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SYMPLECTIC SURFACES IN A FIXED HOMOLOGY CLASS

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1. Introduction

The purpose of this paper is to investigate the following problem:

> For a fixed 2-dimensional homology class α in a simply connected symplectic 4-manifold, up to smooth isotopy, how many connected smoothly embedded symplectic submanifolds represent α?

It has been conjectured in some quarters that such a homology class α should be represented by at most finitely many connected embedded symplectic submanifolds; some have conjectured that such a representative must be unique.

As motivation for this conjecture, suppose one fixes a homology class $\alpha \in H_2(X; \mathbf{Z})$ where X is a Kähler surface and asks, up to smooth isotopy, how many nonsingular complex curves represent this class. If $a \in H^2(X; \mathbf{Z})$ is the Poincaré dual of α, then each complex curve representing α is the zero set of a section of a holomorphic line bundle with $c_1 = a$. Thus we must ask about the preimage of a under the map

$$c_1 : H^1(X; \mathcal{O}_X^*) \to H^2(X; \mathbf{Z}).$$

Equivalently, we study the kernel of c_1. This is an analytic variety, and hence has finitely many connected components. Since the points corresponding to singular curves form a subvariety of complex codimension

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at least one, up to smooth isotopy, there are at most finitely many nonsingular complex curves representing α. In particular, if $H^1(X;\mathbf{Z}) = 0$, then the Picard torus, $\ker(c_1) = 0$; so there is a unique representative of the class in question.

In contrast we shall prove the following theorem in §5.

Theorem. *Let X be a simply connected symplectic 4-manifold which contains a c-embedded symplectic torus T. Then in each homology class $2m\,[T]$, $m \geq 2$, there is an infinite family of smoothly embedded symplectic tori, no two of which are smoothly isotopic.*

To say that a torus T is *c-embedded* means that T is a smoothly embedded homologically essential torus of self-intersection zero which has a a pair of simple curves which generate its first homology and bound vanishing cycles (disks of self-intersection -1) in X. (See [5].) The simplest examples of c-embedded tori are generic fibers of simply connected elliptic fibrations. One can also find c-embedded tori in many surfaces of general type (including Horikawa surfaces) via the process of rational blowdowns [4].

One might then ask for what families of symplectic 4-manifolds are there only finitely many smooth isotopy classes of symplectic surfaces in any fixed homology class. In light of the above theorem, a reasonable conjecture might be that this finiteness condition holds for ruled surfaces or rational surfaces with $c_1^2 > 0$. Siebert and Tian have shown that each symplectic surface in $S^2 \times S^2$ with genus ≤ 3 is smoothly isotopic to a complex curve; so finiteness holds in that situation.

The technique of this paper, described in detail below, is to replace the torus T in its tubular neighborhood, $T = S^1 \times S^1 \subset S^1 \times S^1 \times D^2$, with $S^1 \times B \subset S^1 \times S^1 \times D^2$ where B is a closed braid in $S^1 \times D^2$. In case the braid B has an even number of strands and also represents the unknot in S^3, let L_B denote the 2-component link in S^3 obtained as the preimage of the axis of B under the 2-fold cover of S^3 branched over B. We can then identify the double cover of the symplectic manifold X branched over $S^1 \times B$ as the manifold X_{L_B} of [5]. This manifold has a Seiberg-Witten invariant which was computed in [5]; it is related to the Alexander polynomial of the link L_B. Note that if B has $2m$ strands, $S^1 \times B$ is homologous to $2mT$. In order to obtain infinitely many nonisotopic such tori homologous to $2mT$, we will utilize a braid construction of Birman and Menasco [2], [3] to construct infinitely many such braids which are distinguished by their Alexander polynomials.

The construction of this paper contrasts with an older construction of the authors that produced (under mild hypotheses) infinitely many non-smoothly isotopic embedded surfaces, all topologically ambiently isotopic to a given embedded surface [6]. This older construction replaced an annulus $S^1 \times I \times \{0\} \subset S^1 \times I \times D^2$ with $S^1 \times K \subset S^1 \times I \times D^2$ where K is the result of tying a knot in the core $I \times \{0\}$ of the cylinder $I \times D^2$. If Σ is a symplectic surface of positive genus and nonnegative self-intersection, and Σ_K is the result of performing this knotting operation, it is shown in [6] that Σ_K is *not* smoothly isotopic to a symplectic submanifold as long as the Alexander polynomial of K is nontrivial.

There have also been informal conjectures asserting the finiteness of the number of smooth simply connected symplectic 4-manifolds in a fixed homeomorphism type which admit symplectic Lefschetz fibrations with a fiber of fixed genus. In the last section of this paper we produce counterexamples, stemming from our examples, to these conjectures. Other, more easily obtained, counterexamples are given in [7].

It is interesting to ask whether the nonfiniteness results of this paper are a general phenomenon applying to surfaces of arbitrary genus or whether they are unique to tori. The authors have general constructions which apply to surfaces of higher genus, but they have been unable to determine whether or not the resulting surfaces are smoothly isotopic.

Finally, the authors wish to express appreciation to Bill Menasco for (e-mail) conversations concerning his joint work with Joan Birman, and to Gang Tian whose interest stimulated this work.

2. Braids

In this section we shall describe a sequence of families of closed $2m$-strand braids $B_{2m,k}$ in S^3, $m, k = 1, 2, \dots,$ whose corresponding double branched covers yield fibered 2-component links in S^3. We begin by describing a construction of fibered 2-component links due to D. Goldsmith [8]. Let B be a closed $2m$-strand braid in S^3 with axis A. I.e., B is a braid in an unknotted solid torus $V = S^1 \times D^2$ in S^3, and A is the core of the complementary unknotted solid torus. We may think of A as a fibered knot, whose fibers are the disks $\{t_0\} \times D^2$ of V. Each such disk contains $2m$ points of the braid B.

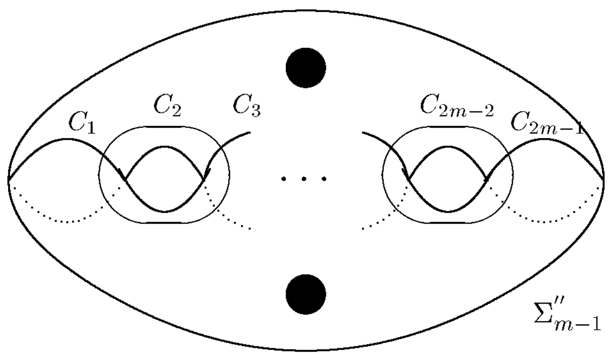

FIGURE 1

Now suppose further that B represents an unknotted circle in S^3, that is to say, B can be isotoped to the unknot in S^3 when one allows it to pass through A. The double branched cover of S^3 branched over B is then S^3 again, and since A links B an even number of times, it lifts to a 2-component link L_B in the cover S^3. This link is fibered, and its fibers are simply the double branched covers of the fibers of the unknot A. These are twice-punctured surfaces of genus $m-1$. So we have

$$S^3 \setminus L_B = S^1 \times_\varphi \Sigma''_{m-1},$$

where Σ''_{m-1} is the surface of genus $m-1$ with two boundary components. The monodromy map φ can be calculated from the braid B. The braid group on $2m$ strands is generated by the elementary braid transpositions $\beta_1, \dots, \beta_{2m-1}$, where β_i denotes a right-hand crossing of the ith strand over the $(i+1)$st. In the double branched cover each such crossing contributes a Dehn twist. (See e.g.[1, p.172].) If we write the braid group element corresponding to B as a word in the $\{\beta_i\}$, it follows that the monodromy will be the product of Dehn twists about the simple closed curves $\{C_i\}$ as shown in Figure 1.

Our next task is to construct for each integer $n \geq 4$ a family of closed n-braids $\{B_{n,k}\}$, k a nonnegative integer, with the properties that each $B_{n,k}$ is unknotted in S^3 and, for fixed m, the Alexander polynomial of the 2-component link $L_{B_{2m,k}}$ is distinguished by the integer k. It is these braids which will be used to construct our examples of symplectic submanifolds.

We begin with the 4-strand braid $B_{4,0}$, shown in Figure 2, first constructed by Birman and Menasco [2]. For us, the key property of this braid is that it represents the unknot in S^3.

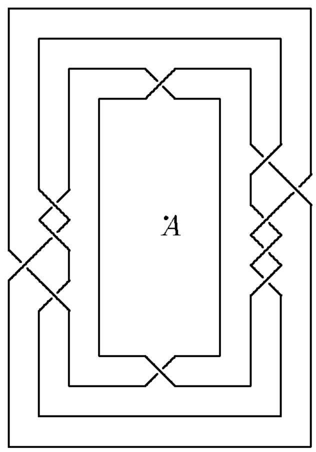

FIGURE 2. $B_{4,0}$

Using the integer j as shorthand for the braid transposition β_j and $\bar{j}$ for β_j^{-1}, the braid $B_{4,0}$ is given by the expression

$$B_{4,0} = (\bar{2}.\bar{2}.1.\bar{2}).3.(2.2.2.\bar{1}.2).\bar{3}.$$

We define braids $B_{m,0}$ inductively as follows. Assume that $B_{m,0}$ is given by

$$B_{m,0} = \Phi_m.(m-1).\Psi_m.(\overline{m-1}),$$

where Φ_m and Ψ_m are expressions involving only the braid transpositions $j < m-1$ and their inverses. Define

$$B_{m+1,0} = (m-2).(m-1).\Phi_m.(\overline{m-1}).m.(\overline{m-1}).\Psi_m.(m-1).\bar{m}.$$

Thus

$$\Phi_{m+1} = (m-2).(m-1).\Phi_m.(\overline{m-1})$$

and

$$\Psi_{m+1} = (\overline{m-1}).\Psi_m.(m-1).$$

A schematic is given in Figure 3.

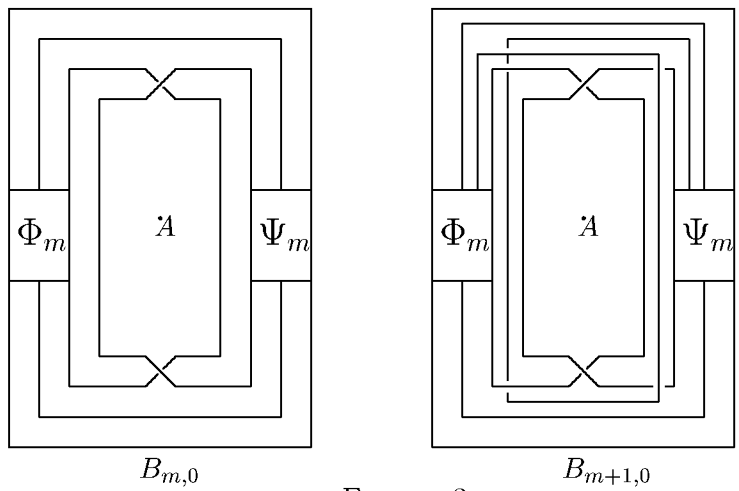

FIGURE 3

Lemma 2.1. *When the braids $B_{m,0}$ ($m \geq 4$) are considered as knots in S^3, they are unknotted.*

Proof. Figure 4 shows how $B_{m+1,0}$ is isotopic to $B_{m,0}$ in S^3, and this completes the proof since $B_{4,0}$ is unknotted. q.e.d.

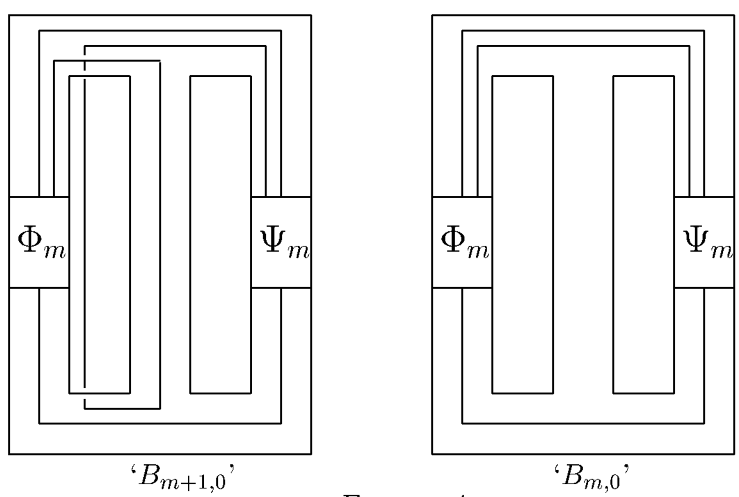

FIGURE 4

In [3], Birman and Menasco introduced an operation on m-strand braids of the form $B = \Phi.(m-1).\Psi.(\overline{m-1})$ where Φ and Ψ are expressions in braid transpositions $j < m-1$ and their inverses. This operation is pictured in Figure 5.

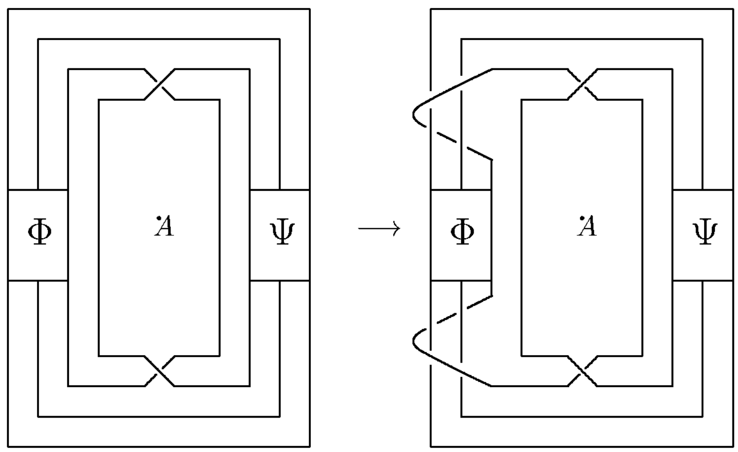

FIGURE 5

The Birman-Menasco operation preserves the link-type of the braid (as a link in S^3). Formally, the Birman-Menasco operation is:

$$\Phi.(m-1).\Psi.(\overline{m-1}) \quad \longrightarrow \quad \Gamma_{m-2}^{-1}.\Phi.\Gamma_{m-2}.(m-1).\Psi.(\overline{m-1}),$$

where

$$\Gamma_r = r.(r-1).\cdots.2.1.1.2.\cdots.(r-1).r.$$

Our family of m-strand braids is $\{B_{m,k}\}$ where $B_{m,k}$ is the result of applying the Birman-Menasco operation k times to the braid $B_{m,0}$. Hence

$$B_{m,k} = \Gamma_{m-2}^{-k}.\Phi_m.\Gamma_{m-2}^{k}.(m-1).\Psi_m.(\overline{m-1})$$

or $B_{m,k} = \Phi_{m,k}.(m-1).\Psi_m.(\overline{m-1})$ where $\Phi_{m,k} = \Gamma_{m-2}^{-k}.\Phi_m.\Gamma_{m-2}^{k}$. From Lemma 2.1 and the fact (easily seen in Figure 5) that the Birman-Menasco operation preserves the link type of the braid, it follows that the braids $B_{m,k}$ all represent unknots in S^3.

3. The double covering links

In this section we shall study the 2-component links $L_{2m,k}$ which result from taking the preimage $\pi^{-1}(A)$ of the axis in the double cover of S^3 branched over $B_{2m,k}$. Recall from §2 that $L_{2m,k}$ is a fibered link and its fiber is the twice-punctured surface Σ''_{m-1} of genus $m-1$. We are interested in the monodromy of this fibration. As was discussed in §2, this monodromy is a product of Dehn twists given by the braid transpositions which describe $B_{2m,k}$ as an element of the braid group on $2m$-strands. Each transposition β_j corresponds to the Dehn twist about the curve C_j of Figure 1. We orient these curves so that their intersection numbers are

$$\begin{aligned} C_i \cdot C_j &= 0, \quad j \neq i \pm 1, \\ C_{i-1} \cdot C_i &= 1, \\ C_i \cdot C_{i+1} &= -1. \end{aligned}$$

In homology, the Dehn twist corresponding to β_k is given by $a \to a + (a \cdot C_k)\, C_k$. Thus the matrix representing this Dehn twist on $H_1(\Sigma''_{m-1}; \mathbf{Z})$ is $D_{2m,k} = I_{2m-1} + J_{2m-1,k}$ where I_{2m-1} is the identity matrix of rank $2m-1$, and $J_{2m-1,k}$ is the $(2m-1) \times (2m-1)$ matrix whose entries are all 0 except for $(J_{2m-1,k})_{k,k-1} = 1$ and $(J_{2m-1,k})_{k,k+1} = -1$.

Denote the matrix representing the homology monodromy of $L_{2m,k}$ by $\Omega_{2m,k}$. For example, $B_{4,0} = (\bar{2}.\bar{2}.1.\bar{2}).3.(2.2.2.\bar{1}.2).\bar{3}$; so

$$\Omega_{4,0} = D_{4,2}^{-1} \cdot D_{4,2}^{-1} \cdot D_{4,1} \cdot D_{4,2}^{-1} \cdot D_{4,3} \cdot D_{4,2} \cdot D_{4,2} \cdot D_{4,2} \cdot D_{4,1}^{-1} \cdot D_{4,2} \cdot D_{4,3}^{-1}.$$

This matrix is

$$\Omega_{4,0} = \begin{pmatrix} -10 & -17 & 11 \\ 46 & 73 & -46 \\ 7 & 10 & -6 \end{pmatrix}.$$

In order to save notation we shall denote by Φ_n, $\Phi_{n,k}$, and Ψ_n the rank $n-1$ square matrices corresponding to the product of Dehn twists resulting from the braid group elements with the same name. (For these purely combinatorial expressions, there is no need to assume that n is even.) Similarly, we let $\Gamma_{n,r}$ be the rank $n-1$ square matrix corresponding to Γ_r. An easy inductive argument gives:

Lemma 3.1. *For any integer k, the matrix power $\Gamma_{n,n-2}^k$ is given by:*

$$\begin{pmatrix} & & & 2k & 0 \\ & & & 0 & 0 \\ & & & 2k & 0 \\ & & & 0 & 0 \\ & I_{n-3} & & \vdots & \vdots \\ & & & 2k & 0 \\ & & & 0 & 0 \\ & & & 2k & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} k \textit{ even},$$

$$\begin{pmatrix} & & & -2k & 2 \\ & & & 0 & 0 \\ & & & -2k & 2 \\ & & & 0 & 0 \\ & -I_{n-3} & & \vdots & \vdots \\ & & & -2k & 2 \\ & & & 0 & 0 \\ & & & -2k & 2 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} k \textit{ odd}.$$

Recall that the matrices $\Phi_{n,k}$ are recursively defined by the formulas

$$\begin{aligned} \Phi_{n,k} &= \Gamma_{n,n-2}^{-k}.\Phi_n.\Gamma_{n,n-2}^{k}, \\ \Phi_{n+1} &= (n-2).(n-1).\Phi_n.(\overline{n-1}). \end{aligned}$$

Thus

$$\begin{aligned} \Phi_{2m,k} &= \Gamma_{2m,2m-2}^{-k}.\Phi_{2m}.\Gamma_{2m,2m-2}^{k}, \\ \Phi_{2m+2} &= (2m-1).(2m).(2m-2).(2m-1).\Phi_{2m}.(\overline{2m-1}).(\overline{2m}). \end{aligned}$$

Since we have $\Phi_4 = \begin{pmatrix} 2 & -1 & -1 \\ -5 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, we obtain the following closed formulas for $\Phi_{2m,k}$ (and consequently for $\Phi_{2m} = \Phi_{2m,0}$):

Lemma 3.2. *For $m \geq 3$, the matrices $\Phi_{2m,k}$ are given by*

$$\begin{pmatrix}
10k+2 & 6k & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 20k^2-8k-1 & -10k-1 \\
2 & 1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2k-11 & -1 \\
10k+2 & 6k+1 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 20k^2-8k-1 & -10k-1 \\
2 & 1 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 2k-11 & -1 \\
10k+2 & 6k+1 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 20k^2-8k-1 & -10k-1 \\
2 & 1 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 2k-11 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\
10k+2 & 6k+1 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 20k^2-8k-1 & -10k-1 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2k-11 & -1 \\
10k+2 & 6k+1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 20k^2-8k-2 & -10k-1 \\
-5 & -3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -10k+6 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Also $\Psi_{2m} = (2m-2).(2m-3).\Psi_{2m-2}.(\overline{2m-3}).(\overline{2m-2})$ and $\Psi_4 = \begin{pmatrix} 2 & 1 & -1 \\ 7 & 4 & -7 \\ 0 & 0 & 1 \end{pmatrix}$ so we similarly obtain:

Lemma 3.3. *For $m \geq 3$, the matrices Ψ_{2m} are given by*

$$\begin{pmatrix}
2 & 0 & 0 & \cdots & 0 & 1 & -1 \\
7 & -3 & 0 & \cdots & 0 & 7 & -7 \\
-7 & 4 & \ulcorner & & \urcorner & -7 & 7 \\
7 & -4 & & & & 7 & -7 \\
-7 & 4 & & I_{2m-5} & & -7 & 7 \\
\vdots & \vdots & & & & \vdots & \vdots \\
7 & -4 & & & & 7 & -7 \\
-7 & 4 & \llcorner & & \lrcorner & -7 & 7 \\
7 & -4 & 0 & \cdots & 0 & 8 & -7 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}.$$

Finally, since

$$\Omega_{2m,k} = \Gamma_{2m-2}^{-k}.\Phi_{2m}.\Gamma_{2m-2}^{k}.(2m-1).\Psi_{2m}.(\overline{2m-1})$$

we obtain an expression for the monodromy.

Proposition 3.4. *For $m \geq 3$, $\Omega_{2m,k}$ is given by*

$$\begin{pmatrix} a(k) & b(k) & 0 & \cdots & 0 & c(k) & -a(k)+1 \\ 14k+4 & -8k+1 & \ulcorner & & \urcorner & 30k+3 & -14k-3 \\ a(k) & b(k)+1 & & & & c(k) & -a(k)+1 \\ 14k+4 & -8k+1 & & -I_{2m-5} & & 30k+3 & -14k-3 \\ \vdots & \vdots & & & & \vdots & \vdots \\ a(k) & b(k)+1 & & & & c(k) & -a(k)+1 \\ 14k+4 & -8k+1 & \llcorner & & \lrcorner & 30k+3 & -14k-3 \\ a(k) & b(k)+1 & 0 & \cdots & 0 & c(k)+1 & -a(k)+1 \\ 46-70k & -35+40k & 0 & \cdots & 0 & -150k+108 & 70k-46 \\ 7 & -4 & 0 & \cdots & 0 & 14 & -16 \end{pmatrix}$$

where $a(k) = 140k^2 - 64k - 10$, $b(k) = -80k^2 + 54k + 8$, *and* $c(k) = 300k^2 - 156k - 25$.

The Alexander polynomial of the 2-component link $L_{2m,k}$ is a function $\Delta_{L_{2m,k}}(t_1, t_2)$ of 2 variables. The reduced Alexander polynomial is the single variable polynomial defined by $\bar{\Delta}_{L_{2m,k}}(t) = \Delta_{L_{2m,k}}(t,t)$. For a fibered link L with homology monodromy μ whose characteristic polynomial is $p_\mu(t)$ one has

$$\bar{\Delta}_L(t) \cdot (t-1) = p_\mu(t)$$

(see [10]). For our construction it will suffice to compute the reduced Alexander polynomials of the links $L_{2m,k}$. Again this will be an inductive calculation relying on the explicit form for $\Omega_{2m,k}$ given by Proposition 3.4.

Theorem 3.5. *The reduced Alexander polynomials $\bar{\Delta}_{L_{2m,k}}(t)$ for the 2-fold covering links $L_{2m,k}$ are given by:*

(a) for $m = 2$, $\bar{\Delta}_{L_{4,k}}(t) = t^2 - (140k^2 - 174k + 56)t + 1$,

(b) for $m \geq 3$,

$$\begin{aligned} \bar{\Delta}_{L_{2m,k}}(t) =& (t^{2m-2}+1) - (140k^2 - 222k + 92)(t^{2m-3} + t) \\ &+ (136k^2 - 258k + 119) \sum_{j=1}^{m-2} t^{2j} \\ &- (140k^2 - 270k + 128) \sum_{j=1}^{m-3} t^{2j+1}. \end{aligned}$$

Proof. Let $p_{m,k} = \det(\Omega_{2m,k} - tI)$, the characteristic polynomial of $\Omega_{2m,k}$. First, for $m \geq 4$, we give recursive formulas which reduce the calculation of $p_{m,k}$ to that of $p_{8,k}$. According to Lemma 3.4, $\Omega_{2m,k} - tI$ is a rank $2m-1$ matrix of the form

$$\Omega_{2m,k} - tI = \begin{pmatrix} a-t & b & 0 & 0 & \cdots & 0 & c & d \\ x & y-t & -1 & 0 & \cdots & 0 & z & w \\ a & b+1 & -t & -1 & & 0 & c & d \\ x & y & 0 & -t & & 0 & z & w \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ a & b+1 & 0 & 0 & & 0 & c & d \\ x & y & 0 & 0 & & -1 & z & w \\ a & b+1 & 0 & 0 & & -t & c+1 & d \\ \alpha & \gamma & 0 & 0 & \cdots & 0 & \varepsilon - t & \vartheta \\ \beta & \delta & 0 & 0 & \cdots & 0 & \zeta & \kappa - t \end{pmatrix} \tag{1}$$

where we have left out the dependence on k. Expand its determinant by the third column to obtain

$$p_{m,k} = \det(U_{m,k}) - t\det(V_{m,k}). \tag{2}$$

Expanding $\det(V_{m,k})$ twice, each time by the third column, we obtain

$$\det(V_{m,k}) = t^2 \det(V_{m-1,k}). \tag{3}$$

Similarly, always expanding by the third column gives

$$\begin{aligned} \det(U_{m,k}) &= \det(Q_{m,k}) - t\det(R_{m,k}), && (4)\\ \det(Q_{m,k}) &= \det(U_{m-1,k}) - t\det(S_{m,k}), && (5)\\ \det(S_{m,k}) &= t^2\det(S_{m-1,k}), && (6)\\ \det(R_{m,k}) &= t^2\det(R_{m-1,k}). && (7) \end{aligned}$$

Equations (2) – (7) reduce the problem of calculating $p_{m,k}$, $m \geq 4$, to the calculation of the quantities in these equations for $m = 4$, and this is accomplished directly from equation (1). Similarly, $p_{6,k}$ is calculated from (1) and $p_{4,k}$ is easy to calculate as well. The theorem follows by dividing $p_{m,k}$ by $t-1$. q.e.d.

Corollary 3.6. *The two components of the links $L_{2m,k}$ have nonzero algebraic linking number.*

Proof. The algebraic linking number of the 2 components of $L_{2m,k}$ is $\bar{\Delta}_{L_{2m,k}}(1)$, [10]. This is easily calculated from Theorem 3.5:

$$-\bar{\Delta}_{L_{2m,k}}(1) = (4m + 132)k^2 - (12m + 150)k + (9m + 36),$$

and the lemma follows simply from this. q.e.d.

4. Some background on link surgery and Seiberg-Witten invariants

The Seiberg-Witten invariant of a smooth closed oriented 4-manifold X with $b_2^+(X) > 1$ is an integer-valued function which is defined on the set of *spin*^{c} structures over X (cf. [12]). In case $H_1(X; \mathbf{Z})$ has no 2-torsion there is a natural identification of the *spin*^{c} structures of X with the characteristic elements of $H_2(X; \mathbf{Z})$ (i.e., those elements k whose Poincaré duals $\hat{k}$ reduce mod 2 to $w_2(X)$). In this case we view the Seiberg-Witten invariant as

$$\mathrm{SW}_X : \{k \in H_2(X; \mathbf{Z}) | \hat{k} \equiv w_2(TX) \pmod 2)\} \to \mathbf{Z}.$$

The sign of SW_X depends on an orientation of

$$H^0(X; \mathbf{R}) \otimes \det H^2_+(X; \mathbf{R}) \otimes \det H^1(X; \mathbf{R}).$$

If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$ where $\mathrm{e}(X)$ is the Euler number, and $\mathrm{sign}(X)$ is the signature of X.

Now let $\{\pm\beta_1, \ldots, \pm\beta_n\}$ be the set of nonzero basic classes for X. Consider variables $t_\beta = \exp(\beta)$ for each $\beta \in H^2(X; \mathbf{Z})$, which satisfy the relations $t_{\alpha+\beta} = t_\alpha t_\beta$. We may then view the Seiberg-Witten invariant of X as the Laurent polynomial

$$\mathcal{SW}_X = \mathrm{SW}_X(0) + \sum_{j=1}^{n} \mathrm{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_{\beta_j}^{-1}).$$

We next recall the link surgery construction of [5]. This construction starts with an oriented n-component link $L = \{K_1, \ldots, K_n\}$ in S^3 and n pairs (X_i, T_i) of smoothly embedded self-intersection 0 tori in simply connected 4-manifolds. The tori are assumed to be c-embedded, that is, each torus T_i is homologically essential and has a pair of embedded curves which generate its first homology and bound vanishing cycles (disks of self-intersection -1) in X_i. For example, T is c-embedded if it has a neighborhood $N \subset X$ such that the pair (N, T) is diffeomorphic to (N_C, F), where N_C is a neighborhood of a cusp fiber in an elliptic surface and F is a smooth elliptic fiber in N_F. Let $N(K_i)$ be disjoint tubular neighborhoods of the components K_i of L in S^3 and $N(L) = \cup N(K_i)$.

Let $\alpha_L : \pi_1(S^3 \setminus L) \to \mathbf{Z}$ denote the homomorphism characterized by the property that it sends the meridian m_i of each component K_i to 1, and let ℓ_i denote the longitude of K_i. The curves $\gamma_i = \ell_i + \alpha_L(\ell_i) m_i$ on $\partial N(K_i)$ form the boundary of a Seifert surface for the link, and in case L is a fibered 2-component link, the γ_i are given by the boundary components of a fiber.

In $S^1 \times (S^3 \setminus N(L))$ let $T_{m_i} = S^1 \times m_i$, and define the 4-manifold $X(X_1, \dots X_n; L)$ by

$$X(X_1, \dots X_n; L) = (S^1 \times (S^3 \setminus N(L)) \cup \bigcup_{i=1}^{n} (X_i \setminus (T_i \times D^2)),$$

where $S^1 \times \partial N(K_i)$ is identified with $\partial N(T_i)$ so that for each i:

$$[T_{m_i}] = [T_i], \quad \text{and} \quad [\gamma_i] = [\mathrm{pt} \times \partial D^2].$$

We have the following calculation of its Seiberg-Witten invariant:

Theorem 4.1 ([5]). *If each T_i is c-embedded in X_i and each $\pi_1(X \backslash T_i) = 1$, then $X(X_1, \dots X_n; L)$ is simply connected and its Seiberg-Witten invariant is*

$$\mathcal{SW}_{X(X_1,\dots X_n;L)} = \Delta_L^{sym}(t_1, \dots, t_n) \cdot \prod_{j=1}^{n} \mathcal{SW}_{X_j} \cdot (t_j^{1/2} - t_j^{-1/2}),$$

where $t_j = \exp(2[T_j])$, and $\Delta_L^{sym}(t_1, \dots, t_n)$ is the symmetric multivariable Alexander polynomial.

In case each $(X_i, T_i) \cong (X, T)$, a fixed pair, we write

$$X(X_1, \dots X_n; L) = X_L.$$

(We implicitly remember T, but it is removed from the notation.) As an example, consider the case where each $X_i = E(1)$, the rational elliptic surface ($E(1) \cong \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$) and each $T_i = F$ is a smooth elliptic fiber. Since $SW_{E(1)} = (t^{1/2} - t^{-1/2})^{-1}$, we have that

$$\mathcal{SW}_{E(1)_L} = \Delta_L^{sym}(t_1, \dots, t_n). \tag{8}$$

5. Symplectic submanifolds

Let T be a c-embedded symplectic torus in the simply connected symplectic 4-manifold (X, ω). Then T has a tubular neighborhood which may be identified with $N = S^1 \times S^1 \times D^2$, and $T = S^1 \times S^1 \times \{0\}$. The symplectic tubular neighborhood theorem implies that the restriction of ω to this neighborhood is equivalent to the symplectic form $dx \wedge dy + rdr \wedge d\theta$. Let B be a closed $2m$-strand braid contained in an unknotted solid torus in S^3 with axis A. Define T_B to be the torus $T_B = S^1 \times B \subset N$. Then T_B represents the homology class $2m\,[T]$. Furthermore, T_B is a symplectic submanifold of X because its tangent space at each point is spanned by $\partial/\partial x$ and the tangent vector w along the curve B, and w always has a nontrivial $(\partial/\partial y)$-component,

$$(dx \wedge dy + rdr \wedge d\theta)\,(\partial/\partial x, w) = dx \wedge dy\,(\partial/\partial x, w) \neq 0.$$

Given $m \geq 2$, consider our family of braids, $B_{2m,k}$ of § 2. Our examples are the symplectic tori $T_{B_{2m,k}}$. Let us fix $m \geq 2$ and denote $B_{2m,k}$ by B_k and $T_{B_{2m,k}}$ by Σ_k.

It is plausible that one might be able to distinguish the isotopy classes of the tori Σ_k by means of the fundamental groups of their complements. However, in our situation, where T is c-embedded, the following lemma points out that one has to work harder.

Lemma 5.1. *If T is c-embedded, then the complement of Σ_k satisfies $H_1(X \setminus \Sigma_k; \mathbf{Z}) = \mathbf{Z}_{2m}$ and $\pi_1(X \setminus \Sigma_k)$ is independent of k. If also $\pi_1(X \setminus T) = 1$ then $\pi_1(X \setminus \Sigma_k) = \mathbf{Z}_{2m}$.*

Proof. The fundamental group of the complement of T is normally generated by the boundary μ_T of a normal disk to T. The fundamental group of $X \setminus \Sigma_k$ is an amalgamated free product,

$$\pi_1(X \setminus \Sigma_k) = \pi_1(X \setminus T) *_{\pi_1(\partial(T \times D^2))} \pi_1((T \times D^2) \setminus \Sigma_k) \tag{9}$$

and $(T \times D^2) \setminus \Sigma_k$ is the product of a circle with the fiber bundle $S^1 \times_\varphi \Delta$ where Δ is a 2-disk with $2m$ punctures. (Of course, φ depends on k.) Thus

$$\begin{aligned} &\pi_1((T \times D^2) \setminus \Sigma_k) \\ &\quad = \langle \mu_1, \dots, \mu_{2m}, s, t \mid [s, \mu_i] = 1, [s, t] = 1, t\mu_i t^{-1} = \varphi(\mu_i) \rangle. \end{aligned} \tag{10}$$

Since each braid B_k is connected, the action of the monodromy φ is transitive on the μ_i. Also, s and t both lie in the image of $\pi_1(\partial(T \times D^2))$.

If T is c-embedded, then it has a cusp neighborhood N, containing both vanishing cycles, so that the inclusion induces the trivial map $\pi_1(T) \to \pi_1(\partial N)$. Thus s and t are both trivial in the amalgamated free product (9). It follows from (10) that all the μ_i are equal in $\pi_1(X \setminus \Sigma_k)$, and $\mu_T = \mu_1^{2m}$. The lemma now follows directly from (9).

q.e.d.

(Note that the most obvious examples, such as $(X,T) = (E(n), F)$, where $E(n)$ is the elliptic surface over $\mathbf{CP}^1$ without multiple fibers and with holomorphic Euler number n and F is a smooth fiber, have $\pi_1(X \setminus T) = 1$.) We shall show that the symplectic tori Σ_k are not smoothly isotopic in X by considering the double branched covers $\pi_k : \tilde{X}_k \to X$ branched over the Σ_k.

Let $L_k = \pi^{-1}(A)$ be the double branched covering link. This is the link that was denoted $L_{2m,k}$ in § 3. We may write

$$\tilde{X}_k = (X \setminus N) \cup S^1 \times (S^3 \setminus L_k) \cup (X \setminus N), \tag{11}$$

since the double cover is trivial over $X \setminus N$ and

$$(D^2 \times S^1, B_k) = (S^3 \setminus A, B_k).$$

In the branched cover (11), the pieces are glued together so that the boundary circles of the fiber $\Sigma_{m-1} \setminus (D^2 \cup D^2)$ of $S^3 \setminus L_k$ are glued to the boundaries ∂D^2 of $X \setminus N = X \setminus (D^2 \times T)$. Thus $\tilde{X}_k$ is the manifold X_{L_k} of § 4. It follows that:

$$\begin{aligned} \mathcal{SW}_{\tilde{X}_k} = &\Delta^{sym}_{L_k}(t_1, t_2) \cdot \mathcal{SW}_{X_1} \cdot (t_1^{1/2} - t_1^{-1/2}) \\ &\cdot \mathcal{SW}_{X_2} \cdot (t_2^{1/2} - t_2^{-1/2}), \end{aligned} \tag{12}$$

where T_j is a copy of T in the jth ($j = 1, 2$) copy X_j of X, and $t_j = \exp(2[T_j])$.

Assume that there is an isotopy in X which takes Σ_i to Σ_j. This isotopy gives rise to a diffeomorphism $h : X \to X$ satisfying $h(\Sigma_i) = \Sigma_j$ and $h_* = \mathrm{id}$ on homology. There is a lift to a diffeomorphism $\tilde{h} : \tilde{X}_i \to \tilde{X}_j$ of double branched covering spaces. For a fixed homology class $\beta \in H_2(X, \mathbf{Z})$ consider all the basic classes $\tilde{\beta}$ of $\tilde{X}_i$ satisfying $\pi_{i*}(\tilde{\beta}) = \beta$. For any such class, it is also true that ${\pi_j}_* \tilde{h}_*(\tilde{\beta}) = \beta$ since $h_* = \mathrm{id}$. The invariance of the Seiberg-Witten invariant under diffeomorphisms also implies that $\mathrm{SW}_{\tilde{X}_j}(\tilde{h}_*(\tilde{\beta})) = \mathrm{SW}_{\tilde{X}_i}(\tilde{\beta})$. Thus for a fixed $\beta \in H_2(X; \mathbf{Z})$,

$$\sum_{\pi_{i*}(\tilde{\beta}) = \beta} \mathrm{SW}_{\tilde{X}_i}(\tilde{\beta}) = \sum_{{\pi_j}_*(\tilde{\gamma}) = \beta} \mathrm{SW}_{\tilde{X}_j}(\tilde{\gamma}).$$

This equation implies that the Seiberg-Witten invariants of $\tilde{X}_i$ and $\tilde{X}_j$ become equal after applying the projections π_{i*} and π_{j*}. Equivalently, working with the Laurent polynomials as in equation (12), we get

$$\begin{aligned}\bar{\Delta}^{sym}_{L_i}(t)\cdot(\mathcal{SW}_X)^2\cdot(t^{1/2}-t^{-1/2})^2 \\ = \bar{\Delta}^{sym}_{L_j}(t)\cdot(\mathcal{SW}_X)^2\cdot(t^{1/2}-t^{-1/2})^2,\end{aligned} \tag{13}$$

where $t=\exp(2[T])$.

Theorem 5.2. *Let X be a symplectic 4-manifold which contains a c-embedded symplectic torus T. Then each homology class $2m\,[T]$, $m\geq 2$, contains the infinite family $\{T_{B_{2m,k}}\}$ of symplectic tori, no two of which are smoothly isotopic.*

Proof. Fix m and consider the double branched covers $\tilde{X}_k$ of $(X, T_{B_{2m,k}})$. If $T_{B_{2m,i}}$ is smoothly isotopic to $T_{B_{2m,j}}$, then equation (13) follows. However, since X is symplectic, $\mathcal{SW}_X\neq 0$ [11], and it follows from Theorem 3.5 that the $\bar{\Delta}^{sym}_{L_k}(t)$ are all distinct for different k. Thus equation (13) can hold only if $i=j$. q.e.d.

Notice that since we are unable to compute the 2-variable Alexander polynomials for the links L_i, this proof, in itself, does not show that the covers $\tilde{X}_k$ are mutually nondiffeomorphic — only that they can not be made diffeomorphic via a $\mathbf{Z}_2$-equivariant diffeomorphism which covers the identity on $H_2(X;\mathbf{Z})$. Thus, without further information about the Alexander polynomials of the links, we are unable to show via this technique that there is no diffeomorphism of X which throws $T_{B_{2m,i}}$ onto $T_{B_{2m,j}}$ for $i\neq j$. However for the case $X=E(1)$, we get a stronger result.

Theorem 5.3. *Let T denote a smooth elliptic fiber in the rational elliptic surface, $E(1)$. Then each homology class $2m\,[T]$, $m\geq 2$, contains the infinite family $\{T_{B_{2m,k}}\}$ of symplectic tori, no two of which are equivalent under diffeomorphisms of $E(1)$.*

Proof. In this case the double branched covers $\tilde{X}_k$ cannot be diffeomorphic for different k. For, it follows from equations (12) and (8) that $\mathcal{SW}_{\tilde{X}_k}=\Delta^{sym}_{L_k}(t_1,t_2)$. Thus $\Delta^{sym}_{L_k}(1,1)$ is a diffeomorphism invariant of $\tilde{X}_k$, the sum of all its Seiberg-Witten invariants. The calculation of Corollary 3.6 shows that these numbers are different for different k.

q.e.d.

6. Lefschetz fibrations

In this section, we show how our constructions above naturally yield examples of infinite classes of homeomorphic but nondiffeomorphic symplectic manifolds, all of which admit Lefschetz fibrations of fixed fiber genus. (There is a more general construction presented in [7].) For simplicity, we restrict ourselves with the application of this procedure to the rational elliptic surface $E(1)$. Let $T = F$, a generic elliptic fiber in $E(1)$ and let $\tilde{X}_{2m,k}$ be the double branched cover of $E(1)$ with branch set $T_{B_{2m,k}}$. Then $\tilde{X}_{2m,k} \cong E(1)_{L_{2m,k}}$ is a homotopy $K3$ surface.

It is well-known that $E(1)$ admits a genus 0 fibration with 4 singular fibers. This is seen by noting that $E(1)$ is the double branched cover of $S^2 \times S^2$ with branch set equal to 4 disjoint copies of $S^2 \times \{\mathrm{pt}\}$ together with 2 disjoint copies of $\{\mathrm{pt}\} \times S^2$. The resultant branched cover has 8 singular points (corresponding to the double points in the branch set), whose neighborhoods are cones on $\mathbf{RP}^3$. These are desingularized in the usual way, replacing these neighborhoods with cotangent bundles of S^2. The result is $E(1)$. The horizontal and vertical fibrations of $S^2 \times S^2$ pull back to give fibrations of $E(1)$ over $\mathbf{CP}^1$. A generic fiber of the vertical fibration is the double cover of S^2, branched over 4 points — this gives an elliptic fibration on $E(1)$. The generic fiber of the horizontal fibration is the double cover of S^2, branched over 2 points — this gives the genus 0 fibration of $E(1)$. The 4 singular fibers are the preimages of the four $S^2 \times \{\mathrm{pt}\}$'s in the branch set. The generic fiber T of the elliptic fibration meets a generic fiber Σ_0 of the horizontal fibration in 2 points, $\Sigma_0 \cdot T = 2$. This means that $T_{B_{2m,k}}$ meets Σ_0 transversely in $\Sigma_0 \cdot T_{B_{2m,k}} = 4m$ points. Therefore the horizontal fibration on $S^2 \times S^2$ lifts to a fibration on $\tilde{X}_{2m,k}$ whose generic fiber is the double cover of S^2 branched over $4m$ points, that is, a genus $2m-1$ fibration. The definition of a Lefschetz fibration requires the monodromy around each singular fiber to be a Dehn twist. This is not true for these examples, but they can be perturbed to be Lefschetz (see [9]).

One can give an alternative description of this fibration on $\tilde{X}_{2m,k} \cong E(1)_{L_{2m,k}}$. The elliptic fiber T of $E(1)$ meets each genus 0 fiber transversely, and it meets a generic genus 0 fiber Σ_0 twice. Thus $\Sigma_0 \setminus (\Sigma_0 \cap T)$ is an annulus. The construction of

$$E(1)_{L_{2m,k}} = \{E(1) \setminus N(T)\} \cup \{S^1 \times M_{L_{2m,k}} \setminus (N(T_1) \\ \cup N(T_2))\} \cup \{E(1) \setminus N(T)\}$$

preserves the fibrations. In the manifold (with boundary)

$$\{E(1) \setminus N(T)\} \cup \{S^1 \times M_{L_{2m,k}} \setminus (N(T_1) \cup N(T_2))\}$$

a generic fiber of the induced fibration is the union of 2 fibers of the fibration

$$\begin{array}{c} \Sigma''_{m-1} \to S^1 \times M_{L_{2m,k}} \setminus (N(T_1) \cup N(T_2)) = S^1 \times (S^3 \setminus L_{2m,k}) \\ \downarrow \\ S^1 \times S^1 \end{array}$$

together with the annulus $\Sigma_0 \setminus (\Sigma_0 \cap T)$. This is a surface of genus $2m-2$ with 2 boundary components. Adding the second copy of $E(1)$ adds an annulus which closes up the surface, and we obtain a surface of genus $2m-1$.

It is not difficult to see that the fibrations on $\tilde{X}_{2m,k}$ which are described here are actually hyperelliptic, that is, the hyperelliptic involutions on the fibers extend to a global involution of $\tilde{X}_{2m,k}$. In fact, the orbit space of this hyperelliptic involution is $E(1)$, and the image of the fixed point set is just the torus $T_{B_{2m,k}}$.

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