SYMPLECTIC STRUCTURES ON BANACH MANIFOLDS

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1. Normal form. Let M be a Banach manifold. A *symplectic structure* on M is a closed 2-form Ω such that the associated mapping $\bar{\Omega}: T(M) \to T^*(M)$ defined by $\bar{\Omega}(X) = X \lrcorner \Omega$ is a bundle isomorphism.

If M is finite dimensional, Darboux's theorem states that every point in M has a coordinate neighborhood N with coordinate functions $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ such that $\Omega = \sum_{i=1}^n dx_i \wedge dy_i$ on N. Standard proofs of this theorem (e.g. [**4**]) use induction on n, so they do not apply to the infinite-dimensional case. It happens, however, that an idea of J. Moser [**3**] may be adapted to prove a similar result for Banach manifolds.

Since the problem is a local one, it suffices to consider a symplectic structure Ω on a neighborhood of 0 in a Banach space B.

THEOREM. *Let Ω_1 be the symplectic structure on B which is constant with respect to the natural parallelism on B and equal to Ω at* 0. *Then there are neighborhoods U and V of* 0 *and a diffeomorphism $f: U \to V$ such that $f(0) = 0$, $f_*(0)$ is the identity, and $f^*(\Omega_1) = \Omega$.*

The local classification of symplectic structures on a manifold modeled on B is thus reduced to the classification of nonsingular, skew-symmetric, bilinear forms on B. If B is a Hilbert space, every such form is equal to $\sum_{i \in I} \xi_i \wedge \eta_i$ for some basis $\{\xi_i\}_{i \in I} \cup \{\eta_i\}_{i \in I}$ of B^*.

PROOF OF THEOREM. Let $\omega = \Omega_1 - \Omega$, $\Omega_t = \Omega + t\omega$, $t \in [0, 1]$. From the compactness of $[0, 1]$ and the openness of invertibility, it follows that there is a neighborhood U_1 of 0 such that all the Ω_t are symplectic structures on U_1. We may assume that U_1 is star-shaped. By the Poincaré lemma [**2**], there is a 1-form ϕ on U_1 such that $d\phi = \omega$ and $\phi(0) = 0$. The fact that $\Omega_1 = \Omega$ at 0 implies that the first derivative of ϕ vanishes at 0. Let $X_t = -(\bar{\Omega}_t)^{-1}(\phi)$. X_t is a smooth, time-dependent vector field on U_1 which vanishes, together with its first derivative, at 0. X_t may be integrated to a family $\{f_t\}$ of partially defined mappings from U_1 to U_1. The compactness of $[0, 1]$ and the openness (see [**2**]) of the domain of $\{f_t\}$ in $U_1 \times [0, 1]$ imply the existence of

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a neighborhood U of 0 on which all the f_t are defined. Let $V=f_1(U)$, $f=f_1$. The vanishing of X_t with its first derivative at 0 implies that $(0)=0$ and $f_*(0)$ is the identity. Finally, as in [3],

$$d(f_t^*\Omega_t)/dt = f_t^*(d\Omega_t/dt) + f_t^*[d(X_t \lrcorner_t)\Omega + X_t - d\Omega_t]$$

$$= f_t^*(\omega) + f_t^*(-d\phi) = 0,$$

so

$$f^*(\Omega_1) = f_0^*(\Omega_0) = \Omega.$$

2. **Automorphisms.** The method of proof used above also shows that two symplectic structures which agree on a closed submanifold N of a Banach manifold are equivalent on a neighborhood of N. (Proofs for this and other assertions made here will appear in a subsequent paper.) If (M, Ω) is any symplectic manifold, this result can be used to obtain an equivalence between the canonical 2-form on $T^*(M)$ near the zero section and the form $\Omega\times(-\Omega)$ on $M\times M$ near the diagonal. This yields a parametrization of the symplectic automorphisms of (M, Ω) near the identity by closed 1-forms (or, if $H^1(M, \mathbf{R})=0$, by "generating" functions) on M and a proof that the symplectic automorphisms form a smooth submanifold of the group of diffeomorphisms of M. A version of the last result has also been obtained by Ebin and Marsden [1] by an application of Moser's technique to Hilbert manifolds of diffeomorphisms.

References

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