Synchronization between two coupled complex networks

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We study synchronization for two unidirectionally coupled networks. This is a substantial generalization of several recent papers investigating synchronization inside a network. We derive analytically a criterion for the synchronization of two networks which have the same (inside) topological connectivity. Then numerical examples are given which fit the theoretical analysis. In addition, numerical calculations for two networks with different topological connections are presented and interesting synchronization and desynchronization alternately appear with increasing value of the coupling strength.

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I. INTRODUCTION

Presently our knowledge of complex networks is experiencing rapid growth [1–9]. Generally speaking, a complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents [10].

Among all kinds of complex networks, random graph, small-world effect, and scale-free characteristics are mostly noticeable. In early 1960, the theory of random graphs was first constructed by the two Hungarian mathematicians Erdös and Rényi [11]. This model has dominated the mathematical research of complex networks for nearly half a century, mainly due to the absence of supercomputational power and detailed topological information about various large-scale real-world networks. In 1998, the small-world effect was introduced by Watts and Strogatz [1] to investigate the transition from regular networks to random ones. Such networks behave a high degree of clustering as in the regular networks and a small average distance among nodes. Shortly thereafter, Barabási and Albert [2] brought forward scale-free characteristics where the degree of nodes follows a power-law distribution and most nodes have few connections, but only a few nodes have many connections, the hubs.

Networkers mostly focus on modeling, dynamical analysis, and control. Recently, synchronization of complex networks, strictly speaking, "inner synchronization," has attracted much attention. The early work in this regard was given by Wang and Chen [10], where they considered an ideal model

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t), \quad i = 1,2, \ldots ,N, \qquad (1)$$

where $x_i=(x_{i1},x_{i2},\ldots,x_{in})^T \in \mathfrak{R}^n$ are the state variables of node i and N is the number of the network nodes. $f:\mathfrak{R}^n \to \mathfrak{R}^n$ is a continuously differentiable function which determines the dynamical behavior of the nodes. $c>0$ is a coupling strength, and $\Gamma \in \mathfrak{R}^{n\times n}$ is a constant 0-1 matrix linking coupled variables. For simplicity, one often assumes that $\Gamma = \mathrm{diag}(r_1,r_2,\ldots,r_n) \geqslant 0$ is a diagonal matrix. $A=(a_{ij}) \in \mathfrak{R}^{N\times N}$ represents the coupling configuration between nodes of the whole network (it is often assumed that there is at most one connection between node i and a different node j and that there are no isolated clusters; that is, A is an irreducible matrix), whose entries a_{ij} are defined as follows: if there is a connection between node i and node $j(j\neq i)$, then $a_{ij}=1$; otherwise, $a_{ij}=0(j\neq i)$; the diagonal elements of A are defined as $a_{ii}=-\sum_{j=1,j\neq i}^{N} a_{ij}=-\sum_{j=1,j\neq i}^{N} a_{ji}$, and clearly, if the degree of node i is k_i, then $a_{ii}=-k_i, i=1,2,\ldots,N$.

In that paper, they studied the fact that all nodes in the network achieved a synchronous state, which was determined by $\dot{x}(t)=f(x(t))$. Since they considered synchronization in a network, we may regard it as "inner synchronization" of a network. Improved and expanded work in this respect—i.e., introducing weighted connections, time dependence in the coupling matrices, nonlinear coupling function, time delays, etc.—can be found in the literature [12–18] and many references cited therein. Rather than the above case, the synchronous state in a network may be different from the steady state determined by a single node, $\dot{x}(t)=f(x(t))$; i.e., see [19,20], in which bifurcation of a network was also studied.

A natural question is, does synchronization between two coupled networks also happen? We may call it can "outer synchronization" of networks if such a synchronization exists. In effect, outer synchronization exists in our lives. Here we only cite three examples to illustrate that outer synchronization is common in our real world. If such a synchronization is a benefit, we should enhance it, while if it harmful, we should avoid it. From the angle of sociology, the present world can be divided into two networks: developed networks constructed by developed countries and developing networks constructed by developing countries. With the gradual increase of international exchange, the two networks will be synchronized; i.e., the future world will reach "great harmony." In the animal world, synchronization phenomena universally exist if no external intervention appears. In prey and predator communities, for example, the number of prey and that of predators are usually invariant under no outside invasion. An another example is taken from the computer world. All educators' computers form education networks while all researchers' networks compose research networks. These two networks are coupled via the Internet. If all educators and researchers explore the same Internet source, then congestion appears, which is harmful and so should be avoided. So it is very necessary to study the outer synchronization between coupled networks.

In the present paper we study this interesting topic. For more details, a synchronization analysis of two coupled networks with the same connection topologies is given in Sec. II. In Sec. III, two numerical examples are in line with the theoretical analysis derived in Sec. II. In addition, synchronization or desynchronization between two coupled networks with different topologies are also numerically investigated. A conclusion and discussion are included in Sec. IV.

II. MODEL PRESENTATION AND SYNCHRONIZATION ANALYSIS

In [21], the authors used the open-plus-closed-loop method [22] to realize synchronization between the following master and slave systems. The master system is

$$\frac{dx}{dt} = f(x), \quad x \in \Re^n. \tag{2}$$

By their choice, the slave system reads

$$\frac{dy}{dt} = f(y) + \left(H - \frac{\partial f(x)}{\partial x}\right)(y - x), \tag{3}$$

where the matrix H is an arbitrary constant Hurwitz one (a matrix with negative real part eigenvalues) whose elements can be chosen as simple as possible. When $[\partial f(x)/\partial x]_{ik}$ is a constant, we can then choose $H_{ik}=[\partial f(x)/\partial x]_{ik}$ such that $[H-\partial f(x)/\partial x]_{ik}$ is zero. When we cannot find such a Hurwitz matrix, we introduce one or two or more parameters guaranteeing that H is a Hurwitz matrix. From this viewpoint we can conclude that the coupling form may be simpler if $f(x)$ has fewer nonlinear terms.

Now we apply the above simple coupling form to investigate the synchronization between two coupled networks. Here we take the driving network in the form

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t), \quad i = 1,2, \ldots ,N, \tag{4}$$

and the response network as

$$\dot{y}_i(t) = f(y_i(t)) + \left(H - \frac{\partial f(x_i)}{\partial x_i}\right)[y_i(t) - x_i(t)] + c\sum_{j=1}^{N} b_{ij}\Gamma y_j(t), \quad i = 1,2, \ldots ,N, \tag{5}$$

where x_i, y_i, N, f, Γ, and c have the same meanings as those in (1), and $A=(a_{ij})_{N\times N}$ and $B=(b_{ij})_{N\times N}$ are symmetric or asymmetric matrices, each line sum of A and B being equal to zero.

Hereafter, network (4) and network (5) are said to achieve synchronization if

$$\lim_{t\to+\infty} \|y_i(t) - x_i(t)\| = 0, \quad i = 1,2, \ldots ,N. \tag{6}$$

In the following, we study the synchronization between system (4) and system (5), where both have the same topology structures—i.e., $A=B$.

Letting $e_i=y_i-x_i$ and linearizing the error system around x_i, we get

$$\dot{e}_i = He_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j, \quad i = 1,2, \ldots ,N. \tag{7}$$

Equation (7) can be written as

$$\dot{e} = He + c\Gamma eA^T, \tag{8}$$

where T stands for matrix transpose and $e=[e_1,e_2,\ldots,e_N]$ denotes an $n\times N$ matrix. Decompose the coupling matrix according to $A^T=SJS^{-1}$, where J is a Jordan canonical form with complex eigenvalues $\lambda \in \mathcal{C}$ and S contains the corresponding eigenvectors s. Multiplying Eq. (8) from the right with S and denoting $\eta=eS$, we obtain

$$\dot{\eta} = H\eta + c\Gamma\eta J, \tag{9}$$

where J is a block diagonal matrix,

$$\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_h \end{bmatrix},$$

and J_k is a block corresponding to the m_k multiple eigenvalue λ_k of A:

$$\begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}.$$

Let $\eta=[\eta_1,\eta_2,\ldots,\eta_h]$ and $\eta_k=[\eta_{k,1},\eta_{k,2},\ldots,\eta_{k,m_k}]$. Due to the fact that the sum of every line of the matrix A is zero, we can assume $\lambda_1=0$ and J_1 is a 1×1 matrix. If $\lambda_1=0$, we get $\dot{\eta}_1=H\eta_1$. Since H is a Hurwitz matrix, the zero solution $\eta_1=0$ is asymptotically stable. Next, we discuss the cases $k=2,3,\ldots,h$. We can rewrite Eq. (9) in a component form

$$\dot{\eta}_{k,1} = (H + c\lambda_k\Gamma)\eta_{k,1}, \tag{10a}$$

$$\dot{\eta}_{k,p+1} = (H + c\lambda_k\Gamma)\eta_{k,p+1} + c\Gamma\eta_{k,p}, \quad 1 \leqslant p \leqslant m_k - 1. \tag{10b}$$

First, we consider the stability of Eq. (10a). Let $\eta_{k,1}=u_{k,1}+jv_{k,1}$ and $\lambda_k=\alpha_k+j\beta_k$, where $u_{k,1}$, $v_{k,1}$, $\beta_k \in \Re$, $\alpha_k<0$, $k=2,\ldots,h$, and j is the imaginary unit. Equation (10a) can be rewritten as

$$\dot{u}_{k,1} = (H + c\alpha_k\Gamma)u_{k,1} - c\beta_k\Gamma v_{k,1},$$

$$\dot{v}_{k,1} = (H + c\alpha_k\Gamma)v_{k,1} + c\beta_k\Gamma u_{k,1}. \tag{11}$$

We define the Lyapunov function as

$$V(t) = u_{k,1}^T u_{k,1} + v_{k,1}^T v_{k,1}. \tag{12}$$

Then we get

$$\dot{V}(t) = u_{k,1}^T(P^T+P)u_{k,1} + v_{k,1}^T(P^T+P)v_{k,1} + c\beta_k u_{k,1}^T(\Gamma^T - \Gamma)v_{k,1} + c\beta_k v_{k,1}^T(\Gamma^T-\Gamma)u_{k,1} = \begin{bmatrix} u_{k,1} \\ v_{k,1} \end{bmatrix}^T M \begin{bmatrix} u_{k,1} \\ v_{k,1} \end{bmatrix}, \tag{13}$$

where

$$M = \begin{bmatrix} P^T+P & \\ & P^T+P \end{bmatrix} \tag{14}$$

and $P=H+c\alpha_k\Gamma$. If $M<0$—i.e., if this matrix is negative definite—then the zero solution of Eq. (10a) is asymptotically stable.

Second, we study the stability of Eq. (10b). Without loss of generality, we take $p=1$ using the same method. Let $\eta_{k,2}=u_{k,2}+jv_{k,2}$, we get

$$\dot{u}_{k,2} = (H+c\alpha_k\Gamma)u_{k,2} - c\beta_k\Gamma v_{k,2} + c\Gamma u_{k,1},$$

$$\dot{v}_{k,2} = (H+c\alpha_k\Gamma)v_{k,2} + c\beta_k\Gamma u_{k,2} + c\Gamma v_{k,1}. \tag{15}$$

Let the Lyapunov function be

$$V(t) = u_{k,1}^T u_{k,1} + v_{k,1}^T v_{k,1} + u_{k,2}^T u_{k,2} + v_{k,2}^T v_{k,2}. \tag{16}$$

Then we get

$$\dot{V}(t) = \begin{bmatrix} u_{k,1} \\ v_{k,1} \\ u_{k,2} \\ v_{k,2} \end{bmatrix}^T Q \begin{bmatrix} u_{k,1} \\ v_{k,1} \\ u_{k,2} \\ v_{k,2} \end{bmatrix}, \tag{17}$$

where

$$Q = \begin{bmatrix} P^T+P & & c\Gamma & \\ & P^T+P & & c\Gamma \\ c\Gamma & & P^T+P & \\ & c\Gamma & & P^T+P \end{bmatrix}. \tag{18}$$

If $Q<0$, the zero solution to Eq. (10b) is asymptotically stable. From this, it follows that the synchronization between the drive network (4) and the response one (5) is achieved. Now we can conclude that *networks (4) and (5) can be synchronized if two conditions hold: i.e.,* $A=B$ *and* H *is a Hurwitz matrix such that* M *and* Q *are both negative definite*.

In the next section, several illustrative examples are presented, which are in line with the derived theoretical analysis. A natural question happens: even if H is a Hurwitz-type matrix and M and Q are negative definite, can synchronization between (4) and (5) be achieved provided that $A \neq B$? Such a theoretical study is far more difficult. However, by numerical simulations, we find that synchronization between (4) and (5) can be approached under suitable conditions.

III. NUMERICAL EXAMPLES

In the considered networks below, the dynamics at every node follows the well-known Lorenz system

$$\dot{x}_{i1} = \sigma(x_{i2} - x_{i1}),$$

$$\dot{x}_{i2} = \gamma x_{i1} - x_{i1}x_{i3} - x_{i2},$$

$$\dot{x}_{i3} = x_{i1}x_{i2} - b x_{i3}, \tag{19}$$

where σ, γ, and b are parameters. We always use in the following $\sigma=10$, $\gamma=28$, and $b=8/3$; i.e., the system has a chaotic attractor.

We take H as the following form:

$$H = \begin{bmatrix} -\sigma & \sigma & 0 \\ u & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{20}$$

where u is a parameter. If $u<1$, it is easy to check that H is a Hurwitz matrix. Here we always set $\Gamma=\mathrm{diag}(1,1,1)$.

In what follows, we discuss two cases $A=B$ and $A\neq B$, totally including six subcases. The case $A=B$ includes two subcases: i.e., A is symmetric and A is nonsymmetric. And the case $A\neq B$ has four subcases: (i) A and B are symmetric, but are not equal, (ii) A is symmetric, but B is not, (iii) B is symmetric, but A is not, and (iv) A and B are both asymmetric, but are not equal.

A. Identical topological structures ($A=B$)

In this subsection we first consider the case where both networks (4) and (5) have the same topological structures: i.e., $A=B$. This case consists of two subcases: that is, A is symmetric and A is nonsymmetric. The former means that the network connection is undirected, while the latter means that the network connection is directed. For brevity, we analyze networks with 10 nodes; at this time, a combination of systems (4) and (5) is 60 dimensional. For the first subcase, we suppose that $A=B=A_1$, where

$$A_1 = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -5 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -5 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -5 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & -6 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -5 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -5 \end{bmatrix}. \tag{21}$$

Since the coupling matrix is symmetric, we know that its first eigenvalue is zero and the rest are negative. From our analysis, the real parts of the eigenvalues of $H+c\lambda_k\Gamma$ are negative for arbitrary $u<1$ in H. It immediately follows that the synchronization between network (4) and network (5) can be achieved. In the following, the initial values are chosen randomly in (0,1). Let $\|e(t)\|=\max\{\max_{1\leq i\leq 10}|x_{i1}(t)-y_{i1}(t)|, \max_{1\leq i\leq 10}|x_{i2}(t)-y_{i2}(t)|, \max_{1\leq i\leq 10}|x_{i3}(t)-y_{i3}(t)|\}$, for $t\in[0,+\infty)$. Figure 1 plots the synchronization errors for different values of u in H.

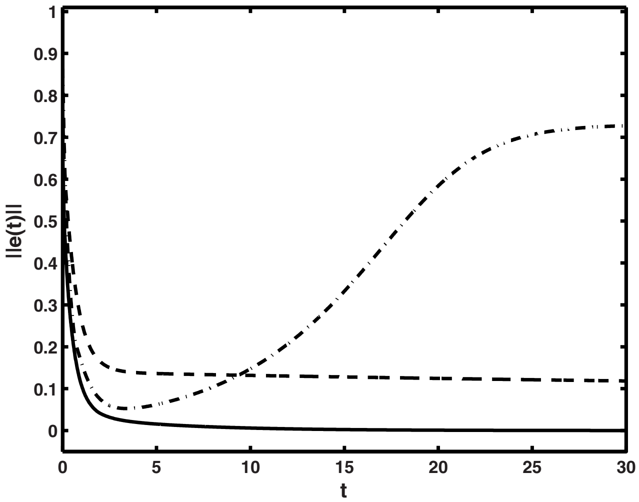

FIG. 1. Synchronization errors between network (4) and network (5) for three values of u in H with $c=0.5$, $A=B=A_1$. The solid line corresponds to $u=0.8$, the dashed line corresponds to $u=1.0$, and the dash-dotted line corresponds to $u=1.2$.

For the second subcase—i.e., where A, B are asymmetric—we set $A=B=A_2$, where

$$A_2=\begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1\\ 1 & 0 & -5 & 0 & 0 & 1 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & -3 & 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -3 & 0 & 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 & 0 & -5 & 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 1 & 1 & 0 & -5 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -4 & 1 & 0\\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & -6 & 1\\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -3 \end{bmatrix}. \tag{22}$$

Through the LMI toolbox in Matlab, we can easily find a pair of (u,c) such that M, $Q<0$—say, $u=-0.5$ and $c=0.5$. The synchronization errors are presented in Fig. 2 with different values of u in H.

From our computations, we find that the values of u in H play an important role in synchronous processes and that the coupling strength c seems not to have a relation to synchronization. In Figs. 1 and 2, c can be chosen at large. This can easily be seen from the structures of (4) and (5), compared to (2) and (3). If we let $L(X)=c(A\otimes\Gamma)X$ and $L(Y)=c(B\otimes\Gamma)Y$, where the signal $\otimes$ denotes the Kronecker product and $A=B$, then (4) and (5) can be rewritten as a compact form

$$\frac{dX}{dt}=F(X)+L(X),$$

$$\frac{dY}{dt}=F(Y)+L(Y)+\left(\Lambda-\frac{\partial F(X)}{\partial X}\right)(Y-X), \tag{23}$$

in which $\Lambda=\mathrm{diag}(H,H,\ldots,H)$. System (23) is a analogy of drive-response system (4) and (5). From Refs. [21,22], the synchronization between systems (4) and (5) depends upon the Hurwitz matrix H. Similarly, the synchronization in (23) also depends on the Hurwitz matrix Λ. Such a diagonal block matrix Λ has the same diagonal element H. This H makes it such that M and Q are negative definite. P and Q have a relation with c, H, Γ, and the real parts of the eigenvalues of coupling matrix A.

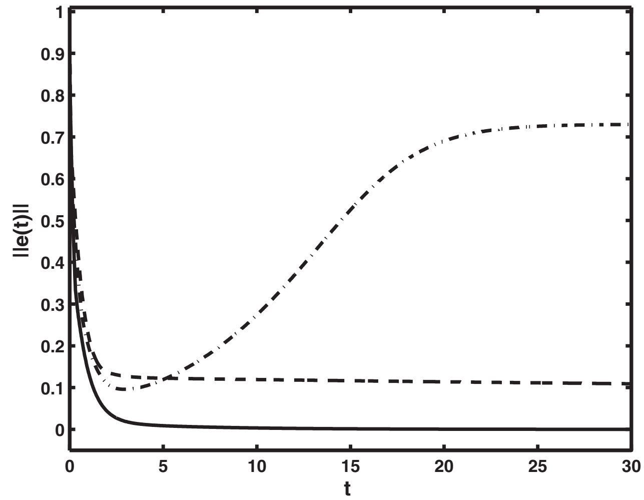

FIG. 2. Synchronization errors between network (4) and network (5) for three values of u in H with $c=0.5$, $A=B=A_2$. The solid line corresponds to $u=0.8$, the dashed line corresponds to $u=1.0$, and the dash-dotted line corresponds to $u=1.2$.

B. Different topological structures ($A\neq B$)

We have analytically given a synchronous criterion between two networks with the same topological connections $A=B$ in Sec. II. Numerical computations are displayed in Sec. III A. However, a criterion of possible synchronization for two networks with different topological connections $A\neq B$ is not easy to derive. In the following subsection, we numerically study the possible synchronization between two networks with different topological structures. This case has four subcases.

(i) A and B are symmetric but are not equal. Say, set $A=A_1$, $B=A_3$, where A_3 is defined as follows:

$$A_3=\begin{bmatrix} -3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & -4 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1\\ 0 & 0 & -4 & 0 & 0 & 1 & 1 & 1 & 0 & 1\\ 1 & 0 & 0 & -4 & 0 & 1 & 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & -5 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 0 & -5 & 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 1 & 0 & -5 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & -5 & 1 & 0\\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & -5 & 0\\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & -4 \end{bmatrix}. \tag{24}$$

(ii) A is symmetric, but B is not. We take $A=A_1$, $B=A_2$ as an example.

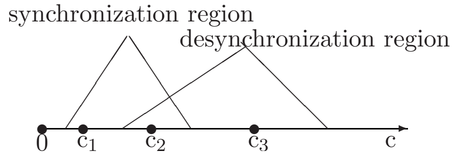

FIG. 3. Synchronization and desynchronization regions with respect to the coupling strength c. $(0,c_1)$ and (c_2,c_3) are synchronization regions, (c_1,c_2) and $(c_3,+\infty)$ are desynchronization regions.

(iii) B is symmetric, but A is not. For instance, let $B=A_1$ and $A=A_2$.

(iv) A and B are both asymmetric, but are not equal. For example, we choose $A=A_2$ and $B=A_4$, where A_4 is given as follows:

$$A_4=\begin{bmatrix} -3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -5 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & -6 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}. \tag{25}$$

We do numerical simulations for the above-mentioned four subcases. In Fig. 3, we plot the synchronization region with respect to the coupling strength c. These four subcases all possess the same properties as depicted in Fig. 3. For subcases (i) and (ii), the values of c_1, c_2, and c_3 approximately equal 0.0001, 0.4, and 4.5, respectively; yet for subcases (iii) and (iv), the values of c_1, c_2, and c_3 are close to 0.0001, 0.3, and 5.5. When the value of the coupling strength c is very small (10^{-4}), synchronization between (4) and (5) with four choices of A and B is very easily to obtain, and this fits our intuition; with the increasing value of c from c_1 to c_2, the coupling strength c starts to result in desynchronization; if the coupling strength c goes into a interval (c_2,c_3), a new synchronization happens between them; when it exceeds the threshold value c_3, synchronization fails. This numerical phenomenon is very interesting. A theoretical analysis seems to be more difficult than expected.

In (4) and (5), if $A\neq B$, set $L_A(X)=c(A\otimes\Gamma)X$ and $L_B(Y)=c(B\otimes\Gamma)Y$; Eqs. (4) and (5) can simply read as

$$\frac{dX}{dt}=F(X)+L_A(X),$$

$$\frac{dY}{dt}=F(Y)+L_B(Y)+\left(\Lambda-\frac{\partial F(X)}{\partial X}\right)(Y-X), \tag{26}$$

where $\Lambda=\mathrm{diag}(H,H,\ldots,H)$. The drive-response system (26) is coupled by two different systems since $L_A(X)\neq L_B(X)$ due to $A\neq B$. A theoretical analysis regarding the synchronization between two different systems via coupled connections is far too difficult to obtain. Comments in the respect can be found in [23]. Present studies in this regard mainly rely on numerical simulations.

C. Asymmetric coupling ($H\rightarrow H_i$)

Sections III A and III B are devoted to numerical simulations of systems (4) and (5), where the H in this master-slave system is always same. If we let $H\rightarrow H_i$ in (5), the new response system is given as follows:

$$\dot{y}_i(t)=f(y_i(t))+\left(H_i-\frac{\partial f(x_i)}{\partial x_i}\right)[y_i(t)-x_i(t)] + c\sum_{j=1}^{N} b_{ij}\Gamma y_j(t), \quad i=1,2,\ldots,N, \tag{27}$$

where $(b_{ij})=(a_{ij})$.

In the following, we give some numerical results and a theoretical explanation for systems (4) and (5). At first we give a simple theoretical analysis since this analysis is a direct and easy generalization of Sec. II with the case $(a_{ij})_{N\times N}=(b_{ij})_{N\times N}$. For systems (4) and (27), the variation equation (first order approximation) near x_i reads

$$\dot{e}_i=H_ie_i+c\sum_{j=1}^{N}a_{ij}\Gamma e_j, \quad i=1,2,\ldots,N, \tag{28}$$

in which $e_i=y_i-x_i\in R^n$. If we let $e^T=(e_1^T,e_2^T,\ldots,e_N^T), e\in\Re^{nN}$, Eq. (28) has the following compact form:

$$\dot{e}=(\bar{H}+cA\otimes\Gamma)e, \tag{29}$$

where $\bar{H}=\mathrm{diag}(H_1,H_2,\ldots,H_N)$ and $H_i, i=1,2,\ldots,N$, are n-dimensional matrices. If the real parts of all the eigenvalues of $\bar{H}+cA\otimes\Gamma$ are negative, then the zero solution Eq. (29) is asymptotically stable; i.e., synchronization between networks (4) and (27) can be realized.

In the present slave network (27), we need not restrict H_i to be Hurwitz matrices. But synchronous conditions should be imposed on $\bar{H}+cA\otimes\Gamma\in R^{nN}\times R^{nN}$; i.e., the real parts of all the eigenvalues of $\bar{H}+cA\otimes\Gamma$ are negative. This guarantees that Eqs. (4) and (27) can be synchronized.

Next, we do some numerical simulations between the drive network (4) and the response network (27). Owing to the varieties of H_i, we can simply take

$$H_i=\begin{bmatrix} -\sigma & \sigma & 0 \\ u_i & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{30}$$

where u_i, $1\leqslant i\leqslant 10$, are chosen at random in $(-10,10)$. We investigate the synchronization between (4) and (27) with $H_i\neq H_j$ for $i\neq j$. But no synchronizations between the drive and response networks (4) and (27) happen for $A=B=A_1,A_2,A_3,A_4,\Gamma=\mathrm{diag}(1,1,1), c\in(10^{-3},10)$. For the same coupled networks (4) and (27), the case $A\neq B$ is omitted here. The synchronization between them might depend on the chosen form of the matrix H_i, the topological connection, and the coupling strength c.

IV. CONCLUSIONS

In this article, synchronization *between* two coupled complex networks, which is independent of intranetwork synchronization, is studied. Generally there are a lot of interactive forms between networks; here, we choose a simple master-slave action one to discuss. We theoretically and numerically show that when driving-response networks have identical connection topologies, then synchronization between them can be achieved. For this case, the structure of driving-response networks (4) and (5) with $(a_{ij})_{N\times N} = (b_{ij})_{N\times N}$ is almost the same as that of the single master-slave systems (2) and (3), but the former is much higher dimensional. Since the scales of networks and the connection topology influence the synchronization, to study synchronization between coupled networks is more difficult, but interesting and challenging.

On the other hand, a synchronization analysis between two different systems with a coupled connection is not available, let alone coupled networks (4) and (5) with case $A \neq B$, or equivalently, a compact form (23). To establish a synchronization criterion for such a drive-response system is a long-term task. In this article, we do some numerical simulations for case $A \neq B$ and find an interesting result. This result—that synchronization and desynchronization alternately appear with increasing value of the coupling strength c—is derived. In accordance with our intuition, the threshold values of the coupling strength depend on the size of networks, the node state function, and the types of connection topology. Future studies will be done in this respect. Another interesting topic is to study synchronization between coupled networks with dilution of the connections between A and B. We hope that such reports will appear elsewhere.

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