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by

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Synchronization analysis of linearly coupled systems described by differential equations with a coupling delay *

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Abstract

When a transmission delay occurs in the interconnection of linearly coupled systems described by ordinary differential equations (LCODEs), both synchronization and the final synchronized state will vary. In this paper, mathematical analysis is presented on the synchronization phenomena of LCODEs with a single coupling delay. Criteria are derived for both local and global synchronization. It is known that addition to the dynamical behaviors of the underlying uncoupled system and the coupling configuration, the coupling strength and the coupling delay also play key roles on the stability of synchronization. Both theoretical and numerical analysis indicate that under some conditions, if the coupling strength is large enough, the coupled system can be completely synchronized for any coupling delays. On the other hand, in some case, the coupled system can be synchronized if the coupling delay is small enough.

Key words: Synchronization space, Transverse space, Coupling delay, Local

synchronization, Global synchronization

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1 Introduction and model description

Models of complex networks have been widely used to describe systems in science, engineering, and nature. As an implicit assumption, these systems are regarded as a set of interconnected individuals, in which a node is a fundamental unit with specific contents. For each node, its behavior is determined by two factors: its own characteristics and the influence of other nodes interconnecting it. Typical examples of complex networks include, the Internet, World Wide Web (WWW), food webs, cellular and metabolic networks, etc [1–3].

Linearly coupled ordinary differential equations (LCODEs) provide a large class of models that can be used to describe coupled systems with continuous time and state values, as well as discrete spatial states. This class of dynamical systems have been intensively investigated as theoretical models of spatiotemporal phenomena in complex networks [4,5]. The dynamical behavior of a network is governed by the following two mechanisms: the intrinsic nonlinear dynamics of each node and the diffusion due to the spatial coupling among nodes. The LCODEs can be described as follows:

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + c\sum_{j=1,j\neq i}^{m} a_{ij}\Gamma\Big[x^j(t) - x^i(t)\Big],\ i = 1,\cdots,m, \tag{1}$$

where $x^i(t) \in \mathbb{R}^n$ denotes the state variable vector of the i-th node, $i = 1,\cdots,m$, $f(\cdot)$:

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$\mathbb{R}^n \rightarrow \mathbb{R}^n$ is a differentiable function denoting the intrinsic dynamics of the uncoupled system at each node, the scalar c is the coupling strength, $a_{ij} \geq 0$ denotes the coupling coefficient from node j to node i for $i, j = 1, \cdots, m$, $i \neq j$, and the n-dimensional diagonal matrix $\Gamma = diag\{\gamma_1, \cdots, \gamma_n\}$ denotes the inner connection at each node with $\gamma_i \geq 0$, for all $i = 1, \cdots, n$ (see [3,6,7]). It means that two nodes are connected by their i-th component where $\gamma_i > 0$.

Many complicated dynamical behaviors of coupled oscillators have been studied [8], where the synchronization phenomenon has been a focal topic for research [9]. In mathematics, there are various concepts of synchronization, for example, phase synchronization, lag synchronization, partial synchronization, generalized synchronization (see [9,10]). In this paper, we only consider complete synchronization, defined as follows: if $\lim_{t\to\infty} |x^j(t) - x^i(t)| = 0$ holds for all $i, j = 1, \cdots, m$, where $| \cdot |$ denotes some norm, then the coupled system is said to be completely synchronized. For simplicity, this is referred to synchronization in this paper.

Recently, synchronization of coupled dynamical networks has attracted increasing attention in various research fields. In [11–13], the authors presented the master stability function based on the transverse Lyapunov exponents to study local synchronization. In [14–16], the authors investigated global synchronization of coupled nonlinear dynamical systems by introducing a distance to synchronization with some structural matrix. In [17], the authors presented an approach to define Lyapunov function by the distance from $[x^1, x^2, \cdots, x^m]$ to its projection on the synchronization space. Synchronization of randomly connected complex networks such as small-world and scale-free networks were studied in [6,7].

Due to the finiteness of signal transmission and switching speeds, coupling delay in a real network is inevitable [18,19]. Synchronization of two chaotic systems in the master-slaver configuration with coupling delays has been widely studied in the last two decades [20–25]. In [26], the authors studied synchronization of three bidirectionally globally coupled hyperchaotic systems. In [27,28], the authors extended the master stability function methodology to investigate synchronization of the following coupled oscillators with coupling delays:

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + c\sum_{j=1}^{m} a_{ij}H(x^j(t-\tau)). \tag{2}$$

where $H(\cdot)$ can be some coupling function and τ denotes the coupling delay. Moreover, in [27], the authors revealed the phenomenon that delay can enhance synchronization with a small coupling strength.

With a coupling delay, the synchronization must be considered as a problem no longer in a finite dimensional space. Instead, it should be studied in the infinite dimensional Banach space $C([-\tau, 0], \mathbb{R}^n)$. Attracting invariant manifold of delayed differential equations has been studied in [29–31]. The authors proved that the differential equations with small delays have Lipschitz inertial smooth manifolds under some mild conditions. General references for invariant manifold of delayed differential equations can be referred to [32,33].

As formulated in [26], in this paper, we consider the case that a coupling delay occurs when the signals from the each nodes are transmitted to its interconnected nodes. In this case, the LCODEs with a coupling delay are described as follows:

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + c\sum_{j=1, j\neq i}^{m} a_{ij}\Gamma\Big[x^j(t-\tau) - x^i(t)\Big]. \tag{3}$$

We define the set $\mathbf{S} = \Big\{x = [x^{1^\top}(\theta), \cdots,$ $x^{m^\top}(\theta)]^\top : x^i(\theta) \in C_\tau, x^i(\theta) = x^j(\theta),\ i, j = 1, 2\ldots,\ and\ \theta \in [-\tau, 0]\Big\}$ as the **synchronization space** for the LCODEs with a coupling delay τ, where $\top$ denotes the matrix transpose and C_τ denotes the Banach space $C([-\tau, 0], \mathbb{R}^n)$.

In case that the LCODEs reach synchronization, i.e., $x^1(t) = x^2(t) = \cdots = x^m(t) = s(t)$, we have the following synchronized state equation:

$$\frac{ds(t)}{dt} = f(s(t)) + ca_i\Gamma\Big[s(t-\tau) - s(t)\Big], \quad i = 1, \cdots, m \tag{4}$$

where $a_i = \sum\limits_{j=1,j\neq i}^{m} a_{ij}$. Obviously, the synchronized state $s(t)$ is uniform, i.e., the synchronization space $\mathbf{S}$ is invariant for the coupled system (3), if and only if $a_1 = a_2 = \cdots = a_m$. Therefore, to realize complete synchronization of the coupled system (3), the assumption $a_1 = a_2 = \cdots = a_m$ must be imposed.

By these explanations, in the sequel, we study the following coupled system:

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + c \sum_{j=1,j\neq i}^{m} a_{ij}\Gamma\Big[x^j(t-\tau) - x^i(t)\Big], \quad i = 1, \cdots, m \tag{5}$$

where $x^i(t) = [x_1^i(t), \cdots, x_n^i(t)]^\top \in \mathbb{R}^n$ denotes the n-dimensional state variable of the i-th node, $i = 1, \cdots, m$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a differential function of the intrinsic system, c is the coupling strength, $\Gamma = diag\{\gamma_1, \cdots, \gamma_n\}$ is the inner connection diagonal matrix with $\gamma_i \geq 0, t = 1, \cdots, m, a_{ij} \geq 0$, for all $i, j = 1, \cdots, m, i \neq j$, is the coupling coefficient from node j to node i, and $\tau \geq 0$ is the coupling delay. We assume that $\sum\limits_{j=1,j\neq i}^{m} a_{ij} = 1$, for all $i = 1, \cdots, m$ and put the uniform row sum a_i into the coupling strength c. We should point out here why parameters c and Γ are necessary. First, since on each node, the dynamical system is also high dimensional, $\gamma_j > 0$ indicates that each node is connected to its neighbors by the i-th component. Second, we separate the row sum aside a_{ij} to emphasize the sense of the coupling strength c. As the coupling matrix $A = (a_{ij})$ denotes the coupling configuration of the dynamical network, c denotes the coupling strength which can be adjusted out of the structure of the network. Thus, c has important engineering and physical sense if we consider the control and physical problem of synchronization [6,11].

Let $a_{ii} = -1$ for all $i = 1, \cdots, m$ and we have the following equivalent form of the coupled system (5):

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + c\sum_{j=1}^{m} a_{ij}x^j(t-\tau) + c\Big[x^i(t-\tau) - x^i(t)\Big], \quad i = 1, \cdots, m \tag{6}$$

Without assuming symmetry and irreducibility for the coupling configuration, we study the synchronization problem of the LCODE systems (6) with a coupling delay. In the synchronization space $\mathbf{S}$, we define a suitable manifold, which is different from the intrinsic invariant manifold studied in [29–31]. We extend the master stability function methodology to infinite dimensional Banach space. The corresponding transverse Lyapunov spectra can be utilized to analyze local synchronization. Also, we derive qualitative criteria guaranteeing local synchronization. From these criteria, one can see that a large coupling strength and a small coupling delay may often imply synchronization easily. However, it is not always so since the dynamical behaviors of the synchronized state also depend on the parameters c and τ.

Furthermore, we extend the methodology introduced in [17] to the delayed coupled systems. With the left eigenvector of the coupling matrix associated with eigenvalue 0, we define a projection of the spatial states on the synchronization space and some special functionals of the difference between the spatial states and its projection. These functionals can be regarded as potentials of the spatial states to the synchronization space. If this functional is a Lyapunov functional, then we can obtain global synchronization. Based on these ideas, for some class of functions $f(\cdot)$, we present criteria guaranteeing global synchronization. Some criteria are independent of the coupling delay τ. That is, under some conditions, the coupled system can be globally synchronized no matter how large the coupling delay is. Other criteria depend on τ. That is, for some smaller c, the coupled system can be globally synchronized for a small enough τ. These two phenomena can also be observed by numerical examples.

We organize this paper in the following way. In section 2, we present some definitions, lemmas, and denotations that will be useful throughout the paper. We then investigate the local and global synchronization in section 3 and 4, respectively. We present some numerical illustrations in section 5 to verify the theoretical results and conclude the paper in section 6.

2 Preliminaries

In this section, we present some definitions, notations and lemmas, which will be useful throughout the paper. The vector-valued function $x(t,\phi) = [x^{1^\top}(t), \cdots, x^{m^\top}(t)]^\top \in \mathbb{R}^{nm}$ is used to denote the solution of system (6) satisfying initial conditions

$$x^i(\theta) = \phi^i(\theta), \quad for\ i = 1, 2, \dots, m \ \ \theta \in [-\tau, 0] \tag{7}$$

where $\phi = [\phi^1(\theta), \cdots, \phi^m(\theta)]$ with $\phi^i(\theta) \in C_\tau$. Sometimes, $x(t,\phi)$ is denoted by $x(t)$. $\mathbf{C}_\tau$ denotes $C_\tau \times C_\tau \times \cdots \times C_\tau$, the Cartesian product of m C_τ. And, $x_t(\theta) = x(t+\theta)$ denotes an element in $\mathbf{C}_\tau$. $|v|$ denotes some norm of a vector v; especially, $|v|_2$ denotes the 2-norm of v by $|v|_2 = \sqrt{\sum\limits_{k=1}^{n} v_k^2}$. Then, the norm of matrix $|A|$ can be induced by vector norm. $\|\phi\|$ denotes the maximum norm of a vector-value function belonging to C_τ by $\|\phi\| = \sup\limits_{\theta\in[-\tau,0]} |\phi(\theta)|$. And, I_n denotes the identity matrix with dimension n and $\mathbf{C}_\tau$ denotes $C_\tau \times C_\tau \times \cdots \times C_\tau$, the Cartesian product of m C_τ.

Definition 1 *The coupled system (6) is locally exponentially synchronized, if there exist a compact set $\mathcal{A} \subset C_\tau$, $\delta > 0$, $\alpha > 0$, $T > 0$, $M > 0$ such that for any $\phi = [\phi^1, \cdots, \phi^m]$ satisfying $dist(\phi_i, \mathcal{A}) \le \delta$, where $dist(\cdot,\cdot)$ denotes the Hausdorff distance, and $\|\phi^i - \phi^j\| \le \delta$ for all $i, j = 1, \cdots, m$, we have*

$$\|x_t^i - x_t^j\| \le Me^{-\alpha t}, \ for\ all\ i, j = 1, \cdots, m, \ t > T. \tag{8}$$

Definition 2 *The coupled system (6) is globally exponentially synchronized, if there exists $\alpha > 0$ such that for any initial condition $\phi^i \in C_\tau$, $i = 1, \cdots, m$, there exists $M > 0$ satisfying*

$$\|x_t^i - x_t^j\| \le Me^{-\alpha t} \tag{9}$$

for all $t \ge 0$ and $i, j = 1, \cdots, m$.

Definition 3 *Function class $Quad(P, D, \alpha)$: We say $f \in Quad(P, D, \alpha)$, where $P = diag\{p_1, \cdots, p_n\}$ is a positive definite diagonal matrix, $D = diag\{d_1, \cdots, d_n\}$ is a diago-*

nal matrix, and $\alpha > 0$, if and only if

$$(x-y)^\top P[f(x)-f(y)-Dx+Dy] \le -\alpha(x-y)^\top(x-y)$$

holds for any $x, y \in \mathbb{R}^n$.

Definition 4 *Function class $H(M)$: We say $f \in H(M)$, where $M > 0$, if and only if*

$$\Big[f(x)-f(y)\Big]^\top \Big[f(x)-f(y)\Big] \le M^2(x-y)^\top(x-y)$$

holds for any $x, y \in \mathbb{R}^n$.

For the coupling matrix A in system (6), we give the following constraints.

Definition 5 *We say $A \in \mathbf{A}1$, if matrix $A = (a_{ij}) \in \mathbb{R}^{m,m}$ satisfies: (1). $a_{ij} \ge 0$, for all $i \ne j$; $a_{ii} = -1$; (2). $\sum\limits_{j=1}^{m} a_{ij} = 0$, for all $i = 1, \cdots, m$; (3). $rank(A) = m-1$.*

As indicated in [15], the coupling matrix A can be regarded as a directed weighted graph $G = [V, E, W]$. V denotes the node set which can be ordered by $1, 2, \cdots, m$; edge $e(i, j) \in E$ denotes the edge from node j to i which is supposed to be simple (without loops and multiple edges); $W = (w_{ij})$ denotes the weight set: $w_{ij} = a_{ij}$, for all $i \ne j$. Thus, the coupling matrix A can be regarded the Laplacian of weighted graph G. For example, normalized Laplacian of a graph: $I - D^{-1}B$ where D denotes the degree diagonal matrix and B denotes the adjacent matrix of a graph. And, $A \in \mathbf{A}1$ if and only if the corresponding graph must have a spanning tree [15].

Definition 6 *We say $A \in \mathbf{A}2$, if matrix $A = (a_{ij}) \in \mathbb{R}^{m,m}$ satisfies: (1). $a_{ij} \ge 0$, for all $i \ne j$; $a_{ii} = -1$; (2). $\sum\limits_{j=1}^{m} a_{ij} = 0$, for all $i = 1, \cdots, m$; (3). A is irreducible.*

It can be seen that $\mathbf{A1} \supset \mathbf{A2}$. For the matrix class $\mathbf{A}1$, by Gershgorin disc theorem [34] and Perron-Frobenius theory [35], we have the following lemma.

Lemma 1 *Suppose $A \in \mathbf{A}1$. Then*

(1) $[1, 1, \cdots, 1]^\top$ is the right eigenvector of A corresponding to eigenvalue 0 with multi-

plicity 1*, and the other eigenvalues* λ_i *satisfy* $\mathcal{R}e(\lambda_i) < 0$ *and* $|1+\lambda_i| \leq 1$*;*

(2) the left eigenvector of A *corresponding to eigenvalue* 0*:* $\xi = [\xi_1, \xi_2, \cdots, \xi_m] \in \mathbb{R}^m$ *has the following properties: it is non-zero (without loss of generality, assume* $\sum_{i=1}^{m} \xi_i = 1$*) and its multiplicity is* 1*; all* $\xi_i \geq 0$*,* $i = 1, \cdots, m$*; more precisely,*

(a) A *is irreducible if and only if* $\xi_i > 0$ *for all* $i = 1, \cdots, m$*;*

(b) A *is reducible if and only if for some* i*,* $\xi_i = 0$*, and in this case, by suitable rearrangement, one can assume that* $\xi^\top = [\xi_+^\top, \xi_0^\top]$*, where* $\xi_+ = [\xi_1, \xi_2, \cdots, \xi_p]^\top \in \mathbb{R}^p$ *with all* $\xi_i > 0$*,* $i = 1, \cdots, p$*;* $\xi_0 = [\xi_{p+1}, \xi_{p+2}, \cdots, \xi_m]^\top \in \mathbb{R}^{m-p}$ *with all* $\xi_j = 0$*,* $p+1 \leq j \leq m$*, and* A *can be rewritten as* $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$*, where* $A_{11} \in \mathbb{R}^{p,p}$ *is irreducible and* $A_{12} = 0$*.*

Let $\xi = [\xi_1, \cdots, \xi_m] \in \mathbb{R}^m$ be the left eigenvector of the coupling matrix $A \in \mathbf{A}1$ corresponding to eigenvalue 0, and satisfy $\sum_{i=1}^{m} \xi_i = 1$, $\xi_i \geq 0$, for all $i = 1, \cdots, m$. We define a weighted average $\bar{x}(t) = \sum_{i=1}^{m} \xi_i x^i(t)$ for $\{x^1(t), \cdots, x^m(t)\}$ with ξ. We also denote $\Delta x^i(t) = x^i(t) - \bar{x}(t)$ and

$$\hat{x}(t) = [\bar{x}^\top(t), \bar{x}^\top(t), \cdots, \bar{x}^\top(t)]^\top \in \mathbb{R}^{nm}, \quad \Delta x(t) = [\Delta x^{1^\top}(t), \cdots, \Delta x^{m^\top}(t)] \in \mathbb{R}^{nm}$$

It can be seen that $x(t) = \hat{x}(t) + \Delta x(t)$ and $\hat{x}(t) \in \mathbf{S}$. By the definition of $\bar{x}(t)$, we have $\sum_{i=1}^{m} \xi_i \Delta x^i(t) = 0$. Thus, we define the **transverse space**: $\mathbf{L} = \left\{ [\phi^{1^\top}, \cdots, \phi^{m^\top}]^\top : \phi^i \in C_\tau \ and \ \sum_{i=1}^{m} \xi_i \phi^i(\theta) = 0 \ holds \ for \ all \ i = 1, \cdots, m \ and \ \theta \in [-\tau, 0] \right\}$, which is used to describe the distance between x_t and the synchronization space $\mathbf{S}$. It is clear that $\Delta x_t \in \mathbf{L}$. This implies that $\mathbf{C}_\tau = \mathbf{S} + \mathbf{L}$. If we can prove Δx_t converges to zero when $t \to \infty$ (equivalently $\Delta x(t) \to 0$), then the synchronization is guaranteed. This is the main methodology for analyzing the synchronization used in this paper. In particular, for $\mathbb{R}^m$, we denote $L = \left\{ [u_1, u_2, \cdots, u_m]^\top \in \mathbb{R}^m : \sum_{i=1}^{m} \xi_i u_i = 0 \right\}$ as a subspace of $\mathbb{R}^m$.

3 Local Synchronization

In this section, we discuss local synchronization of the LCODEs with a coupling delay. Let $x^i(t) = s(t)$, $i = 1, \cdots, m$, and we can write a synchronized state as

$$\frac{ds(t)}{dt} = f(s(t)) + c\Gamma\Big[s(t-\tau) - s(t)\Big] \tag{10}$$

Suppose system (10) has an attractor $\mathcal{A} \in C_\tau$. Equivalently,

(1). $\mathcal{A}$ is a compact set in the Banach space C_τ;

(2). $\mathcal{A}$ is attracting, i.e, for any initial $\phi \in C_\tau$ near $\mathcal{A}$ enough, the trajectory will converge into $\mathcal{A}$;

(3). $\mathcal{A}$ is invariant, i.e, any trajectory with initial condition in $\mathcal{A}$ will never go out of $\mathcal{A}$;

(4). $\mathcal{A}$ is minimal, i.e., $\mathcal{A}$ does not have any non-trivial closed attracting set.

Generally, $\mathcal{A}$ can be a fixed point, a limit circle, or a chaotic attractor defined for infinite dimensional systems [36]. Here, we extend this concept to the synchronization space $\mathbf{S}$ by defining the (diagonal) synchronization manifold $\mathcal{U} = \{[\phi^\top, \phi^\top, \cdots, \phi^\top]^\top : \phi \in \mathcal{A}\}$. Hence, the local synchronization can be defined by the way that the synchronization manifold $\mathcal{U}$ is asymptotically stable for the space $\mathbf{C}_\tau$.

First, we present following proposition, which guarantees the existence of a global attractor for the delayed system (10).

Proposition 1 *Suppose $\Gamma = I_n$. If there exists $K > 0$ such that for any $\epsilon > 0$, there there exists $\delta(\epsilon) > 0$ such that*

$$x^\top f(x) < -\delta(\epsilon) \tag{11}$$

holds for all $|x| > K + \epsilon$, then the system (10) has a global compact attractor.

This proposition comes from theorem 18 in [37] and the proof will be given in Appendix 1.

Suppose that system (10) has an attractor and $s(t)$ is a solution of (10) located in this attractor. By the linearization technique, the variational equation with $\delta x^i = x^i(t) - s(t)$ is written as follows:

$$\frac{d\delta x^i(t)}{dt} = Df(s(t))\delta x^i(t) + c\sum_{j=1}^{m} a_{ij}\Gamma\delta x^j(t-\tau) + c\Gamma\Big[\delta x^i(t-\tau) - \delta x^i(t)\Big], \qquad (12)$$

where $Df(s(t))$ is the Jacobian matrix of f at $s(t)$ and $i = 1, \cdots, m$. Denote $\delta X(t) = [\delta x^1(t), \cdots, \delta x^m(t)] \in \mathbb{R}^{n,m}$. Then

$$\frac{d\delta X(t)}{dt} = Df(s(t))\delta X(t) + c\Gamma\delta X(t-\tau)A^\top + c\Gamma\Big[\delta X(t-\tau) - \delta X(t)\Big] \qquad (13)$$

Let $A^\top = VJV^{-1}$ be the Jordan decomposition of A, where

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{bmatrix}$$

and Jordan blocks, $J_1, J_2, \cdots, J_s$ are associated with the eigenvalues $\lambda_1 = 0, \lambda_2, \cdots, \lambda_m$ of A. Let $\delta Y(t) = \delta X(t)V = \{\delta y^1(t), \cdots, \delta y^m(t)\}$. Then, the variational equation of $\delta Y(t)$ is

$$\frac{d\delta Y(t)}{dt} = \Big[Df(s(t)) - c\Gamma\Big]\delta Y(t) + c\Gamma\delta Y(t-\tau)(J + I_m) \qquad (14)$$

Since the first column of V is the left eigenvector of A, $\delta y^1(t)$ can be regarded as variation near the synchronization manifold $\mathcal{U}$. Extending from the concept of the master stability function introduced in [11], here the transverse Lyapunov exponents of the following variational systems in Banach space C_τ:

$$\frac{d\varphi(t)}{dt} = \left[Df(s(t)) - c\Gamma\right]\varphi(t) + c(\lambda_k + 1)\Gamma\varphi(t-\tau), \quad k = 2, \cdots, m \tag{15}$$

are utilized to analyze the local stability of the synchronization manifold.

Similar to the Lyapunov exponents defined in finite dimensional systems, we embed C_τ to Hilbert space $L^2([-\tau, 0], \mathbb{R}^n)$. Let $T^t_{k,s}\varphi_0 = \varphi(t+\theta)$, $\varphi \in C_\tau$, be the flow through the variational system (15) associated with eigenvalue λ_k with initial value $\varphi(\theta) = \varphi_0(\theta) \in C_\tau$,$\theta \in [-\tau, 0]$, and $s(\theta) \in \mathcal{A}$. Define $\lim_{t\to\infty}(T^{t*}_{k,s}T^t_{k,s})^{1/2t} = \Lambda_{k,s}$, where * denotes the conjugate of a linear operator in the Hilbert space $L^2([-\tau, 0], \mathbb{R}^n)$. Then, the spectra of the operator $\Lambda_{k,s}$ denote the Lyapunov spectra of the variational system (15). Since C_τ is separable, it was shown in [38] that under some conditions, $\Lambda_{k,s}$ has discrete spectra $\mu^k_{1,s} > \mu^k_{2,s} > \cdots > \mu^k_{p_k(s),s}$, where $p_k(s)$ depends on the trajectory $s(t)$ and can be $+\infty$. For $k = 1$, the corresponding Lyapunov spectra are just those of the synchronized state system (10). Let $mle = \sup_{s\in\mathcal{A}}\sup_{k\geq 2} log\{\mu^k_{1,s}\}$ denote the largest Lyapunov exponent of the variational system (15) in the transverse space. The negativity of mle guarantees that all the transverse eigenmodes are stable. In other words, the coupled system (6) is locally synchronized if $mle < 0$. It is clear that mle is a function of λ_k, $k = 2, \cdots, m$, the coupling delay τ, and the coupling strength c and can be regarded as the master stability function of the LCODEs with a coupling delay (6).

In [36], the author presented a numerical method to compute the Lyapunov spectra of a delayed variational system by splitting function $\phi \in C_\tau$ into a vector $[\phi(t-(N-1)\Delta t), \phi(t-(N-2)\Delta t), \cdots, \phi(t)]$, where $\Delta t = \tau/(N-1)$. Thus, the evolution of the infinite dimensional system can be transformed into a difference system with high dimension. By this way, we can compute the Lyapunov exponent of a delayed system.

Despite that transverse Lyapunov exponents provide a viable numerical method to study stability of the synchronization manifold, it is difficult qualitatively to show how synchronization depends on parameters c, τ. Here, we introduce the following qualitative analysis to explore this dependence.

Theorem 1 *Suppose that system (10) has an asymptotically stable attractor $\mathcal{A}$ and $s(t)$ is a solution of the coupled system (10) included in $\mathcal{A}$. Let $Df(s(t))$ be the Jacobian of $f(s(t))$, $\mu = \overline{\lim_{t\to\infty}}|Df(s(t))|_2$, where $< \cdot >$ denotes the time average, $k_1 = \overline{\lim_{t\to\infty}}|Df(s(t)) - cI_n|_2$, and $k_2 = c\max_{k\geq 2}|1+\lambda_k|$. If the following inequality holds:*

$$c > \frac{\mu}{1 - \max_{k\geq 2}|1+\lambda_k|} \tag{16}$$

or the following two inequalities

$$\begin{cases} \beta = c\min_{k\geq 2}|\mathcal{R}e(\lambda_k)| - < |Df(s(t))|_2 >> 0 \\ e^{2\beta\tau}\tau < \frac{\beta}{c(k_1+k_2)\max_{k\geq 2}|1+\lambda_k|} \end{cases} \tag{17}$$

hold. Then the coupled system (6) is locally synchronized.

Proof is given in Appendix 2.

In our previous work [17], we pointed out that if $s(t)$ is not included in $\mathcal{A}$, the reasoning above might fail. So, generally, without knowing the existence of the asymptotically stable attractor $\mathcal{A}$ for the synchronized state system (10), we can replace $s(t)$ by the weighted average $\bar{x}(t)$ defined in section 2. Then, we have

Theorem 2 *Suppose $Df(\bar{x}(t))$ is the Jacobian of f at $\bar{x}(t)$, where $\bar{x}(t)$ is defined in section 2. Let $\mu' = \overline{\lim_{t\to\infty}}|Df(\bar{x}(t))|_2$, $k_1' = \overline{\lim_{t\to\infty}}|Df(\bar{x}(t)) - cI_n|_2$, and $k_2 = c\max_{k\geq 2}|1+\lambda_k|$. If the following inequality holds:*

$$c > \frac{\mu'}{1 - \max_{k\geq 2}|1+\lambda_k|} \tag{18}$$

or the following two inequalities

$$
\begin{cases}
\beta' = c\min_{k\geq 2}|\mathcal{R}e(\lambda_k)| - < |Df(\bar{x}(t))|_2 >> 0 \\
e^{2\beta'\tau}\tau < \frac{\beta'}{c(k_1'+k_2)\max\limits_{k\geq 2}|1+\lambda_k|}
\end{cases}
\tag{19}
$$

hold. Then the coupled system (6) is locally synchronized.

In fact, by the definition, we have

$$
\frac{d\bar{x}(t)}{dt} = \sum_{k=1}^{m}\xi_k\frac{dx^k(t)}{dt} = \sum_{k=1}^{m}\xi_k f(x^k(t,t)) + c\Gamma\Big[\bar{x}(t-\tau) - \bar{x}(t)\Big]
$$

Since $\sum_{i=1}^{m}\xi_i\Delta x_i = 0$, we have

$$
\begin{aligned}
&\sum_{i=1}^{m}\xi_i f(x^i,t) - f(\bar{x},t) = \sum_{i=1}^{m}\xi_i[f(x^i,t) - f(\bar{x},t)] \\
=&\sum_{i=1}^{m}\xi_i[f'(\bar{x},t) + o(1)]\Delta x_i = f'(\bar{x},t)\sum_{i=1}^{m}\xi_i\Delta x_i + o(\|\Delta x\|) = o(\|\Delta x\|)
\end{aligned}
$$

Then, by the linearization technique, the variational equation of Δx^i can be written as follows:

$$
\frac{d\Delta x^i(t)}{dt} = Df(\bar{x}(t),t)\Delta x^i(t) + c\sum_{j=1}^{m}a_{ij}\Gamma\Delta x^j(t-\tau) + c\Gamma\Big[\Delta x^i(t-\tau) - \Delta x^i(t)\Big] \tag{20}
$$

which is of the same form with (13). The remaining is just a repetition of the proof of theorem 1.

Remark 1

(1) *In case $A \in \mathbf{A}1$. By lemma 1, we have $|1+\lambda_k| \leq 1$ for all $k \geq 2$. If $f(\cdot)$ is a globally Lipschitz function, which implies μ can be estimated independent on c and τ, then for sufficient large strength c, the coupled system can be synchronized with an arbitrary coupling delay τ.*

(2) *From the inequality (16), one can use the quantity*

$$
cap := 1 - \max_{k=2,m}|1+\lambda_k|
$$

as an index to measure the synchronizability of a coupling configuration with an arbitrary delay. The larger the cap is, the smaller the coupling strength c is needed to synchronize the coupled system (6) no matter how large the coupling delay τ is.

(3) *Because $|1 + \lambda_k| \geq 1 - |\mathcal{R}e(\lambda_k)|$ holds for all $k \geq 2$, the first inequality of (17) is easier to be satisfied than (16). Furthermore, if $\tau = 0$, one can see that inequalities (17) can hold if c is large enough because of the existence of synchronized compact attractor. According to the continuous dependence (see [32] for details), despite that c could not guarantee the synchronization for any delays, it can synchronize the coupled system with a small delay.*

Remark 2 *In fact, the synchronization of the coupled system (6) is of more complexity. For example, as reported in [27], a large coupling delay might enhance synchronization in coupled Hindmarsh-Rose (HR) neurons with coupling delay. This phenomenon is because the synchronized state system (10) itself depends on parameters c and τ. Different delays indicate different possible dynamics of the synchronization manifold. Further discussions will be given in section 5 by the numerical way.*

4 Global Synchronization

In section 3, we studied the local synchronization, i.e., the trajectory $x(t)$ initiated near the synchronization manifold $\mathcal{U}$ will move to $\mathcal{U}$. Here, we should consider the case that the trajectory will converge to the synchronization space $\mathbf{S}$ from any initial position. This is named by global synchronization. First, we consider the case $A \in \mathbf{A}2$, which implies that the left eigenvector $\xi = [\xi_1, \xi_2, \cdots, \xi_m]$ of A associated with the eigenvalue 0 can be assumed $\xi_i > 0$ for all $i = 1, \cdots, m$ as well as $\sum_{i=1}^{m} \xi_i = 1$, as mentioned in lemma 1.

4.1 Irreducible coupling matrix

In this section, we assume that the coupling matrix is irreducible. The equation of $\Delta x^i(t) = x^i(t) - \bar{x}(t)$ can be written without linearlization as follows:

$$\begin{aligned}\frac{d\Delta x^i(t)}{dt} &= f(x^i(t)) - f(\bar{x}(t)) - \sum_{k=1}^{m} \xi_k \Big[f(x^k(t)) - f(\bar{x}(t)) \Big] \\ &\quad + c\sum_{j=1}^{m} a_{ij}\Gamma \Delta x^j(t-\tau) + c\Gamma \Big[\Delta x^i(t-\tau) - \Delta x^i(t) \Big], \quad i = 1, \cdots, m. \end{aligned} \tag{21}$$

Let

$$\Delta x(t) = \begin{bmatrix} \Delta x^1(t) \\ \vdots \\ \Delta x^m(t) \end{bmatrix}, \quad \Delta F(t) = \begin{bmatrix} f(x^1(t)) - f(\bar{x}(t)) \\ \vdots \\ f(x^m(t)) - f(\bar{x}(t)) \end{bmatrix}, \quad \Xi = \begin{bmatrix} \xi_1 & \cdots & \xi_m \\ \vdots & \vdots & \vdots \\ \xi_1 & \cdots & \xi_m \end{bmatrix}$$

$P = diag\{p_1, \cdots, p_n\}$ be some positive definite diagonal matrix, $D = diag\{d_1, \cdots, d_n\}$ be a diagonal matrix, $\Lambda = diag\{\xi_1, \cdots, \xi_m\}$, and $\otimes$ be the Kronecker product. Moreover, let

$$\begin{aligned} &\mathbf{\Xi} = \Xi \otimes I_n \quad \mathbf{A} = A \otimes \Gamma \quad \mathbf{\Lambda} = \Lambda \otimes I_n \\ &\mathbf{P} = I_m \otimes P \quad \mathbf{D} = I_m \otimes D, \quad \mathbf{I} = I_m \otimes I_n, \quad \mathbf{\Gamma} = I_m \otimes \Gamma. \end{aligned}$$

The equation (21) can be rewritten in the following matrix form:

$$\frac{d\Delta x(t)}{dt} = (\mathbf{I} - \mathbf{\Lambda})\Delta F(t) + c\mathbf{A}\Delta x(t-\tau) + c\mathbf{\Gamma}\Big[\Delta x(t-\tau) - \Delta x(t)\Big]. \tag{22}$$

Next, we introduce another expression for components of $\Delta x(t)$.

Let $\Delta x^i(t) = [\Delta x_1^i, \cdots, \Delta x_n^i(t)]^\top \in \mathbb{R}^n$ and define

$$\Delta\tilde{x}^j(t) = [\Delta x_j^1(t), \cdots, \Delta x_j^m(t)]^\top \in \mathbb{R}^m, \quad \Delta\tilde{x}(t) = \begin{bmatrix} \Delta\tilde{x}^1(t) \\ \vdots \\ \Delta\tilde{x}^n(t). \end{bmatrix}$$

It should be emphasized that $\Delta x_t \in \mathbf{L}$, which implies $\Delta\tilde{x}^j(t) \in L$, holds for all $j = 1, \cdots, m$, where $\mathbf{L}$ and L are defined in section 2. Thus, the discussion of synchronization can be focused on the component $\Delta\tilde{x}^j(t)$ in the transverse subspace L. We will investigate under which conditions it converges to zero.

Now, define a linear operator C mapping $\mathbb{R}^{m-1}$ to L by

$$C = \begin{bmatrix} & I_{m-1} & \\ -\frac{\xi_1}{\xi_m} & \cdots & -\frac{\xi_{m-1}}{\xi_m} \end{bmatrix} \in \mathbb{R}^{m,m-1}.$$

Then, we have $C\mathbb{R}^{m-1} = L$. The following theorem concerns the global synchronization with an arbitrary delay τ.

Theorem 3 *Suppose $A \in \mathbf{A2}$. If there exist a positive definite diagonal matrix $P = diag\{p_1, \cdots, p_n\}$, a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$, and a positive constant α, such that $f \in Quad(D, P, \alpha)$, and for any $j = 1, \cdots, n$, there exists a positive definite matrix Q_j respectively such that the following linear matrix inequalities (LMIs)*

$$\begin{bmatrix} 2(d_j - c\gamma_j)C^\top \Lambda C + C^\top Q_j C & c\gamma_j C^\top \Lambda (A + I_m) C \\ c\gamma_j C^\top (A^\top + I_m)\Lambda C & -C^\top Q_j C \end{bmatrix} \leq 0 \tag{23}$$

holds for all $j = 1, \cdots, n$, then the coupled system (6) is globally exponentially synchronized.

The Proof is given in Appendix 3. In particular, we have

Corollary 1 *Assume that there exist a positive definite diagonal matrix $P = diag\{p_1, \cdots, p_n\}$ and a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$ such that $f \in Quad(D, P, \alpha)$. Suppose σ is a constant such that the following linear matrix inequality*

$$(I_m + A^\top)\Lambda(A + I_m) \leq \sigma^2\Lambda. \tag{24}$$

holds on the subspace L. And

$$\begin{cases} d_j \leq 0 & if\ \gamma_j = 0 \\ c \geq \frac{d_j}{\gamma_j(1-\sigma)} & if\ \gamma_j > 0 \end{cases} \tag{25}$$

Then, the coupled system (6) is globally exponentially synchronized.

Proof: From theorem 3 and the Shur complement [39], the LMIs (23) are equivalent to the following linear matrix equalities holding on the transverse subspace L:

$$N_j = 2(d_j - c\gamma_j)\Lambda + Q_j + \gamma_j^2 c^2 (I_m + A^\top)\Lambda Q_j^{-1}\Lambda(A + I_m) < 0, \quad j = 1, \cdots, n$$

If $\gamma_j = 0$ and $d_j \leq 0$, then letting $Q_j = 0$ we can obtain inequality (23) holds. Instead, if $\gamma_j > 0$, we let $Q_j = \rho_j\Lambda$. Then,

$$\begin{aligned} N_j &= 2(d_j - c\gamma_j)\Lambda + \rho_j\Lambda + \rho_j^{-1}c^2\gamma_j^2(I_m + A^\top)\Lambda(A + I_m) \\ &\leq \left[2(d_j - c\gamma_j) + \rho_j + \rho_j^{-1}c^2\gamma_j^2\sigma^2\right]\Lambda = \left[2d_j - 2c\gamma_j(1-\sigma)\right]\Lambda \end{aligned}$$

picking $\rho_j = c\sigma\gamma_j$. Therefore, if $c \geq \frac{d_j}{\gamma_j(1-\sigma)}$, then the LMI (23) can hold. The corollary is a direct consequence from theorem 2.

Remark 3 *From corollary 1, one can see that if $\sigma < 1$, then the coupled system can be globally exponentially synchronized for any coupling delays if c is large enough.*

Next, we address how the global synchronization depends on the delay τ.

Rewrite equation (21) as follows:

$$\frac{d\Delta x(t)}{dt} = (\mathbf{I}-\Xi)\Delta F(t) + c\mathbf{A}\Delta x(t) + c(\mathbf{A}+\mathbf{I})\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}ds. \tag{26}$$

Theorem 4 *Suppose $A \in \mathbf{A2}$. If there exist a positive definite diagonal matrix $P = diag\{p_1, \cdots, p_n\}$, a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$, $\alpha > 0$, and $M > 0$, such that*

(1) $f \in Quad(D, P, \alpha)$;

(2) $f \in H(M)$;

(3) $\begin{cases} d_j \le 0 & if\ \gamma_j = 0 \\ c > \max\limits_{u\in L,\ u\neq 0} \frac{d_j|\Lambda^{1/2}u|}{-\gamma_j u^\top \Lambda A u} & if\ \gamma_j > 0 \end{cases}$;

(4) the delay $\tau < \sqrt{\frac{a_3}{a_1+a_2a_3}}$, where

$$\begin{aligned} a_1 &= 3c^2|\Lambda|_2^2|A+I_m|_2^2|P|_2^2\Big(M^2|I_m-\Xi|_2^2 + c^2|A|_2^2\Big) \\ a_2 &= 3c^2|A+I_m|_2^2 \quad a_3 = \alpha^2(\min_i \xi_i)^2. \end{aligned}$$

Then, the coupled system (6) is globally exponentially synchronized.

The proof is given in Appendix 4.

Remark 4 *It can be seen (see [15]) that $\Lambda A + A^\top \Lambda$ is negative definite in the subspace L since $A \in \mathbf{A2}$. Therefore, if $A \in \mathbf{A2}$ and $d_j, \gamma_j > 0$, then $\max\limits_{u\in L\ u\neq 0} \frac{d_j|\Lambda^{1/2}u|}{-\gamma_j u^\top \Lambda A u}$ is a positive number .*

The following proposition explores that the conditions required in theorem 3 is stronger than the third item in theorem 4.

Proposition 2 *If the linear matrix inequalities (23) satisfy, then the inequalities in item 3 of theorem 4 hold.*

Proof: By the Shur complement, the linear matrix inequalities (23) are equivalent to that the following matrix inequalities hold on the subspace L:

$$W_j = 2(d_j - c\gamma_j)\Lambda + Q_j + c^2\gamma_j^2(I_m + A^\top)\Lambda Q_j^{-1}\Lambda(A+I_m) \le 0, \quad j = 1, \cdots, n \tag{27}$$

Also, by matrix inequality $0 \leq (Q^{1/2} - Q^{-1/2}Z)^{\top}(Q^{1/2} - Q^{-1/2}Z)$, where $Z = c\gamma_j\Lambda(A + I_m)$, we have $Q_j + c^2\gamma_j^2(I_m + A^{\top})\Lambda Q_j^{-1}\Lambda(A + I_m) \geq 2c\gamma_j\left[\Lambda(A + I_m)\right]^s$. Hence, $W_j \geq 2(d_j - c\gamma_j)\Lambda + 2c\gamma_j\left[\Lambda(A + I_m)\right]^s = 2\left[\Lambda(c\gamma_j A + d_j I_m)\right]^s$, where $M^s = \frac{1}{2}(M + M^{\top})$ denotes the symmetric part of the matrix M. Therefore, the inequalities in item 3 of theorem 3 hold.

Remark 5 *Proposition 2 implies that it is possible that the coupled system may not be globally synchronized with any coupling delay but can be synchronized for a small coupling delay.*

4.2 Reducible coupling matrix

In many cases, the reducible coupling matrix should be investigated, for example, the master slave system. For any reducible $A \in \mathbf{A}1$, with a proper permutation, we can rewrite A in the following upper-block-triangular matrix form (Frobenius form):

$$A = \begin{bmatrix} A^1 & A^{12} & \cdots & A^{1m} \\ 0 & A^2 & \cdots & A^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^q \end{bmatrix} \tag{28}$$

where A^q is irreducible and $-A^j$, for $j = 1, 2, \cdots, q-1$, are all M-matrices (see [15]).

Denote the subsystem with respect to coupling matrix A^l by subsystem N_l which has m_l nodes, $l = 1, 2, \cdots, q$.

Noticing that subsystem N_q can be globally synchronized if the conditions in theorem 3 or 4 are satisfied. Synchronization of the whole coupled system with a reducible coupling matrix is equivalent to that each subsystem with respect to coupling matrix A^l, for $l < q$, synchronizes with system N_q. More precisely, by induction, suppose that

(1) subsystem N_q can be exponentially globally synchronized,

(2) for each $1 \le l \le q-1$, the collection of subsystems: $\Big\{N_j:\ j=l+1,l+2,\cdots,q\Big\}$, are all exponentially globally synchronized,

Then, we can conclude that the whole coupled system (5) can be globally exponentially synchronized. In the following, we present some criteria for global synchronization of a coupled system with a reducible coupling matrix. For subsystem N_l, assuming $s(t)$ is the synchronized state of subsystem N_j, $j=l+1,l+2,\cdots,q$, satisfying

$$\frac{ds(t)}{dt} = f(s(t)) + c\Gamma\Big[s(t-\tau)-s(t)\Big] + O(e^{-\epsilon t})$$

for some positive constant ϵ. Then, we have

$$\begin{aligned}\frac{d\Big[x^i(t)-s(t)\Big]}{dt} &= f(x^i(t)) - f(s(t)) + c\sum_{j\neq i} a_{ij}\Gamma[x^j(t-\tau)-x^i(t)] \\ &\quad -c\Gamma\Big[s(t-\tau)-s(t)\Big] + O(e^{-\epsilon t}) \\ &= \Big[f(x^i(t))-f(s(t))\Big] + c\sum_{j\in N_l, j\neq i} a_{ij}\Gamma\Big[x^j(t-\tau)-s(t-\tau)\Big] \\ &\quad -c\Gamma\Big[x^i(t)-s(t)\Big] + O(e^{-\epsilon t}), \quad i\in N_l. \end{aligned} \tag{29}$$

Define an $m_l \times m_l$ matrix $\hat{A}_l$ as follows: $(\hat{A}_l)_{ij} = \begin{cases} (A_l)_{ij} & i\neq j \\ 0 & i=j \end{cases}$.

Theorem 5 *Suppose that the coupling matrix $A \in \mathbf{A}1$ has the form of (28), and the following conditions are all satisfied:*

(1) there exist a positive definite diagonal matrix $P = diag\{p_1,\cdots,p_n\}$, a diagonal matrix $D = diag\{d_1,\cdots,d_n\}$, and a positive constant α, such that $f \in Quad(D,P,\alpha)$;

(2) for each $j=1,\cdots,n$, there exists a positive definite matrix Q_j^q such that

$$\begin{bmatrix} 2(d_j - c\gamma_j)C^{q^\top}\Lambda^q C^q + C^{q^\top} Q_j^q C^q & c\gamma_j C^{q^\top}\Lambda^q (A_q + I_m)C^q \\ c\gamma_j C^{q^\top}(A_q^\top + I_m)\Lambda^q C^q & -C^{q^\top} Q_j^q C^q \end{bmatrix} < 0$$

where $\Lambda^q = diag\{\xi_1^q, \cdots, \xi_{m_q}^q\}$ *is the positive definite diagonal matrix composing of the components of the left eigenvector* $\xi^q = [\xi_1^q, \cdots, \xi_{m_q}^q]$ *of* A_q *associated with eigenvalue* 0*, satisfying* $\sum_{i=1}^{m_q} \xi_i^q = 1$ *and* $\xi_i^q > 0$*,* $i = 1, 2, \cdots, m_q$*, and*

$$C^q = \begin{bmatrix} & I_{m_q-1} & \\ -\frac{\xi_1^q}{\xi_{m_q}^q} & \cdots & -\frac{\xi_{m_q-1}^q}{\xi_{m_q}^q} \end{bmatrix} \in \mathbb{R}^{m_q, m_q-1}$$

is a linear operator transforming $\mathbb{R}^{m_q-1}$ *to the transverse subspace* $L^q = \Big\{[v_1, \cdots, v_{m_q}]^\top \in \mathbb{R}^{m_q} : \sum_{i=1}^{m_q} \xi_i^q v_i = 0\}$*;*

(3) for each $l = 1, 2, \cdots, q-1$*, and each* $j = 1, \cdots, n$*, there exist a positive definite symmetric matrix* Q_j^l *and a positive definite diagonal matrix* $G^l = diag\{G_1^l, \cdots, G_{m_l}^l\}$ *such that the following LMIs hold:*

$$\begin{bmatrix} 2(d_j - c\gamma_j)G^l + Q_j^l & c\gamma_j G^l \hat{A}_l \\ c\gamma_j \hat{A}_l^\top G^l & -Q_j^l \end{bmatrix} < 0.$$

Then, the coupled system (6) with the reducible coupling matrix A *is global exponentially synchronized.*

The proof is given in Appendix 5.

Remark 6 *(1) All the conditions in item 2 guarantee that the subsystem* N_l *can be globally exponentially synchronized;*

(2) The condition in item 3 guarantees the subsystem N_l *can be synchronized to subsystems* N_q *step by step globally and exponentially.*

Futhermore, equation (45) can be rewritten as:

$$\frac{d\Delta x^i(t)}{dt} = \Big[f(x^i(t)) - f(s(t))\Big] + c\sum_{j\in N_l, j\neq i} a_{ij}\Gamma\Delta x^j(t) - c\Gamma\Delta x^i(t)$$
$$-c\sum_{j\in N_l, j\neq i} a_{ij}\Gamma\int_{t-\tau}^{t}\frac{d\Delta x^j(s)}{ds}ds, \quad i\in N_l.$$

Define an $m_l \times m_l$ matrix $\breve{A}_l$ by

$$(\breve{A}_l)_{ij} = \begin{cases} (A_l)_{ij} & i\neq j \\ -1 & i=j \end{cases}.$$

Similar to theorem 4, we give the following criteria for global synchronization depending on the coupling delay τ.

Theorem 6 *Suppose the reducible matrix $A \in \mathbf{A2}$ has the form of (28). Assume that the following conditions are satisfied:*

(1) there exists a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$, a positive definite diagonal matrix $P = diag\{p_1, \cdots, p_n\}$, and a positive constant α, such that $f \in Quad(D, P, \alpha)$;

(2) there exists an $M > 0$ such that $f \in H(M)$;

(3) for each $j = 1, \cdots, n$,

$$C^\top\{\Lambda(c\gamma_j A_q + d_j I_{m_q})\}^s C < 0$$

where $\Lambda = diag\{\xi_1^q, \cdots, \xi_{m_q}^q\}$,

$$C = \begin{bmatrix} & I_{m_q-1} & \\ -\frac{\xi_1}{\xi_{m_q}} & \cdots & -\frac{\xi_{m_q-1}}{\xi_{m_q}} \end{bmatrix} \in \mathbb{R}^{m_q, m_q-1}$$

and $\xi^q = [\xi_1^q, \cdots, \xi_{m_q}^q]^\top \in \mathbb{R}^{m_q}$ satisfying $\sum_{j=1}^{m_q}\xi_j^q = 1$, $\xi_j^q > 0$, for $j = 1, \cdots, m_q$, and

$\xi^{q\top} A_q = 0$;

(4) *for each* $l = 1, 2, \cdots, q-1$ *and* $j = 1, \cdots, n$, *there exists a positive definite diagonal matrix* $G^l = diag\{G^l_1, \cdots, G^l_{m_l}\}$ *such that*

$$\{G^l(c\gamma_j \breve{A}_l + d_j I_{m_l})\}^s < 0;$$

(5) *the coupling delay* τ *is small enough.*

Then, the coupled system (6) with the reducible coupling matrix A *is globally exponentially synchronized.*

Since the proof is similar to that of theorems 4 and 5, it is omitted here and the delay boundedness can be estimated in the way similar to that used in the proof of theorem 4.

5 Numerical Examples

In this section, we present several simulations to show how to apply the theoretical results obtained above to judge whether a coupled system with coupling delay can be synchronized and how the coupling delay and coupling strength influence the synchronization.

Example 1: Final synchronized state

The synchronized state equation is described as follows:

$$\frac{ds(t)}{dt} = f(s(t)) + c\Gamma\Big[s(t-\tau) - s(t)\Big] \tag{30}$$

This is quite different from the case without coupling delay, where the synchronized state equation is

$$\frac{ds(t)}{dt} = f(s(t))$$

One can see that both the coupling strength and the coupling delay heavily influence the final state if the coupled system is synchronized.

Here, we use Lorenz system [40]

$$\begin{cases} \frac{dx_1}{dt} = \sigma[x_2 - x_1], \ \ \sigma = 16.0 \\ \frac{dx_2}{dt} = (r - x_3)x_1 - x_2, \ \ r = 40 \\ \frac{dx_3}{dt} = -bx_3 + x_1 x_2, \ \ b = 4.0 \end{cases} \tag{31}$$

R$\ddot{o}$ssler system [41]

$$\begin{cases} \frac{dx_1}{dt} = (x_2 + x_3) \\ \frac{dx_2}{dt} = x_1 + \frac{1}{5}x_2 \\ \frac{dx_3}{dt} = \frac{1}{5} + x_3(x_1 - \mu), \ \ \mu = 5.7 \end{cases} \tag{32}$$

and Chua's circuit [42]

$$\begin{cases} \frac{dx}{dt} = 9[y - \frac{2}{7}x - \frac{3}{14}(|x+1| - |x-1|)] \\ \frac{dy}{dt} = x - y + z \\ \frac{dz}{dt} = -\frac{2}{7}y \end{cases} \tag{33}$$

as the uncoupled system on each node and pick $\Gamma = I_3$.

Figures 1, 2, and 3 show the dynamical behaviors of the coupled system (30) with different coupling strength and delays. One can see that the dynamical behaviors of the coupled system (30) depend heavily on the parameters c and τ. Some of these attractors are first observed and need further investigation.

Example 2: Local synchronization

In this simulation, we couple three nodes globally with a coupling delay as follows:

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + \frac{c}{2}\sum_{j=1,j\neq i}^{3}\Gamma[x^j(t-\tau) - x^i(t)] \quad i = 1,2,3 \tag{34}$$

where $x^i(t) = [x_1^i(t), x_2^i(t), x_3^i(t)]^\top \in \mathbb{R}^3$, $i = 1,2,3$, $f(\cdot)$ is defined in equation (31), (32), or (33), respectively, and $\Gamma = I_3$. We use the largest transverse Lyapunov exponent mle to investigate how the local synchronization depends on the parameters c and τ.

In figures 4, 5, and 6, we use different colors to indicate the values of mle with different coupling strengths and delays, where the intrinsic system is Lorenz system (31), Rössler system (32), or Chua's circuits (33), respectively.

It can be seen in figures 4 and 5 that a larger coupling strength c does not always indicate synchronization. In figure 4 and more clearly in figure 5, one can see that $mle < 0$ only for small delays in case $c \geq 10$. If $f(\cdot)$ is global Lipschitzian like Chua's circuits, a larger c might imply synchronization easier as indicated in figure 6. Moreover, if c is large enough as $c \geq 2$ shown in figure 6, the coupled system can be synchronized in a large region of the coupling delays. Interaction delay means communication asymmetry between connected nodes. Intuitively, delay might prevent synchronization. However, the phenomena that delay enhances synchronization occur in the region $c \in [0, 1.5]$ and $\tau \in [9, 10]$ as indicated in the figure 5, which was also reported in [27]. This should be also tightly connected to the dynamical behaviors of the synchronized state (30). In figure 4, in the region $c \in [10, 18]$, synchronization and de-synchronization intersect, which might imply that synchronization is very sensitive with respect to parameters. As our actual computation, the mle is very near zero in these regions.

Example 3: Global synchronization with an arbitrary delay for an irreducible coupling

In this part, we will illustrate how to apply theorem 3 to judge whether the coupled system can be globally synchronized with any delays.

Consider four coupled Chua's circuits with a coupling delay:

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + c \sum_{j=1, j\neq i} a_{ij}\Gamma[x^j(t-\tau) - x^i(t)], \quad i = 1,2,3,4 \tag{35}$$

where $x^i(t) = [x^i_1(t), x^i_2(t), x^i_3(t)]^\top \in \mathbb{R}^3$, $i = 1,2,3,4$, $\Gamma = I_3$, $f(\cdot)$ is defined in (33), and the coupling matrix $A \in \mathbb{R}^{4,4}$ is

$$A = \begin{bmatrix} -1 & 0.2046 & 0.4560 & 0.3394 \\ 0.5761 & -1 & 0.1636 & 0.2603 \\ 0.2204 & 0.4171 & -1 & 0.3625 \\ 0.2945 & 0.3636 & 0.3418 & -1 \end{bmatrix}.$$

Its left eigenvector of A associated with eigenvalue 0 is (after normalization) $\xi = [0.2668, 0.2452, 0.2449, 0.2431]^\top$.

First, for the function $f(\cdot)$ defined in (33), we have

$$[Df^\top(x) + Df(x)]/2 \leq R = \begin{bmatrix} 1.2857 & 5.0000 & 0 \\ 5.0000 & -1.0000 & -6.6429 \\ 0 & -6.6429 & 0 \end{bmatrix}$$

of which eigenvalues are $-8.6325, 0.8107, 8.1075$. Therefore, letting $P = I_3$, $D = d \cdot I_3$, where $d > 8.1075$, we conclude that $f \in Quad(I_3, D, \alpha)$ for some positive constant α.

Next, we search the low bound of the coupling strength c_{cr} such that the coupled system (35) can be synchronized by verifying the linear matrix equalities (23). It is equivalent to solve the following Linear Matrix Inequalities (LMIs) (for more details about LMIs, see [39]):

$$\begin{array}{ll} minimize \ c & \\ subject\ to & \begin{cases} there\ exists\ a\ positive\ definite\ matrix\ Q \\ Z_1 = \begin{bmatrix} 2(d-c)C^\top \Lambda C + C^\top Q C & cC^\top \Lambda (A + I_m) C \\ cC^\top (A^\top + I_m) \Lambda C & -C^\top Q C \end{bmatrix} < 0 \end{cases} \end{array}$$

where $\Lambda = diag\{\xi\}$, and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\xi_1}{\xi_4} & -\frac{\xi_2}{\xi_4} & -\frac{\xi_3}{\xi_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.0974 & -1.0082 & 1.0071 \end{bmatrix}.$$

By the Matlab LMI and Control Toolboxes, we obtain $c_{cr} = 14.1150$ and

$$Q = 1.0 \times 10^3 \begin{bmatrix} 1.5047 & 1.3810 & 1.3793 & 1.3696 \\ 1.3810 & 1.2698 & 1.2674 & 1.2585 \\ 1.3793 & 1.2674 & 1.2676 & 1.2570 \\ 1.3696 & 1.2585 & 1.2570 & 1.2497 \end{bmatrix}.$$

Thus, the eigenvalues of Z_1 are $-11.1226, -3.8707, -2.7903, -0.9655, -0.0113, -0.0003$, which implies that the conditions of theorem 3 are satisfied. Therefore, if $c > c_{cr}$ fixed, the coupled Chua's circuits (35) can be globally synchronized with any coupling delay. We use the following quantity

$$err(t) = \frac{1}{t}\int_0^t \sum_{j=1}^{m} \log |x^j(\theta) - \bar{x}(\theta)|_2 d\theta$$

to measure the synchronization, where $\bar{x}(t)$ is the weighted average as defined before and the integration is computed by discrete method. Picking $c = 14.1200 > c_{cr}$, figure 7 shows the values of $err(t)$. One can see that $err(t)$ converges to zero for any delay $\tau \in [0, 15]$. That is, the coupled system (35) can be synchronized for any delay $\tau \in [0, 15]$.

Example 4: Global synchronization depending on the coupling delay

In this simulation, the coupled system (35) is used here to verify the effectiveness of Theorem 4. We will show that for some coupling strength c, the coupled system (35) can be globally synchronized for small coupling delays.

By careful estimations, we have $f \in Quad(P, D, \alpha)$ with $P = I_3$, $D = 10 \cdot I_3$ and $\alpha = 0.6218$. Moreover, $f \in H(M)$ with $M = 16.9754$.

To apply Theorem 4, we need to solve the following LMI:

$$\begin{array}{l} \min\ c \\ subject\ to\ C^\top\{\Lambda(cA + D)\}C < 0. \end{array}$$

By the Matlab LMI and Control Toolboxes, we obtain $c_{\min} = 7.9335$, which is much less than c_{cr} obtained in example 3. This implies that we can estimate the delay upper-bound by Theorem 4. Figure 8 indicates the variance of theoretically permitted delay upper-bounds for different coupling strength c obtained by Theorem 3. We use the following quantity to measure global synchronization:

$$syn = err(T), \;\; T = 200$$

The initial data are randomly chosen in a ball of radius 10. Figure 9 indicates that for some smaller c, the coupled system (35) can be globally synchronized only for a small delay τ.

Theorem 4 explores some aspects how the global synchronization depends on the parameters c and τ. However, it is inaccurate as shown in the figures 8 and 9. The region, where synchronization can be realized is much bigger than that given by Theorem 4. It is an interesting open problem to give more accurate estimation of the region rigorously.

Example 5: Global synchronization with reducible coupling matrix

In this part, we show how to apply theorem 5 for a reducible coupling matrix B. The coupling matrix is assumed as

$$B = \begin{bmatrix} A_{11} & A_{22} \\ 0 & A \end{bmatrix}$$

where A is the same as that examples 3 and 4, and

$$A_{11} = \begin{bmatrix} -1 & 0.0368 \\ 0.1285 & -1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0.1651 & 0.3459 & 0.4360 & 0.0162 \\ 0.3035 & 0.5043 & 0.0390 & 0.0247 \end{bmatrix}.$$

The corresponding terms in theorem 4 are given by $G^l = I_2$ and

$$\hat{A} = \begin{bmatrix} 0 & 0.0368 \\ 0.1285 & 0 \end{bmatrix}.$$

The uncoupled systems are Hopfield neural networks, which can be modelled as follows:

$$\frac{dv}{dt} = f(v) = -Dv + Tg(v) \tag{36}$$

where $v = [v_1, v_2, v_3]^\top \in \mathbb{R}^3$,

$$T = \begin{bmatrix} 1.2500 & -3.200 & -3.200 \\ -3.200 & 1.1000 & -4.4000 \\ -3.200 & 4.4000 & 1.000 \end{bmatrix}.$$

$D = I_3$, and $g(v) = [g(v_1), g(v_2), g(v_3)]$ where $g(s) = (|s+1| - |s-1|)/2$. As indicated in [43], system (36) has a double-scrolling chaotic attractor with initial condition: $x_1(0) = x_2(0) = x_3(0) = 0.1000$. Based on a result in [17] (simulation 4 in [17]), we have $f \in Quad(I_3, D, \alpha)$ with $D = 5.5685 \cdot I_3$ and some $\alpha > 0$.

To apply theorem 4, we solve the following LMIs:

$$\begin{array}{l} \min\ c \\ subject\ to \begin{cases} Q > 0 \\ Z_1 = \begin{bmatrix} 2(\delta - c)C^\top \Lambda C + C^\top QC & cC^\top \Lambda(A + I_m)C \\ cC^\top(A^\top + I_m)\Lambda C & -C^\top QC \end{bmatrix} < 0 \\ Q_1 > 0 \\ Z_2 = \begin{bmatrix} 2(\delta - c)I_2 + Q_1 & c\gamma_j \hat{A} \\ c\hat{A}^\top & -Q_1 \end{bmatrix} < 0 \end{cases} \end{array}.$$

Using Matlab LMI and Control Toolboxes, we obtain the minimal value $c^* = 9.7259$. That is, the coupled system can be globally synchronized with the reducible coupling matrix B if $c > 9.7259$. Figure 10 shows that $err(t)$ converges to zero with the coupling strength $c = 9.7260$ and coupling delay $\tau \in [0, 15]$.

6 Conclusions

In this paper, we provide a methodology based on theoretical analysis to judge whether a coupled system with a coupling delay can be synchronized. These methodologies can be realized by numerical way. For local synchronization, the largest transverse Lyapunov exponents in the transverse directions can be computed numerically by the method introduced in the previous work. For global synchronization, the LMI tools of mathematical software such as Matlab are useful to verify these obtained criteria. These theoretical analysis also provides some aspects how synchronization depends on the coupling strength c and the coupling delay τ. In case that the function $f(\cdot)$ is global Lipschitzian, the coupled system can be synchronized for any delay if the coupling strength c is large enough; on the other hand, if the coupled system can not be synchronized for any delay, it still can be synchronized at least for some small delays. These theoretical results are validated by numerical illustrations.

Furthermore, illustrations also reveal some unexpected phenomena. Large coupling strength c might not enhance synchronization. Instead, for some small coupling strength c, large delays might enhance synchronization The theoretical analysis of these phenomena is one of our future research topics.

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Appendices

Appendix 1.

Proof of proposition 1: Firstly, we prove that for any $M > K$, the bounded set $\mathcal{M} = \{\phi \in C_\tau : \|\phi\| = \phi^\top \phi \le M\}$ is invariant through the dynamical system (10). Here, the norm $|\cdot|_2$ denotes the 2-norm.

Suppose $\phi \in \mathcal{M}$, i.e. $\phi^\top \phi \le M$. Denote the solution $s(t) = s(t, \phi)$ is a solution of the delayed system (10) with the initial value ϕ.

We claim that under the conditions of Proposition 1, $|s(t)| \le M$ holds for all $t > 0$. In fact, $\|s(t)\| = \|\phi\| \le M$ at $t = 0$, which means $|s(t)| \le M$ for $-\tau \le t \le 0$. Now, suppose that $|s(t)| \le M$ for all $t < t_1$ and $|s(t_1)| = M$. Then, we have

$$
\begin{aligned}
&\frac{1}{2}\{\frac{d}{dt} s^\top(t)s(t)\}_{t=t_1} = s^\top(t_1) f(s(t_1)) + c\Big[s^\top(t_1)s(t_1-\tau) - s^\top(t_1)s(t_1)\Big] \\
&\le s^\top(t_1) f(s(t_1)) + c\Big[\frac{1}{2}s^\top(t_1)s(t_1) + \frac{1}{2}s^\top(t_1-\tau)s(t_1-\tau) - s^\top(t_1)s(t_1)\Big] \\
&\le s^\top(t_1) f(s(t_1)) \le -\delta
\end{aligned} \tag{37}
$$

which implies that $\|s(t)\|$ is non-increasing at point t_1. Therefore, $s(t)$ never exceeds M, i.e. $|s(t)| \le M$ for all $t > 0$ and $\mathcal{M}$ is invariant through the evolution.

Secondly, we will prove that the set $\mathcal{K} = \{\phi \in C_\tau, \|\phi\| \le K\}$ is globally attractive.

Define $V(s_t) = \frac{1}{2}s(t)^\top s(t) + \frac{c}{2}\int_{t-\tau}^{t} s^\top(\theta)s(\theta)d\theta$. Then, we have

$$\begin{aligned}\frac{dV(s_t)}{dt} &= s^\top(t)f(s(t)) + c\Big[s^\top(t)s(t-\tau) - s^\top(t)s(t) + \frac{1}{2}s^\top(t)s(t) - \frac{1}{2}s^\top(t-\tau)s(t-\tau)\Big] \\ &= s^\top(t)f(s(t)) - \frac{c}{2}\Big[s(t) - s(t-\tau)\Big]^\top\Big[s(t) - s(t-\tau)\Big] \\ &\le s^\top(t)f(s(t))\end{aligned}$$

By LaSalle principle [32], we conclude that the trajectory $s(t)$ converges to the maximum invariant set of $\{z \in \mathbb{R}^n : \ z^\top f(z) = 0\} \subset \mathcal{K}$. This implies that s_t will converge to the set $\mathcal{K}$. Proposition 1 is proved.

Appendix 2.

Proof of theorem 1: We will prove that under the assumptions of theorem 1, all the following systems

$$\frac{d\varphi(t)}{dt} = \Big[Df(s(t)) - cI_n\Big]\varphi(t) + c(\lambda_k + 1)\varphi(t-\tau), \quad k \ge 2 \tag{38}$$

are globally asymptotically stable.

Define $\mu = \overline{\lim_{t\to\infty}}|Df(s(t))|_2$ and $w(t) = \varphi(t)^\top\varphi(t)$, $w(t-\tau) = \varphi(t-\tau)^\top\varphi(t-\tau)$. Then, for a sufficient large t,

$$\begin{aligned}\frac{1}{2}\dot{w}(t) &= \varphi^\top(Df(s(t)) - cI_n)\varphi + c(1+\lambda_k)\varphi^\top\varphi_\tau \\ &\le (-c + \mu + \frac{c}{2}|1+\lambda_k|)w(t) + \frac{c}{2}|1+\lambda_k|w(t-\tau)\end{aligned}$$

By the similar arguments used in the proof of the previous theorems (or the Lyapunov-Krasovskii theorem [32,33]), we conclude that if

$$c > \frac{\mu}{1 - \max\limits_{k\ge 2}|1+\lambda_k|}, \tag{39}$$

then all the coupled systems in (38) are stable, which implies that the coupled system (6) is locally synchronized.

On the other hand, we rewrite the variational system (15) as

$$\frac{d\varphi(t)}{dt} = \Big[Df(s(t))) + c\lambda_k I_n\Big]\varphi(t) + c(1+\lambda_k)\int_t^{t-\tau}\dot{\varphi}(\theta)d\theta, \quad k \geq 2 \tag{40}$$

First, consider the following system:

$$\frac{d\psi}{dt} = \Big[Df(s(t)) + c\lambda_k I_n\Big]\psi. \tag{41}$$

One of its solution is

$$\psi(t) = e^{c\lambda_k t}\psi(0) + \int_0^t e^{c\lambda_k(t-\theta)}Df(s(\theta))\psi(\theta)d\theta.$$

This implies:

$$|\psi(t)| \leq |e^{c\lambda_k t}\psi(0)| + |e^{c\lambda_k t}|\int_0^t |e^{-c\lambda_k\theta}||Df(s(\theta))||\psi(\theta)|d\theta.$$

By Gronwall's inequality [44], we obtain

$$|\psi(t)| \leq C|\psi(0)|e^{\int_0^t [|Df(s(\theta))|+c\mathcal{R}e(\lambda_k)]d\theta}$$

where C is some positive constant. So, let $U(t,0)$ be the basic solution matrix of system (41). Then, we have $|U(t,0)| \leq Me^{-\beta t}$ where

$$\beta \geq c\mathcal{R}e(\lambda_k) - \overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t |Df(s(\theta))|d\theta = c\mathcal{R}e(\lambda_k) - < |Df(s(t))| > .$$

Here, $\mathcal{R}e(\cdot)$ denotes the real part of a complex number and $< \cdot >$ denotes the supper-limit of time average. So, we write the solution of the system (40) as follows:

$$\varphi(t) = U(t,0)\varphi(0) + c(1+\lambda_k)\int_0^t d\theta U(t,\theta)\int_\theta^{\theta-\tau}\dot{\varphi}(\alpha)d\alpha$$

Then, noting that $|\dot{\varphi}| \leq k_1|\varphi(t)| + k_2|\varphi(t-\tau)|$, where $k_1 = \overline{\lim_{t\to\infty}}|Df(s(t))| = \mu$ and $k_2 = c\max_{k\geq 2}|1+\lambda_k|$, we have

$$|\varphi(t)| \leq |U(t,0)||\varphi(0)| + c|1+\lambda_k|\tau \int_0^t |U(t,\theta)|(k_1+k_2)\chi(\theta)d\theta$$

where $\chi(t) = \max_{\theta\in[-2\tau,0]}|\varphi(t+\theta)|$. This implies that

$$\chi(t) \leq C_1 e^{-\beta t} + C_2 e^{-\beta t}\int_0^t e^{\beta\theta}\chi(\theta)d\theta$$

where $C_1 = |\varphi(0)|e^{2\beta\tau}$ and $C_2 = c|1+\lambda_k|\tau(k_1+k_2)e^{2\beta\tau}$. By Gronwall's inequality, we can conclude that

$$\chi(t) \leq Ce^{(-\beta+C_2)t}$$

holds for some constant $C > 0$. In other words, if

$$\begin{aligned} &\beta = c\min_{k\geq 2}\mathcal{R}e(\lambda_k) - <|Df(s(t))|>> 0, \\ &e^{2\beta\tau}\tau < \frac{\beta}{c(k_1+k_2)\max_{k\geq 2}|1+\lambda_k|}, \end{aligned} \tag{42}$$

Appendix 3.

Proof of theorem 3: Because of the matrix inequality (23), there exists a positive constant ϵ satisfying $-2\alpha I_n + \epsilon P < 0$ and

$$Z_j = \begin{bmatrix} 2(d_j - c\gamma_j)\Lambda + Q_j e^{\epsilon\tau} & c\gamma_j\Lambda(A+I_m) \\ c\gamma_j(A^\top + I_m\Lambda) & -Q_j \end{bmatrix} < 0$$

holds on the transverse subspace $L \times L$.

Define

$$L(t)=\Delta x^{\top}(t)\mathbf{P}\mathbf{\Lambda}\Delta x(t)e^{\epsilon t}+\sum_{j=1}^{n}\int_{t-\tau}^{t}p_j\Delta\tilde{x}^{j\top}(s)Q_j\Delta\tilde{x}^{j}(s)e^{\epsilon(s+\tau)}ds$$

Differentiating $L(t)$, we have

$$\begin{aligned}
\frac{dL(t)}{dt}&=\epsilon e^{\epsilon t}\Delta x^{\top}(t)\mathbf{P}\mathbf{\Lambda}\Delta x(t)+2e^{\epsilon t}\Delta x^{\top}(t)\mathbf{P}\mathbf{\Lambda}\bigg[(\mathbf{I}-\mathbf{\Xi})\Delta F(t)-\mathbf{D}\Delta x(t)\\
&+\bigg(\mathbf{D}-c\mathbf{I}\bigg)\Delta x(t)+c\bigg(\mathbf{I}+\mathbf{A}\bigg)\Delta x(t-\tau)\bigg]+\sum_{j=1}^{n}p_j\Delta\tilde{x}^{j\top}(t)Q_j\Delta\tilde{x}^{j}(t)e^{\epsilon(t+\tau)}\\
&-\sum_{j=1}^{n}p_j\Delta\tilde{x}^{j\top}(t-\tau)Q_j\Delta\tilde{x}^{j}(t-\tau)e^{\epsilon t}\\
&=e^{\epsilon t}\epsilon\sum_{i=1}^{m}\xi_i\Delta x^{i\top}(t)P\Delta x^{i}(t)\\
&+2e^{\epsilon t}\sum_{i=1}^{m}\xi_i\Delta x^{i\top}(t)P\bigg[f(x^{i}(t),t)-Dx^{i}(t)-f(\bar{x}(t),t)+D\bar{x}(t)\bigg]\\
&+2e^{\epsilon t}\Delta x^{\top}(t)\mathbf{P}\mathbf{\Lambda}(\mathbf{D}-c\mathbf{\Gamma})\Delta x(t)+2ce^{\epsilon t}\Delta x^{\top}(t)\mathbf{P}\mathbf{\Lambda}(\mathbf{I}+\mathbf{A})\Delta x(t-\tau)\\
&+\sum_{j=1}^{n}p_j\Delta\tilde{x}^{j\top}(t)Q_j\Delta\tilde{x}^{j}(t)e^{\epsilon(t+\tau)}-\sum_{j=1}^{n}p_j\Delta\tilde{x}^{j\top}(t-\tau)Q_j\Delta\tilde{x}^{j}(t-\tau)e^{\epsilon t}\\
&=e^{\epsilon t}\sum_{i=1}^{m}\xi_i\Delta x^{i\top}(t)\bigg(-2\alpha I_n+\epsilon P\bigg)\Delta x^{i}(t)\\
&+e^{\epsilon t}\sum_{j=1}^{n}p_j\bigg\{\Delta\tilde{x}^{j\top}(t)\bigg[2(d_j-c\gamma_j)\Lambda+Q_je^{\epsilon\tau}\bigg]\Delta\tilde{x}^{j}(t)\\
&+2c\gamma_j\Delta\tilde{x}^{j\top}(t)\Lambda(A+I_m)\Delta\tilde{x}^{j}(t-\tau)+\Delta\tilde{x}^{j\top}(t-\tau)Q_j\Delta\tilde{x}^{j}(t-\tau)\bigg\}\\
&=e^{\epsilon t}\sum_{i=1}^{m}\xi_i\Delta x^{i\top}(t)\bigg(-2\alpha I_n+\epsilon P\bigg)\Delta x^{i}(t)\\
&+e^{\epsilon t}\sum_{j=1}^{n}p_j[\Delta\tilde{x}^{j\top}(t),\Delta\tilde{x}^{j\top}(t-\tau)]Z_j\begin{bmatrix}\Delta\tilde{x}^{j}(t)\\ \Delta\tilde{x}^{j}(t-\tau)\end{bmatrix}\le 0
\end{aligned}$$

since $\Delta\tilde{x}^{j}(\theta)\in L$. Therefore, $L(t)\le L(0)$, which implies that $\Delta x(t)^{\top}\mathbf{P}\mathbf{\Lambda}\Delta x(t)\le L(0)e^{-\epsilon t}$. Hence, $\Delta x(t)$ converges to zero exponentially with rate $O(e^{-\frac{\epsilon}{2}t})$. Theorem 3 is proved.

Appendix 4.

Proof of theorem 4: It follows from the conditions listed in item 4 that there exists an $\varepsilon > 0$ such that

$$
\begin{aligned}
&(\varepsilon\|P\|_2 - 2\alpha)\min_i \xi_i + \sigma\left(c\tau\|\Gamma\|_2\|P\|_2\|\Lambda\|_2^2\|A + I_m\|_2^2\right) \\
&+3\eta k\left(M^2\|I_m - \Xi\|_2^2 + c^2\|A\|_2^2\|\Gamma\|_2^2\right)\sigma^{-1} < 0 \qquad (43)\\
&k > \frac{c\|\Gamma\|_2\|P\|_2}{1 - 3c^2\eta\tau\|A + I_m\|_2^2\|\Gamma\|_2^2} \qquad (44)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta &= \frac{e^{\varepsilon\tau} - 1}{\varepsilon} \\
\sigma &= \sqrt{\frac{3M^2\|I_m - \Xi\|_2^2 + 3c^2\|A\|_2^2\|\Gamma\|_2^2}{\|\Lambda\|_2^2\|A + I_m\|_2^2\left(1 - 3c^2\tau\eta\|A + I_m\|_2^2\|\Gamma\|_2^2\right)}}
\end{aligned}
$$

Denote

$$
\begin{aligned}
L_1(t) &= e^{\varepsilon t}\Delta x^\top(t)\mathbf{\Lambda}\mathbf{P}\Delta x(t) \\
L_2(t) &= k\int_{t-\tau}^{t} e^{\varepsilon(s+\tau)}ds\int_s^t \frac{d\Delta x(t)}{d\theta}^\top \frac{d\Delta x(t)}{d\theta}d\theta \\
\bar{L}(t) &= L_1(t) + L_2(t)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{dL_1(t)}{dt} = \varepsilon e^{\varepsilon t}\Delta x^\top(t)\mathbf{\Lambda P}\Delta x(t) + 2e^{\varepsilon t}\mathbf{\Lambda P}\Bigg[(\mathbf{I}-\mathbf{\Xi})\Delta F(t) \\
&-\mathbf{D}\Delta x(t) + (c\mathbf{A}+\mathbf{D})\Delta x(t) + c(\mathbf{A}+\mathbf{\Gamma})\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}ds\Bigg] \\
&\leq e^{\varepsilon t}(\varepsilon\|P\|_2 - 2\alpha)(\min_i \xi_i)\Delta x^\top(t)\Delta x(t) \\
&+ 2e^{\varepsilon t}\sum_{j=1}^{n} p_j \Delta\tilde{x}^{j\top}(t)\Lambda(cA\gamma_j + d_j I_m)\Delta\tilde{x}^j(t) \\
&+ 2ce^{\varepsilon t}\int_{t-\tau}^{t}\Delta x^\top(t)\mathbf{\Lambda P}(\mathbf{A}+\mathbf{\Gamma})\frac{d\Delta x(s)}{ds}ds
\end{aligned}
$$

By item 3, we know that $\left\{\Lambda(cA\gamma_j + d_j I_m)\right\}^s$ is negative definite in the transverse space L, i.e.,

$$
\Delta\tilde{x}^{j\top}(t)\left\{\Lambda(cA\gamma_j + d_j I_m)\right\}^s\Delta\tilde{x}^j(t) \leq 0
$$

Therefore,

$$
\begin{aligned}
&\frac{dL_1(t)}{dt} \leq e^{\varepsilon t}(\varepsilon\|P\|_2 - 2\alpha)(\min_i \xi_i)\Delta x^\top(t)\Delta x(t) \\
&+ c\tau\|\mathbf{\Lambda}\|_2|\mathbf{P}\|_2\left(\sigma\|\mathbf{A}+\mathbf{\Gamma}\|_2^2\Delta x^\top(t)\Delta x(t) + \sigma^{-1}\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^\top\frac{d\Delta x(s)}{ds}ds\right) \\
&= e^{\varepsilon t}(\varepsilon\|P\|_2 - 2\alpha)(\min_i \xi_i)\Delta x^\top(t)\Delta x(t) \\
&+ c\tau\|\Lambda\|_2|P\|_2\left(\sigma\|\Gamma\|_2^2\|A+I_m\|_2^2\Delta x^\top(t)\Delta x(t) + \sigma^{-1}\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^\top\frac{d\Delta x(s)}{ds}ds\right)
\end{aligned}
$$

Differentiating $L_2(t)$, we have

$$
\begin{aligned}
\frac{dL_2(t)}{dt} &= -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}ds + k\eta e^{\varepsilon t}\frac{d\Delta x(t)}{dt}^{\top}\frac{d\Delta x(t)}{dt}\\
&= -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}ds + k\eta e^{\varepsilon t}\bigg[(\mathbf{I}-\boldsymbol{\Xi})\Delta F(t) + c\mathbf{A}\Delta x(t)\\
&\quad +c(\mathbf{A}+\boldsymbol{\Gamma})\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}ds\bigg]^{\top}\bigg[(\mathbf{I}-\boldsymbol{\Xi})\Delta F(t) + c\mathbf{A}\Delta x(t)\\
&\quad +c(\mathbf{A}+\boldsymbol{\Gamma})\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}ds\bigg]\\
&\le -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}ds + 3k\eta e^{\varepsilon t}\bigg[\Delta F^{\top}(t)(\mathbf{I}-\boldsymbol{\Xi})^{\top}(\mathbf{I}-\boldsymbol{\Xi})\Delta F(t)\\
&\quad +c^2\Delta x^{\top}(t)\mathbf{A}^{\top}\mathbf{A}\Delta x(t)\\
&\quad +c^2\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}ds(\mathbf{A}+\boldsymbol{\Gamma})^{\top}(\mathbf{A}+\boldsymbol{\Gamma})\int_{t-\tau}^{t}\frac{d\Delta x(\theta)}{d\theta}d\theta\bigg]
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}ds\int_{t-\tau}^{t}\frac{d\Delta x(\theta)}{d\theta}d\theta = \int_{t-\tau}^{t}ds\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(\theta)}{d\theta}d\theta\\
&\le \frac{1}{2}\bigg(\int_{t-\tau}^{t}ds\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}d\theta + \int_{t-\tau}^{t}ds\int_{t-\tau}^{t}\frac{d\Delta x(\theta)}{d\theta}^{\top}\frac{d\Delta x(\theta)}{d\theta}d\theta\bigg)\\
&= \tau\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}ds
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{dL_2(t)}{dt} \le -ke^{\varepsilon t}\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}ds + 3k\eta e^{\varepsilon t}\bigg[M^2\|I_m-\Xi\|_2^2\Delta x^{\top}(t)\Delta x(t)\\
&+c^2\|A\|_2^2\|\Gamma\|_2^2\Delta x^{\top}(t)\Delta x(t) + c^2\tau\|A+I_m\|_2^2\|\Gamma\|_2^2\int_{t-\tau}^{t}\frac{d\Delta x(s)}{ds}^{\top}\frac{d\Delta x(s)}{ds}ds\bigg]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d\bar{L}(t)}{dt} \leq e^{\varepsilon t}\Bigg[&(\varepsilon\|P\|_2 - 2\alpha)(\min_i \xi_i) + c\sigma\tau\|\Lambda\|_2\|P\|_2\|A + I_m\|_2^2\|\Gamma\|_2^2 \\
&+3k\eta M^2\|I_m - \Xi\|_2^2 + 3k\eta c^2\|A\|_2^2\|\Gamma\|_2^2\Bigg]\Delta x^\top(t)\Delta x(t) \\
&+e^{\varepsilon t}\Bigg[c\|\Lambda\|_2\|P\|_2\sigma^{-1} - k + 3k\eta\tau c^2\|A + I_m\|_2^2\|\Gamma\|_2^2\Bigg]
\end{aligned}
$$

From inequalities (44), we have

$$
\frac{d\bar{L}(t)}{dt} \leq 0
$$

which implies that $L(t) \leq L(0)$. Therefore, $\Delta x(t)$ converges to zero exponentially with rate $O(e^{-\frac{\varepsilon}{2}t})$. The theorem is proved.

Appendix 5.

Proof of theorem 5: In view of the comments in section 4.2, letting $\Delta x^i(t) = x^i(t) - s(t)$, we only need to prove that for system (29), $\Delta x^i(t)$ will globally exponentially converge to zero. Omitting the term $O(e^{-\epsilon t})$, we have

$$
\frac{d\Delta x^i(t)}{dt} = \Bigg[f(x^i(t),t) - f(s(t),t)\Bigg] + c\sum_{j\in N_l, j\neq i} a_{ij}\Gamma\Delta x^j(t-\tau) - c\Gamma\Delta x^i(t), \quad i \in N_l \tag{45}
$$

Let

$$
L_3(t) = \sum_{i\in N_l} e^{\epsilon t} G_i^l \Delta x_i^\top(t) P \Delta x^i(t) + \sum_{j=1}^{n} p_j \int_{t-\tau}^{t} \Delta\tilde{x}^{j\top}(s) Q_j^l \Delta\tilde{x}^j(s) e^{\epsilon(s+\tau)} ds
$$

where ϵ satisfies

$$
Z_j^l = \begin{bmatrix} 2(d_j - c\gamma_j)G^l + Q_j^l e^{\epsilon\tau} & c\gamma_j G^l A_l \\ c\gamma_j A_l^\top G^l & -Q_j^l \end{bmatrix} < 0
$$

on the corresponding transverse subspace $L^q = \left\{(v_1, \cdots, v_{m_q})^\top \in R^{m_q} : \sum_{i=1}^{m_q} \xi_i^q v_i = 0\right\}$, with $\epsilon P - 2\alpha I_n < 0$. Let $\Delta\tilde{x}^j(t) = [\Delta x_j^1(t), \cdots, \Delta x_j^{m_l}(t)]^\top \in \mathbb{R}^{m_l}$.

Differentiating $L_3(t)$, we have

$$
\begin{aligned}
\frac{dL_3(t)}{dt} =& \epsilon e^{\epsilon t} \sum_{i \in N_l} G_i^l \Delta x_i^\top(t) P \Delta x^i(t) + 2e^{\epsilon t} \sum_{i \in N_l} G_i^l \Delta x^i(t) P \Bigg\{ \Bigg[f(x^i(t), t) - f(s(t), t) \\
& -D\Delta x^i(t) \Bigg] + c \sum_{j \in N_l, j \neq i} a_{ij} \Gamma \Delta x^j(t-\tau) + D\Delta x^i(t) - c\Gamma \Delta x^i(t) \Bigg\} \\
& + e^{\epsilon t} \sum_{j=1}^{n} p_j \Delta\tilde{x}^{j\top}(t) Q_j^l \Delta\tilde{x}^j(t) e^{\epsilon\tau} - e^{\epsilon t} \sum_{j=1}^{n} p_j \Delta\tilde{x}^{j\top}(t-\tau) Q_j^l \Delta\tilde{x}^j(t-\tau) \\
\le & e^{\epsilon t} \sum_{i \in N_l} e^{\epsilon t} G_i^l \Delta x_i^\top(t) (\epsilon P - 2\alpha I_n) \Delta x^i(t) \\
& + e^{\epsilon t} \sum_{j=1}^{n} p_j \Bigg\{ \Delta\tilde{x}^{j\top}(t) \Bigg[2(d_j - c\gamma_j) G^l + Q_j^l e^{\epsilon\tau} \Bigg] \Delta\tilde{x}^j(t) \\
& + 2c\gamma_j \Delta\tilde{x}^{j\top}(t) \hat{A} \Delta\tilde{x}^j(t-\tau) - \Delta\tilde{x}^{j\top}(t-\tau) Q_j^l \Delta\tilde{x}_j(t-\tau) \Bigg\} \\
\le & e^{\epsilon t} \sum_{j=1}^{n} p_j [\Delta\tilde{x}^{j\top}(t), \Delta\tilde{x}^{j\top}(t-\tau)] Z_j^l \begin{bmatrix} \Delta\tilde{x}^j(t) \\ \Delta\tilde{x}^j(t-\tau) \end{bmatrix} \\
\le & 0
\end{aligned}
$$

By induction and theorem 4, we conclude the global exponential synchronization of the coupled system (6) with the reducible coupling matrix A. The theorem is proved.

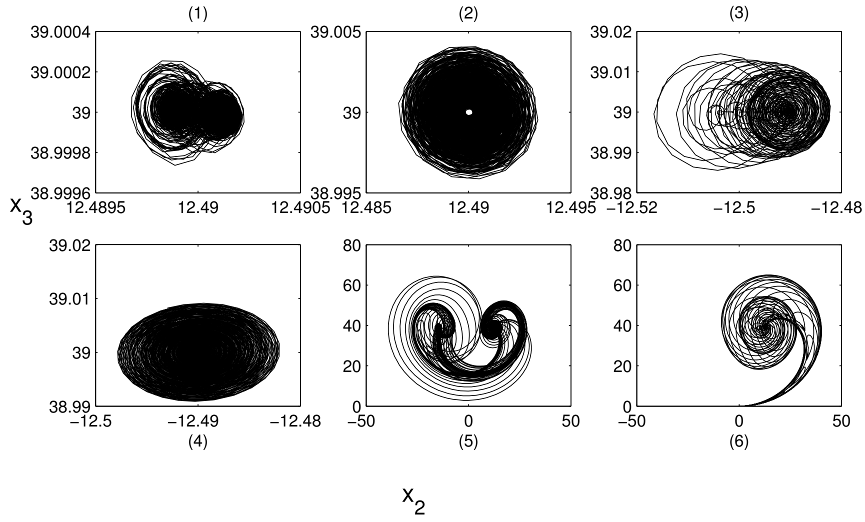

Fig. 1. The attractors of Lorenz system with a delay term $c[s(t-\tau)-s(t)]$: (1) $c=5$, $\tau=0.5$; (2) $c=5$, $\tau=5$; (3) $c=5$, $\tau=15$; (4) $c=10$, $\tau=0.5$; (5) $c=10$, $\tau=5$; (6) $c=10$, $\tau=15$.

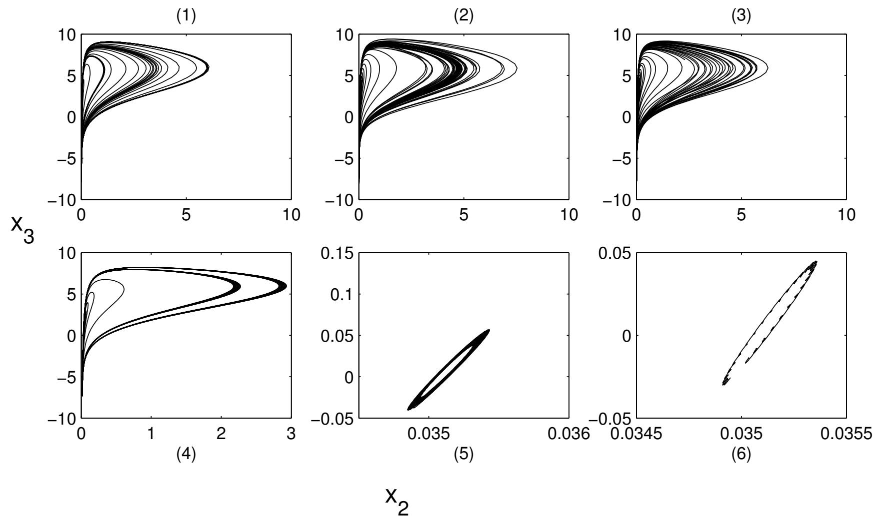

Fig. 2. The attractors of Rossel system with a delay term $c[s(t-\tau)-s(t)]$: (1) $c=5$, $\tau=0.5$; (2) $c=5$, $\tau=5$; (3) $c=5$, $\tau=15$; (4) $c=10$, $\tau=0.5$; (5) $c=10$, $\tau=5$; (6) $c=10$, $\tau=15$.

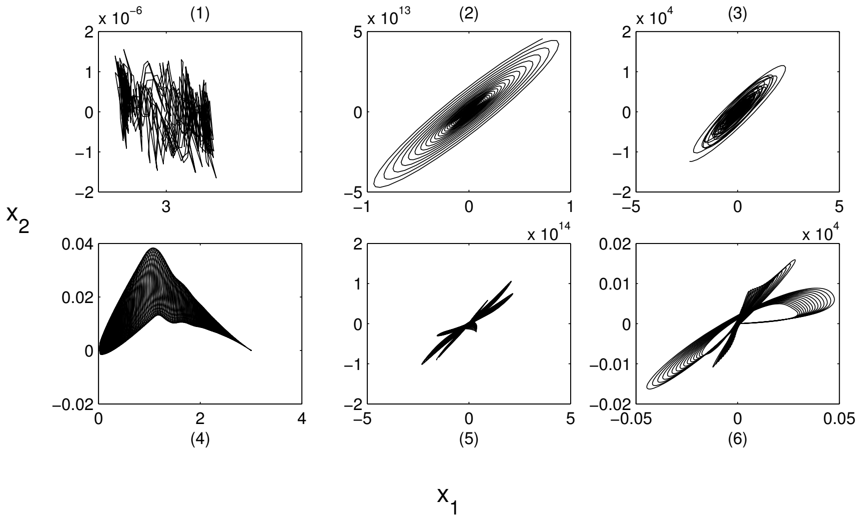

Fig. 3. The attractors of Chua's circuits with a delay term $c[s(t-\tau)-s(t)]$: (1) $c=5$, $\tau=0.5$; (2) $c=5$, $\tau=5$; (3) $c=5$, $\tau=15$; (4) $c=10$, $\tau=0.5$; (5) $c=10$, $\tau=5$; (6) $c=10$, $\tau=15$.

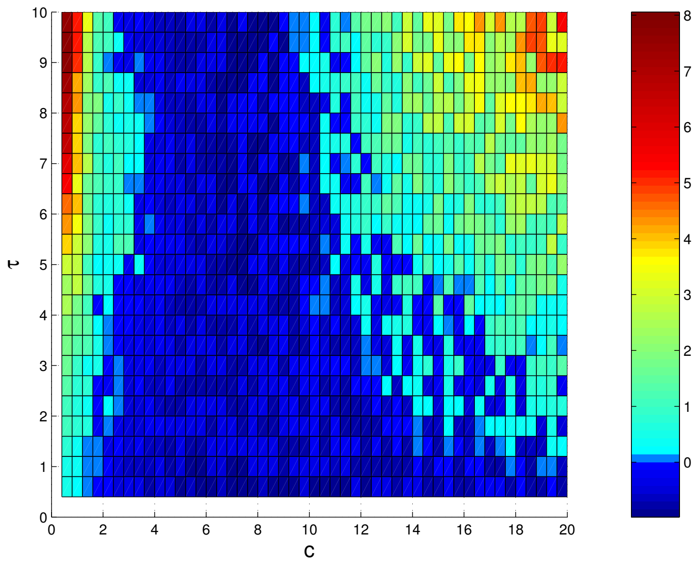

Fig. 4. Variance of mle of coupled Lorenz systems with respect to various coupling strength c and coupling delay τ.

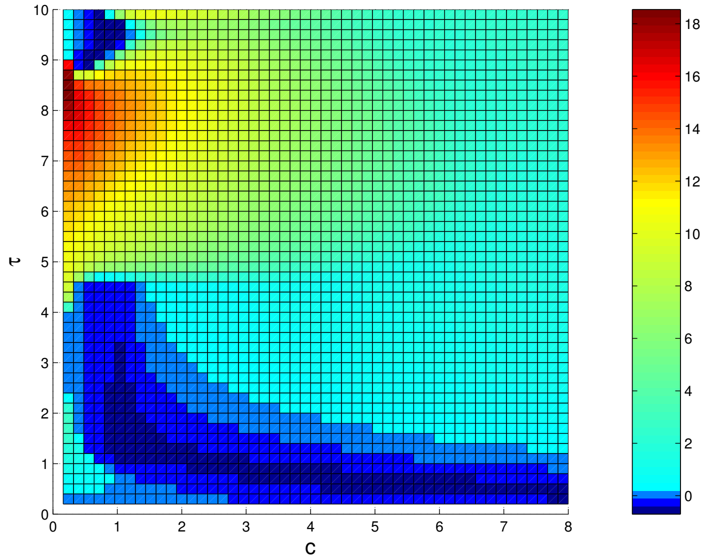

Fig. 5. Variance of mle of coupled Rossel systems with respect to various coupling strength c and coupling delay τ.

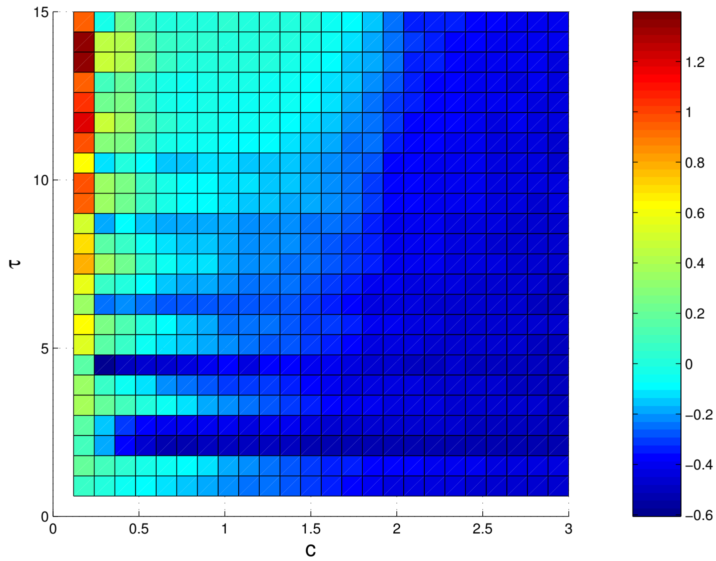

Fig. 6. Variance of mle of coupled Chua's circuits with coupling delay for various coupling strength c and coupling delay τ.

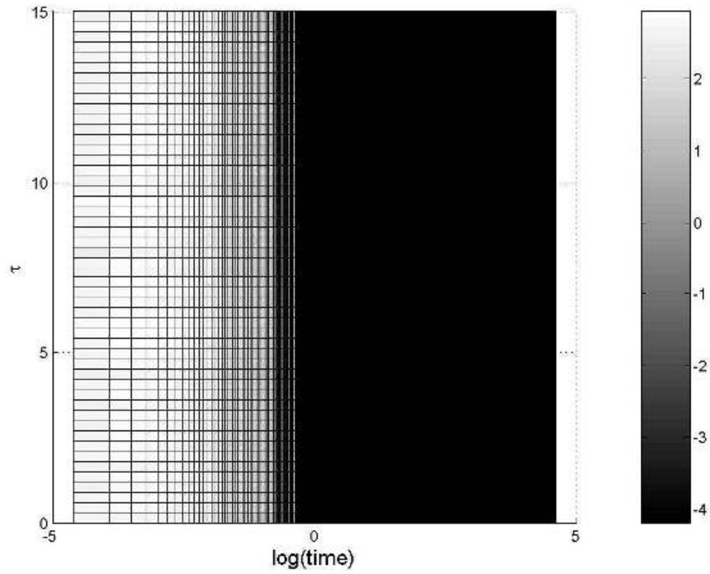

Fig. 7. Variance of $err(t)$ indicated by color-grayness for coupled Chua's circuits (35) with respect to the logarithm of time and the coupling delay, picking c=14.1200.

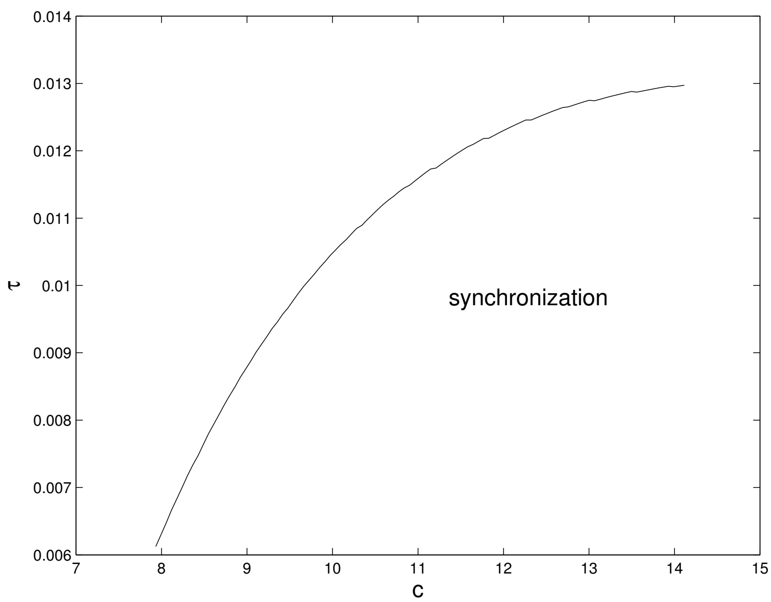

Fig. 8. Synchronization region obtained by theorem 4 in the plane (c, τ).

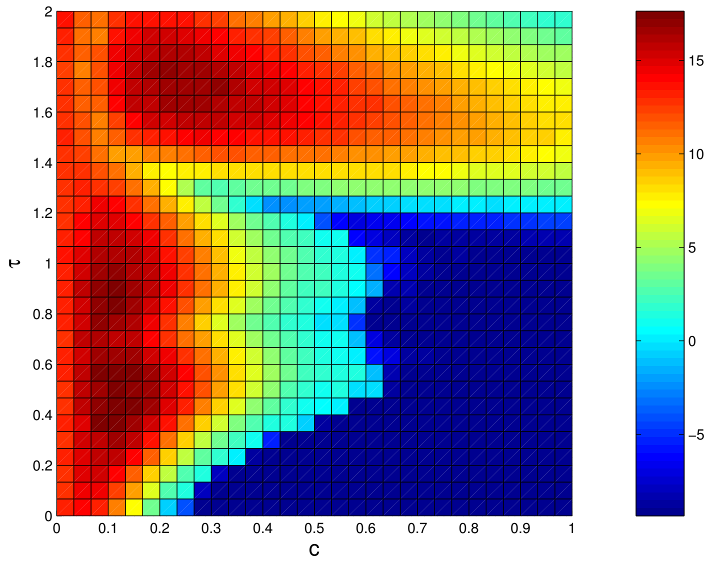

Fig. 9. Variance of syn with respect to τ and c for coupled Chua's circuits (35).

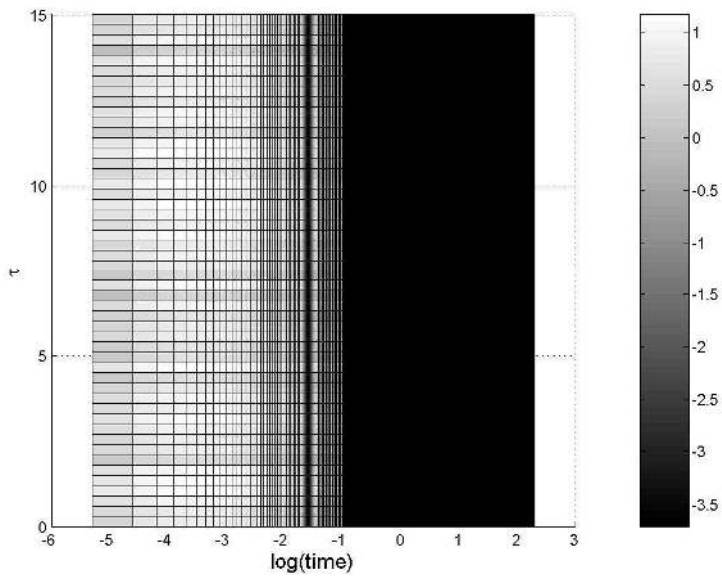

Fig. 10. Variance of $err(t)$ indicated by color grayness with respect to the logarithm of time and the coupling delay τ, picking $c = 14.1150$.