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SYNCHRONIZATION AND GRAPH TOPOLOGY

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Dedicated to Leonid P. Shilnikov on the Occasion of his 70th Birthday.

This paper clarifies the relation between synchronization and graph topology. Applying the Connection Graph Stability method developed by Belykh *et al.* [2004a] to the study of synchronization in networks of coupled oscillators, we show which graph properties matter for synchronization. In particular, while we explicitly link the stability of synchronization with the average path length for a wide class of coupling graphs, we prove by a simple argument that the average path length is not always the crucial quantity for synchronization. We also show that synchronization in scale-free networks can be described by means of regular networks with a star-like coupling structure. Finally, by considering an example of coupled Hindmarsh–Rose neuron models, we demonstrate how global stability of synchronization depends on the parameters of the individual oscillator.

Keywords: Synchronization; connection graph; path length.

1. Introduction

Synchronization of two limit-cycle systems traditionally means that their time evolution is periodic, with the same period and, perhaps, the same phase. Synchrony in periodic systems is often divided into two classes: synchronization of an oscillator by an external force (external synchronization) and mutual synchronization of two coupled nonlinear oscillators. Classical mathematical studies of external synchronization are due to van der Pol [1927], Andronov and Vitt [1930], Cartwright and Littlewood [1945]. Discovered by Huygens around 1665, mutual synchronization of two periodic oscillators was first analytically studied by Maier [1935] (for an extensive review on classical synchronization, see [Pikovsky *et al.*, 2001]).

From a mathematical point of view we understand the synchronization of two periodic oscillators as a bifurcation transition from quasiperiodic motion (beating) to regular periodic behavior for the system as a whole. The quasiperiodic behavior is associated with the existence of an ergodic torus. The synchronization regime corresponds to the existence of a stable periodic orbit on the (resonant) torus, in a synchronization zone known as Arnold's tongue. Therefore, the main problem in the studies of classical synchronization is (i) to find a regime of

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synchrony in the parameter space, and (ii) to determine possible routes to the loss of synchronization on the boundaries of this region. Within the framework of bifurcation theory, this problem was extensively studied by Afraimovich and Shilnikov [1974, 1977, 1991]. In particular, they rigorously described different scenarios of a torus break-down (for a review, see [Shilnikov *et al.*, 2004; Shilnikov *et al.*, 1998, 2001]).

More recently, synchronization of chaotic systems has been discovered [Fujisaka & Yamada, 1983; Afraimovich *et al.*, 1986; Pecora & Carroll, 1990] and since then it has become an important research topic in mathematics, physics, and engineering (for reviews on chaos synchronization, see, e.g. [Pecora *et al.*, 1997; Kurths *et al.*, 2003]).

The strongest form of synchrony in chaotic systems is complete synchronization when all oscillators of the network acquire identical chaotic behaviors. Similar, to some extent, to classical synchronization, the complete chaotic synchronization implies that the dynamics in the phase space is restricted to a lower dimensional manifold. If the oscillators are identical, this submanifold is a hyperplane. This leads to two fundamental considerations in studying chaos synchronization; (i) finding the synchronous solution (hyperplane), if synchronization is not complete and (ii) determining its stability. In this context, the central question about complete synchronization in networks of oscillators is: When is such synchronous behavior stable, especially in regard to coupling strengths and coupling configurations of the network?

Most known methods for determining stability for synchronized chaotic systems are based on the calculation of the eigenvalues of the coupling matrix for different regular coupling schemes and a term depending on the dynamics of the individual oscillators, see, e.g. [Belykh *et al.*, 1992; Wu & Chua, 1996; Pecora & Carroll, 1998; Pecora, 1998; Pogromsky & Nijmeijer, 2001]. A general approach to the local synchronization of chaotic systems for any linear coupling scheme, called the Master Stability function, was developed by Pecora and Carroll [1998]. Stronger, but more conservative, global stability results for synchronization in networks of coupled chaotic systems were also obtained [Wu & Chua, 1996; Pogromsky & Nijmeijer, 2001; Wu, 2002].

In recent papers, Belykh *et al.* [2004a, 2005b] have developed an alternative method for proving complete synchronization in networks of coupled limit-cycle or chaotic oscillators with arbitrary connection graphs. This approach, called the Connection Graph Stability method, combines the Lyapunov function approach with graph theoretical reasoning. The method directly links synchronization with graph theory and allows one to avoid calculating the eigenvalues of the coupling matrix. It is also applicable to time-dependent networks.

The purpose of this paper is to proceed with the application of the Connection Graph Stability method to the study of global synchronization in regular and complex networks and to elucidate the relation between network dynamics and graph theory. The layout of this paper is as follows. First, in Sec. 2, we state the problem under consideration. We briefly review the main known methods for determining stability for synchronized chaotic systems and touch upon their strengths and weaknesses. We revisit the Connection Graph Stability method and, in Sec. 3, apply it to regular and complex networks of oscillators. Finally, in the Appendix, as an illustrative example we derive a proof for global stability of synchronization between coupled Hindmarsh–Rose models.

2. Network Synchronization

2.1. *Network considered*

We consider a network of n identical oscillators that are linearly and mutually coupled:

$$\dot{x}_i = F(x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t) P x_j, \quad i = 1, \ldots, n. \tag{1}$$

Here, x_i is the d-vector containing the coordinates of the ith oscillator, and $F(x_i)$ is a nonlinear vector function defining the dynamics of the individual oscillator (whether periodic or chaotic). The nonzero elements of the $d \times d$ matrix P determine which variables couple the oscillators. The connectivity (Laplacian) matrix $G = (\varepsilon_{ij}(t))$ is an $n \times n$ symmetric matrix with zero row-sums and non-negative off-diagonal elements. The matrix G is assumed to have one zero eigenvalue λ_1 and $n-1$ negative eigenvalues $\lambda_n < \cdots < \lambda_3 < \lambda_2 < 0$. The matrix G defines a connected graph with n vertices and m edges. The vertices of the graph correspond to the individual oscillators, and the graph has an edge between nodes i and j if $\varepsilon_{ij} = \varepsilon_{ji} > 0$.

The completely synchronous state of system (1) is defined by the linear invariant manifold $D = \{x_1 = x_2 = \cdots = x_n\}$, often called the synchronization manifold. The eigenvector with eigenvalue $\lambda_1 = 0$ corresponds to the longitudinal direction along the synchronization manifold. Being negative, all the other eigenvalues $\lambda_2, \ldots, \lambda_n$ correspond to transversal directions and are crucial for the transversal stability of D.

Typically, in networks of continuous time oscillators (1), the synchronization manifold becomes stable when the coupling strengths between the oscillators exceed a critical value ε^*. This threshold ε^* depends on the individual oscillator dynamics (limit-cycle oscillators are easier to synchronize than chaotic systems) and on the graph topology. Indeed, locally coupled networks are very resistant to synchronization, whereas global couplings facilitate synchronization and can significantly lower the synchronization threshold.

2.2. *Eigenvalue methods*

An important step in the research on network synchronization is the Wu–Chua conjecture [Wu & Chua, 1996]. Although it is not true in general [Pecora, 1998; Pogromsky & Nijmeijer, 2001], it shows how the second largest eigenvalue λ_2 of the connectivity matrix G influences the synchronization condition.

2.2.1. *The Wu–Chua conjecture*

This conjecture involves a relation between the eigenvalue λ_2 and the synchronization threshold in networks of various sizes. The conjecture is as follows. If a network with n_1 number of oscillators synchronizes at the coupling $\varepsilon^*_{n_1}$, then another similarly coupled network of size n_2 will synchronize at $\varepsilon^*_{n_2}$, such that the following relation is preserved:

$$\varepsilon^*_{n_1}\lambda_2(n_1) = \varepsilon^*_{n_2}\lambda_2(n_2),$$

where $\lambda_2(n_1)$ and $\lambda_2(n_2)$ are the second largest eigenvalues of G for the n_1 and n_2 networks, respectively. In other words, it claims that the synchronization threshold ε^* in a network of an arbitrary size can be predicted from synchronization in two coupled oscillators (since $\lambda_2(2) = -2$):

$$\varepsilon^*_n = \frac{2\varepsilon^*_2}{|\lambda_2(n)|}, \tag{2}$$

where ε^*_2 is the coupling threshold for synchronization in network (1) composed of two oscillators.

Wu and Chua examined the stability of the least stable transversal mode associated with the eigenvalue λ_2, assuming that when this mode was stable, all other transversal modes would remain stable. For coupled systems which undergo short-wave bifurcations [Pecora, 1998] and desynchronize with increasing coupling, this assumption can fail. Examples include x-coupled Rössler oscillators. Belykh *et al.* [2000] have linked these desynchronization bifurcations to the equilibria disappearance and the presence of different invariant sets lying outside the synchronization manifold.

The conjecture is true if applied to individual transversal modes. Therefore, in networks of limit-cycle and chaotic oscillators (e.g. Lorenz, double-scroll, and Hodgkin–Huxley-type models) where there is no desynchronization with increasing coupling, it serves as a guide to global synchronization and correctly predicts the synchronization threshold for different coupling schemes and number of oscillators n (for a detailed discussion, see [Pecora, 1998]).

2.2.2. *Master stability function*

A more universal method called the Master Stability Function [Pecora & Carroll, 1998] takes into account not only the second eigenvalue λ_2, but the eigenratio λ_2/λ_n, where λ_n is the lowest eigenvalue of G. This method shows that when considering the stability of synchronization for the entire network, one must examine the stability diagrams corresponding to different eigenvalues (not only to λ_2) and make sure that all modes are stable at the chosen coupling.

This powerful approach to the local stability of the synchronization manifold is based on the calculation of the maximum transversal Lyapunov exponent and is widely used in studies of synchronization in complex networks (see, e.g. [Barahona & Pecora, 2002; Nishikawa *et al.*, 2003]).

2.3. *Connection graph stability method*

The eigenvalue methods are difficult to apply analytically for irregular graphs. Moreover, for networks of oscillators with a time-varying coupling (where the connectivity matrix is time-dependent), the use of methods based on the eigenvalues of the connectivity matrix and the Lyapunov exponents is often impossible (the linearized system becomes time-dependent and may fail to provide the stability results).

Recently, Belykh *et al.* [2004a, 2004b] have developed a new approach, the Connection Graph

Stability (CGS) method, based on the calculation of the path lengths in the networks. The main step of the method is to choose a path $P_{i,j}$ from node i to node j, for any pair of nodes (i, j) and then to calculate the total length of all chosen paths passing through an edge k on the network connection graph. The coupling constant that guarantees complete synchronization is proportional to this sum, or rather, the maximal value of this sum when k varied.

Theorem 1 [Belykh *et al.*, 2004a] (sufficient conditions). *The synchronization manifold of the system (1) is globally asymptotically stable if*

$$\varepsilon_k(t) > \frac{a}{n}\, b_k(n, m) \quad \text{for } k = 1, \ldots, m \text{ and for all } t. \tag{3}$$

Here, $a = 2\varepsilon_2^*$ is the double coupling strength sufficient for global synchronization of two oscillators.[1] *The quantity $b_k(n, m) = \sum_{j>i;\, k\in P_{ij}}^{n} |P_{ij}|$ is the sum of the lengths of all chosen paths P_{ij} which pass through a given edge k that belongs to the coupling graph.*

The first step of the method is to calculate the parameter a and to prove that two coupled oscillators globally synchronize when exceeding a. This condition has to be proven for each particular situation (for the concrete individual system and the matrix P).

The details of this proof for coupled Lorenz systems can be found in [Belykh *et al.*, 2003; 2004a]. As an illustrative example, in the Appendix, we derive the proof for global stability of synchronization between Hindmarsh–Rose neuron models and calculate an upper bound for the synchronization threshold ε_2^*.

The second step is to calculate $b_k(n, m)$. To do so, we first choose a set of paths $\{P_{ij}|i, j = 1, \ldots, n,\ j > i\}$, one for each pair of vertices i, j, and determine their lengths $|P_{ij}|$, the number of edges in each P_{ij}. Then, for each edge k of the connection graph we calculate the sum $b_k(n, m)$ of the lengths of all P_{ij} passing through k.

For a given choice of paths P_{ij} we obtain for each ε_k a lower bound (3). If, in a given coupled network, all lower bounds on the coupling strengths ε_k are satisfied, Theorem 1 guarantees complete synchronization. Often, all coupling strengths in a network are equal, i.e. $\varepsilon_k = \varepsilon$ for all k. In this case the lower bound for ε is $\varepsilon^* = \max_k(a/n)\, b_k(n, m)$. This amounts to determining the edge k such that the sum of the lengths of all paths through k is maximal (the weakest link).

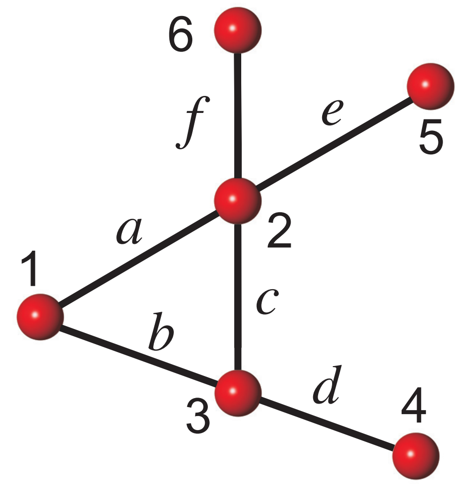


Fig. 1. Example of a network for which the choice of the shortest paths between the nodes is not optimal for calculating b_k.

Clearly, the bound ε^* we obtain by this method depends on the choice of the paths P_{ij}. The number of possible choices of paths is normally huge. However, most of these choices are clearly suboptimal. Usually, one takes for P_{ij} the shortest path from vertex i to vertex j. Sometimes, however, a different choice of paths can lead to lower bounds (3). The following example supports this claim and shows how to apply Theorem 1 to a concrete network.

Example 1. Consider the network (1) shown in Fig. 1 and calculate b_k. Choose the shortest paths between the nodes: $P_{12} = a$, $P_{13} = b$, $P_{14} = bd$, $P_{15} = ae$, $P_{16} = af$, $P_{23} = c$, $P_{24} = cd$, $P_{25} = e$, $P_{26} = f$, $P_{34} = d$, $P_{35} = ce$, $P_{36} = cf$, $P_{45} = dce$, $P_{46} = dcf$, $P_{56} = fe$. Calculate the sum of path lengths passing through edge

$$a:\ b_a = |P_{12}| + |P_{15}| + |P_{16}| = 1 + 2 + 2 = 5$$

$$b:\ b_b = |P_{13}| + |P_{14}| = 1 + 2 = 3$$

$$c:\ b_c = |P_{23}| + |P_{24}| + |P_{35}| + |P_{36}| + |P_{45}| + |P_{46}| = 13$$

[1] More precisely, under the condition (3), $\sum_{j>i} \|Px_j - Px_i\|^2$ is a global Lyapunov function, where in (3) $a = 2\varepsilon_2^*$ and ε_2^* is the minimum coupling strength between two oscillators such that $\|Px_2 - Px_1\|^2$ is a global Lyapunov function.

$$d: \; b_d = |P_{14}| + |P_{24}| + |P_{34}| + |P_{45}| + |P_{46}| = 11$$
$$e: \; b_e = |P_{15}| + |P_{25}| + |P_{35}| + |P_{45}| + |P_{56}| = 10$$
$$f: \; b_d = |P_{16}| + |P_{26}| + |P_{36}| + |P_{46}| + |P_{56}| = 10$$

and take the maximum $b_c = 13$ as an upper bound. Now change P_{23} from c to ab and recalculate the sum of path lengths (this change only affects edges a, b and c). Hence,

$$a: \; b_a = |P_{12}| + |P_{15}| + |P_{16}| + |P_{23}| = 7$$
$$b: \; b_b = |P_{13}| + |P_{14}| + |P_{23}| = 5$$
$$c: \; b_c = |P_{24}| + |P_{35}| + |P_{36}| + |P_{45}| + |P_{46}| = 12.$$

Thus, the maximum sum has been reduced to $b_c = 12$, and consequently it gives a lower upper bound ε^* for the synchronization threshold. In other words, by redistributing the chosen paths, we have decreased traffic load on the most loaded edge c. Here, the load of a node quantifies the communication traffic passing through it.

Changing P_{36} from cf to abf, one gets even lower bounds: $b_a = 10$; $b_b = 8$, $b_c = 10$, $b_f = 11$ such that the final synchronization threshold becomes $\varepsilon^* = a \cdot 11/6$. This also shows that homogeneity of load distribution often (but not always) favors synchronization.

Note that if we admit coupling coefficients ε_k that depend on the edge k, then a "synchronization threshold" would be a vector $\varepsilon_1^*, \ldots, \varepsilon_m^*$ such that for all vectors $\varepsilon_1, \ldots, \varepsilon_m$ with $\varepsilon_1 > \varepsilon_1^*, \ldots, \varepsilon_m > \varepsilon_m^*$ complete synchronization is achieved. In this setting, synchronization thresholds are not unique. Indeed, the two choices of paths lead to two bounds on synchronization thresholds that are not comparable. The first one is

$$\varepsilon_a^* = \frac{5a}{6}, \quad \varepsilon_b^* = \frac{3a}{6}, \quad \varepsilon_c^* = \frac{13a}{6},$$
$$\varepsilon_d^* = \frac{11a}{6}, \quad \varepsilon_e^* = \frac{10a}{6}, \quad \varepsilon_f^* = \frac{10a}{6},$$

whereas the second is

$$\varepsilon_a^* = \frac{10a}{6}, \quad \varepsilon_b^* = \frac{8a}{6}, \quad \varepsilon_c^* = \frac{10a}{6},$$
$$\varepsilon_d^* = \frac{11a}{6}, \quad \varepsilon_e^* = \frac{10a}{6}, \quad \varepsilon_f^* = \frac{10a}{6}.$$

It is also worth noticing that the value b_k/n can be replaced by $1/|\lambda_2|$ coming from the eigenvalue method. In this context, b_k/n can be thought of as a graph-based estimate of $1/|\lambda_2|$. In fact, bound (2) is the best possible for networks admitting global synchronization and an analytical derivation of a (cf. Theorem 1). Coming from the sum of the path lengths, the estimate given in Theorem 1 is somewhat suboptimal, but it has the advantage that it can be analytically obtained not only for regular, but also for quite irregular networks. Also, it allows one to link explicitly the conditions for the stability of synchronization with the average path length of a wide class of the coupling graphs.

As the study of Laplacian eigenvalues pervades many mathematical disciplines that use graphs, there is no surprise that there exists an embedding approach to estimate λ_2 [Guattery & Miller, 2000]. The essence of this embedding method is that a complete graph H is embedded into the connection graph G, and both H and G have the same number of nodes. Any edge from node i to j in G is expressed as path(s) in H, thus forming the embedding matrix Γ. The matrix Γ depends on the choice of these paths. The main step is to choose the paths in the embedding, according to Kirchoff's laws. The error of λ_2 estimation is proportional to $\log^2(n)$ in the worst case such that the approximation is not tight but still reasonable. A nice aspect of this graph method, closely related to the CGS method, is that it allows us to interpret synchronization in terms of Kirchoff's laws.

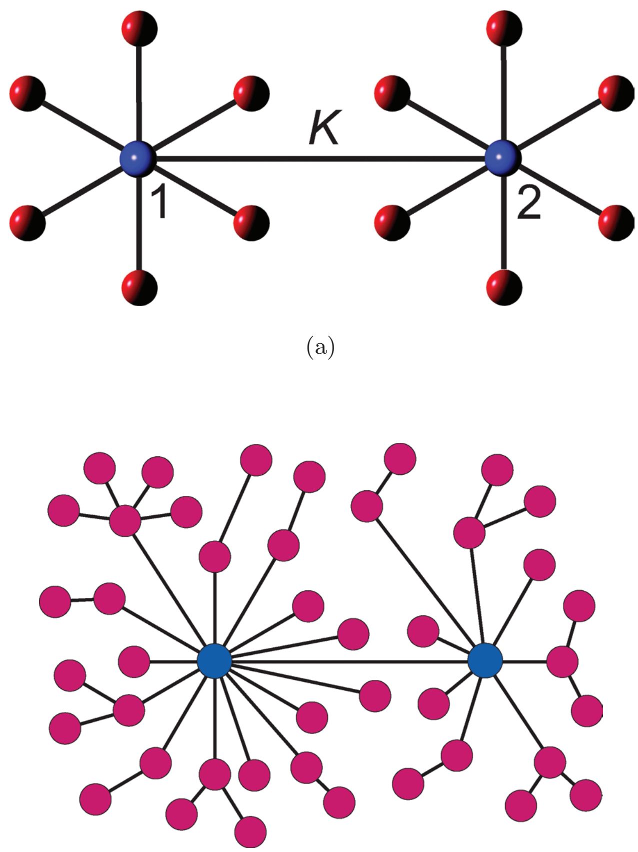
3. Application of the CGS Method

3.1. *Regular networks*

In [Belykh *et al.*, 2004a, 2004b] the method was applied to networks with different coupling topologies. Examples include star coupling, $2K$-nearest neighbor coupling in a ring, and a globally coupled network obtained from a ring of $2K$ nearest neighbor coupled oscillators by adding a weak global coupling. Especially the last two examples are highly nontrivial, but the method allowed one to achieve an excellent result with only moderate effort.

Here, we show how to determine b_k/n for networks with other coupling configurations. Then, we summarize the results.

Example 2. We start with a coupling scheme composed of n nodes, where two star-coupled networks are connected by one edge through their hubs. Figure 2(a) illustrates this network topology. To find an upper bound for the synchronization thresholds, we shall follow the steps of the above study. Here, the choice of paths between the nodes is



(a)

(b)

Fig. 2. (a) Graph with n nodes, where two star-coupled networks are connected by one edge through their hubs. (b) Instance of a scale-free graph, grown by attaching new nodes at random (one node at each time step) to previously existing nodes. Blue color indicates the two nodes with the most links.

unique. The most heavily loaded edge K linking the two central nodes gives the maximum value of b_k/n. Calculate now the total length of all shortest paths passing through edge K. The path from node 1 to node 2 is comprised of one edge. The path from node 1 (2) to a secondary node of the right (left) star consists of two edges. The length of the path between any two secondary nodes of the right and left stars equals 3. Therefore, we obtain

$$b_K = 1 + 2 \cdot 2\left(\frac{n}{2} - 1\right) + 3\left(\frac{n}{2} - 1\right)^2 = \frac{3n^2}{4 - n}.$$

Finally, we have $b_K/n = 3n/4 - 1$ and, consequently, $\varepsilon^* = a(3n/4 - 1)$ as an upper bound for the network synchronization.

Similarly, we derive b_k/n for a network of two stars connected by one edge through their secondary nodes, and for a one-dimensional array with zero-flux boundary conditions (path graph). Table 1 summarizes the results.

Table 1. Graph theoretical quantity b_k/n calculated for main coupling configurations and showing how graph topology influences the synchronization threshold ε^*.

Connection Graph	$\max_k \frac{b_k}{n}$
path graph (n is even)	$\frac{n^2}{8}$
path graph (n is odd)	$\frac{n^2}{8} - \frac{3n^2}{8} + \frac{n}{4}$
ring graph (n is even)	$\frac{n^2}{24}$
ring graph (n is odd)	$\frac{n^2}{24} - \frac{1}{24}$
star graph	$\frac{2n-3}{n}$
2 connected stars (center to center)	$\frac{3n}{4} - 1$
2 connected stars (through two secondary nodes)	$\frac{5n}{4} - 3$
$2K$ nearest neighbor coupled graph (any K)	$\frac{\left(\frac{n}{2K}\right)^3\left(1 + \frac{65K}{4n}\right)}{n}$
complete graph	$\frac{1}{n}$

Note that b_k/n becomes close to the optimal value $1/|\lambda_2|$ (the eigenvalue method) when the number of oscillators n is large. For example, the path graph is associated with the second largest eigenvalue $\lambda_2 = -4\sin^2(\pi/2n)$. When n is large, the value $1/|\lambda_2| \cong n^2/\pi^2 \cong n^2/9.87$ is nearly identical to $b_k/n = n^2/8$ (see Table 1).

It is clear that among tree graphs, the path graph has the largest b_k/n and, consequently, the worst synchronization properties, and the star graph has the lowest b_k/n and synchronization threshold. Naturally, the complete graph providing direct information flow between any two nodes has the lowest b_k/n for all graphs with n nodes. In general, adding an edge to the connection graph G with fixed n will always result in decreasing or constant b_k. The proof is trivial, adding an edge creates more possible paths between the nodes and thus a clever choice of the paths will lead to decreasing or constant b_k.

The upper bound $\varepsilon^* > ab_k/n$ for the ring of $2K$-nearest neighbor nodes (cf. Table 1) is particularly interesting in the context of the relation between synchronization and graph topology. The bound is $\varepsilon^* = (a/n) \cdot (n/2K)^3(1 + (65/4)(K/n))$, therefore ε^* is in direct proportion to $L = n/2K$, the average path length of the coupling graph for the ring of $2K$-nearest neighbor coupled oscillators.

3.2. *Scale-free networks*

In any large-scale complex network, some nodes are more highly connected than the others (for a review on complex networks, see [Strogatz, 2001]). Empirical studies show that the connectivity distributions in many real-world networks have the power law form $P(k) \sim k^{-\gamma}$, where $P(k)$ is the fraction of nodes that have k links, and γ is a positive real number. The power laws have no characteristic scale, therefore such networks were called "scale-free" by Barabási and Albert [1999]. Barabási and Albert [1999] proposed a way of creating a growing scale-free graph by attaching new nodes at random to previously existing nodes (starting from a network of m_0 nodes and adding m_1 nodes at every time step). The probability of attachment is proportional to the number of links to the target node. The result is that richly connected nodes get richer, leading to the formation of hubs [see Fig. 2(b)].

Synchronization in scale-free networks has been numerically studied, by calculating the eigenvalues of the connectivity matrix [Wang & Chen, 2002; Nishikawa *et al.*, 2003]. In particular, it was shown that the synchronizability of a scale-free dynamical network is robust against random removal of nodes, but it is fragile to specific removal of the most highly connected nodes [Wang & Chen, 2002]. The eigenvalues for scale-free (random) matrices are hard to find analytically. Typically, one randomly generates a large number of scale-free networks and translates the statistics of the connection matrices into the statistics of the synchronization thresholds.

The CGS method promises to allow an analytical (probabilistic) treatment of synchrony in scale-free networks. This amounts to finding a combinatorial algorithm which maps a set of chosen paths in the growing network from one step to another, starting from the network with m_0 nodes. However, this problem requires a separate careful study and remains open. Here, we consider another problem: Can the synchronization properties of a scale-free network be estimated by means of networks with a regular connection graph with the same number of nodes?

Generating a large number of scale-free graphs, grown from a network with $m_0 = 2$ by attaching one node ($m_1 = 1$) at each step, we notice that their structures are, to some extent, close to a graph of a few star-networks, connected from center to center. Therefore, we can use $b_k/n = (3n/4-1)$, calculated for a graph of two center-to-center connected stars (cf. Example 2), as a rough estimate for the averaged b_k/n in the scale-free networks with $m_0 = 2$ and $m_1 = 1$. These specific scale-free graphs are trees. Hence, the choice of paths between the nodes is unique, and the computational complexity of calculating b_k is practically equal to determining the second largest eigenvalue λ_2. Thus, an optimal value b_k/n can be calculated numerically with moderate effort and compared with $b_k/n = (3n/4 - 1)$.

In Fig. 3, we see the function b_k/n calculated numerically for the scale-free networks for different n and averaged at each step over 20 realizations (when n is small and only a few different

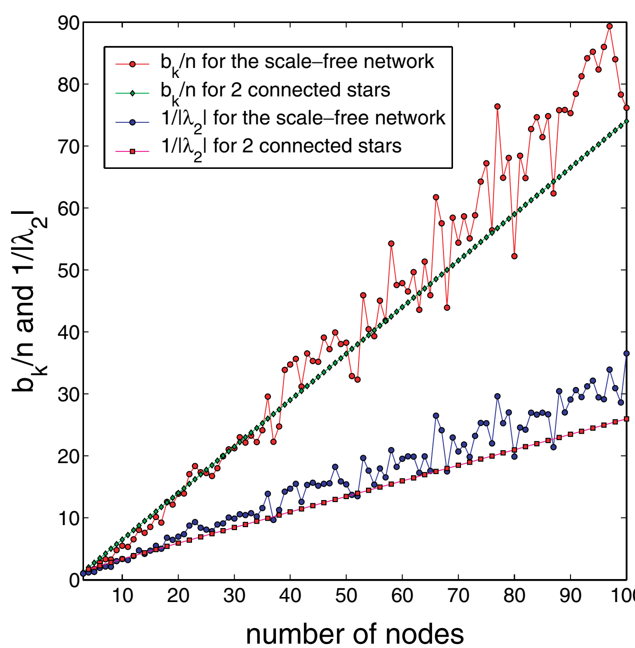


Fig. 3. Averaged b_k/n calculated numerically for scale-free networks for different n (red circles). Analytical curve $b_k/n = (3n/4 - 1)$ for the network of two connected stars (green curve). The two curves are close for $n < 90$. This shows that synchronization properties of scale-free networks are close to those of a regular network with the same number of nodes and composed of several coupled stars. Numerically calculated curves $1/|\lambda_2|$ for the two networks give an additional support to this statement.

networks can be generated, b_k/n is calculated and averaged over all possible networks). These values b_k are depicted by red circles. The curve $b_k/n = (3n/4 - 1)$ for the network of two-connected starts closely follows the numerical curve b_k/n for the scale-free networks in the region where n is not too large ($n < 90$). This shows that the synchronization properties of scale-free networks of intermediate size are to a high degree of precision the same as the regular networks of connected starts. In additional support of this claim, we have applied the eigenvalue value approach to synchronization in such scale-free networks and two-connected stars and obtained the qualitatively same result (see Fig. 3).

Note that a regular network composed of three stars that are all connected from center to center through a unique, additional node gives a better approximation for the scale-free networks.

3.3. *Role of the average path length*

The semi-random graphs having small-world or scale-free properties have a smaller average path length, when compared to locally coupled networks. This property is often associated with the heterogeneity in the scale-free distribution of connectivities (it is known that such heterogeneity tends to reduce the average path length, see, e.g. [Nishikawa *et al.*, 2003]). Intuition says, the smaller the average path length of a network, the more efficient is the communication between oscillators, and therefore the lower the synchronization threshold will be. As we have shown in the Sec. 3.1, this is often true and in specific cases can be made rigorous [Pogromsky *et al.*, 2001]. Globally coupled networks have a very short average path length $L = 1$ and are easy to synchronize. Whereas, the locally coupled systems with $L = n^3/8$ (for a ring of oscillators) are very resistant to synchronization. Then the following question arises: is the average path length always important for synchronization, especially for complex networks? Nishikawa *et al.* [2003] have observed the opposite to hold. Using the Master Stability Function method and numerically calculating the second largest eigenvalue λ_2, they have shown that scale-free networks with a homogeneous distribution of connectivity were more synchronizable than heterogeneous ones, even though the average path length was larger. The authors have explained this intriguing behavior by examining the load distribution on nodes. It is stated that increased concentration of load on center nodes, or hubs, results in loss of synchronization. However, this plausible statement is *not* always correct: the star-graph can grow unbounded such that for large n the central node (hub) becomes heavily loaded, but the synchronization threshold remains the same (as $\lambda_2 = 1$ for a star graph).

A simple example can show that the average path length is not always important for synchronization. Consider a complete graph with 1000 nodes. For this network, the synchronization threshold ε^* is very low: $b_k/n = 1/1000$ for any edge k and the average path length is short, $L = 1$. Now add one node with only one edge K to this graph. The average path length for this network, $L = 1 + (2/n) \cong 1$, did *not* substantially change. At the same time, $b_K/(n+1) = 2 - (1/(n+1)) \cong 2$ becomes close to b_k for the star-graph (2 instead of 1/1000) such that addition of one node completely changes synchronization behavior of the network. This is a proof that the average path length is unimportant in this case, and what always matters for synchronization is *the weakest link*. Therefore, like in a famous TV game, we always have to find the Weakest Link. In our study, this is the link having the maximum traffic load on it. This edge is a bottle neck for synchronization of the whole network and requires the maximum coupling strength $\varepsilon^* = \max_k \varepsilon_k$ to synchronize all the oscillators of the network (this edge is associated with the most unstable transversal mode defined by λ_2).

4. Conclusions

In this paper, we have proceeded with the application of the CGS method to the study of global synchronization in regular and complex networks. This method is based on graph theoretical reasoning and reveals a clear connection between synchronization and graph theory. It gives upper bounds for global synchronization thresholds in networks of mutually coupled oscillators. The dependence of the bounds on the graph topology and number of cells is to a high degree of precision the same as the dependence of the real limit of complete synchronization that is determined by numerical simulation. Only the multiplicative factor a that is mainly related to the dynamics of the individual cells is higher in the rigorous bounds of the CGS method with respect to the factor obtained by numerical simulations.

Note that we can also use the eigenvalues of the connectivity matrix for the Lyapunov function approach. In this way we may obtain, in the case of constant connection matrices, a better bound for the global synchronization threshold than with the CGS method. However, this eigenvalue based method may be difficult to apply for irregular graphs, it gives a less direct relation to graph theoretical quantities and in general, it fails to provide an analytically derived bound for time-dependent coupling coefficients. The CGS method is directly applicable to networks of slightly nonidentical oscillators (up to $10-15\%$ parameter mismatch.) In this case, perfect synchronization cannot exist anymore, but approximate synchronization is still possible and therefore similar global stability conditions of approximate synchronization can be derived by means of the CGS method and the technique developed by Belykh *et al.* [2003].

Applying both eigenvalue and CGS methods, we have shown that synchronization in scale-free networks can often be described by means of a regular network with the same number of nodes and composed of several coupled stars. We have also discussed the role of the average path length in the network synchronization and proved by a simple argument that the average path length is not always important. What is always important for efficient synchrony is a balance between the average path length and load distributions (the sum of the chosen paths passing through each edge, resulting in the concept of the weakest link).

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Appendix

In this appendix, we show how to calculate analytically an upper bound ε_2^* for global synchronization in the network (1) of two x-coupled Hindmarsh–Rose (HR) neuron models [Hindmarsh & Rose, 1984].

The single model reads

$$\begin{cases} \dot{x} = y + \alpha x^2 - x^3 - z + I \\ \dot{y} = 1 - dx^2 - y \\ \dot{z} = \mu(r(x-c) - z), \end{cases} \tag{A.1}$$

where x represents the membrane potential, and y and z are associated with fast and slow currents, respectively. α, d, r, I, c, μ are parameters, and μ is small.

The HR model displays chaotic dynamics in a wide region of parameters. The typical parameters for the chaotic bursting's appearance are $\alpha = 3$, $I = 3.281$, $d = 5$, $r = 4$, $c = -1.6$ and $\mu = 0.002$ [Huerta *et al.*, 1997].

Consider two x-coupled HR models. The equations of motion are the following:

$$\begin{cases} \dot{x}_1 = y_1 + \alpha x_1^2 - x_1^3 - z_1 + I + \varepsilon(x_2 - x_1) \\ \dot{y}_1 = 1 - dx_1^2 - y_1 \\ \dot{z}_1 = \mu(r(x_1 - c) - z_1) \\ \dot{x}_2 = y_2 + \alpha x_2^2 - x_2^3 - z_2 + I + \varepsilon(x_1 - x_2) \\ \dot{y}_2 = 1 - dx_2^2 - y_2 \\ \dot{z}_2 = \mu(r(x_2 - c) - z_2) \end{cases} \tag{A.2}$$

To test global stability of the synchronization manifold $D = \{x_1 = x_2,\ y_1 = y_2;\ z_1 = z_2\}$, we have to consider the difference equations and study their stability.

Introducing the notation for the differences

$$X = x_2 - x_1, \quad Y = y_2 - y_1, \quad Z = z_2 - z_1$$

and using simple algebraic expressions

$$x_2^3 - x_1^3 = \frac{(x_2 - x_1)((x_2 - x_1)^2 + 3(x_2 + x_1)^2)}{4},$$

$$x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1),$$

we obtain the system for the difference variables

$$\begin{cases} \dot{X} = Y + \left(\alpha U - \dfrac{3U^2}{4} - 2\varepsilon\right) X - \dfrac{X^3}{4} - Z \\ \dot{Y} = -dUX - Y \\ \dot{Z} = \mu(rX - Z), \end{cases} \tag{A.3}$$

where $U = x_2 + x_1$.

The origin ($X = 0$, $Y = 0$, $Z = 0$) is an equilibrium of system (A.3). Its stability amounts to the stability of the synchronization manifold.

The proof that the origin can be globally stable involves the construction of a Lyapunov function, a smooth, positive definite function that decreases along trajectories of system (A.3).

Consider the Lyapunov function

$$W = \frac{1}{2}X^2 + \frac{\gamma}{2}Y^2 + \frac{1}{2\mu r}Z^2, \qquad (A.4)$$

where γ is a positive auxiliary parameter to be defined.

Calculating the time derivative of W with respect to system (A.3), we obtain

$$\dot{W} = -(AX_1^2 - BX_1Y_1 + \gamma Y_1^2) - \frac{X_1^4}{4} - \frac{Z_1^2}{r}, \quad (A.5)$$

where $A = (3U^2/4 - \alpha U + 2\varepsilon)$ and $B = (1 - \gamma dU)$.

The derivative $\dot{W}$ is negative along trajectories if the quadratic form $S = AX^2 - BXY + \gamma Y^2$ is positive definite. This holds true if the following two conditions are satisfied:

(i) $A = 3U^2/4 - \alpha U + 2\varepsilon > 0$. The roots of the equation $A = 0$ are: $U^{1,2} = 2(\alpha \pm \sqrt{\alpha^2 - 6\varepsilon})/3$. Hence, the equation $A = 0$ has no solutions for

$$\varepsilon > \frac{\alpha^2}{6}.$$

Therefore, under this condition, $A > 0$. Note that in this stability condition, we have managed to get rid of the variable U, corresponding to the x-coordinate of the attractor in the regime of synchronous bursting. This gives the conditions which explicitly depend on the system parameters.

(ii) $\gamma A - B^2/4 = (3 - \gamma d^2)U^2 - 2(\alpha - d)U + 8\varepsilon - 1/\gamma > 0$. This is true under the conditions

$$\varepsilon > \frac{(\alpha - d)^2}{8(3 - \gamma d^2)} + \frac{1}{8\gamma}, \quad \gamma < \frac{3}{d^2}.$$

Collecting all the conditions, we obtain an upper bound on the coupling strength sufficient to achieve globally stable synchronization in system (A.2)

$$\varepsilon_2^* = \max\left(\frac{\alpha^2}{6},\ \frac{(\alpha - d)^2}{8(3 - \gamma d^2)} + \frac{1}{8\gamma}\right), \quad \gamma < \frac{3}{d^2}. \qquad (A.6)$$

The auxiliary parameter γ is chosen such that $\gamma < 3/d^2$. In a chaotic region, $d = 5$, therefore one can choose $\gamma = 0.1$.

Coming from sufficient conditions, the bound (A.6) gives an overestimate for the real synchronization threshold: 2.25 (predicted) versus 0.5 (actual) for the above mentioned parameters. However, it guarantees that synchronization arises and remains globally stable in a concrete network with increasing coupling. This makes the use of the CGS method possible and allows us to predict the synchronization thresholds in the network (1)–(A.1) with an arbitrary connection graph.

Using similar arguments based on the construction of a Lyapunov function for the difference variables, one can prove global stability of synchronization for many other coupled limit-cycle or chaotic systems.