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Symplectic Small Deformations of Special Instanton Bundle on $\mathbb{P}^{2n+1}$ (*).

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Abstract. – *Let $MI_{\text{Simp}, \mathbb{P}^{2n+1}}(k)$ be the moduli space of stable symplectic instanton bundles on $\mathbb{P}^{2n+1}$ with second Chern class $c_2 = k$ (it is a closed subscheme of the moduli space $MI_{\mathbb{P}^{2n+1}}(k)$). We prove that the dimension of its Zariski tangent space at a special (symplectic) instanton bundle is $2k(5n-1)+4n^2-10n+3$, $k \geq 2$.*

Introduction.

Symplectic instanton bundles on $\mathbb{P}^{2n+1}$ are holomorphic bundles of rank 2n (see [1], [4] and [6]) that correspond to the self-dual solutions of Yang-Mills equations on $\mathbb{P}^n(\mathbb{H})$. They are given by some monads (see section 2 for precise definitions) and their only topological invariant is $c_2 = k$.

At present the dimension of their moduli space $MI_{\text{Simp}, \mathbb{P}^{2n+1}}(k)$ is not known except in the cases $n = 1$, where the dimension is $8k - 3$ (see [3]), and in few other cases corresponding to small values of k.

$MI_{\text{Simp}, \mathbb{P}^{2n+1}}(k)$ is a closed subscheme of $MI_{\mathbb{P}^{2n+1}}(k)$ and this last scheme parametrizes stable instanton bundles with structural group $GL(2n)$.

The class of special instanton bundles was introduced in [7].

Let $E \in MI_{\mathbb{P}^{2n+1}}(k)$ be a special symplectic instanton bundle. The tangent dimension $h^1(\text{End}(E))$ was computed in [6] and it is equal to $4(3n-1)k + (2n-5)(2n-1)$.

The Zariski tangent space of $MI_{\text{Simp}, \mathbb{P}^{2n+1}}(k)$ at E is isomorphic to $H^1(S^2E)$ and in this paper we prove that

$$h^2(S^2E) = \binom{k-2}{2}\binom{2n-1}{2}, \qquad \forall\, k \geq 2 . \tag{1}$$

By the Hirzebruch-Riemann-Roch formula, since $h^0(S^2E) = 0$ and $h^i(S^2E) = 0$,

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$\forall\, i \geq 3$, it follows that:

$$\chi(S^2E) = h^2(S^2E) - h^1(S^2E) = 2n^2 + n + \frac{1}{2}\left[k^2\binom{2n-1}{2} - k(10n^2 - 5n - 1)\right]$$

and

THEOREM 0.1. – *Let E be a special symplectic instanton bundle. Then*

$$h^1(S^2E) = 2k(5n-1) + 4n^2 - 10n + 3\,, \qquad k \geq 2$$

(for $n = 1$ it is well known that $h^1(S^2E) = 8k - 3$) and, in the real case, for $n = 2$ the dimension $18k - 1$ has been found in [5] with different techniques).

Now, since by the Kuranishi map $H^2(S^2E)$ is the space of obstructions to the smoothness at E of $MI_{\mathrm{Simp},\, \mathbb{P}^{2n+1}}(k)$, we obtain

COROLLARY 0.1. – *$\forall k \geq 2$ the dimension of any irreducible component of $MI_{\mathrm{Simp},\, \mathbb{P}^{2n+1}}(k)$, containing a special symplectic instanton bundle is bounded by the value $2k(5n-1) + 4n^2 - 10n + 3$ (linear in k).*

COROLLARY 0.2. – *$\forall n$ $MI_{\mathrm{Simp},\, \mathbb{P}^{2n+1}}(3)$ is smooth at a special instanton bundle E, and the dimension of any irreducible component containing E is $4n^2 + 20n - 3$.*

The main remark of this paper is that it is easier to compute $H^2(S^2E)$ and $H^2(\overset{2}{\wedge} E)$ together as $SL(2)$-modules (although this second cohomology space has a geometrical meaning only for orthogonal bundles) than to compute $H^2(S^2E)$ alone.

1. – Preliminaries.

Throughout this paper $\boldsymbol{K}$ denotes an algebraically closed field of characteristic zero. U denotes a 2-dimensional $\boldsymbol{K}$ vector space ($U = \langle s, t\rangle$), $S_n = S^n U$ its n-th symmetric power

$$(S_n = \langle s^n, s^{n-1}t, \ldots, t^n\rangle), \qquad V_n = U \otimes S_n \ (V_n = \langle s\otimes s^n, s \otimes s^{n-1}t, \ldots s\otimes t, \ldots t \otimes t^n\rangle)$$

and $\mathbb{P}^{2n+1} = \mathbb{P}(V_n)$.

DEFINITION 1.1. – *A vector bundle E on $\mathbb{P}^{2n+1}$ of rank $2n$ is called an* instanton bundle *of quantum number k if:*

— *E has Chern polinomial $c_t(E) = (1 - t^2)^{-k}$;*

— *$E(q)$ has natural cohomology in the range $-(2n+1) \leq q \leq 0$, that is $H^i(E(q)) \neq 0$ for at most one $i = i(q)$.* ∎

By [5], [2], the Definition 1.1 is equivalent to:

i) E is the cohomology bundle of a monad:

$$0 \to O(-1)^k \to \Omega^1(1)^k \to O^{2n(k-1)} \to 0$$

or ii) E is the cohomology bundle of a monad:

$$0 \to O(-1)^k \xrightarrow{A} O^{2n+2k} \xrightarrow{B^t} O(1)^k \to 0$$

(where, after we have fixed a coordinate system, A and B can be identified with matrices in the space $\mathrm{Mat}(k, 2n+2k, S_1)$)

DEFINITION 1.2. – *An instanton bundle E is called* symplectic *if there is an isomorphism $\varphi: E \to E^\vee$ satisfying $\varphi = -\varphi^\vee$.* ∎

DEFINITION 1.3. – *An instanton bundle is called* special *if it arises from a monad where the morfism B^t is defined in some system of homogeneous coordinates $x_0, \dots x_n, y_0 \dots y_n$ on $\mathbb{P}^{2n+1}$ by the trasposed of the matrix:*

$$B = \begin{bmatrix} x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n & 0 & \dots & 0 \\ 0 & x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n & 0 \\ 0 & \dots & 0 & x_0 & \dots & x_n & 0 & \dots & 0 & y_0 & \dots & y_n \end{bmatrix}. \quad \blacksquare$$

The following lemma is well known (and easy to prove)

LEMMA 1.1.

$$H^0(O(1)) \cong V^\vee,$$

$$H^0(\Omega^1(2)) \cong \bigwedge^2 V^\vee,$$

$$H^i(\mathbb{P}^n, S^2\Omega^1(1)) = \begin{cases} 0, & \text{se } i \neq 1, \\ \bigwedge^2 V^\vee, & \text{se } i = 1. \end{cases}$$

2. – Existence of a special symplectic instanton bundle.

There is a natural exact sequence of $GL(U)$-equivariant maps for any $k, n \geq 1$ (Clebsch-Gordan sequence):

$$0 \to \bigwedge^2 U \otimes S_{k-1} \otimes V_{n-1} \xrightarrow{\beta} S_k \otimes V_n \xrightarrow{\mu} V_{k+n} \to 0, \tag{2}$$

where μ is the multiplication map and β is defined by $(s \wedge t) \otimes f \otimes g \to (sf \otimes tg - tf \otimes sg)$.

We can define (see [6]) the morphism

$$\tilde{b}: S_{k-1}^\vee \otimes \Omega^1(1) \to \bigwedge^2 U^\vee \otimes S_{k-2}^\vee \otimes V_{n-1}^\vee \otimes O$$

and it is induced the complex

$$A\otimes O(-1)\xrightarrow{\tilde{a}} S^{\vee}_{k-1}\otimes\Omega^1(1)\xrightarrow{\tilde{b}} \overset{2}{\wedge} U^{\vee}\otimes S^{\vee}_{k-2}\otimes V^{\vee}_{n-1}\otimes O\,, \tag{3}$$

where A is a k-dimensional subspace of $S^{\vee}_{2n+k-1}\otimes \overset{2}{\wedge} U^{\vee}$ such that (3) is a monad and the cohomology bundle E is a special symplectic instanton bundle. It was proved in [6] that

$$H^2(\mathrm{End}\,E)\cong \mathrm{Ker}\,(\Phi^{\vee})^{\vee},$$

where

$$\Phi^{\vee}: S^{\otimes 2}_{k-2}\otimes V^{\otimes 2}_{n-1}\to S^{\otimes 2}_{k-1}\otimes \overset{2}{\wedge} V_n$$

and there is an isomorphism of $SL(2)$-representations

$$\varepsilon: S^{\vee}_{k-3}\otimes S^{\vee}_{k-3}\otimes S^2 V^{\vee}_{n-2}\to \mathrm{Ker}\,(\Phi^{\vee})\,.$$

3. – How to identify $H^2(S^2E)$ and $H^2(\overset{2}{\wedge} E)$.

PROPOSITION 3.1. – *Let E be special symplectic instanton bundle, cohomology of monad* (3) *and* $N=\mathrm{Ker}\,\tilde{b}$. *Then*

(i) $H^2(S^2E)\cong H^2(S^2N)$,
(ii) $H^2(\overset{2}{\wedge} E)\cong H^2(\overset{2}{\wedge} N)$.

PROOF. – We denote $B:=S^{\vee}_{k-1}$ and $C:=\overset{2}{\wedge} U^{\vee}\otimes S^{\vee}_{k-2}\otimes V^{\vee}_{n-1}$.
The result follows from the two exact sequences given by monad (3):

$$0\to N\to B\otimes\Omega^1(1)\to C\otimes O\to 0\,, \tag{4}$$

$$0\to A\otimes O(-1)\to N\to E\to 0\,. \tag{5}$$

In fact, by performing the second symetric and alternating power of sequence (4), we have

$$\begin{array}{c} 0\to S^2N\to \tilde{A}\to B\otimes C\otimes\Omega^1(1)\to \overset{2}{\wedge} C\otimes O\to 0\\ \searrow\ \nearrow\\ M^1\\ \nearrow\ \searrow\\ 0\qquad 0\end{array} \tag{6}$$

where $\tilde{A}:=S^2(B\otimes\Omega^1(1))=(S^2B\otimes S^2(\Omega^1(1)))\oplus(\overset{2}{\wedge} B\otimes\Omega^2(2))$ and

$$\begin{array}{c} 0\to \overset{2}{\wedge} N\to \bar{A}\to B\otimes C\otimes\Omega^1(1)\to O\otimes S^2C\to 0\\ \searrow\ \nearrow\\ M\\ \nearrow\ \searrow\\ 0\qquad 0\end{array} \tag{7}$$

where $\bar{A}:=\overset{2}{\wedge}(B\otimes\Omega^1(1))=(\overset{2}{\wedge} B\otimes S^2(\Omega^1(1)))\oplus(S^2B\otimes\Omega^2(2))$. ∎

3.1. *Identifying* $H^2(S^2N)$ *and* $H^2(\overset{2}{\wedge}N)$. – i) Diagram (6) gives the following two exact sequences:

$$(8)\quad O\to H^0(M^1)\to H^1(S^2N)\to H^1(\tilde{A})\to H^1(M^1)\to H^2(S^2(N))\to H^2(\tilde{A})\to\dots,$$

$$(9)\quad O\to H^0(M^1)\to B\otimes C\otimes \underset{\underset{0}{\|}}{H^0(\Omega^1(1))}\to \overset{2}{\wedge}C\to H^1(M^1)\to B\otimes C\otimes \underset{\underset{0}{\|}}{H^1(\Omega^1(1))}\to\dots.$$

Sequence (9) implies:

$$H^0(M^1)=0 \quad\text{and}\quad \mathrm{H}^1(\mathrm{M}^1)\cong \overset{2}{\wedge}\mathrm{C}\,.$$

Then, by using the two formulas:

$$H^1(\tilde{A})=(S^2B\otimes H^1(S^2\Omega^1(1)))\oplus(\overset{2}{\wedge}B\otimes H^1(\Omega^2(2)))=S^2B\otimes\overset{2}{\wedge}V^\vee$$

and:

$$H^2(\tilde{A})=(S^2B\otimes H^2(S^2\Omega^1(1)))\oplus(\overset{2}{\wedge}B\otimes H^2(\Omega^2(2)))=0\,,$$

sequence (8) becomes:

$$0\to H^1(S^2N)\to H^1(\tilde{A})\to H^1(M^1)\to H^2(S^2(N))\to 0\,,$$

i.e.

$$0\to H^1(S^2N)\to S^2B\otimes\overset{2}{\wedge}V^\vee\overset{\tilde{\Phi}}{\to}\overset{2}{\wedge}C\to H^2(S^2N)\to 0$$

$$\Rightarrow\quad H^2(S^2N)\cong \mathrm{Coker}(\tilde{\Phi})=(\mathrm{Ker}(\tilde{\Phi}^\vee))^\vee.$$

Then:

$$H^2(S^2N)^\vee=\mathrm{Ker}\,[\overset{2}{\wedge}(S_{k-2}\otimes V_{n-1})\overset{\tilde{\Phi}^\vee}{\to}S^2(S_{k-1})\otimes\overset{2}{\wedge}V_n].$$

ii) Diagram (7) gives the following two exact sequences:

$$(10)\quad O\to H^0(M)\to H^1(\overset{2}{\wedge}N)\to H^1(\overline{A})\to H^1(M)\to H^2(\overset{2}{\wedge}N)\to H^2(\overline{A})\to\dots$$

$$(11)\quad O\to H^0(M)\to B\otimes C\otimes \underset{\underset{0}{\|}}{H^0(\Omega^1(1))}\to S^2C\otimes \underset{\underset{S^2C}{\|}}{H^0(O)}\to H^1(M)\to 0\to\dots$$

and, from sequence (11), we get $H^0(M)=0$ and $H^1(M)\simeq S^2C$.

Then, since:

$$H^1(\overline{A})=\left(H^1(S^2(\Omega^1(1))\otimes\overset{2}{\wedge}B)\right)\oplus(S^2B\otimes H^1(\Omega^2(2)))=\overset{2}{\wedge}B\otimes\overset{2}{\wedge}V^\vee$$

and $H^2(\overline{A})=0$,

sequence (10) becomes:

$$O \to H^0(M) \to H^1(\overset{2}{\wedge} N) \to H^1(\overline{A}) \to H^1(M) \to H^2(\overset{2}{\wedge} N) \to 0$$
$$H^0(M) = 0$$

i.e.

$$0 \to H^1(\overset{2}{\wedge} N) \to \overset{2}{\wedge} B \otimes \overset{2}{\wedge} V^\vee \overset{\overline{\Phi}}{\to} S^2 C \to H^2(\overset{2}{\wedge} N) \to 0$$
$$\Rightarrow \quad H^2(\overset{2}{\wedge} N) \cong \mathrm{Coker}(\overline{\Phi}) = (\mathrm{Ker}(\overline{\Phi}^\vee))^\vee$$

Then we obtain:

$$(H^2(\overset{2}{\wedge} N))^\vee = \mathrm{Ker}\left[S^2(S_{k-2} \otimes V_{n-1}) \overset{\overline{\Phi}^\vee}{\to} \overset{2}{\wedge} S_{k-1} \otimes \overset{2}{\wedge} V_n\right].$$

3.2. *Identifying* $H^2(S^2E)$. – We have

$$H^2(S^2E)^\vee \cong \mathrm{Ker}\ \widetilde{\Phi}^\vee$$

where $\widetilde{\Phi}^\vee \colon \overset{2}{\wedge}(S_{k-2} \otimes V_{n-1}) \to S^2 S_{k-1} \otimes \overset{2}{\wedge} V_n$ is explicitly given by

$$\widetilde{\Phi}^\vee((g \otimes v) \wedge (g^1 \otimes v^1)) = sg \cdot sg^1 \otimes (tv \wedge tv^1) - sg \cdot tg^1 \otimes (tv \wedge sv^1) +$$
$$- tg \cdot sg^1 \otimes (sv \wedge tv^1) + tg \cdot tg^1 \otimes (sv \wedge sv^1),$$

i.e. $\widetilde{\Phi}^\vee = \tilde{p} \circ (\overset{2}{\wedge} \beta)$, where $\beta \colon \overset{2}{\wedge} U \otimes S_{k-2} \otimes V_{n-1} \to S_{k-1} \otimes V_n$ is such that

$$(s \wedge t) \otimes (g \otimes v) \mapsto (sg \otimes tv) - (tg \otimes sv)$$

and

$$\tilde{p} \colon \overset{2}{\wedge}(S_{k-1} \otimes V_n) \to S^2 S_{k-1} \otimes \overset{2}{\wedge} V_n$$
$$\overset{2}{\wedge}(S_{k-1} \otimes V_n) = (\overset{2}{\wedge} S_{k-1} \otimes S^2 V_n) \oplus (S^2 S_{k-1} \otimes \overset{2}{\wedge} V_n)$$

is such that

$$(f \otimes u) \wedge (f' \otimes u^1) \mapsto f \cdot f' \otimes u \wedge u^1.$$

Now, we consider the $SL(2)$-equivariant morphism:

$$\tilde{\varepsilon}^1 \colon \overset{2}{\wedge}(S_{k-3} \otimes V_{n-2}) \to \overset{2}{\wedge}(S_{k-2} \otimes V_{n-1})$$

where, up to the order of factors, the map $\tilde{\varepsilon}^1 := \beta^1 \wedge \beta^1$ and $\beta^1 \colon S_{k-3} \otimes V_{n-2} \to S_{k-2} \otimes \otimes V_{n-1}$ is defined as β. Hence, $\tilde{\varepsilon}^1$ is injective.

Finally, we define

$$\tilde{\varepsilon} \colon \overset{2}{\wedge} S_{k-3} \otimes S^2 V_{n-2} \to \overset{2}{\wedge}(S_{k-2} \otimes V_{n-1})$$

as $\tilde{\varepsilon} = \tilde{\varepsilon}^1 \circ \tilde{i}$, where $\tilde{i}\colon \overset{2}{\wedge} S_{k-3} \otimes S^2 V_{n-2} \to \overset{2}{\wedge}(S_{k-3} \otimes V_{n-2})$ such that

$$f \wedge f' \otimes u \cdot u^1 \mapsto (f \otimes u) \wedge (f' \otimes u^1) + (f \otimes u^1) \wedge (f' \otimes u)$$

is an injective map. Then, also $\tilde{\varepsilon}$ is injective.

LEMMA 3.1. – Im $\tilde{\varepsilon} \subset \operatorname{Ker} \tilde{\Phi}^{\vee}$.

PROOF. – Straightforward computation. ∎

3.3. *Identifying* $H^2(\overset{2}{\wedge} E)$. – We have

$$H^2(\overset{2}{\wedge} E)^{\vee} \cong \operatorname{Ker} \overline{\Phi}^{\vee},$$

where $\overline{\Phi}^{\vee}\colon S^2(S_{k-2} \otimes V_{n-1}) \to \overset{2}{\wedge} S_{k-1} \otimes \overset{2}{\wedge} V_n$ is explicity given by

$$\overline{\Phi}^{\vee}((g \otimes v) \cdot (g^1 \otimes v^1)) = sg \wedge sg^1 \otimes (tv \wedge tv^1) - sg \wedge tg^1 \otimes (tv \wedge sv^1) - \\ - tg \wedge sg^1 \otimes (sv \wedge tv^1) + (tg \wedge tg^1) \otimes (sv \wedge sv^1),$$

i.e. $\overline{\Phi}^{\vee} = \overline{p} \circ (S^2 \beta)$ where

$$\begin{array}{l} \overline{p}\colon S^2(S_{k-1} \otimes V_n) \to \overset{2}{\wedge} S_{k-1} \otimes \overset{2}{\wedge} V_n \\ \qquad \| \\ (\overset{2}{\wedge} S_{k-1} \otimes \overset{2}{\wedge} V_n) \oplus (S^2 S_{k-1} \otimes S^2 V_n) \end{array}$$

is such that

$$\overline{p}((f \otimes u) \cdot (f' \otimes u^1)) = f \wedge f' \otimes u \wedge u^1 .$$

We consider the $SL(2)$-equivariant morphism:

$$\overline{\varepsilon}^1\colon S^2(S_{k-3} \otimes V_{n-2}) \to S^2(S_{k-2} \otimes V_{n-1}),$$

such that:

$$\overline{\varepsilon}^1((f \otimes u) \cdot (f' \otimes u^1)) = (sf \otimes tu) \cdot (sf' \otimes tu^1) - (sf \otimes su) \cdot (tf' \otimes tu^1) + \\ - (tf \otimes tu) \cdot (sf' \otimes su^1) + (sf \otimes tu) \cdot (sf' \otimes tu^1)$$

($\overline{\varepsilon}^1 = S^2 \beta^1$ hence $\overline{\varepsilon}^1$ is injective). Finally, we define

$$\overline{\varepsilon}\colon S^2 S_{k-3} \otimes S^2 V_{n-2} \to S^2(S_{k-2} \otimes V_{n-1})$$

as $\overline{\varepsilon} = \overline{\varepsilon}^1 \circ \overline{i}$ where $\overline{i}\colon S^2 S_{k-3} \otimes S^2 V_{n-2} \to S^2(S_{k-3} \otimes V_{n-2})$ such that

$$f \cdot f' \otimes uu^1 \mapsto (f \otimes u)(f' \otimes u^1) + (f \otimes u^1)(f' \otimes u)$$

is an injective map. Then, also $\overline{\varepsilon}$ is injective

LEMMA 3.2. – Im $\overline{\varepsilon} \subset \operatorname{Ker} \overline{\Phi}^{\vee}$.

PROOF. – Straightforward computation. ■

THEOREM 3.1. – *For any special symplectic instanton bundle E*

$$H^2(S^2E) \simeq \bigwedge^2 (S_{k-3})^\vee \otimes S^2(V_{n-2})^\vee .$$

PROOF. – By lemma 3.1 and 3.2 we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^2(\bigwedge^2 N)^\vee & \to & S^2(S_{k-2}\otimes V_{n-1}) & \xrightarrow{\overline{\Phi}^\vee} & \bigwedge^2 S_{k-1}\otimes \bigwedge^2 V_n & \to & H^1(\bigwedge^2 N)^\vee & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^2(N\otimes N)^\vee & \to & S_{k-2}^{\otimes 2}\otimes V_{n-1}^{\otimes 2} & \xrightarrow{\Phi^\vee} & S_{k-1}^{\otimes 2}\otimes \bigwedge^2 V_n & \to & H^1(N\otimes N)^\vee & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^2(S^2 N)^\vee & \to & \bigwedge^2 (S_{k-2}\otimes V_{n-1}) & \xrightarrow{\tilde{\Phi}^\vee} & S^2 S_{k-1}\otimes \bigwedge^2 V_n & \to & H^1(S^2 N)^\vee & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

It was shown in [7] that:

$$H^2(\operatorname{End} E) \simeq \operatorname{Ker} \Phi^\vee = H^2(N\otimes N)^\vee \simeq S_{k-3}^{\otimes 2}\otimes S^2 V_{n-2}.$$

We have proved that there are two injective maps:

$$\tilde{\varepsilon} : \bigwedge^2 (S_{k-3}) \otimes S^2 V_{n-2} \to \operatorname{Ker} \tilde{\Phi}^\vee \simeq H^2(S^2 N)^\vee \simeq H^2(S^2 E)^\vee$$

$$\bar{\varepsilon} : S^2(S_{k-3}) \otimes S^2 V_{n-2} \to \operatorname{Ker} \overline{\Phi}^\vee \simeq H^2(\bigwedge^2 N)^\vee \simeq H^2(\bigwedge^2 E)^\vee$$

Then, we can consider the following diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & S^2 S_{k-3}\otimes S^2 V_{n-2} & \to & S_{k-3}^{\otimes 2}\otimes S^2 V_{n-2} & \to & \bigwedge^2 S_{k-3}\otimes V_{n-2} & \to 0\\
 & \downarrow \bar{\varepsilon} & & \downarrow \varepsilon & & \downarrow \tilde{\varepsilon} & \\
0 \to & H^2(\bigwedge^2 E)^\vee & \to & H^2(\operatorname{End} E)^\vee & \to & H^2(S^2 E)^\vee & \to 0\\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}$$

and by the *Snake-Lemma* there is the exact sequence:

$$\begin{array}{ccccccccccccc} 0 \to & \mathrm{Ker}\ \bar{\varepsilon} & \to & \mathrm{Ker}\ \varepsilon & \to & \mathrm{Ker}\ \tilde{\varepsilon} & \to & \mathrm{Coker}\ \bar{\varepsilon} & \to & \mathrm{Coker}\ \varepsilon & \to & \mathrm{Coker}\ \tilde{\varepsilon} & \to 0 \\ & \| & & \| & & \| & & \| & & & & & \\ & 0 & & 0 & & 0 & & 0 & & & & & \end{array}$$

$\Rightarrow$ Coker $\bar{\varepsilon} = 0 \Rightarrow \bar{\varepsilon}$ is an isomorphism $\Rightarrow \tilde{\varepsilon}$ is an isomorphism.

Thus:

$$H^2(S^2E)^\vee \cong \overset{2}{\wedge}(S_{k-3}) \otimes S^2(V_{n-2})$$

i.e. $H^2(S^2E) \simeq \overset{2}{\wedge}(S_{k-3})^\vee \otimes S^2(V_{n-2})^\vee$ as we wanted. ■

REMARK 3.1. – *By this theorem formula* 1 *and theorem* 0.1 *are easily proved.*

REFERENCES

[1] V. ANCONA, *I fibrati istantoni sugli spazi proiettivi*, quaderno INDAM, Roma (1996).

[2] V. ANCONA - G. OTTAVIANI, *Stability of special instanton bundles on* $\mathbb{P}^{2n+1}$, Trans. Amer. Math. Soc., Vol. **341**, n. 2 (1994), pp. 677-693.

[3] A. HIRSCHOWITZ - M. S. NARASIMHAN, *Fibrés de't Hooft spéciaux et applications* (Proc. Nice Conf. 1981), Birkhäuser, Basel and Boston (1982), pp. 143-163.

[4] M. MAMONE CAPRIA - S. M. SALAMON, *Yang Mills fields on quaternionic spaces*, Nonlinearity, **1** (1988), pp. 517-530.

[5] M. MAMONE CAPRIA, *Teoria di Yang Mills e strutture Geometriche*, Doctoral thesis (1990).

[6] C. OKONEK - H. SPINDLER, *Mathematical instanton bundles on* $\mathbb{P}^{2n+1}$, J. Reine Angew. Math., **364** (1986), pp. 35-60.

[7] G. OTTAVIANI - G. TRAUTMANN, *The Tangent space at a special symplectic instanton bundle on* $\mathbb{P}^{2n+1}$, Manuscripta Math., 85 (1994), pp. 97-107.

[8] H. SPINDLER - G. TRAUTMANN, *Special instanton bundles on* $\mathbb{P}^{2n+1}$, *their geometry and their moduli*, Math. Ann., **286** (1990), pp. 559-592.