Symplectic structures and dynamical symmetry groups

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Apart from the total energy, the two-dimensional isotropic harmonic oscillator possesses three independent constants of motion which, with the standard symplectic structure, generates a dynamical symmetry group isomorphic to SU(2). We show that, by suitably redefining the symplectic structure, any of these three constants of motion can be used as a Hamiltonian, and that the remaining two, together with the total energy, generate a dynamical symmetry group isomorphic to SU(1,1). We also show that the standard energy levels of the quantum two-dimensional isotropic harmonic oscillator and their degeneracies are obtained making use of the appropriate representations of SU(1,1), provided that the canonical commutation relations are modified according to the new symplectic structure. Whereas in classical mechanics the different symplectic structures lead to equivalent formulations of the equations of motion, in quantum mechanics the modifications of the commutation relations should be accompanied by modifications in the interpretation of the formalism in order to obtain results equivalent to those found with the common relations.

Keywords: Dynamical symmetry groups; symplectic structures; quantization.

Aparte de la energía total, el oscilador armónico bidimensional isótropo posee tres constantes de movimiento independientes las cuales, con la estructura simpléctica estándar, generan un grupo de simetría dinámica isomorfo a SU(2). Mostramos que, definiendo adecuadamente la estructura simpléctica, cualquiera de estas tres constantes de movimiento puede ser usada como hamiltoniana y que las dos restantes, junto con la energía total, generan un grupo de simetría dinámica isormorfo a SU(1,1). Mostramos también que los niveles de energía usuales del oscilador armónico bidimensional isótropo cuántico y sus degeneraciones se obtienen haciendo uso de las representaciones apropiadas de SU(1,1), si las relaciones de conmutación canónicas se modifican de acuerdo con la nueva estructura simpléctica. Mientras que en la mecánica clásica las diferentes estructuras simplécticas llevan a formulaciones equivalentes de las ecuaciones de movimiento, en la mecánica cuántica, la modificación de las relaciones de conmutación debe estar acompañada de modificaciones en la interpretación del formalismo para obtener resultados equivalentes a los que se hallan con las relaciones usuales.

Descriptores: Grupos de simetría dinámica; estructuras simplécticas; cuantización.

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1. Introduction

In the Hamiltonian formulation of classical mechanics, the constants of motion of an autonomous system are generators of canonical transformations that leave the Hamiltonian function invariant. The set of constants of motion is closed under the Poisson bracket and, in some cases, one can find a finite number of constants of motion that form a basis of a Lie algebra.

In the standard procedure, the Hamiltonian function and the relationship between the canonical momenta and the velocities are obtained starting from the Lagrangian function, which usually is taken as the kinetic energy minus the potential energy. However, the equations of motion can be written in Hamiltonian form in infinitely many ways, using any nontrivial constant of motion as Hamiltonian, by defining appropriately the Poisson bracket, which is not fixed by the choice of the Hamiltonian [1,2].

The fact that a given function defined on the phase space is a constant of motion only depends on the equations of motion, not on the choice of the Hamiltonian or of the Poisson bracket, but any change in the definition of the Poisson bracket may change the value of the Poisson bracket of two constants of motion, which will always be a constant of motion (see the examples below); therefore, a set of constants of motion that form a basis of a Lie algebra with some Poisson bracket need not be a basis for a Lie algebra with another Poisson bracket or may be a basis for a different Lie algebra.

As it is well known, the two-dimensional isotropic harmonic oscillator (TIHO) possesses three constants of motion, S_1, S_2, S_3, that, with respect to the usual Poisson bracket, form a basis of a Lie algebra isomorphic to su(2) (see, *e.g.*, Refs. 3 to 6). The operators corresponding to these constants of motion in the quantum-mechanical version of the TIHO form a basis of a Lie algebra isomorphic to su(2) with the commutator, and this fact can be employed to find the energy levels and their degeneracies [5] (*cf.* also Refs. 7 and 8). In this paper we show that by taking any of the constants of motion S_1, S_2, S_3, as Hamiltonian, the corresponding Poisson bracket can be chosen in such a way that the remaining two, together with the total energy, form a basis of a Lie algebra isomorphic to su(1,1). We also show that, by imposing the appropriate commutation relations, the operators corresponding to these constants of motion form a basis of a Lie algebra isomorphic to su(1,1) and we derive the standard results for the spectrum of the total energy.

2. Hamilton's equations

The Hamilton equations of motion of a mechanical system with n degrees of freedom and forces derivable from a potential are usually written in the form

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \tag{1}$$

($i = 1, 2, \ldots, n$), where H is the Hamiltonian function and $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are the canonical coordinates on the phase space. Then, for any differentiable function, f, defined on the phase space, Eqs. (1) imply that

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t}+\frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}=\frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i}=\{f,H\}, \tag{2}$$

where $\{\ ,\ \}$ is the Poisson bracket,

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{3}$$

and there is a summation over repeated indices. The Poisson bracket (3) satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0,$$

and if $(x^1, x^2, \ldots, x^{2n})$ is an arbitrary coordinate system on the phase space, the Poisson bracket can be expressed as

$$\begin{aligned}\{f,g\} &= \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu}\left(\frac{\partial x^\mu}{\partial q^i}\frac{\partial x^\nu}{\partial p_i} - \frac{\partial x^\mu}{\partial p_i}\frac{\partial x^\nu}{\partial q^i}\right)\\ &= \sigma^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu},\end{aligned} \tag{4}$$

($\mu, \nu, \ldots = 1, 2, \ldots, 2n$) with

$$\sigma^{\mu\nu} \equiv \{x^\mu, x^\nu\}. \tag{5}$$

Owing to the Jacobi identity, the functions $\sigma^{\mu\nu}$ satisfy

$$\sigma^{\rho\lambda}\frac{\partial\sigma^{\mu\nu}}{\partial x^\rho} + \sigma^{\rho\mu}\frac{\partial\sigma^{\nu\lambda}}{\partial x^\rho} + \sigma^{\rho\nu}\frac{\partial\sigma^{\lambda\mu}}{\partial x^\rho} = 0, \tag{6}$$

and one can verify that

$$\{f,g\} = \sigma^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} \tag{7}$$

satisfies the Jacobi identity if Eqs. (6) hold. The fact that the x^μ form a coordinate system implies that $\det(\sigma^{\mu\nu}) \neq 0$. Thus, in terms of an arbitrary system of coordinates, the Hamilton equations are expressed in the form

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \sigma^{\mu\nu}\frac{\partial H}{\partial x^\nu}. \tag{8}$$

A function, f, is a constant of motion if $\{f, H\} = 0$ [see Eq. (2)]. As a consequence of Eq. (2) and of the antisymmetry of the Poisson bracket, H is a constant of motion. It turns out that *any* nontrivial constant of motion can be used as Hamiltonian, with an appropriate choice for the functions $\sigma^{\mu\nu}$; furthermore, for a given Hamiltonian, there are infinitely many sets of functions $\sigma^{\mu\nu}$ satisfying Eqs. (6) and (8), if the number of degrees of freedom is greater that 1 (see Refs. 1 and 9 and the examples below).

3. Dynamical symmetry groups of the two-dimensional isotropic harmonic oscillator

In what follows we shall consider the two-dimensional isotropic harmonic oscillator (TIHO), which corresponds to the equations of motion

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{p}_x = -m\omega_0^2 x, \quad \dot{p}_y = -m\omega_0^2 y, \tag{9}$$

where m is the mass of the particle and ω_0 is the angular frequency of the oscillations. (Note that x, y, p_x, and p_y are the only coordinates on the phase space that *need not* be canonical variables.) As can be readily verified making use of Eqs. (9), the functions

$$\begin{aligned}S_0 &\equiv \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega_0^2(x^2 + y^2),\\ S_1 &\equiv \frac{p_x p_y}{m} + m\omega_0^2 xy,\\ S_2 &\equiv \frac{p_y^2 - p_x^2}{2m} + \frac{1}{2}m\omega_0^2(y^2 - x^2),\\ S_3 &\equiv \omega_0(xp_y - yp_x),\end{aligned} \tag{10}$$

are constants of motion. These functions are related through

$$S_1^2 + S_2^2 + S_3^2 = S_0^2 \tag{11}$$

and S_0 is the usual Hamiltonian of the TIHO. (The functions S_1, S_2, S_3 are normalized here in such a way that S_0, S_1, S_2, and S_3 have dimension of energy, and that Eq. (11) holds with all numerical coefficients equal to 1.)

Even if we employ the usual Hamiltonian, S_0, the Poisson bracket is not fixed by the equations of motion (9). Substituting Eqs. (9) into Eqs. (8) with $H = S_0$ and

$$(x^1, x^2, x^3, x^4) = (x, y, p_x, p_y),$$

we obtain

$$\begin{aligned}\frac{p_x}{m} &= \sigma^{12}m\omega_0^2 y + \sigma^{13}\frac{p_x}{m} + \sigma^{14}\frac{p_y}{m},\\ \frac{p_y}{m} &= -\sigma^{12}m\omega_0^2 x + \sigma^{23}\frac{p_x}{m} + \sigma^{24}\frac{p_y}{m},\\ -m\omega_0^2 x &= -\sigma^{13}m\omega_0^2 x - \sigma^{23}m\omega_0^2 y + \sigma^{34}\frac{p_y}{m},\\ -m\omega_0^2 y &= -\sigma^{14}m\omega_0^2 x - \sigma^{24}m\omega_0^2 y - \sigma^{34}\frac{p_x}{m},\end{aligned} \tag{12}$$

where we have taken into account the fact that $(\sigma^{\mu\nu})$ is antisymmetric. These equations allow us to express, *e.g.*, σ^{14}, σ^{23}, and σ^{34} in terms of σ^{12}, σ^{13}, and σ^{24}

$$\begin{aligned}\sigma^{14} &= -m^2\omega_0^2 \frac{y}{p_y}\sigma^{12} - \frac{p_x}{p_y}(\sigma^{13}-1),\\ \sigma^{23} &= m^2\omega_0^2 \frac{x}{p_x}\sigma^{12} - \frac{p_y}{p_x}(\sigma^{24}-1),\\ \sigma^{34} &= m^4\omega_0^4 \frac{xy}{p_x p_y}\sigma^{12} + m^2\omega_0^2\frac{x}{p_y}(\sigma^{13}-1)\\ &\quad - m^2\omega_0^2\frac{y}{p_x}(\sigma^{24}-1).\end{aligned} \tag{13}$$

However, σ^{12}, σ^{13}, and σ^{24} cannot be chosen arbitrarily since, apart from the condition $\det(\sigma^{\mu\nu}) \neq 0$, the functions $\sigma^{\mu\nu}$ must satisfy Eqs. (6). Nevertheless, there are infinitely many nonsingular matrices $(\sigma^{\mu\nu})$ satisfying Eqs. (6) and (12) (the general solution of Eqs. (6) and (12) involves three arbitrary functions of three variables [1]). The usual Poisson bracket is given by $\sigma^{12}=0$, $\sigma^{13}=\sigma^{24}=1$ (hence, $\sigma^{14}=\sigma^{23}=\sigma^{34}=0$).

The constants of motion S_1, S_2, S_3, are known to generate a Lie algebra isomorphic to su(2) with the usual Poisson bracket. In fact, when $\sigma^{13}=1$ and $\sigma^{24}=1$ are the only nonvanishing independent components of $(\sigma^{\mu\nu})$, one finds that

$$\{S_i, S_j\} = 2\omega_0 \varepsilon_{ijk} S_k, \tag{14}$$

$(i,j,\ldots = 1,2,3)$. Moreover, S_1, S_2, S_3 generate an action of SU(2) on the phase space that leaves invariant both the Hamiltonian and the Poisson bracket (see, *e.g.*, Refs. 3 to 6, and 10). Surprisingly, for any choice of $\sigma^{\mu\nu}$ reproducing the equations of motion with $H=S_0$, the Poisson brackets among S_1, S_2, S_3 are given by expressions similar to Eq. (14). Without specifying the functions $\sigma^{\mu\nu}$, using the definitions (10) and the relations (13), a straightforward computation yields

$$\{S_i, S_j\} = 2\omega_0 \varepsilon_{ijk}(\sigma^{13}+\sigma^{24}-1) S_k. \tag{15}$$

This equation shows that if $(\sigma^{\mu\nu})$ satisfies the conditions (6) and (12), then $\sigma^{13}+\sigma^{24}$ must be a constant of motion and that, in order for S_1, S_2, S_3, to generate a Lie algebra isomorphic to su(2), the Poisson bracket must be chosen in such a way that $\sigma^{13}+\sigma^{24}$ is a real number different from 1.

As pointed out above, any constant of motion, such as S_1, S_2, and S_3, can be used as Hamiltonian. For instance, taking $H=S_1$ and $(x^1,x^2,x^3,x^4)=(x,y,p_x,p_y)$, Eqs. (8) read

$$\begin{aligned}\frac{p_x}{m} &= \sigma^{12} m\omega_0^2 x + \sigma^{13}\frac{p_y}{m} + \sigma^{14}\frac{p_x}{m},\\ \frac{p_y}{m} &= -\sigma^{12} m\omega_0^2 y + \sigma^{23}\frac{p_y}{m} + \sigma^{24}\frac{p_x}{m},\\ -m\omega_0^2 x &= -\sigma^{13} m\omega_0^2 y - \sigma^{23} m\omega_0^2 x + \sigma^{34}\frac{p_x}{m},\\ -m\omega_0^2 y &= -\sigma^{14} m\omega_0^2 y - \sigma^{24} m\omega_0^2 x - \sigma^{34}\frac{p_y}{m},\end{aligned} \tag{16}$$

where we have taken into account Eqs. (9) and the fact that $(\sigma^{\mu\nu})$ is antisymmetric. Using these equations we can express σ^{13}, σ^{24}, and σ^{34} in terms of σ^{12}, σ^{14}, and σ^{23}

$$\begin{aligned}\sigma^{13} &= -m^2\omega_0^2 \frac{x}{p_y}\sigma^{12} - \frac{p_x}{p_y}(\sigma^{14}-1),\\ \sigma^{24} &= m^2\omega_0^2 \frac{y}{p_x}\sigma^{12} - \frac{p_y}{p_x}(\sigma^{23}-1),\\ \sigma^{34} &= -m^4\omega_0^4 \frac{xy}{p_x p_y}\sigma^{12} - m^2\omega_0^2\frac{y}{p_y}(\sigma^{14}-1)\\ &\quad + m^2\omega_0^2\frac{x}{p_x}(\sigma^{23}-1).\end{aligned} \tag{17}$$

By inspection, one finds that a particular choice for $\sigma^{\mu\nu}$ that satisfies Eqs. (6) and (17) is such that $\sigma^{14}=\sigma^{23}=1$ are the only nonvanishing independent components of $\sigma^{\mu\nu}$, *i.e.*,

$$\{x, p_y\} = 1, \qquad \{y, p_x\} = 1 \tag{18}$$

then, the Poisson brackets among the constants of motion S_0, S_2, and S_3 are given by

$$\{S_i, S_j\} = 2\omega_0 \varepsilon_{ijk} g^{kl} S_l \qquad (i,j,\ldots = 0,2,3), \tag{19}$$

where (g^{kl}) is a diagonal matrix with $g^{00}=-1$, $g^{22}=g^{33}=1$, and ε_{ijk} is totally antisymmetric with $\varepsilon_{023}=1$. This means that S_0, S_2, and S_3 generate a Lie algebra isomorphic to su(1,1). It may be remarked that, by contrast with the usual Hamiltonian S_0, S_1 is not positive definite.

Making use of the two-component spinor notation (see, *e.g.*, Refs. 6 and 11), we can see explicitly that S_0, S_2, and S_3 generate an action of SU(1,1) on the phase space. Indeed, letting

$$\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} \equiv \begin{pmatrix} p_x + p_y + \mathrm{i}m\omega_0(x+y)\\ p_x - p_y + \mathrm{i}m\omega_0(x-y)\end{pmatrix} \tag{20}$$

using Eqs. (18), we find that the only nonvanishing Poisson brackets among ψ^1, ψ^2, and their conjugates are given by $\{\psi^1, \overline{\psi^1}\} = 4\mathrm{i}m\omega_0$, $\{\psi^2, \overline{\psi^2}\} = -4\mathrm{i}m\omega_0$, where the bar denotes complex conjugation. It is convenient to make use of the *mate* of ψ^A, which is defined by [6,11]

$$\widehat{\psi}^A = -\mathrm{i}\eta^{AB}\overline{\psi_B}, \tag{21}$$

where

$$(\eta^{AB}) = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}. \tag{22}$$

The spinor indices $A, B, \ldots$, take the values 1, 2 and are raised or lowered according to $\psi^A = \psi_B \varepsilon^{BA}$, $\psi_A = \varepsilon_{AB}\psi^B$, with

$$(\varepsilon_{AB}) = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} = (\varepsilon^{AB}). \tag{23}$$

Hence, $\widehat{\psi}^1 = \mathrm{i}\overline{\psi^2}$, $\widehat{\psi}^2 = \mathrm{i}\overline{\psi^1}$. According to the foregoing definitions

$$\begin{aligned}\{\psi^A, \psi^B\} = 0, \qquad \{\widehat{\psi}^A, \widehat{\psi}^B\} = 0,\\ \{\psi^A, \widehat{\psi}^B\} = -4m\omega_0\varepsilon^{AB}\end{aligned} \tag{24}$$

and

$$S_1 = \frac{\mathrm{i}}{4m} \varepsilon_{AB} \widehat{\psi}^A \psi^B. \tag{25}$$

The group SU(1,1) is formed by the unimodular 2×2 complex matrices, $(U^A{}_B)$, such that

$$U_B{}^A \overline{U_D{}^C} \eta^{BD} = \eta^{AC}. \tag{26}$$

If the components of a spinor ψ^A transform according to $\psi^A \mapsto \psi'^A = U^A{}_B \psi^B$, then using Eqs. (21) and (26)

$$\begin{aligned} \widehat{\psi}'^A &= -\mathrm{i}\eta^{AB} \overline{U_{BC}} \overline{\psi^C} = \mathrm{i}\eta^{AB} \overline{U_B{}^C} \overline{\psi_C} \\ &= -\mathrm{i}\eta^{BC} U^A{}_B \overline{\psi_C} = U^A{}_B \widehat{\psi}^B, \end{aligned}$$

which means that $\widehat{\psi}^A$ transforms in the same way as ψ^A. Since

$$\varepsilon_{AB} U^A{}_C U^B{}_D = \varepsilon_{CD},$$

if and only if $\det(U^A{}_B) = 1$, from Eqs. (24) and (25) it follows that the Poisson bracket and the Hamiltonian are invariant under the SU(1,1) transformations $\psi^A \mapsto U^A{}_B \psi^B$.

The group SU(1,1) is a double covering group of $\mathrm{SO}_0(2,1)$, the connected component of the identity in SO(2,1), in the same way as SU(2) is a double covering group of SO(3). The fact that a SU(1,1) matrix $(U^A{}_B)$ and its negative give two different transformations $\psi^A \mapsto \psi'^A$ implies that we are dealing with an action of SU(1,1) and not of SO(2,1).

The constants of motion S_0, S_2, and S_3 can be expressed in a form analogous to Eq. (25), namely

$$S_i = \frac{1}{4m} \sigma_{iAB} \widehat{\psi}^A \psi^B, \qquad (i = 0, 2, 3) \tag{27}$$

with

$$(\sigma_{0AB}) = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad (\sigma_{2AB}) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

$$(\sigma_{3AB}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28}$$

(The matrices $\sigma_i \equiv (\sigma_i{}^A{}_B)$ form a basis for the Lie algebra of SU(1,1) [6]).

Any differentiable function defined on the phase space, G, is the generator of a one-parameter group of canonical transformations whose orbits are the solutions of

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \{x^\mu, G\} = \sigma^{\mu\nu} \frac{\partial G}{\partial x^\nu}.$$

[*cf.* Eq. (8)]. Then, using the components ψ^A as complex coordinates on the phase space, according to Eqs. (24) and (27), the one-parameter group of canonical transformations generated by S_i $(i = 0, 2, 3)$ is given by

$$\begin{aligned} \frac{\mathrm{d}\psi^A}{\mathrm{d}s} &= \{\psi^A, S_i\} = \frac{1}{4m} \sigma_{iBC} \{\psi^A, \widehat{\psi}^B \psi^C\} \\ &= \frac{1}{4m} \sigma_{iBC} (-4m\omega_0 \varepsilon^{AB}) \psi^C = \omega_0 \sigma_i{}^A{}_C \psi^C, \end{aligned}$$

hence,

$$\begin{pmatrix} \psi^1(s) \\ \psi^2(s) \end{pmatrix} = \exp(s\omega_0 \sigma_i) \begin{pmatrix} \psi^1(0) \\ \psi^2(0) \end{pmatrix}.$$

Since σ_i belongs to the Lie algebra of SU(1,1), $\exp(s\omega_0\sigma_i)$ belongs to SU(1,1).

While the usual symplectic structure is invariant under rotations, the symplectic structure defined by Eqs. (18) is not; in other words, the rotations in the xy-plane are not canonical transformations with respect to this symplectic structure. Equations (18) mean that p_y is the infinitesimal generator of translations along the x-axis while p_x is the infinitesimal generator of translations along the y-axis.

If, instead of assuming (18), we take any other nonsingular matrix $(\sigma^{\mu\nu})$ satisfying Eqs. (6) and (16) we obtain

$$\{S_i, S_j\} = 2\omega_0 \varepsilon_{ijk} g^{kl} (\sigma^{14} + \sigma^{23} - 1) S_l \tag{29}$$

(i,j, ... = 0,2,3), [*cf.* Eqs. (15) and (19)]. Thus, when S_1 is the Hamiltonian, $\sigma^{14} + \sigma^{23}$ is a constant of motion and, if $\sigma^{14} + \sigma^{23}$ is a number different from 1, $\{S_0, S_2, S_3\}$ generates a Lie algebra isomorphic to su(1,1).

It can be readily seen that taking S_2 or S_3 as the Hamiltonian leads to results similar to those obtained with S_1.

4. The Heisenberg equations of motion for the TIHO and the energy eigenvalues

Now we shall consider the variables x, y, p_x, and p_y appearing in Eqs. (9) as operators in the Heisenberg picture with the commutation relations

$$[x, p_y] = \mathrm{i}\hbar, \qquad [y, p_x] = \mathrm{i}\hbar, \tag{30}$$

which follow from Eqs. (18), applying the usual rule of replacing Poisson brackets by commutators divided by $\mathrm{i}\hbar$, and all the other commutators among the operators x, y, p_x, and p_y not given by Eqs. (30) equal to zero. Then it can be readily verified that the Heisenberg equations of motion

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[f, H],$$

reproduce Eqs. (9) with $H = S_1$ and S_1 defined by Eqs. (10).

Since S_3, defined by Eqs. (10), involves products of noncommuting operators we replace the expression for S_3 given there by

$$S_3 = \frac{\omega_0}{2}(xp_y + p_y x - yp_x - p_x y).$$

Then, making use of Eqs. (30), one finds that the last expression can be rewritten as

$$S_3 = \omega_0(xp_y - yp_x). \tag{31}$$

A straightforward computation shows that S_1 commutes with S_0, S_2, and S_3, which are, therefore, constants of motion and

$$[S_0, S_2] = 2\mathrm{i}\hbar\omega_0 S_3, \qquad [S_2, S_3] = -2\mathrm{i}\hbar\omega_0 S_0,$$
$$[S_3, S_0] = 2\mathrm{i}\hbar\omega_0 S_2, \tag{32}$$

[*cf.* Eqs. (19)]. Hence $\{S_0, S_2, S_3\}$ generate a Lie algebra isomorphic to su(1,1). Taking into account Eqs. (30) one finds that, instead of Eq. (11), now we have

$$S_1^2 + S_2^2 + S_3^2 = S_0^2 + \hbar^2\omega_0^2. \tag{33}$$

(It may be noticed that, assuming that the usual commutation relations hold, one obtains $S_1^2 + S_2^2 + S_3^2 = S_0^2 - \hbar^2\omega_0^2$ [5].) From Eqs. (32) it follows that

$$C \equiv S_0^2 - S_2^2 - S_3^2 \tag{34}$$

commutes with S_0, S_2, and S_3 and that

$$[S_0, S_\pm] = \pm 2\hbar\omega_0 S_\pm, \qquad [S_+, S_-] = -4\hbar\omega_0 S_0, \tag{35}$$

where $S_\pm \equiv S_2 \pm \mathrm{i}S_3$ (see also Refs. 10, 12). Hence, there exist common eigenvectors of C and S_0, $|j\,\mu\rangle$, such that

$$\begin{aligned} C|j\,\mu\rangle &= j(j+1)(2\hbar\omega_0)^2|j\,\mu\rangle, \\ S_0|j\,\mu\rangle &= \mu 2\hbar\omega_0|j\,\mu\rangle, \end{aligned} \tag{36}$$

where the constant factors $(2\hbar\omega_0)^2$ and $2\hbar\omega_0$ are introduced for later convenience. Then,

$$S_\pm|j\,\mu\rangle = c_\pm(j,\mu)|j, \mu \pm 1\rangle, \tag{37}$$

where $c_\pm(j,\mu)$ are some constants. Since

$$\begin{aligned} C &= S_0^2 + 2\hbar\omega_0 S_0 - S_- S_+ \\ &= S_0^2 - 2\hbar\omega_0 S_0 - S_+ S_-, \end{aligned}$$

assuming that the vectors $|j\,\mu\rangle$ are normalized and that x, y, p_x, p_y (and, hence, S_0, S_1, S_2, S_3) are Hermitian operators, we have

$$\begin{aligned} |c_+(j,\mu)|^2 &= \langle j\,\mu|S_- S_+|j\,\mu\rangle \\ &= \langle j\,\mu|(S_0^2 + 2\hbar\omega_0 S_0 - C)|j\,\mu\rangle \\ &= (2\hbar\omega_0)^2[\mu(\mu+1) - j(j+1)] \end{aligned}$$

and, similarly,

$$|c_-(j,\mu)|^2 = (2\hbar\omega_0)^2[\mu(\mu-1) - j(j+1)].$$

Thus,

$$\mu(\mu \pm 1) \geq j(j+1) \tag{38}$$

and we can choose the eigenvectors $|j\,\mu\rangle$ in such a way that

$$S_\pm|j\,\mu\rangle = 2\hbar\omega_0\sqrt{\mu(\mu \pm 1) - j(j+1)}\,|j, \mu \pm 1\rangle. \tag{39}$$

Furthermore, since S_0 is the sum of squares of Hermitian operators,

$$\mu \geq 0 \tag{40}$$

[see Eqs. (36)]. Hence, for a fixed value of j, there exists a value of μ, $\mu_{\min}$, such that $\mu_{\min}(\mu_{\min} - 1) = j(j+1)$, *i.e.*,

$$\mu_{\min} = -j \qquad \text{or} \qquad j+1. \tag{41}$$

However, the foregoing relations do not fix the values of μ and j.

Owing to Eqs. (33), (34), and (36), $|j\,\mu\rangle$ is also an eigenvector of S_1^2

$$S_1^2|j\,\mu\rangle = [(j + \tfrac{1}{2})2\hbar\omega_0]^2|j\,\mu\rangle. \tag{42}$$

Since S_1 and S_0 commute, there exists a basis formed by common eigenvectors of S_1 and S_0, which would also be automatically eigenvectors of S_1^2; hence, we can assume that $|j\,\mu\rangle$ is an eigenvector of S_1 with

$$S_1|j\,\mu\rangle = (j + \tfrac{1}{2})2\hbar\omega_0|j\,\mu\rangle. \tag{43}$$

(This is related to the fact that the eigenvalue of C is unchanged if we replace $j + 1/2$ by $-(j + 1/2)$, see Eq. (36).)

Introducing the operators

$$K_+ \equiv \frac{p_x^2 - p_y^2}{2m} + \frac{1}{2}m\omega_0^2(y^2 - x^2) + \mathrm{i}\omega_0(xp_x - yp_y) \tag{44}$$

and $K_- \equiv K_+^\dagger$, making use of Eqs. (30) one finds that

$$[S_0, K_\pm] = 0, \qquad [S_1, K_\pm] = \pm 2\hbar\omega_0 K_\pm,$$
$$[K_+, K_-] = 4\hbar\omega_0 S_1, \tag{45}$$

which implies that $K_\pm|j\,\mu\rangle \propto |j \pm 1, \mu\rangle$ and that $\{K_1/2\omega_0, K_2/2\omega_0, S_1/2\omega_0\}$, with $K_\pm = K_1 \pm \mathrm{i}K_2$, obey the commutation relations of the standard basis of su(2) (*i.e.*, $[J_i, J_j] = \mathrm{i}\hbar\varepsilon_{ijk}J_k$). (The operator K_+ is essentially the product $\psi^1\psi^2$ of the components of the spinor defined in Eq. (20).) Furthermore,

$$K_1^2 + K_2^2 + S_1^2 = S_0^2 - \hbar^2\omega_0^2 \tag{46}$$

hence [see Eqs. (36)]

$$(K_1^2 + K_2^2 + S_1^2)|j\,\mu\rangle = (\mu - \tfrac{1}{2})(\mu + \tfrac{1}{2})(2\hbar\omega_0)^2|j\,\mu\rangle.$$

Thus, applying the standard results about the representations of su(2), it follows that for a fixed value of μ, j can take $2(\mu - 1/2) + 1 = 2\mu$ values, in integral steps, such that $-(\mu - 1/2) \leq j + \frac{1}{2} \leq \mu - 1/2$ or, equivalently,

$$-\mu \leq j \leq \mu - 1, \tag{47}$$

and that $\mu - 1/2$ can take the values $0, 1/2, 1, 3/2, \ldots$. Taking into account the second equation in (36), these results reproduce the spectrum of the total energy of the TIHO obtained by means of the standard approach [5]. Note that, according to Eqs. (36) and (37), the operators $S_\pm$ raise or lower the eigenvalue of the total energy operator by $2\hbar\omega_0$ just as the operators $T_\pm$ defined in Sec. 34 of Ref. 10 for the one-dimensional harmonic oscillator. In the terminology of Ref. 10, S_+, S_-, together with their commutator generate a symmetry group SU(1,1) for the TIHO with the commutation relations (30), while in the case of the one-dimensional harmonic oscillator, T_+, T_-, and their commutator do not generate a symmetry group but a dynamical group SU(1,1). Similarly, $\{K_1, K_2, S_1\}$ generate a dynamical group SU(2) for the TIHO when the commutation relations (30) are imposed.

5. Concluding remarks

In the context of classical mechanics, S_1, S_2, S_3, or any other nontrivial constant of the motion can be used as Hamiltonian, leading to a consistent formulation. However, in the quantum-mechanical version, things are not so clear especially regarding the physical implications that commutation relations like (30) can have in connection with the simultaneous measurability of the observables.

In the example considered here, we have shown that, by suitably modifying the commutation relations, the compact symmetry group SU(2) of the TIHO is substituted by the non-compact symmetry group SU(1,1) when the usual commutation relations are replaced by (30). Nevertheless, the spectrum of the total energy operator is left unchanged by this replacement.

In classical mechanics, the alternative symplectic structures lead, by construction, to equivalent formulations of the equations of motion. However, a relevant fact is that, as in the case considered in Sec. 3, rigid translations or rotations in configuration space may not correspond to canonical transformations with respect to the alternative structures, which might be seen as a drawback or, at least, as a reason to prefer the symplectic structure obtained in the usual way. Nevertheless, when the configuration space is curved and does not possess symmetries, it might be more convenient to employ one of the many alternative symplectic structures.

As pointed out above, in quantum mechanics, the modification of the commutation relations leads to deeper questions. It seems that, simultaneously with the modification of the commutation relations, one would have to suitably modify the interpretation of the formalism since the predictions of the theory should not depend on which constant of motion we want to employ to express the evolution equations.

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