Synchronization in an array of linearly stochastically coupled networks with time delays

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Abstract

In this paper, the complete synchronization problem is investigated in an array of linearly stochastically coupled identical networks with time delays. The stochastic coupling term, which can reflect a more realistic dynamical behavior of coupled systems in practice, is introduced to model a coupled system, and the influence from the stochastic noises on the array of coupled delayed neural networks is studied thoroughly. Based on a simple adaptive feedback control scheme and some stochastic analysis techniques, several sufficient conditions are developed to guarantee the synchronization in an array of linearly stochastically coupled neural networks with time delays. Finally, an illustrate example with numerical simulations is exploited to show the effectiveness of the theoretical results.

Key Words - Stochastic coupling, adaptive synchronization, mean square asymptotic stability, LaSalle invariance principle, delayed neural networks, coupling delay.

I. Introduction

Since chaos synchronization in an array of linearly coupled systems was studied by Wu and Chua in [1], in the past two decades, the synchronization problem in coupled dynamical systems has attracted considerable attention from researchers in different fields. Delayed neural networks, as a special class of complex networks, have been found to exhibit complex and unpredictable behaviors [16]–[17] such as synchronization, in addition to the traditional stability and periodic oscillation that have been intensively studied in the past years [12]–[15]. Hence, there has been a great deal of activities on investigating the synchronization problem in arrays of coupled

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delayed neural networks.

In the research area of neural network synchronization, several results have appeared in the literature. In [5], via Hermitian matrix theory and Lyapunov functional method, Chen, Zhou and Liu derived a sufficient condition for global asymptotical synchronization of coupled identical delayed neural networks. A sufficient condition was proposed in [6] for global exponential synchronization of coupled connected neural networks with constant delays by using linear matrix inequality techniques, where the connection matrix is not assumed to be symmetric and irreducible. Wang and Cao [7] studied synchronization in arrays of linearly coupled identical connected neural networks with time-varying delay. In addition, Li and Chen [8] investigated local and global robust adaptive synchronization of uncertain dynamical networks via adaptive techniques. In [9], global exponential synchronization in arrays of coupled identical delayed neural networks (DNNs) with constant and delayed coupling was considered. Chen and Zhou [10] studied synchronization of general complex networks via the adaptive feedback control scheme. Zhou, Lu and Lü [11] investigated locally and globally adaptive synchronization of an uncertain complex dynamical network. For other relevant results, we refer the readers to Refs. [2]–[4], [24]–[28].

However, in all the literature mentioned above, only linear or nonlinear deterministic coupling has been considered. Another important phenomenon, namely, stochastic coupling, has not drawn any research attention despite the fact that in reality the network coupling often occurs in a random fashion and therefore stochastic coupling is ubiquitous in the evolution of complex networks. In fact, the signal transmitted between the neurons of one subsystem is inevitably subjected to noisy disturbances from the environment, which could lead to probabilistic loss of the information in the transmission process. Therefore, the transmitted signals may not be fully detected and received by the other neurons of the system. As a conclusion, in order to investigate and simulate more realistic complex networks, the noise's effect should be taken into account in modeling the networks.

It should be mentioned that, recently, some initial research efforts have been devoted to the noises' influence on the synchronization of chaos dynamical systems, and several preliminary results have just been reported. For example, Lin and He [20] investigated the complete synchronization between unidirectionally coupled Chua's circuits within stochastic perturbation via feedback control techniques. In Lin and Chen [21], the white noise was taken to enhance synchronization of coupled chaotic systems, and it was analytically proved that chaos synchronization could be achieved with probability one merely via white-noise-based coupling. Inspired by these recently published works, in this paper, we model the array of coupled delayed neural networks through introducing the *linear stochastic coupling* strategies, in order that the dynamical behaviors of networks could reflect the real-world complexity in a more realistic way.

Motivated by the above discussion, the main objective of this paper is to investigate the synchronization problem in an array of linearly stochastically coupled delayed neural networks. Based on the invariance principle of stochastic differential equations, several theoretical results guaranteeing the synchronization of the stochastically coupled systems are developed by using adaptive feedback control techniques. Moreover, the derived theoretical results are illustrated by simulations, which can reveal the true dynamics of the stochastically coupled systems.

The rest of this paper is arranged as follows. In the following section, an array of linearly stochastically coupled delayed neural networks model is presented, and some definitions and necessary preliminaries are provided. In Section 3, via the adaptive feedback control techniques, several sufficient conditions ensuring complete synchronization of the stochastically coupled systems are developed. In Section 4, an example with numerical simulations is provided to demonstrate the effectiveness of the analytical results obtained. Finally, in Section 5, our paper is completed with a conclusion and some discussions.

II. Notations and preliminaries

Notations: For any matrix A, $A > 0$ means that A is symmetric positive definite; $E\{\cdot\}$ stands for the mathematical expectation operator; $\|x\|$ is used to denote a vector norm defined by $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$; 'T' represents the transpose of the matrix; I is an identical matrix; and $W(t)$ is a one-dimensional Brownian motion defined on the probability space.

In [7], Wang and Cao considered an array of linearly coupled delayed neural networks with coupling delay as follows

$$\begin{aligned} dx_i(t) &= [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) \\ &\quad + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t) + \sum_{j=1}^{N} G_{ij}\Gamma_\tau x_j(t-\tau)]dt, \ i = 1, 2, \cdots, N, \end{aligned} \tag{1}$$

where $x_i(t) = \left[x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)\right]^{\mathrm{T}} \in \mathbb{R}^n$ $(i = 1, 2, \cdots, N)$ is the state vector of the ith delayed neural networks; $C = \mathrm{diag}(c_1, c_2, \cdots, c_n) > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external inputs; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ represent the connection weight matrix and the delayed connection weight matrix, respectively; $f(x_i(t)) = [f_1(x_{i1}(t)), f_2(x_{i2}(t)), \cdots, f_n(x_{in}(t))]^{\mathrm{T}} \in \mathbb{R}^n$ is the activation function and $\tau > 0$ is the transmission delay; The matrices $\Gamma \in \mathbb{R}^{n\times n}$ and $\Gamma_\tau \in \mathbb{R}^{n\times n}$ describe inner coupling of the neural networks at time t and $t-\tau$, respectively. Matrix $G = (G_{ij})_{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the networks, we do not need nonnegative condition on its off-diagonal elements which just satisfies the following conditions [27]:

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij}. \tag{2}$$

As discussed in the introduction, since the complex networks itself is inevitably subjected to noise and disturbance in the environment, the *stochastic* behaviors of the complex networks cannot be properly reflected by only considering the *linear coupling* with or without *coupling delays*. A seemingly natural way is therefore to introduce the *stochastic coupling term* to model the arrays of linearly coupled delayed neural networks, which can exhibit more realistic dynamical behaviors of complex networks.

In this paper, the array of linearly stochastically coupled identical neural networks with time delays are proposed as follows:

$$\begin{aligned} dx_i(t) &= [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t)))]dt \\ &\quad + c_i \sum_{j=1}^{N} G_{ij}\Gamma x_j(t) dW_{i1}(t) + d_i \sum_{j=1}^{N} G_{ij}\Gamma_\tau x_j(t-\tau(t)) dW_{i2}(t) + U_i dt, \ i = 1, 2, \cdots, N, \end{aligned} \tag{3}$$

where $W_i = [W_{i1}, W_{i2}]^{\mathrm{T}}$ are arrays of two-dimensional Brownian motions; c_i and d_i are the noise intensity, respectively; U_i is the control input, $i = 1, 2, \cdots, N$; $0 \leq \tau(t) \leq \tau$ is the time-varying delay.

The initial conditions associated with system (3) are given in the following form:

$$x_i(s) = \xi_i(s), \quad -\tau \le s \le 0, \quad i = 1, 2, \cdots, N,$$

for any $\xi_i \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$, where $L^2_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ is the family of all $\mathcal{F}_0$-measurable $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$-valued random variables satisfying $\sup\limits_{-\tau \le s \le 0} E|\xi_i(s)|^2 < \infty$, and $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued functions $\xi_i(s)$ on $[-\tau, 0]$ with the norm $\|\xi_i\| = \sup\limits_{-\tau \le s \le 0} |\xi_i(s)|$.

Based on the definition in [27], [28], we develop the corresponding definition in the sense of mean square.

Definition 1: Let $x_i(t; t_0, X_0)(1 \le i \le N)$ be a solution of the array of linearly stochastically coupled delayed neural networks (3), where $X_0 = (x_1^0, x_2^0, \cdots, x_N^0)$. If there is a nonempty subset $\Omega \subseteq \mathbb{R}^n$, with $x_i^0 \in \Omega(1 \le i \le N)$, such that $x_i(t; t_0, X_0) \in \mathbb{R}^n$ for all $t \ge t_0, 1 \le i \le N$, and

$$\lim_{t \to \infty} E\|x_i(t; t_0, X_0) - s(t; t_0, x_0)\|^2 = 0, \quad i = 1, 2, \cdots, N, \tag{4}$$

where $s(t) \in \mathbb{R}^n$ is a solution of an isolated node

$$ds(t) \;=\; [-Cs(t) + Af(s(t)) + Bf(s(t - \tau(t)))]dt, \tag{5}$$

with $x_0 \in \mathbb{R}^n$, then the array of networks (3) is said to realize synchronization.

Note that, when the stochastically coupled delayed neural networks (3) achieves synchronization, the stochastic coupling term and the control input should vanish, that is

$$c_i \sum_{j=1}^{N} G_{ij} \Gamma x_j(t) dW_{i1}(t) + d_i \sum_{j=1}^{N} G_{ij} \Gamma_\tau x_j(t - \tau(t)) dW_{i2}(t) = 0 \quad \text{and} \quad U_i = 0. \tag{6}$$

Throughout this paper, the following hypotheses are needed:

(H_1) $f_i(x)$ satisfies the Lipschitz condition. That is, for each $i = 1, 2, \cdots, n$, there is a constant $\beta_i > 0$ such that

$$|f_i(x) - f_i(y)| \le \beta_i |x - y|, \quad \forall\, x, y \in \mathbb{R}.$$

(H_2) $\tau(t)$ is a continuous differentiable function on $\mathbb{R}^+$, i.e., $\dot{\tau}(t) \le \mu < 1$.

(H_3) $f(0) \equiv 0$.

Now, let $e_i(t) = x_i(t) - s(t)$ be the error signal, and it then follows from (3), (5) and the condition (2) that

$$\begin{aligned} de_i(t) \;=\; & [-Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t - \tau(t)))]dt \\ & + c_i \sum_{j=1}^{N} G_{ij} \Gamma e_j(t) dW_{i1}(t) + d_i \sum_{j=1}^{N} G_{ij} \Gamma_\tau e_j(t - \tau(t)) dW_{i2}(t) + U_i dt, \; i = 1, 2, \cdots, N, \end{aligned} \tag{7}$$

where $g(e_i(t)) = f(e_i(t) + s(t)) - f(s(t))$ and $g(e_i(t - \tau(t))) = f(e_i(t - \tau(t)) + s(t - \tau(t))) - f(s(t - \tau(t)))$.

Under the hypotheses (H_1) and (H_3), it is easy to have

$$\|g(e_i(t))\| = \|f_i(e_i(t) + s(t)) - f_i(s(t))\| \le \|\Sigma e_i(t)\|, \tag{8}$$

and $g(0) = 0$, where $\Sigma = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n)$. Hence, it follows from [18] that the error system (7) admits a trivial solution $e_i(0) \equiv 0, \; i = 1, 2, \cdots, N$.

Then the objective of the controller U_i is to guarantee system (3) to be synchronized, that is the trivial solution of the error system (7) is asymptotically stable in mean square, i.e.,

$$\lim_{t \to \infty} E\|e_i(t)\|^2 = \lim_{t \to \infty} E\|x_i(t) - s(t)\|^2 = 0, \quad i = 1, 2, \cdots, N. \tag{9}$$

3. Main results

In this section, based on a simple adaptive feedback control scheme, we will derive several sufficient conditions for the synchronization in an array of linearly stochastically coupled neural networks with time delays.

Theorem 1: Under the hypotheses $(H_1)-(H_3)$, the linearly stochastically coupled delayed neural networks (3) is synchronized with the controller $U_i = -k_i e_i(t)$, where the time-varying feedback gains $k_i = \mathrm{diag}(k_{i1}, k_{i2}, \cdots, k_{in})$ with the update law are taken as:

$$\dot{k_{ij}} = \alpha_{ij} e_{ij}^2(t), \tag{10}$$

for $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, n$, and $\alpha_{ij} > 0$ are arbitrary constants, respectively.

Proof: Construct the following non-negative function

$$V(t, e(t)) = \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N}\int_{t-\tau(t)}^{t} e_i^{\mathrm{T}}(s)Pe_i(s)ds + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{1}{\alpha_{ij}}(k_{ij}-L)^2, \tag{11}$$

where P is a positive definite matrix and L is a constant, and P and L are to be determined.

By Itô's differential rule, the stochastic derivative of V along trajectories of error system (7) can be obtained as follows

$$\begin{aligned} dV(t, e(t)) &= \mathcal{L}V(t, e(t))dt \\ &\quad + \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\left[c_i\sum_{j=1}^{N} G_{ij}\Gamma x_j(t)dW_{i1}(t) + d_i\sum_{j=1}^{N} G_{ij}\Gamma_\tau x_j(t-\tau(t))dW_{i2}(t)\right], \end{aligned} \tag{12}$$

where the weak infinitesimal operator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}V(t, e(t)) &= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\big[-Ce_i(t) + Ag(e_i(t)) + Bg(e_i(t-\tau(t))) - k_i e_i(t)\big] + \sum_{i=1}^{N}\big[e_i^{\mathrm{T}}(t)Pe_i(t) \\ &\quad -(1-\dot{\tau}(t))e_i^{\mathrm{T}}(t-\tau(t))Pe_i(t-\tau(t))\big] + \sum_{i=1}^{N}\sum_{j=1}^{n}(k_{ij}-L)e_{ij}^2(t) \\ &\quad + \frac{c_i^2}{2}\sum_{i=1}^{N}\big[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\big]^{\mathrm{T}}\big[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\big] \\ &\quad + \frac{d_i^2}{2}\sum_{i=1}^{N}\big[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau(t))\big]^{\mathrm{T}}\big[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau(t))\big] \\ &= \sum_{i=1}^{N}\Big\{-e_i^{\mathrm{T}}(t)Ce_i(t) + e_i^{\mathrm{T}}(t)Ag(e_i(t)) + e_i^{\mathrm{T}}(t)Bg(e_i(t-\tau(t))) + e_i^{\mathrm{T}}(t)Pe_i(t) \\ &\quad -(1-\dot{\tau}(t))e_i^{\mathrm{T}}(t-\tau(t))Pe_i(t-\tau(t)) - Le_i^{\mathrm{T}}(t)e_i(t) \\ &\quad + \frac{c_i^2}{2}\big[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\big]^{\mathrm{T}}\big[\sum_{j=1}^{N} G_{ij}\Gamma e_j(t)\big] \\ &\quad + \frac{d_i^2}{2}\big[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau(t))\big]^{\mathrm{T}}\big[\sum_{j=1}^{N} G_{ij}\Gamma_\tau e_j(t-\tau(t))\big]\Big\}. \end{aligned} \tag{13}$$

It follows from condition (8) and an elementary inequality that

$$\begin{aligned} e_i^{\mathrm{T}}(t)Ag(e_i(t)) &\le \frac{1}{2}e_i^{\mathrm{T}}(t)A^{\mathrm{T}}Ae_i(t)+\frac{1}{2}g^{\mathrm{T}}(e_i(t))g(e_i(t)) \\ &\le \frac{1}{2}e_i^{\mathrm{T}}(t)A^{\mathrm{T}}Ae_i(t)+\frac{1}{2}e_i^{\mathrm{T}}(t)\Sigma^{\mathrm{T}}\Sigma e_i(t), \end{aligned} \tag{14}$$

$$\begin{aligned} e_i^{\mathrm{T}}(t)Bg(e_i(t-\tau(t))) &\le \frac{1}{2}e_i^{\mathrm{T}}(t)B^{\mathrm{T}}Be_i(t)+\frac{1}{2}g^{\mathrm{T}}(e_i(t-\tau(t)))g(e_i(t-\tau(t))) \\ &\le \frac{1}{2}e_i^{\mathrm{T}}(t)B^{\mathrm{T}}Be_i(t)+\frac{1}{2}e_i^{\mathrm{T}}(t-\tau(t))\Sigma^{\mathrm{T}}\Sigma e_i(t-\tau(t)), \end{aligned} \tag{15}$$

where $\Sigma=\mathrm{diag}(\beta_1,\beta_2,\cdots,\beta_n)$. The last two terms in (13) can be estimated by

$$\begin{aligned} \big[\sum_{j=1}^{N}G_{ij}\Gamma e_j(t)\big]^{\mathrm{T}}\big[\sum_{j=1}^{N}G_{ij}\Gamma e_j(t)\big] &\le N\sum_{j=1}^{N}G_{ij}^2e_j^{\mathrm{T}}(t)\Gamma^T\Gamma e_j(t) \\ &\le N\lambda_{\max}(\Gamma^{\mathrm{T}}\Gamma)\sum_{j=1}^{N}G_{ij}^2e_j^{\mathrm{T}}(t)e_j(t), \end{aligned} \tag{16}$$

$$\begin{aligned} \big[\sum_{j=1}^{N}G_{ij}\Gamma_\tau e_j(t-\tau(t))\big]^{\mathrm{T}}\big[\sum_{j=1}^{N}G_{ij}\Gamma_\tau e_j(t-\tau(t))\big] &\le N\sum_{j=1}^{N}G_{ij}^2e_j^{\mathrm{T}}(t-\tau(t))\Gamma_\tau^T\Gamma_\tau e_j(t-\tau(t)) \\ &\le N\lambda_{\max}(\Gamma_\tau^{\mathrm{T}}\Gamma_\tau)\sum_{j=1}^{N}G_{ij}^2e_j^{\mathrm{T}}(t-\tau(t))e_j(t-\tau(t)). \end{aligned} \tag{17}$$

Substituting (14), (15), (16) and (17) into (13) yields

$$\begin{aligned} \mathcal{L}V(t,e(t)) &\le \sum_{i=1}^{N}\Big\{-e_i^{\mathrm{T}}(t)Ce_i(t)+\frac{1}{2}e_i^{\mathrm{T}}(t)A^{\mathrm{T}}Ae_i(t)+\frac{1}{2}e_i^{\mathrm{T}}(t)\Sigma^{\mathrm{T}}\Sigma e_i(t)+\frac{1}{2}e_i^{\mathrm{T}}(t)B^{\mathrm{T}}Be_i(t) \\ &\quad +\frac{1}{2}e_i^{\mathrm{T}}(t-\tau(t))\Sigma^{\mathrm{T}}\Sigma e_i(t-\tau(t))+e_i^{\mathrm{T}}(t)Pe_i(t)-(1-\mu)e_i^{\mathrm{T}}(t-\tau(t))Pe_i(t-\tau(t)) \\ &\quad -Le_i^{\mathrm{T}}(t)e_i(t)+cN\Lambda\lambda_{\max}(\Gamma^{\mathrm{T}}\Gamma)\sum_{j=1}^{N}e_j^{\mathrm{T}}(t)e_j(t) \\ &\quad +dN\Lambda\lambda_{\max}(\Gamma_\tau^{\mathrm{T}}\Gamma_\tau)\sum_{j=1}^{N}e_j^{\mathrm{T}}(t-\tau(t))e_j(t-\tau(t))\Big\} \\ &= \sum_{i=1}^{N}\Big\{e_i^{\mathrm{T}}(t)\big[-LI-C+\frac{1}{2}A^{\mathrm{T}}A+\frac{1}{2}\Sigma^{\mathrm{T}}\Sigma+\frac{1}{2}B^{\mathrm{T}}B+P+cN^2\Lambda\lambda_{\max}(\Gamma^{\mathrm{T}}\Gamma)\big]e_i(t) \\ &\quad +e_i^{\mathrm{T}}(t-\tau(t))\big[\frac{1}{2}\Sigma^{\mathrm{T}}\Sigma-(1-\mu)P+dN^2\Lambda\lambda_{\max}(\Gamma_\tau^{\mathrm{T}}\Gamma_\tau)\big]e_i(t-\tau(t))\Big\}, \end{aligned} \tag{18}$$

where $\Lambda=\max_{1\le i,j\le N}\{G_{ij}^2\}, c=\max_{1\le i\le N}\{c_i\}$ and $d=\max_{1\le i\le N}\{d_i\}$.

Letting

$$\begin{aligned} P &= \frac{1}{2(1-\mu)}\big(\Sigma^{\mathrm{T}}\Sigma + 2dN^2\Lambda\lambda_{\max}(\Gamma_\tau^{\mathrm{T}}\Gamma_\tau)I\big), \\ L &= cN^2\Lambda\lambda_{\max}(\Gamma^{\mathrm{T}}\Gamma) + \frac{1}{1-\mu}dN^2\Lambda\lambda_{\max}(\Gamma_\tau^{\mathrm{T}}\Gamma_\tau) + \lambda_{\max}(-C) \\ &\quad + \frac{1}{2}\lambda_{\max}(A^{\mathrm{T}}A) + \frac{2-\mu}{2(1-\mu)}\lambda_{\max}(\Sigma^{\mathrm{T}}\Sigma) + \frac{1}{2}\lambda_{\max}(B^{\mathrm{T}}B) + 1, \end{aligned}$$

we can derive that

$$\mathcal{L}V(t,e(t)) \;\leq\; -e^{\mathrm{T}}(t)e(t). \tag{19}$$

Based on the LaSalle invariance principle of stochastic differential equation, which was developed in [19], we have $e(t) \to 0$, which in turn illustrates that $E\|e(t;\xi)\|^2 \to 0$, and at the same time $k_{ij}(t) \to k_{ij}$ (constants), $i = 1, 2, \cdots, N, j = 1, 2, \cdots, n$. This completes the proof.

Remark 1: Inspired by the key idea in [22] and [11], in this paper, the adaptive feedback scheme is used to realize the synchronization in an array of *linearly stochastically coupled* delayed neural networks (3), and a sufficient condition is derived. The stochastic coupling term is introduced to model the array of coupled delayed neural networks, which can not only reflect more realistic dynamical behaviors of the networks, but also make it possible to simulate more complicate dynamical behaviors. The interesting phenomenon can be found from the simulations provided in the next section.

Remark 2: For some special cases, it can be found by simulations that synchronization in an array of linearly stochastically coupled delayed neural networks can be achieved only by the stochastic coupling, which can somewhat support the analytical results developed by Lin and Chen [21]. Unfortunately, the condition needed in [21] is sharper and the coupling strength is very large, which make it difficult to implement the neural networks in the real world once the coupling structure is determined.

Remark 3: In fact, the stochastically coupled delayed neural networks model considered in this paper could be more general. For example, the linear stochastic coupling can be the general stochastic coupling or the unknown general stochastic coupling. Moreover, the coupled system (3) itself can be a general complex network that is not restricted to the delayed neural network. One can extend the results derived in this paper to more general cases without major difficulties.

Remark 4: It should be noted that the stochastic disturbance considered in our paper is modeled as a scalar Brownian motion, which means that the signal transmitted by one of the subsystems is influenced by the same noise. Such an assumption might not be realistic in the true environment of networks because each cell of one node is always influenced by the different multidimensional noise. One of our future research topics is to extend the results obtained to the neural networks with multidimensional noise disturbances.

If the time-varying delay in the stochastically coupled delayed neural networks (3) becomes the constant delay, we can derive the following results.

Theorem 2: Under the hypotheses (H_1)–(H_3), the array of linearly stochastically coupled delayed neural networks with constant delay is synchronized with the controller $U_i = -k_i e_i(t)$, where the time-varying feedback gains

$k_i = \text{diag}(k_{i1}, k_{i2}, \cdots, k_{in})$ with the update law are taken as:

$$\dot{k_{ij}} = \alpha_{ij} e_{ij}^2(t), \tag{20}$$

for $i = 1, 2, \cdots, N, j = 1, 2, \cdots, n$, where $\alpha_{ij} > 0$ are arbitrary constants, respectively.

Proof: Construct the following non-negative function as

$$V(t, e(t)) = \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N}\int_{t-\tau}^{t} e_i^{\mathrm{T}}(s)Pe_i(s)ds + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{1}{\alpha_{ij}}(k_{ij} - L)^2, \tag{21}$$

where P is a positive definite matrix and L is a constant to be determined, respectively. The rest of the proof is similar to that of Theorem 1, which is omitted here.

When there is no coupling delay, the stochastically coupled system (3) turns into

$$dx_i(t) = [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t - \tau(t)))]dt + c_i\sum_{j=1}^{N} G_{ij}\Gamma x_j(t)dW_i(t) + U_i dt, \tag{22}$$

and the following results are obtained readily.

Corollary 1: Under the hypotheses $(H_1) - (H_3)$, the linearly stochastically coupled delayed neural networks without coupling delay (22) is synchronized with the controller $U_i = -k_i e_i(t)$, where the time-varying feedback gains $k_i = \text{diag}(k_{i1}, k_{i2}, \cdots, k_{in})$ with the update law are taken as:

$$\dot{k_{ij}} = \alpha_{ij} e_{ij}^2(t), \tag{23}$$

for $i = 1, 2, \cdots, N, j = 1, 2, \cdots, n$, where $\alpha_{ij} > 0$ are arbitrary constants, respectively.

If we remove the *stochastic coupling term* from system (3) and only consider the model with general linear coupling and coupling delay, the network reduces to

$$\begin{aligned} dx_i(t) &= [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t - \tau(t))) + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t) \\ &\quad + \sum_{j=1}^{N} G_{ij}\Gamma_\tau x_j(t - \tau(t)) + U_i]dt, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{24}$$

The following results are true.

Corollary 2: Under the hypotheses $(H_1) - (H_3)$, the linearly coupled delayed neural networks with coupling delay (24) is synchronized with the controller $U_i = -k_i e_i(t)$, where the time-varying feedback gains $k_i = \text{diag}(k_{i1}, k_{i2}, \cdots, k_{in})$ with the update law are taken as:

$$\dot{k_{ij}} = \alpha_{ij} e_{ij}^2(t), \tag{25}$$

for $i = 1, 2, \cdots, N, j = 1, 2, \cdots, n$, where $\alpha_{ij} > 0$ are arbitrary constants, respectively.

Remark 5: In [8], Li and Chen studied the robust adaptive synchronization of unknown networks, but the nonlinear controller in their paper seems a bit too complex; hence difficult to implement. While our results not only take into account the coupling delay and therefore generalize their results, but also the adaptive feedback controller is easy to be realized in order to achieve the synchronization of the coupled systems.

Remark 6: In [10], Chen and Zhou investigated the synchronization problem of general complex networks by using adaptive feedback techniques. Although the coupling form in their paper is allowed to be unknown and nonlinear, the impact from the network delays has been neglected. In this case, our results complement and improve those in [10].

If there is no coupling delay in (24), the model degenerates into

$$dx_i(t) = [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t) + U_i]dt, i = 1, 2, \cdots, N, \quad (26)$$

and we arrive at the following results.

Corollary 3: Under the hypotheses $(H_1) - (H_3)$, the linearly coupled delayed neural networks without coupling delay (26) is synchronized with the controller $U_i = -k_i e_i(t)$, where the time-varying feedback gains $k_i = \text{diag}(k_{i1}, k_{i2}, \cdots, k_{in})$ with the update law are taken as:

$$\dot{k_{ij}} = \alpha_{ij} e_{ij}^2(t), \quad (27)$$

for $i = 1, 2, \cdots, N, j = 1, 2, \cdots, n$, where $\alpha_{ij} > 0$ are arbitrary constants, respectively.

Remark 7: If the linear coupling term is removed from the coupled delayed neural networks (26), in the case that $N = 1$, complete synchronization of delayed neural networks can also be realized via the adaptive feedback scheme, and the results derived here are equivalent to the main theorem in [23].

Remark 8: When the time-varying delays in (22), (24) and (26) are specialized to the constant delays, several similar results corresponding to Corollaries 1, 2 and 3 can be derived without difficulties, which generalize and improve some of existing results.

4. Illustrative example

In this section, an example is provided to illustrate the effectiveness of the obtained results.

Example. Consider the following chaotic delayed neural networks [17]:

$$dx(t) = [-Cx(t) + Af(x(t)) + Bf(x(t-\tau))]dt, \quad (28)$$

where $f(x) = \tanh(x)$,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 4.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix}.$$

Here, $\tau = 1$. In the case that the initial condition is chosen as $x_1(s) = 0.3, x_2(s) = 0.5, \forall\ s \in [-1, 0]$, the chaotic attractor can be seen in Fig. 1.

In order to verify the developed results, the *linearly stochastically coupled* identical delayed neural networks model is considered as follows:

$$\begin{aligned} dx_i(t) = & [-Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau))]dt + c_i \sum_{j=1}^{12} G_{ij}\Gamma x_j(t) dW_{i1}(t) \\ & + d_i \sum_{j=1}^{12} G_{ij}\Gamma_\tau x_j(t-\tau) dW_{i2}(t) - k_i e_i(t)dt, i = 1, 2, \cdots, 12, \end{aligned} \quad (29)$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t)]^{\mathrm{T}}$ are the state variables of the ith delayed neural network, the number of the nodes is $N = 12$, and the coupling matrix is chosen as the ***nearest neighbor coupling***

$$G = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}_{12\times 12}.$$

For convenience, the inner linking matrices are taken as

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_\tau = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

respectively, and the noise intensity is taken as $c_i = 0.9, d_i = 0.2$, for $i = 1, 2, \cdots, 12$.

In the simulations, the Euler-Maruyama numerical scheme is used to simulate the arrays of linearly stochastically coupled delayed neural networks (29). Some initial parameters are given as follows: $T = 200$ and time step size is $\delta t = 0.02$. The initial conditions of the networks, $x_{ij}(s)(\forall\ s \in [-1, 0])$; the initial conditions of the adaptive feedback gains, $k_{ij}(s), \forall\ s \in [-1, 0]$; as well as the parameters α_{ij} are presented in Table 1.

TABLE I

INITIAL CONDITIONS OF SYSTEM (29) $x_{ij}(s)$ AND FEEDBACK GAINS $k_{ij}(s)$, AND THE PARAMETERS α_{ij}.

N	1	2	3	4	5	6	7	8	9	10	11	12
$x_{i1}(s)$	0.4	-4	5	0.4	-1	3	4	-2	4	0.5	-3	-2
$x_{i2}(s)$	-0.6	12	0.6	-6	1	-0.6	6	-12	1.6	-0.6	3	0.1
$k_{i1}(s)$	1	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.2
$k_{i2}(s)$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1.0	1.2
α_{i1}	5	4.5	4	3.5	3	2.5	2	1.5	1	1	3	2
α_{i2}	0.5	1	1.5	2	2.5	3	3.5	4	4.5	1	0.5	2

The simulation results can be described as follows. Fig. 2(a) and Fig. 2(b) depict the synchronization errors e_{i1} and e_{i2} $(i = 1, 2, \cdots, 12)$ between the stochastically coupled delayed neural networks (29) and the isolated chaotic delayed neural networks (28), respectively. Fig. 3(a) and Fig. 3(b) are the evolution of the adaptive feedback gains k_{i1} and k_{i2} $(i = 1, 2, \cdots, 12)$, respectively. From these simulations, one can find that synchronization in arrays of linearly stochastically coupled system (29) is realized via the adaptive feedback scheme, and these simulations match the theoretical results perfectly.

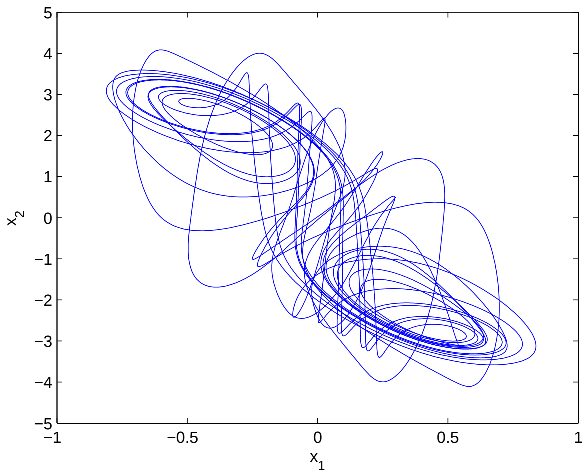


Fig. 1 Chaotic trajectory of model (17).

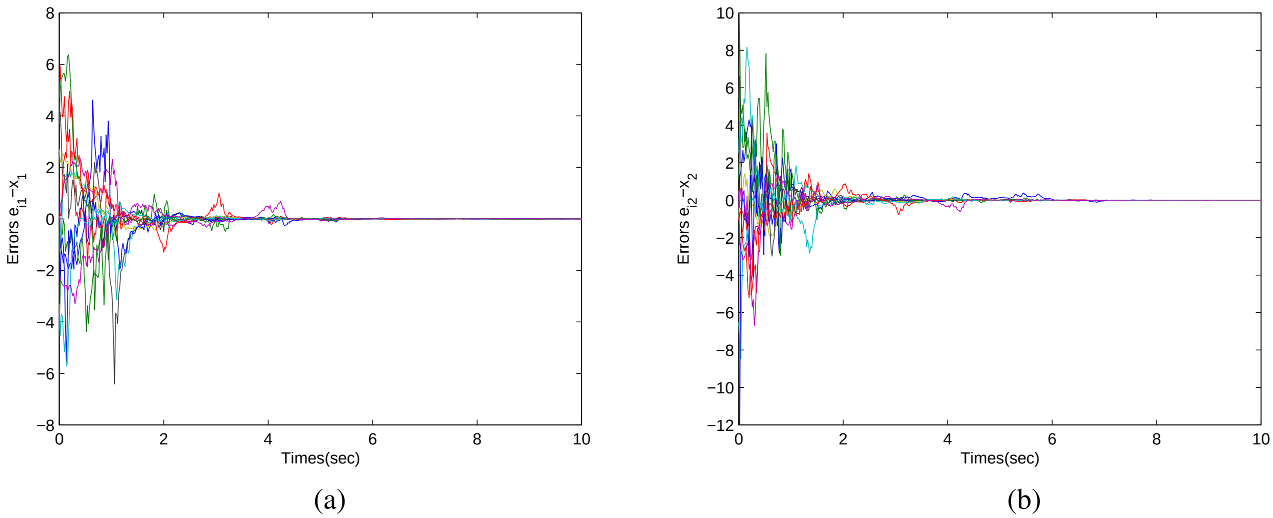
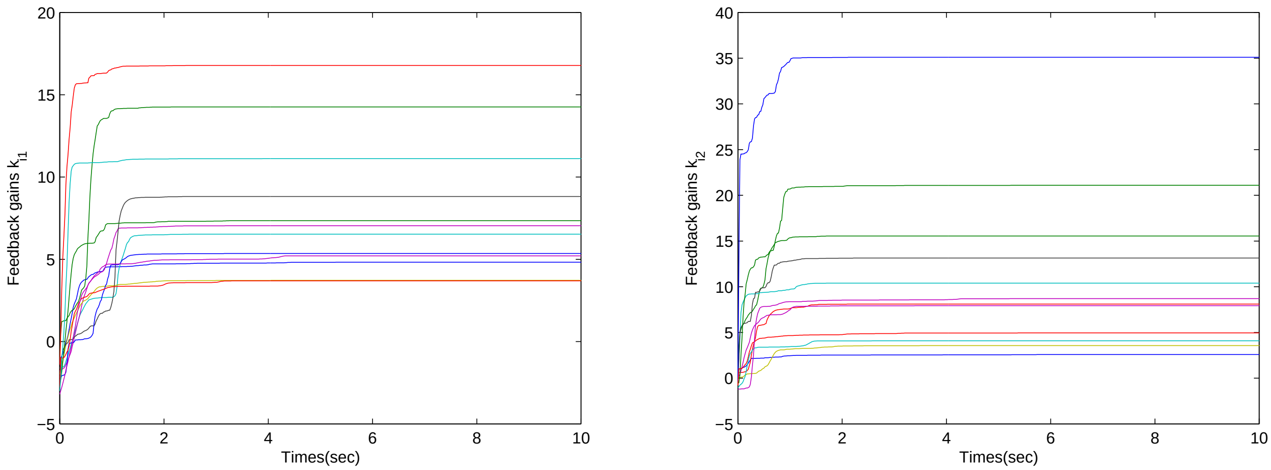


Fig. 2. (a) Synchronization errors of $e_{i1}, i = 1, 2, \cdots, 12$; (b) Synchronization errors of $e_{i2}, i = 1, 2, \cdots, 12$.

5. Conclusions

In this paper, we have introduced the *stochastic coupling term* to model an array of linearly coupled delayed networks, hence the dynamical behaviors of coupled delayed neural networks can be more realistic and complicated. Via the adaptive feedback scheme, several theoretical results ensuring complete synchronization in an array of linearly stochastically coupled delayed networks with or without coupling delay have been obtained. The derived results greatly improve and generalize some earlier results. Some computer simulations results provided later can illustrate the effectiveness of the derived results. Moreover, we believe that our initial results about synchronization in arrays of linearly stochastically coupled delayed networks can help enlighten more research branches on complex networks with stochastic noises, which is undoubtedly important for the study of complex network dynamics.



(a) (b)

Fig. 3. (a) Feedback gains $k_{i1}, i = 1, 2, \cdots, 12$; (b) Feedback gains $k_{i2}, i = 1, 2, \cdots, 12$.

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