Synchronization Control for Nonlinear Stochastic Dynamical Networks: Pinning Impulsive Strategy

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***Abstract*— In this paper, a new control strategy is proposed for the synchronization of stochastic dynamical networks with nonlinear coupling. Pinning state feedback controllers have been proved to be effective for synchronization control of state-coupled dynamical networks. We will show that pinning impulsive controllers are also effective for synchronization control of the above mentioned dynamical networks. Some generic mean square stability criteria are derived in terms of algebraic conditions, which guarantee that the whole state-coupled dynamical network can be forced to some desired trajectory by placing impulsive controllers on a small fraction of nodes. An effective method is given to select the nodes which should be controlled at each impulsive constants. The proportion of the controlled nodes guaranteeing the stability is explicitly obtained, and the synchronization region is also derived and clearly plotted. Numerical simulations are exploited to demonstrate the effectiveness of the pinning impulsive strategy proposed in this paper.**

***Index Terms*— Nonlinear coupling, pinning impulsive control, state-coupled dynamical network, synchronization.**

I. Introduction

COMPLEX dynamical networks are composed of a large number of interconnected dynamical nodes, in which each node is a unit with specific contents [1]–[3]. Typical examples of complex networks include the Internet, the World Wide Web, neural networks, food webs, cellular and metabolic networks, etc., [4], [5]. Since the seminal papers on "small-world" and "scale-free" properties [4], [6], complex networks have become a focus of research and have received increasing attention from various fields of science and engineering. Complex networks often exhibit complex and interesting dynamical behavior including synchronization [3], [7], consensus [8], flocking etc. As one of the most interesting and important collective behavior in dynamical networks, synchronization has attracted special attention of researchers in different fields [9]–[13].

Synchronization in dynamical networks is realized via a sufficient information exchange among the nodes' interconnections [8], [14], [15], which makes the final synchronous state difficult to predict. However, for many biological, physical and social dynamical networks, there exists a common requirement to regulate the behavior of large ensembles of interacting units. Some regulatory mechanisms have been uncovered in the context of biological, physiological, and cellular processes [16], which are fundamental to guarantee the correct functioning of the whole network. Examples include the control of the respiratory rhythm played by synaptically coupled pacemaker neurons in the medulla in physiology [17], and opinion leader in social networks. Hence, in many cases, controllers are necessary to be designed to force the unpredicted final synchronous state into a certain required objective trajectory [18]–[22].

It has been revealed that, in the process of controlling various networks, feedback control serves as a simple and effective strategy for stabilization and synchronization. Different kinds of effective methods, including adaptive controllers [23], [24], impulsive controllers [25]–[27] and pinning state feedback controllers [19], [20], have been designed for the stabilization and synchronization of complex dynamical networks. In [23], [24], the feedback strength is asymptotically enhanced according to a certain update law for the stabilization and synchronization of dynamical networks. In [26], distributed impulsive controllers are properly designed for the synchronization control of dynamical networks. Pinning state feedback controllers were first proposed to control multi-mode laser systems in [28], and have recently been used for the synchronization of complex networks by controlling a small fraction of nodes [19], [20], [22]. These methods have been shown to be effective for the synchronization control of networks.

For many realistic networks, the state of nodes is often subject to instantaneous perturbations and experience abrupt change at certain instants which may be caused by switching

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phenomena, frequency change or sudden noise, i.e., it exhibits impulsive effects. On the other hand, each individual node in dynamical networks is often subject to various types of noise and uncertainty, which can have a great influence on the behavior of dynamical networks. Impulsive control strategy has also been shown to be an effective control strategy in many fields due to its potential advantages over general continuous control schemes [26], [29]. However, in previous studies, when impulsive controllers are designed for the synchronization control of dynamical networks with state-coupling, all of the nodes should be controlled, which means that the controlling cost is very high. Pinning state feedback control, which means that only a small fraction of nodes is directly controlled, has been proved to be effective for the synchronization of dynamical networks with state-coupling. Then one may ask: 1) can the stochastic dynamical network be synchronized by *impulsively* controlling a small fraction of nodes; 2) that is, can we also design a certain *pinning impulsive control strategy* for the synchronization of stochastic dynamical networks; and 3) this paper is devoted to solving this problem. Some genetic criteria are given to judge whether dynamical networks can be globally exponentially forced to a desired equilibrium by impulsively controlling a small fraction of nodes. Numerical examples are finally given to demonstrate the effectiveness of the proposed impulsive strategy.

Notations: The standard notations will be used in this paper. I_n is the identity matrix of order n. $\lambda_{\max}(\cdot)$ is used to denote the maximum eigenvalue of a real symmetric matrix. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times n}$ are $n\times n$ real matrices. $\|x\|$ denotes the Euclidean norm of vector $x \in \mathbb{R}^n$. Let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, 3, \ldots\}$. The superscript "$T$" represents the transpose. For any random variable ζ, let $E(\zeta)$ be the expectation value of ζ. $\#G$ denotes the number of elements of a finite set G.

II. Preliminaries

In this paper, we consider the following stochastic dynamical network with nonlinear coupling:

$$dx_i(t) = \left[Cx_i(t) + B\tilde{f}(x_i(t))\right]dt + \tilde{g}(t, x_i(t))dw(t) + c\sum_{j=1}^{N} a_{ij}\Gamma\tilde{h}(x_j(t))dt, \quad i = 1, 2, \ldots, N \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the i-th node at time t, $C \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $w(t) \in \mathbb{R}^m$ is an m-dimensional Brownian motion, $\tilde{f}(x_i(t)) = [\tilde{f}_1(x_{i1}(t)), \tilde{f}_2(x_{i2}(t)), \ldots, \tilde{f}_n(x_{in}(t))]^T$ satisfying $\tilde{f}(\mathbf{0}) = \mathbf{0}$, $\tilde{g}:\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the noise intensity function matrix satisfying $\tilde{g}(t, \mathbf{0}) = \mathbf{0}^{n\times m}$. The nonlinear function $\tilde{h}(x_j(t)) = (\tilde{h}(x_{j1}(t)), \tilde{h}(x_{j2}(t)), \ldots, \tilde{h}(x_{jn}(t)))^T$ satisfies the following conditions: $[(\tilde{h}(u) - \tilde{h}(v))/(u - v)] \geq \vartheta > 0$ for any $u, v \in \mathbb{R}$. The configuration coupling matrix $A = (a_{ij})_{N\times N}$ is defined as follows: if there is a connection between node i and node j $(j \neq i)$, then $a_{ij} = a_{ji} > 0$, otherwise, $a_{ij} = a_{ji} = 0$, and the diagonal elements are defined as $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$. $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\} > 0$ is the inner coupling positive definite matrix between two connected nodes i and j, and c is the coupling strength of the network.

We have the following assumptions and lemma for the derivation of the main results.

Assumption 1: The nonlinear function $\tilde{f}(\cdot)$ is assumed to satisfy a Lipschitz condition, that is, there exists a constant $\kappa > 0$ such that $\|\tilde{f}(u) - \tilde{f}(v)\| \leq \kappa\|u - v\|$ holds for any $u, v \in \mathbb{R}^n$.

Assumption 2: Assume that the noise intensity function matrix $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is uniformly Lipschitz continuous in terms of the norm induced by the trace inner product on the matrices

$$\mathrm{trace}\left[\left(g(t, u) - g(t, v)\right)^T \cdot \left(g(t, u) - g(t, v)\right)\right] \leq \|M(u - v)\|^2 \qquad \forall u, v \in \mathbb{R}^n \quad (2)$$

where M is a known constant matrix with compatible dimensions.

Lemma 1 ([30]): Consider the following stochastic system with impulses:

$$\begin{cases} dx(t) = \phi(t, x(t))dt + \eta(t, x(t))dw(t), \ t \geq t_0, \quad t \neq t_k, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)) \quad k \in \mathbb{N}. \end{cases} \quad (3)$$

Assume that there exist a Lyapunov function $V(t, x(t))$, and functions φ, ψ_k with $\varphi(t, 0) = \psi_k(0) = 0$ for any $t \geq 0$, $k \in \mathbb{N}$, such that:

1) there exist positive constants c_1 and c_2 such that for all $t \geq t_0$, $c_1\|x(t)\| \leq V(t, x(t)) \leq c_2\|x(t)\|$;
2) there exists continuous function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, and $\varphi(t, s)$ is concave on s for each $t \in \mathbb{R}^+$, such that $\mathcal{L}V(t, x) \leq \varphi(t, V(t, x))$, where the operator $\mathcal{L}$ is defined as $\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)\phi(t, x) + (1/2)\mathrm{trace}[\eta^T(t, x)V_{xx}\eta(t, x)]$;
3) there exist continuous and concave functions $\psi_k : \mathbb{R}^+ \to \mathbb{R}^+$, $k \in \mathbb{N}$, such that $V(t_k^+, x(t_k^+)) \leq \psi_k(V(t_k^-, x(t_k^-)))$

then the exponential stability of the trivial solution of the following comparison systems:

$$\begin{cases} \dot{w}(t) = \varphi(t, w(t)), \ t \geq t_0, \ t \neq t_k, \\ w(t_k^+) = \psi_k(w(t_k^-)), \quad k \in \mathbb{N}, \\ w(t_0) = E(V(t_0, x_0)) \end{cases} \quad (4)$$

implies the exponential stability of the trivial solution of the stochastic impulsive system (3).

Let $s(t)$ be a solution of an isolated node described by

$$ds(t) = \left[Cs(t) + B\tilde{f}(s(t))\right]dt + \tilde{g}(t, s(t))dw(t) \quad (5)$$

with initial condition $s_0 \in \mathbb{R}^n$. In this paper, we want to control the nonlinear dynamical network (1) into the desired trajectory $s(t)$.

Let $e_i(t) = x_i(t) - s(t)$ be the error state of the node i. In order to force the whole network (1) into the desired trajectory $s(t)$, the following impulsive controllers are designed for l nodes:

$$I_i(t) = \begin{cases} \sum_{k=1}^{+\infty} \mu e_i(t)\delta(t - t_k), \ i \in \mathfrak{D}(t_k), \ \#\mathfrak{D}(t_k) = l, \\ 0, \ i \notin \mathfrak{D}(t_k) \end{cases} \quad (6)$$

where the constant $\mu \in (-2, 0)$, $\delta(\cdot)$ is the Dirac delta function, the time series $\{t_1, t_2, t_3, \ldots\}$ is a sequence of strictly increasing impulsive instants satisfying $\lim_{k\to\infty} t_k = +\infty$, and the index set of $\mathfrak{D}(t_k)$ is defined as follows: at time instant t_k, for the vectors $e_1(t_k), e_2(t_k), \ldots, e_N(t_k)$, one can reorder the states of the nodes such that $\|e_{p1}(t_k)\| \geq \|e_{p2}(t_k)\| \geq \cdots \geq \|e_{pl}(t_k)\| \geq \|e_{p,l+1}(t_k)\| \geq \cdots \geq \|e_{pN}(t_k)\|$. Then the index set of l controlled nodes $\mathfrak{D}(t_k)$ is defined as $\mathfrak{D}(t_k) = \{p_1, p_2, \ldots, p_l\}$, and $\#\mathfrak{D}(t_k) = l$.

Since $c\sum_{j=1}^{N} a_{ij}\Gamma h(s(t)) = 0$, after adding the pinning impulsive controllers (6) to the dynamical network (1), one can obtain the following impulsively controlled dynamical network:

$$\begin{cases} de_i(t) = [Ce_i(t) + Bf(e_i(t))]dt + g(t, e_i(t))dw(t) \\ \qquad\qquad + c\sum_{j=1}^{N} a_{ij}\Gamma h(e_j(t))dt, \ t \neq t_k, \ k \in \mathbb{N}, \\ e_i(t_k^+) = e_i(t_k^-) + \mu e_i(t_k^-), \quad i \in \mathfrak{D}(t_k), \ \#\mathfrak{D}(t_k) = l \end{cases} \tag{7}$$

where $f(e_i(t)) = \tilde{f}(x_i(t)) - \tilde{f}(s(t))$, $g(t, e_i(t)) = \tilde{g}(t, x_i(t)) - \tilde{g}(t, s(t))$, $h(e_i(t)) = \tilde{h}(x_i(t)) - \tilde{h}(s(t))$. Since $[(\tilde{h}(u) - \tilde{h}(v))/(u - v)] \geq \vartheta > 0$, we have $[(h(u) - h(v))/(u - v)] \geq \vartheta > 0$ for any $u, v \in \mathbb{R}$.

Throughout this paper, we always assume that $e_i(t)$ is left-hand continuous at $t = t_k$, i.e., $e(t_k) = e(t_k^-)$. Therefore, the solutions of (7) are piecewise left-hand continuous functions with discontinuities at $t = t_k$ for $k \in \mathbb{N}$.

Definition 1: The trivial solution of the dynamical system (7) is said to be exponentially mean square stable if for any initial condition $e_i(t_0)$ $(i = 1, 2, \ldots, N)$, there exist positive constants W_0 and ω such that $E\{\sum_{i=1}^{N} \|x_i(t)\|^2\} \leq W_0 e^{-\omega(t-t_0)}$.

By referring to the concept of average dwell time [31], [32], a new concept named *average impulsive interval* has been proposed by the authors to describe wider class of impulsive signal, and has been utilized for the derivation of a unified synchronization criterion of dynamical networks in [7]. Since $\mu \in (-2, 0)$, which means that the impulsive effects are stabilizing, the frequency of impulses should not be too low. In order to guarantee that the frequency of impulses is not too low, the following definition is presented.

Definition 2 ([7] average impulsive interval): The *average impulsive interval* of the impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is *less than* T_a, if there exist a positive integer N_0 and a positive number T_a, such that

$$N_\zeta(T, t) \geq \frac{T - t}{T_a} - N_0 \qquad \forall T \geq t \geq 0 \tag{8}$$

where $N_\zeta(T, t)$ denotes the number of impulsive times of the impulsive sequence ζ in the time interval (t, T).

Remark 1: According to Definition 2, there is no strict requirement for the impulsive sequence on the upper bound of the impulsive intervals, which is normally necessary in the references concerning impulsive control. For very large $\varsigma > 0$ and any $T_a > 0$, many impulsive sequences $\{t_1, t_2, \ldots\}$ can be constructed such that the upper bound of the impulsive intervals is not less than to ς and simultaneously the average impulsive intervals are less than T_a. Let $k = \lceil \varsigma / T_a \rceil$ and $\epsilon > 0$ very small. One simple example is $\zeta^* = \{t_0 + T_a + \epsilon, t_0 + T_a + 2\epsilon, \ldots, t_0 + T_a + k\epsilon, t_0 + T_a + k\epsilon + \varsigma, t_0 + T_a + (k+1)\epsilon + \varsigma, \ldots, t_0 + T_a + 2k\epsilon + \varsigma, t_0 + T_a + 2k\epsilon + 2\varsigma, \ldots\}$. For the impulsive sequence ζ^*, the upper bound of the impulsive interval is ς, which can be very large. Since the upper bound is used to represent the frequency of the impulsive sequence in [26], and [33]–[35], or identical impulsive interval is used in [36], the results obtained in these references are not available for the impulsive sequence ζ^* with very large upper bound of impulsive intervals, for which our results may be applicable.

Remark 2: By using the special example ζ^* presented in Remark 1, the idea behind this concept can be explained as follows: low-density impulses (such as "$t_0 + T_a + k\epsilon, t_0 + T_a + k\epsilon + \varsigma$") are allowed to happen in a certain interval, and high-density impulses (such as "$t_0 + T_a + (k+1)\epsilon + \varsigma, \ldots, t_0 + T_a + 2k\epsilon + \varsigma$") should follow for compensation.

III. Main Results

In this section, we will derive the main results about our pinning impulsive strategy for synchronization control of the stochastic dynamical network (1) with nonlinear coupling. Based on the above-mentioned assumptions and definitions, we can obtain the following theorem to show that the state-coupled dynamical network can be successfully stabilized to an objective state by only impulsive controlling a small fraction of nodes.

Theorem 1: Consider the controlled dynamical network (7) with an irreducible coupling matrix A. Let $\#\mathfrak{D}(t_k) = l$, $\rho = 1 + (l/N) \cdot \mu(\mu + 2) \in (0, 1)$ and $\delta = \lambda_{\max}(C + C^T + M^T M) + 2\sqrt{\lambda_{\max}(B^T B)}\kappa$. Suppose that Assumptions 1 and 2 hold, and the average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is less than T_a. Then, the controlled dynamical network (7) is globally exponentially stable in mean square, if

$$\frac{ln\rho}{T_a} + \delta < 0. \tag{9}$$

It means that the nonlinear stochastic dynamical network (1) can be exponentially controlled to the objective trajectory $s(t)$ by using pinning impulsive controllers (6).

Proof: Consider the following Lyapunov functions:

$$V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t). \tag{10}$$

For $t \in (t_{k-1}, t_k]$, $k \in \mathbb{N}$, we have

$$\begin{aligned} &\mathfrak{L}V(t) \\ &= 2\sum_{i=1}^{N} e_i^T(t)\big[Ce_i(t) + Bf(e_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma h(e_j(t))\big] \\ &\quad + \sum_{i=1}^{N} \mathrm{trace}\big[g^T(t, e_i(t))g(t, e_i(t))\big] \\ &= 2\sum_{i=1}^{N} \big[e_i^T(t)Ce_i(t) + e_i^T(t)Bf(e_i(t))\big] \\ &\quad + 2c\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}e_i^T(t)\Gamma h(e_j(t)) \\ &\quad + \sum_{i=1}^{N} \mathrm{trace}\big[g^T(t, e_i(t))g(t, e_i(t))\big]. \end{aligned} \tag{11}$$

By Assumptions 1 and 2, the following inequalities can be obtained:

$$\begin{aligned}&2e_i^T(t)Bf(e_i(t))\\&\leq 2\|e_i(t)\|\cdot\|Bf(e_i(t))\|\\&\leq 2\|e_i(t)\|\cdot\sqrt{\lambda_{\max}(B^TB)}\cdot\|f(e_i(t))\|\\&\leq 2\sqrt{\lambda_{\max}(B^TB)}\kappa\|e_i(t)\|^2\\&=2\sqrt{\lambda_{\max}(B^TB)}\kappa e_i^T(t)e_i(t)\end{aligned}\tag{12}$$

and

$$\begin{aligned}\text{trace}\big[g^T(t,e_i(t))g(t,e_i(t))\big]&\leq\|Me_i(t)\|^2\\&=e_i^T(t)M^TMe_i(t).\end{aligned}\tag{13}$$

Since $[(h(u)-h(v))/(u-v)]\geq\vartheta>0$, it follows from the diffusive property of symmetric matrix A that

$$\begin{aligned}&2c\sum_{i=1}^N\sum_{j=1}^N a_{ij}e_i^T(t)\Gamma h(e_j(t))\\&=2c\sum_{i=1}^N\sum_{j=1}^N a_{ij}\left[\sum_{\theta=1}^n e_{i\theta}(t)\gamma_\theta h(e_{j\theta}(t))\right]\\&=2c\sum_{\theta=1}^n\gamma_\theta\left[\sum_{i=1}^N\sum_{j=1}^N e_{i\theta}(t)a_{ij}h(e_{j\theta}(t))\right]\\&=-c\sum_{\theta=1}^n\gamma_\theta\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N a_{ij}(e_{i\theta}(t)-e_{j\theta}(t))\\&\quad\times(h(e_{i\theta}(t))-h(e_{j\theta}(t)))\\&\leq -c\sum_{\theta=1}^n\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N\vartheta\gamma_\theta a_{ij}(e_{i\theta}(t)-e_{j\theta}(t))^2\\&\leq 0.\end{aligned}\tag{14}$$

Considering (12)–(14), it follows from (11) that

$$\begin{aligned}&\mathcal{L}V(t)\\&\leq\sum_{i=1}^N e_i^T(t)\big[C+C^T+M^TM\big]e_i(t)\\&\quad+\sum_{i=1}^N 2\sqrt{\lambda_{\max}(B^TB)}\kappa e_i^T(t)e_i(t)\\&\leq\Big(\lambda_{\max}(C+C^T+M^TM)+2\sqrt{\lambda_{\max}(B^TB)}\kappa\Big)\\&\quad\times\sum_{i=1}^N e_i^T(t)e_i(t)\\&=\delta\cdot V(t),\qquad\text{for }\ t\in(t_{k-1},t_k],\quad k\in\mathbb{N}.\end{aligned}\tag{15}$$

For any $k\in\mathbb{N}$, let $\alpha(t_k)=\min\{\|e_i(t_k)\|:i\in\mathfrak{D}(t_k)\}$ and $\beta(t_k)=\max\{\|e_i(t_k)\|:i\notin\mathfrak{D}(t_k)\}$. According to the selection of nodes in set $\mathfrak{D}(t_k)$, we have $\alpha(t_k)\geq\beta(t_k)$. Since $\rho=1+(l/N)\cdot\mu(\mu+2)\in(0,1)$, we get $(1-\rho)(N-l)=[\rho-(1+\mu)^2]l$. Hence, one has

$$\begin{aligned}&(1-\rho)\sum_{i\notin\mathfrak{D}(t_k)}e_i^T(t_k^-)e_i(t_k^-)\\&\leq(1-\rho)(N-l)(\beta(t_k))^2\\&\leq(1-\rho)(N-l)(\alpha(t_k))^2\\&\leq l\big[\rho-(1+\mu)^2\big](\alpha(t_k))^2\\&\leq\big[\rho-(1+\mu)^2\big]\sum_{i\in\mathfrak{D}(t_k)}e_i^T(t_k^-)e_i(t_k^-)\end{aligned}\tag{16}$$

which follows that

$$\begin{aligned}(1+\mu)^2\sum_{i\in\mathfrak{D}(t_k)}e_i^T(t_k^-)e_i(t_k^-)+\sum_{i\notin\mathfrak{D}(t_k)}e_i^T(t_k^-)e_i(t_k^-)\\\leq\rho\sum_{i=1}^N e_i^T(t_k^-)e_i(t_k^-).\end{aligned}\tag{17}$$

Then, for any $k\in\mathbb{N}$, we yield

$$\begin{aligned}&V(t_k^+)\\&=\sum_{i=1}^N e_i^T(t_k^+)e_i(t_k^+)\\&=\sum_{i\in\mathfrak{D}(t_k)}e_i^T(t_k^+)e_i(t_k^+)+\sum_{i\notin\mathfrak{D}(t_k)}e_i^T(t_k^+)e_i(t_k^+)\\&=\sum_{i\in\mathfrak{D}(t_k)}(1+\mu)^2e_i^T(t_k^-)e_i(t_k^-)+\sum_{i\notin\mathfrak{D}(t_k)}e_i^T(t_k^-)e_i(t_k^-)\\&\leq\rho\sum_{i=1}^N e_i^T(t_k^-)e_i(t_k^-)\\&=\rho V(t_k^-).\end{aligned}\tag{18}$$

By (15) and (18), we can obtain the following comparison system (19) for the controlled dynamical network (7):

$$\begin{cases}\dot{w}(t)=\delta w(t),\ t\geq t_0,\ t\neq t_k,\\w(t_k^+)=\rho w(t_k^-),\qquad\rho\in(0,1),\ k\in\mathbb{N},\\w(t_0)=E(V(t_0)).\end{cases}\tag{19}$$

According to (19), for any $t\in\mathbb{R}^+$, one has

$$w(t)=E(V(t_0))\cdot e^{\delta(t-t_0)}\rho^{N_\zeta(t,t_0)}\tag{20}$$

where $N_\zeta(t,t_0)$ means the number of impulses of the impulsive sequence ζ in the time interval (t_0,t).

According to the facts that $\rho\in(0,1)$ and that the average impulsive interval of the impulsive sequence $\zeta=\{t_1,t_2,\ldots\}$ is less than T_a, it follows from Definition 2 that

$$\begin{aligned}w(t)&=E(V(t_0))\cdot e^{\delta(t-t_0)}\rho^{N_\zeta(t,t_0)}\\&\leq E(V(t_0))\cdot e^{\delta(t-t_0)}\rho^{\frac{t-t_0}{T_a}-N_0}\\&=E(V(t_0))\rho^{-N_0}\cdot e^{\delta(t-t_0)}\cdot e^{\frac{ln\rho}{T_a}(t-t_0)}\\&=E(V(t_0))\rho^{-N_0}\cdot e^{(\frac{ln\rho}{T_a}+\delta)(t-t_0)}.\end{aligned}\tag{21}$$

Since $(ln\rho/T_a)+\delta<0$, the trivial solution of the comparison system (19) is exponentially stable. By Lemma 1, we can conclude that the controlled dynamical network (7) is exponential stable, which further implies that the dynamical network (1) can be exponentially stabilized to the objective trajectory $s(t)$ by only impulsively controlling a small fraction of nodes. Theorem 1 is proved. ■

Remark 3: By using the Lyapunov method combined with the comparison principle, the exponential stability criterion of the pinning impulsively controlled dynamical network has been obtained. It means that the state-coupled dynamical network can be efficiently forced to the objective trajectory by using pinning impulsive controllers. Our result displays another kind of effective and relatively cheap control strategy for the synchronization of complex dynamical networks.

Remark 4: The criterion presented in Theorem 1 is closely related to the system parameters, average impulsive interval, impulsive strength, and the proportion of the controlled nodes. The criterion can be easily judged without large computation. In the following, a theorem will be given to explicitly show how many nodes should be controlled for a successful synchronization control of the nonlinear stochastic dynamical network (1).

Theorem 2: Consider the controlled dynamical network (7) with an irreducible coupling matrix A. Let $\delta = \lambda_{\max}(C+C^T+M^TM) + 2\sqrt{\lambda_{\max}(B^TB)}\kappa$. Suppose that Assumptions 1 and 2 hold, and the average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is less than T_a. Then, the controlled dynamical network (7) is globally exponentially stable in mean square, if

$$\frac{l}{N} > \frac{1}{\mu(\mu+2)}\left(e^{-\delta T_a} - 1\right) \tag{22}$$

where $l \ll N$ is the number of nodes to be controlled.

Proof: Since $\rho = 1 + (l/N) \cdot \mu(\mu+2) \in (0, 1)$, this theorem can be proved by using Theorem 1. The detailed proof is omitted here. ■

Remark 5: Since $l \ll N$ is the number of nodes to be controlled, $(l/N) \ll 1$ is the proportion of the controlled nodes. The numerical example illustrates that the stochastic dynamical network can be successfully synchronized to a certain objective trajectory by impulsively controlling 10% of the nodes. It means that our pinning impulsive strategy is effective for the synchronization of networks with a small fraction of nodes controlled.

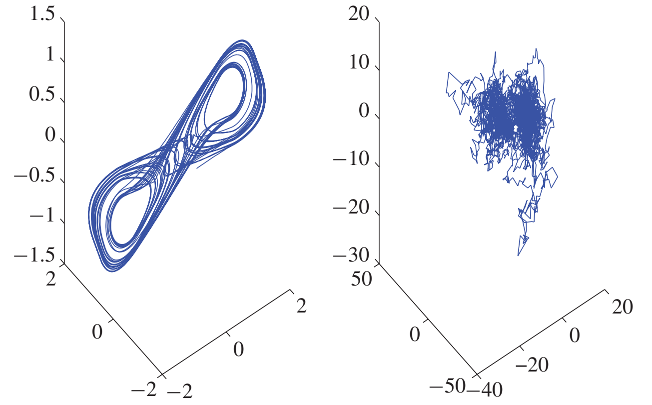
Remark 6: Similar with Theorem 2, we can conclude that: the stochastic dynamical network (1) can be forced to the objective trajectory $s(t)$ by pinning impulsive controllers (6) if one of the following inequalities is satisfied:

1) $T_a < -\frac{1}{\delta}ln(\mu(\mu+2)\cdot\frac{l}{N}+1)$;
2) $-\sqrt{\frac{e^{-\delta T_a}-1}{l/N}+1}-1 < \mu < \sqrt{\frac{e^{-\delta T_a}-1}{l/N}+1}-1$.

Remark 7: The synchronization problem for discrete-time stochastic dynamical networks has drawn much research attention [36], [37]. Our pinning impulsive strategy obtained in this paper is also applicable to the case of discrete-time dynamical networks.

IV. Numerical Example

In this section, numerical example will be given to demonstrate the effectiveness of our main results. A chaotic system with Brownian noise is selected as the isolated node of the dynamical network, and the i-th node is described as follows:



(a) (b)

Fig. 1. Phase trajectories of single dynamical system (a) without noise and (b) with noise.

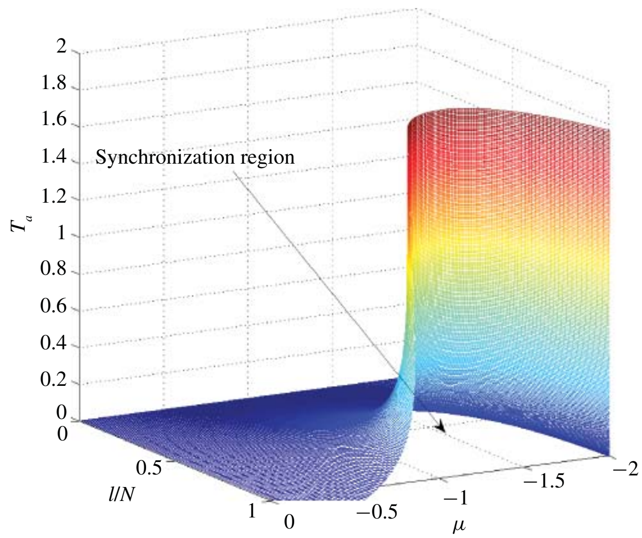


Fig. 2. Estimation of boundaries of synchronization region with respect to μ, T_a, and (l/N).

$$dx_i(t) = \left[Cx_i(t) + B\tilde{f}(x_i(t))\right]dt + \tilde{g}(t, x_i(t))dw_i(t) \tag{23}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \in \mathbb{R}^3$, $dw_i(t)$ is an 3-D Brownian motion, and the parameters are given as $C = -1.2 \cdot I_3$, $B = \begin{pmatrix} 1.16 & -1.5 & -1.5 \\ -1.5 & 1.16 & -2.0 \\ -1.2 & 2.0 & 1.16 \end{pmatrix}$, nonlinear function $\tilde{f}(x_i(t)) = (\tanh(x_{i1}), \tanh(x_{i2}), \tanh(x_{i3}))^T$, and the noise intensity function matrix $\tilde{g}(t, x_i(t)) = 0.5 \cdot \|x_i(t)\| \cdot I_3$. Then we have $\kappa = 1$ and $M = 0.5 \cdot I_3$ for Assumptions 1 and 2. System (23) without Brownian motion noise has a chaotic attractor [38] with initial value [0.3, −0.1, −0.4] as shown in Fig. 1.

In this example, a Newman-Watts small-world network with 100 nodes will be considered [39]. The small-world network is generated by taking initial neighboring nodes $k = 4$ and the edge adding probability $p = 0.1$. The coupling matrix A is defined as follows: if there is a connection between nodes i and j, then $a_{ij} = a_{ji} = 1$, otherwise $a_{ij} = a_{ji} = 0$. The nonlinear coupling function $\tilde{h}$ is taken as $\tilde{h}(z) = z + tanh(z)$

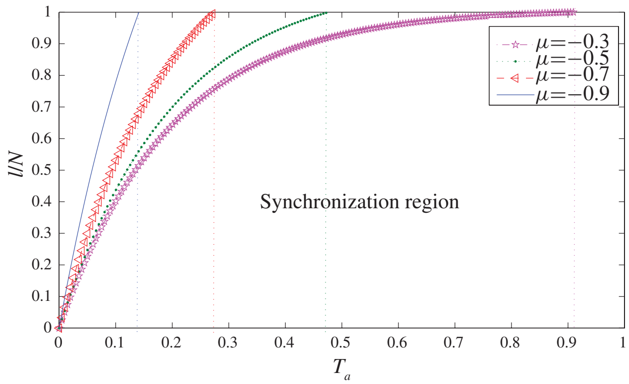


Fig. 3. Estimation of the synchronization region about T_a and (l/N) with different μ.

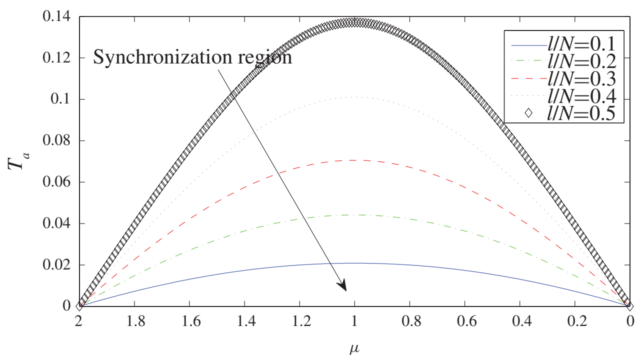


Fig. 4. Estimation of the synchronization region about μ and T_a with different (l/N).

with $\vartheta = 1$, coupling strength $c = 1$ and the inner coupling matrix $\Gamma = I_3$. By some simple calculations, we can obtain that $\delta = 5.0542$. Theorem 2 and Remark 6 have been given to show the explicit relationship between three quantities (l/N), μ and T_a. Fig. 2 shows the synchronization region of the controlled dynamical network for μ, T_a, and (l/N). Fig. 3 displays the estimation of boundaries of the synchronization regions for different μ with respect to T_a and (l/N).

Remark 8: According to the property of the impulsive control, one knows that if $\mu = -1$, the error state of the controlled system becomes zero immediately after the impulsive controller. Hence the impulsive interval can be $+\infty$ when $\mu = -1$ [40], [41]. However, since only a small fraction of nodes is controlled at each impulsive instant t_k, the error states of the controlled nodes would become nonzero due to the interconnections with some other uncontrolled nodes even for $\mu = -1$. Therefore, when $\mu = -1$, the average impulsive interval has an upper bound as shown in Fig. 4. In some real applications, one may choose $\mu = -1$ to have the maximum impulsive interval.

Now, we take special values of μ, T_a, and (l/N) for numerical illustration. Let $\mu = -0.9$ and $T_a = 0.02$, one can get that $[1/\mu(\mu + 2)](e^{-\delta T_a} - 1) = 0.0971$. By Theorem 2, it can be concluded that the nonlinear stochastic dynamical network can be synchronized to the objective trajectory if $(l/N) = 10\%$ of the nodes is controlled. In other words, ten nodes should be impulsively controlled in the generated small-world network containing 100 nodes. The trivial point $s(t) = 0$ is taken as the objective trajectory in this example.

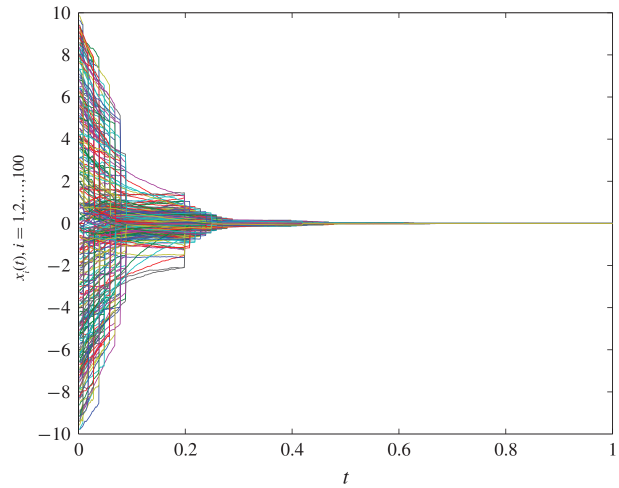


Fig. 5. Pinning impulsive synchronization of small-world coupled dynamical networks by controlling 10% nodes.

Fig. 5 presents the numerical process for the synchronization control, in which the impulsive sequence is generated with $N_0 = 10$ and $\epsilon = 0.01$. All initial values of the dynamical network are uniformly randomly selected from $[-10, 10]$.

V. Conclusion

In this paper, the synchronization control problem of stochastic dynamical networks with nonlinear coupling has been studied by pinning a small fraction of nodes with impulsive controllers. The uncontrolled nodes can be virtually forced to the desired synchronization trajectory by the pinned nodes via the inter-connections. Some stability criteria have been established to guarantee the success of synchronization via pinning controllers, and moreover the stable region can be explicitly revealed and plotted. Numerical examples are also given to demonstrate the effectiveness of our proposed control strategy.

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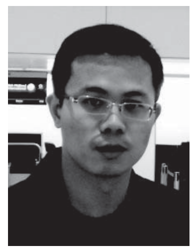

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