let $G$ be a connected semi-simple algebraic group defined over $k_0$ which is decomposable into the direct product of $k_0$-closed subgroups $G^{(i)}$ as follows:

$$G = G^{(1)} \times \dots \times G^{(t)}. \tag{27}$$

We denote by $p^{(i)}$ the projection of $G$ onto $G^{(i)}$. Then, an absolutely irreducible representation $\varrho_1$ of $G$ (defined over $\bar{k}_0$) can be expressed as

$$\varrho_1 = \otimes \varrho_1^{(i)} \circ p^{(i)}, \tag{28}$$

where $\varrho_1^{(i)}$ is an absolutely irreducible representation of $G^{(i)}$. It is clear from the definitions that one has

$$\begin{cases} K_{\varrho_1} = \bigcup K_{\varrho_1^{(i)}} & (= \text{the smallest field containing all } K_{\varrho_1^{(i)}}), \\ \mathfrak{K}_{\varrho_1} \sim \otimes \mathfrak{K}_{\varrho_1^{(i)}} & \text{over } K_{\varrho_1} \end{cases} \tag{29}$$

(cf., also Part II). One puts $\dim \mathfrak{K}_{\varrho_1^{(i)}} = r_i^2$, $\dim \mathfrak{K}_{\varrho_1} = r^2$, $\prod r_i = ru$, and fixes a system of matrix-units $\{\varepsilon'_{ij}\}$ in $\otimes \mathfrak{K}_{\varrho_1^{(i)}}$ giving a matrix representation of it in $\mathfrak{K}_{\varrho_1}$. Let $\varrho_1 = \theta_1 \circ \mathsf{P}_1$, $\varrho_1^{(i)} = \theta_1^{(i)} \circ \mathsf{P}_1^{(i)}$, and let $\mathbf{V}_1^{(i)}$ be an $n_i$-dimensional right $\mathfrak{K}_{\varrho_1^{(i)}}$-space giving the absolutely irreducible representation $\mathsf{P}_1^{(i)}$. Then $\otimes \mathbf{V}_1^{(i)}$ can be regarded as a right vector-space over $\otimes \mathfrak{K}_{\varrho_1^{(i)}} = \mathfrak{M}_u(\mathfrak{K}_{\varrho_1})$, and $\mathbf{V}_1 = (\otimes \mathbf{V}_1^{(i)}) \varepsilon'_{11}$ is a right vector-space over $\mathfrak{K}_{\varrho_1}$ of dimension $n = (\prod n_i) u$, giving the representation $\mathsf{P}_1$. More precisely, one has $\mathsf{P}_1 = \Theta \circ (\otimes \mathsf{P}_1^{(i)})$, where $\Theta$ denotes the natural isomorphism $\mathcal{E}(\otimes \mathbf{V}_1^{(i)} / \otimes \mathfrak{K}_{\varrho_1^{(i)}}) \to \mathcal{E}(\mathbf{V}_1 / \mathfrak{K}_{\varrho_1})$.

Now the representation $\varrho_1$ is of type (a) if and only if all the $\varrho_1^{(i)}$ are of type (a). Supposing this to be the case, let $\iota_0$, $\iota_0^{(i)}$ be involutions of the first kind of $\mathfrak{K}_{\varrho_1}$ and $\mathfrak{K}_{\varrho_1^{(i)}}$, respectively, and for $\otimes \xi_i = (\zeta_{jk})$ $(\xi_i \in \mathfrak{K}_{\varrho_1^{(i)}}, \zeta_{jk} \in \mathfrak{K}_{\varrho_1})$ put

$$\otimes \xi_i^{\iota_0^{(i)}} = J_0^{-1} (\zeta_{kj}^{\iota_0}) J_0 \tag{30}$$

with $J_0 \in \mathfrak{M}_u(\mathfrak{K}_{\varrho_1})$, $J_0^{\iota_0} = \varepsilon'_0 J_0$, $\varepsilon'_0 = \pm 1$. Then, to a system of $\mathsf{P}_1^{(i)}(G^{(i)})$-invariant $\varepsilon^{(i)}$-hermitian forms $\mathbf{F}_1^{(i)}$ on $\mathbf{V}_1^{(i)}$ (with respect to $(\mathfrak{K}_{\varrho_1^{(i)}}, \iota_0^{(i)})$) defined over $K_{\varrho_1^{(i)}}$ $(1 \leqslant i \leqslant t)$, there corresponds a $\mathsf{P}_1(G)$-invariant $\varepsilon$-hermitian form $\mathbf{F}_1$ on $\mathbf{V}_1$ (with respect to $(\mathfrak{K}_{\varrho_1}, \iota_0)$) defined over $K_{\varrho_1}$, where $\varepsilon = \varepsilon'_0 \prod \varepsilon^{(i)}$, by the relation

$$J_0^{-1}(\mathbf{F}_1(x\varepsilon'_{j1}, y\varepsilon'_{k1})) = \otimes \mathbf{F}_1^{(i)}(x_i, y_i) \quad \text{for } x = \otimes x_i,\ y = \otimes y_i,\ x_i, y_i \in \mathbf{V}_1^{(i)} \tag{31}$$

and *vice versa*.

Applying this to the case $k_0 = \mathbf{Q}$ (and $\mathbf{R}$), and by a similar argument as in [9], 2.5, one can prove the following

PROPOSITION 8. *Let $G$ be a connected semi-simple algebraic group defined over $\mathbf{Q}$, of hermitian type, which is a direct product of $\mathbf{Q}$-closed subgroups $G^{(i)}$ $(1 \leqslant i \leqslant t)$. Then, in the above notation, the group $R_{K_{\varrho_1}/\mathbf{Q}}(G_1')$, $G_1' = SU(\mathbf{V}_1/\mathfrak{K}_{\varrho_1}, \mathbf{F}_1)$, is of hermitian type* (II), (III) *and the representation $R_{K_{\varrho_1}/\mathbf{Q}}(\mathsf{P}_1): G_{\mathbf{R}} \to R(G_1')_{\mathbf{R}}$ satisfies the condition* $(\mathrm{H}_2)$, *if and only if, for all $1 \leqslant i \leqslant t$, the group $R_{K_{\varrho_1^{(i)}}/\mathbf{Q}}(G_1'^{(i)})$, $G_1'^{(i)} = SU(\mathbf{V}_1^{(i)}/\mathfrak{K}_{\varrho_1^{(i)}}, \mathbf{F}_1^{(i)})$, is of hermitian type* (II), (III), *the representation $R_{K_{\varrho_1^{(i)}}/\mathbf{Q}}(\mathsf{P}_1^{(i)}): G_{\mathbf{R}}^{(i)} \to R(G_1'^{(i)})_{\mathbf{R}}$ satisfies the condition* $(\mathrm{H}_2)$, *and for each $\tau \in \mathcal{G}(\mathbf{Q})$ there is at most one non-compact group among the $(G'^{(i)\tau})_{\mathbf{R}}$ $(1 \leqslant i \leqslant t)$.*

Thus our problem of determining $\mathsf{P}_1$ such that $R(\mathsf{P}_1)$ satisfies $(\mathrm{H}_2)$ can again be reduced essentially to the case where $G$ is $\mathbf{Q}$-simple. But, without assuming the condition (9), the actual determination of such representations for a $\mathbf{Q}$-simple $G$ would be rather complicated, so that we donot enter this problem any further.

*Example.* $G = G^{(1)} \times \ldots \times G^{(t)}$, where $G^{(i)} = R_{k_i/\mathbf{Q}}(G_1^{(i)})$, $G_1^{(i)} = SU(\mathbf{V}_1^{(i)}/\mathfrak{K}^{(i)}, \mathbf{F}_1^{(i)})$ (type (II) or (III.2)), $\mathfrak{K}^{(i)}$ being a quaternion algebra defined over a totally real number-field $k_i$, and $\mathbf{F}^{(i)}$ an quaternionic $\varepsilon^{(i)}$-hermitian form on $\mathbf{V}_1^{(i)}$, both defined over $k_i$. One supposes that, for every $\tau \in \mathcal{G}(\mathbf{Q})$, there exists at most one index $i$ such that either $(\mathfrak{K}^{(i)\tau})_{\mathbf{R}} \nsim 1$, $\varepsilon_i = -1$, or $(\mathfrak{K}^{(i)\tau})_{\mathbf{R}} \sim 1$, $\varepsilon_i = 1$. Then, taking $\mathsf{P}_1^{(i)}$ to be the projection of $G^{(i)}$ on to the factor $G_1^{(i)}$, one obtains a solution of our problem.

## Appendix. Construction of analytic families of polarized abelian varieties

1. We shall explain here (after Kuga) how to construct a family of polarized abelian varieties from a given symplectic representation. By a *polarized abelian variety* we shall understand here a triple $P = (V/L, I, A)$ where $V$ is a $2n$-dimensional real vector-space, $L$ a lattice in $V$ (i.e. a discrete submodule of rank $2n$), $I$ a complex structure on $V$ (i.e. a linear transformation of $V$ with $I^2 = -1$), and $A$ a non-degenerate alternating form on $V$, such that $A(x, Iy)$ $(x, y \in V)$ is symmetric and positive-definite and that one has $A(x, y) \in \mathbf{Z}$ for all $x, y \in L$. By the existence of such an alternating form (called a "Riemann form"), the complex torus $(V/L, A)$ becomes actually an abelian variety, on which $A$ determines a polarization. Let $P' = (V'/L', I', A')$ be another polarized abelian variety. $P$ and $P'$ are called *equivalent*, if there exists an ($\mathbf{R}$-)linear isomorphism $\Psi$ of $V$ onto $V'$ satisfying the following conditions:

$$\begin{cases} \Psi(L) = L', \\ \Psi \circ I = I' \circ \Psi, \\ \mu A = {}^t\Psi A' \Psi \quad \text{with } \mu \in \mathbf{Q}^*, \mu > 0; \end{cases} \tag{*}$$

Such a $\Psi$ is also called an isomorphism of $P$ onto $P'$. We denote by $[P]$ the equivalence-class containing a polarized abelian variety $P$. For a fixed $L$ and $A$, one sees easily that a linear automorphism $\gamma$ of $V$ gives an isomorphism of $(V/L, I, A)$ onto $(V/L, I', A)$, if and only if $\gamma$ belongs to $Sp(L, A)=\{\gamma\in GL(V)\,|\,\gamma(L)=L,\ {}^t\gamma A\gamma=A\}$ (Siegel's paramodular group) and one has $I'=\gamma I\gamma^{-1}$.

Now let $\mathfrak{D}$ be a complex analytic manifold. By a *uniformized analytic family of polarized abelian varieties* over $\mathfrak{D}$ we mean a collection of polarized abelian varieties

$$\mathfrak{F}=\{P_z=(V/L, I_z, A)\,|\,z\in\mathfrak{D}\}$$

satisfying the following conditions: Namely, there exists a complex analytic structure $J$ on $(V/L)\times\mathfrak{D}$ such that

(i) the canonical projection $\pi:(V/L)\times\mathfrak{D}\to\mathfrak{D}$ is holomorphic;

(ii) for each $z\in\mathfrak{D}$, $J$ induces the complex structure $I_z$ on the fiber $(V/L)\times z$;

(iii) $V\times\mathfrak{D}$ becomes a complex vector-bundle over $\mathfrak{D}$ with respect to the complex structure on it obtained in a natural way from $J$.

It is known that a complex structure $J$ on $(V/L)\times\mathfrak{D}$ satisfying these conditions is unique and that, for each $v\in V$, the section $z\to(v \text{ mod. } L)\times z$ is holomorphic. Two uniformized analytic families $\mathfrak{F}=\{P_z\,|\,z\in\mathfrak{D}\}$ and $\mathfrak{F}'=\{P'_{z'}\in\mathfrak{D}'\}$ are called *equivalent*, if there exists a biholomorphic map $\psi$ of $\mathfrak{D}$ onto $\mathfrak{D}'$ such that $P_z$ is equivalent to $P'_{\psi(z)}$ for all $z\in\mathfrak{D}$; when $\mathfrak{D}$ is connected, the linear isomorphism $\Psi_z$ of the vector-space $V$ onto $V'$ giving the equivalence of $P_z$ onto $P'_{\psi(z)}$ can be taken to be independent of $z\in\mathfrak{D}$. By means of this equivalence and by the usual method of overlapping neighbourhoods, one can define the notion of an *analytic family of polarized abelian varieties* (or rather Kummer varieties) $\{[P_z]\,|\,z\in\mathfrak{D}\}$, starting from an open covering $\{U_\alpha\}$ of $\mathfrak{D}$ and a collection of uniformized analytic families of polarized abelian varieties $\mathfrak{F}_\alpha$ on each $U_\alpha$ satisfying the usual consistency conditions.

Let $\mathfrak{F}'=\{P'_{z'}\,|\,z'\in\mathfrak{D}'\}$ be a (uniformized) analytic family of polarized abelian varieties over a complex analytic manifold $\mathfrak{D}'$ and let $\psi$ be a holomorphic mapping of $\mathfrak{D}$ into $\mathfrak{D}'$. Then one can construct a (uniformized) analytic family $\mathfrak{F}=\{P'_{\psi(z)}\,|\,z\in\mathfrak{D}\}$ over $\mathfrak{D}$, which is called a family *induced* from $\mathfrak{F}'$ by $\psi$. In particular, when $\mathfrak{D}$ is a submanifold of $\mathfrak{D}'$ and $\psi$ is the inclusion map, we get a *subfamily* $\mathfrak{F}=\mathfrak{F}'\,|\,\mathfrak{D}$ of $\mathfrak{F}'$.

2. Let us now fix $V$ and $A$, and consider the corresponding "Siegel space" $\mathfrak{D}'=\mathfrak{D}(V, A)$, which is, by definition, the space of all complex structures $I$ on $V$ such that $A(x, Iy)$ is symmetric and positive-definite. As is well-known, $\mathfrak{D}'$ has a natural complex structure, and the group of all analytic automorphisms of $\mathfrak{D}'$ can be identified with the symplectic group $G'=Sp(V, A)$ (modulo center) operating transitively on $\mathfrak{D}'$ by $(g, I)\to gIg^{-1}$. There-

fore, if one denotes by $K'$ the stabilizer of any complex structure $I \in \mathfrak{D}'$, then $K'$ is a maximal compact subgroup of $G'$ and $\mathfrak{D}'$ can be identified with the coset-space $G'/K'$ by the correspondence $gK' \leftrightarrow gIg^{-1}$ $(g \in G')$. Taking any lattice $L$ in $V$ such that $A(L, L) \subset \mathbf{Z}$, one obtains a uniformized analytic family of polarized abelian varieties $\{P_{gK'} = (V/L, gIg^{-1}, A) \mid gK' \in \mathfrak{D}'\}$. This is the "Siegel's family" relative to $L$, of which the equivalence-class is uniquely characterized by the elementary divisor of $A$ with respect to the lattice $L$. Considering this family modulo $\Gamma' = Sp(L, A)$, i.e. forming the quotient variety $\Gamma' \backslash (V/L) \times \mathfrak{D}'$, one gets also an analytic family over $\Gamma' \backslash \mathfrak{D}'$.

Siegel's family has the following universality (cf. Kuga [8], § 6): Let

$$\mathfrak{F} = \{P_z = (V/L, I_z, A)\}$$

be a collection of polarized abelian varieties parametrized by a complex analytic manifold $\mathfrak{D}$. Then $\mathfrak{F}$ *becomes a uniformized analytic family over* $\mathfrak{D}$ *if and only if the mapping* $\mathfrak{D} \ni z \to I_z \in \mathfrak{D}' = \mathfrak{D}(V, A)$ *is holomorphic*, and, if that is so, $\mathfrak{F}$ is the family induced from the Siegel's family by this mapping.

3. Shimura [12], [13] has recently considered certain analytic families of polarized abelian varieties, which, roughly speaking, are obtained by prescribing the structure of the endomorphism-rings of the abelian varieties in addition to the data $V$, $A$, $L$. He obtained in this way four kinds of families over symmetric domains (of type (I), (II), (III)). More recently, Kuga [8] has given a more general method of constructing a family of polarized abelian varieties over a symmetric domain, starting from a symplectic representation of a semi-simple Lie group. Namely, let $G$ be a semi-simple Lie group of hermitian type with a finite number of connected components and with a finite center, $K$ a maximal compact subgroup of $G$, and $\mathfrak{D} = G/K$ the corresponding symmetric domain; furthermore let $\Gamma$ be a discrete subgroups of $G$ such that the homogeneous space $\Gamma \backslash G$ has a finite volume. Suppose one has a symplectic representation $\varrho : G \to G' = Sp(V, A)$ such that $\varrho(K) \subset K'$, $\varrho(\Gamma) \subset \Gamma'$ and satisfying the condition $(\mathrm{H}_1)$. Then $\varrho$ induces in a natural manner a mapping of $\mathfrak{D} = G/K$ into $\mathfrak{D}' = G'/K'$ as well as a mapping of $\Gamma \backslash \mathfrak{D}$ into $\Gamma' \backslash \mathfrak{D}'$, which are holomorphic by the assumption. Therefore one obtains a uniformized analytic family of polarized abelian varieties $\mathfrak{F} = \{P_{gK} = (V/L, \varrho(g) I \varrho(g)^{-1}, A) \mid gK \in \mathfrak{D}\}$ over the symmetric domain $\mathfrak{D}$ induced from the Siegel's family by this mapping and also an analytic family over $\Gamma \backslash \mathfrak{D}$ induced from that over $\Gamma' \backslash \mathfrak{D}'$. (Of course, this construction applies also to the case where $\mathfrak{D}$ has no complex structure.) It can be proven, by virtue of Borel's density theorem, that if $\mathfrak{F}' = \{P_{gK} = (V'/L', \varrho'(g) I' \varrho'(g)^{-1}, A') \mid gK \in \mathfrak{D}\}$ is another family over $\mathfrak{D}$ obtained in the similar way from another symplectic representation $\varrho' : G \to Sp(V', A')$, then $\mathfrak{F}$ and $\mathfrak{F}'$ are

equivalent (with $\psi = \text{id.}$) if and only if the restrictions of the representations $\varrho$ and $\varrho'$ on the identity connected component of $G$ are equivalent by a linear isomorphism $\Psi$ of $V$ onto $V'$ satisfying the condition (*).

It is well known that when $G$ is (the group of $\mathbf{R}$-rational points of) a semi-simple algebraic group defined over $\mathbf{Q}$, $\varrho$ is a $\mathbf{Q}$-rational representation of $G$ into $G'$, and $\Gamma$ is a subgroup of $G_{\mathbf{Q}}$ commensurable with $G_{\mathbf{Z}}$, taken sufficiently small, then all the above conditions on $\Gamma$ are satisfied. On the other hand, in case $G$ is connected and algebraic and $\varrho$ is faithful, one can show that the above conditions imply that $G$ has a (unique) structure of an algebraic group defined over $\mathbf{Q}$ such that $\varrho$ is $\mathbf{Q}$-rational and $\Gamma$ is commensurable with $G_{\mathbf{Z}}$.

## References

[1]. ALBERT A. A., Involutorial simple algebras and real Riemann matrices. *Ann. of Math.*, 36 (1935), 886–964.

[2]. —— *Structure of algebras*. Amer. Math. Soc. Colloquium Publ., Vol. 24, 1939.

[3]. CHEVALLEY, C. C., *The algebraic theory of spinors*. Columbia Univ. Press, New York, 1954.

[4]. HAMMOND, W. F., The modular groups of Hilbert and Siegel. *Amer. J. Math.*, 88 (1966), 497–516.

[5]. HELGASON, S., *Differential geometry and symmetric spaces*. Acad. Press, New York and London, 1962.

[6]. JACOBSON, N., Clifford algebras for algebra with involution of type *D*. *J. Algebra*, 1 (1964), 288–300.

[7]. KLINGEN, H., Über einen Zusammenhang zwischen Siegelschen und Hermiteschen Modulfunktionen. *Abh. Math. Sem. Univ. Hamburg*, 27 (1964), 1–12.

[8]. KUGA, M., *Fiber varieties over a symmetric space whose fibers are abelian varieties*, I, II. Lecture Notes, Univ. of Chicago, 1963–64.

[9]. SATAKE, I., Holomorphic imbeddings of symmetric domains into a Siegel space. *Amer. J. Math.*, 87 (1965), 425–461.

[9a]. —— Holomorphic imbeddings of symmetric domains into a Siegel space. *Proc. of Conference on Complex Analysis, Minneapolis* 1964, Springer-Verlag, Berlin, Heidelberg, New York, 1965, 40–48.

[10]. —— Clifford algebras and families of abelian varieties. *Nagoya Math. J.*, 27–2 (1966), 435–446.

[11]. SERRE, J. P., *Corps locaux*. Actualités Sci. et Ind. 1296, Hermann, Paris, 1962.

[12]. SHIMURA, G., On analytic families of polarized abelian varieties and automorphic functions. *Ann. of Math.*, 78 (1963), 149–192.

[13]. ——, On the field of definition for a field of automorphic functions. *Ibid.*, 80 (1964), 160–189; II, *ibid.*, 81 (1965), 124–165.

[14]. WEIL, A., Algebras with involutions and the classical groups. *J. Indian Math. Soc.*, 24 (1960), 589–623.

[15]. —— *Adèles and algebraic groups*. Lecture Notes, Institute for Advanced Study, 1961.

*The following articles are added for the references in Part II:*

[16]. CARTAN, E., Les groupes projectifs continus réels qui ne laissent invariante aucune multiplicité plane. *J. Math. Pures Appl.*, 10 (1914), 149–186.

[17]. KNESER, M., Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern. II. *Math. Z.*, 89 (1965), 250–272.

[18]. MAL'CEV, A. I., On semi-simple subgroups of Lie groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 8 (1944), 143–174; *Amer. Math. Soc. Trans.* Ser. 1, 9 (1962), 172–213.

[19]. SATAKE, I., On the theory of reductive algebraic groups over a perfect field. *J. Math. Soc. Japan*, 15 (1963), 210–235.

*Received April 13, 1966*