International Journal of Bifurcation and Chaos, Vol. 12, No. 1 (2002) 187–192

SYNCHRONIZATION IN SMALL-WORLD DYNAMICAL NETWORKS

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Received March 13, 2001; Revised April 9, 2001

We investigate synchronization in a network of continuous-time dynamical systems with small-world connections. The small-world network is obtained by randomly adding a small fraction of connection in an originally nearest-neighbor coupled network. We show that, for any given coupling strength and a sufficiently large number of cells, the small-world dynamical network will synchronize, even if the original nearest-neighbor coupled network cannot achieve synchronization under the same condition.

1. Introduction

Collective motions of coupled dynamical networks are of significant interest in many fields of science and technology. In particular, synchronization in networks of coupled chaotic dynamical systems has received a great deal of attention in recent years. Most of the existing work on synchronization of coupled networks assumes that the coupling configuration is completely regular (see e.g. [Heagy *et al.*, 1994; Wu & Chua, 1995]), while a few studies address the issue of synchronization in randomly coupled networks [Gade, 1996; Manrubia & Mikhailov, 1999]. However, many biological, technological and social networks are neither completely regular nor completely random. To interpolate between these two extremes, Watts and Strogatz [1998] introduced the interesting concept of small-world networks. The so-called small-world networks have intermediate connectivity properties but exhibit a high degree of clustering as in the regular networks and a small average distance between vertices as in the random networks. They also found that the small-world networks of coupled phase oscillators can synchronize almost as readily as the globally coupled networks, despite the fact that they have much fewer edges [Watts, 1999]. For a review of recent works on small-world networks, see [Newman, 2000]. More recently, Gade and Hu [2000] explored the stability of synchronous chaos in coupled map lattices with small-world connectivity and found that in this case synchronous chaos is possible even in the thermodynamic limit. Lago-Fernandez *et al.* [2000] also investigated the fast response and temporal coherent oscillations in a small-world network of Hodgkin–Huxley neurons.

In this study, we consider synchronization in a network of linearly coupled identical continuous-time dynamical systems. As shown by Wu [1995], for any given number of cells, strong enough mutual diffusive coupling will result in synchronization of the cells. Two commonly studied coupling configurations are the so-called easiest-to-implement nearest-neighbor coupling and the most difficult-to-implement global coupling. It has been shown that for any given coupling strength, if the number of cells is large enough, the globally coupled network will eventually synchronize, while the nearest-neighbor coupled network cannot achieve such synchronization under the same condition. This observation naturally poses the following question: for a nearest-neighbor coupled network with a sufficiently large number of cells and with an arbitrary coupling strength, is it possible to achieve synchronization of the network by a small modification of the nearest-neighbor coupling configuration, for example, by adding a small fraction of connection between some different pairs of cells? In this paper we provide a positive answer to this question based on the small-world network models.

2. Preliminaries

We consider a network of N identical cells, linearly coupled through the first state variable of each cell, with each cell being an n-dimensional dynamical subsystem. The state equations of the entire network are

$$\begin{aligned} \dot{x}_{i1} &= f_1(\mathbf{x}_i) + c\sum_{j=1}^{N} a_{ij}x_{j1} \\ \dot{x}_{i2} &= f_2(\mathbf{x}_i) \\ &\vdots \\ \dot{x}_{in} &= f_n(\mathbf{x}_i) \end{aligned} \qquad i = 1, 2, \ldots, N \tag{1}$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathcal{R}^n$ are the state variables of cell i, $f_i(\mathbf{0}) = 0$, $c > 0$ represents the coupling strength, and $\mathbf{A} = (a_{ij})_{N\times N}$ is the coupling matrix.

In this paper, we only consider symmetric and diffusive coupling. In particular, we assume that

(i) $\mathbf{A}$ is a symmetric and irreducible matrix.
(ii) The off-diagonal elements, a_{ij} $(i \neq j)$ of $\mathbf{A}$, are either 1 or 0 (when a connection between cell i and cell j is absent).
(iii) The elements of $\mathbf{A}$ satisfy

$$a_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}\,, \quad i = 1, 2, \ldots, N \tag{2}$$

The above conditions imply that one eigenvalue of $\mathbf{A}$ is zero, with multiplicity 1, and all the other eigenvalues of $\mathbf{A}$ are strictly negative.

Given the dynamics of an isolated cell and the coupling strength, stability of the synchronization state of the network can be characterized by those nonzero eigenvalues of the coupling matrix. A typical result states that the network will synchronize if these eigenvalues are negative enough [Wu & Chua, 1995].

Lemma 1. *Consider network (1). Let λ_1 be the largest nonzero eigenvalue of the coupling matrix $\mathbf{A}$ of the network. The synchronization state of network (1) defined by $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n$ is asymptotically stable, if*

$$\lambda_1 \le -\frac{T}{c} \tag{3}$$

where $c > 0$ is the coupling strength of the network and $T > 0$ is a positive constant such that zero is an exponentially stable point of the n-dimensional system

$$\begin{aligned} \dot{z}_1 &= f_1(\mathbf{z}) - Tz_1 \\ \dot{z}_2 &= f_2(\mathbf{z}) \\ &\vdots \\ \dot{z}_n &= f_n(\mathbf{z}) \end{aligned} \tag{4}$$

Note that system (4) is actually a single cell model with self-feedback $-Tz_1$. Condition (3) means that the entire network will synchronize provided that λ_1 is negative enough, e.g. it is sufficient to be less than $-T/c$, where T is a constant so that the self-feedback term $-Tz_1$ could stabilize an isolated cell.

As mentioned above, two commonly studied coupling configurations are the nearest-neighbor coupling and the global coupling ones. Experimentally, the nearest-neighbor coupling is perhaps the easiest one to implement and, on the contrary, the global coupling is the most expensive one to implement. The nearest-neighbor coupling configuration consists of cells arranged in a ring and coupled to the nearest neighbors. The corresponding coupling matrix is

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix} \tag{5}$$

The eigenvalues of $\mathbf{A}_{nc}$ are

$$\left\{-4\sin^2\left(\frac{k\pi}{N}\right),\ k = 0, 1, \ldots, N-1\right\} \tag{6}$$

Therefore, according to Lemma 1, the nearest-neighbor coupled network will asymptotically synchronize if

$$4\sin^2\left(\frac{\pi}{N}\right) \geq \frac{T}{c} \tag{7}$$

The global coupled configuration means that any two different cells are connected directly. The corresponding coupling matrix is

$$\mathbf{A}_{gc} = \begin{bmatrix} -N+1 & 1 & 1 & \cdots & 1 \\ 1 & -N+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & -N+1 \end{bmatrix} \tag{8}$$

Matrix $\mathbf{A}_{gc}$ has a single eigenvalue at 0 and all the others equal to $-N$. Hence, Lemma 1 implies that this network will asymptotically synchronize if

$$N \geq \frac{T}{c} \tag{9}$$

In summary, for any given coupling strength $c > 0$, the globally coupled network can synchronize as long as the number of cells N is large enough. On the other hand, since $\sin(\pi/N)$ decreases to zero as N increases, relation (7) cannot hold for sufficiently large N. Simulations also show that the nearest-neighbor coupled network cannot synchronize if the number of cells is sufficiently large. Thus, we have seen a trade-off between these two situations.

3. Synchronization in Small-World Networks

Aiming to describe a transition from a regular network to a random network, Watts and Strogatz [1998] introduced an interesting model, now referred to as the small-world (SW) network. The original SW model can be described as follows. Take a one-dimensional lattice of N vertices arranged in a ring with connections between only nearest neighbors. We "rewire" each connection with some probability, p. Rewiring in this context means shifting one end of the connection to a new vertex chosen at random from the whole lattice, with the constraint that no two different vertices can have more than one connection in between, and no vertex can have a connection with itself.

Note, however, that there is a possibility for the SW model to be broken into unconnected clusters. This problem can be circumvented by a slight modification of the SW model, suggested by Newman and Watts [1999], which is referred to as the NW model hereafter. In the NW model, we do not break any connection between any two nearest neighbors. We add with probability p a connection between each other pair of vertices. Likewise, we do not allow a vertex to be coupled to another vertex more than once, or coupling of a vertex with itself. For $p = 0$, it reduces to the originally nearest-neighbor coupled system; for $p = 1$, it becomes a globally coupled system.

In this paper, we are interested in the NW model with $0 < p < 1$.

From a coupling-matrix point of view, network (1) with small-world connections amount to that, in the nearest-neighbor coupling matrix $\mathbf{A}_{nc}$, if $a_{ij} = 0$, we set $a_{ij} = a_{ji} = 1$ with probability p. Then, we recompute the diagonal elements according to formula (2). We denote the new small-world coupling matrix by $\mathbf{A}_{ns}(p, N)$ and let $\lambda_{1ns}(p, N)$ be its largest nonzero eigenvalue. According to Lemma 1, if

$$\lambda_{1ns}(p, N) \leq -\frac{T}{c} \tag{10}$$

then the corresponding network with small-world connections will synchronize.

Figures 1 and 2 show the numerical values of $\lambda_{1ns}(p, N)$ as a function of the probability p and the number of cells N. In these figures, for each pair of values of p and N, $\lambda_{1ns}(p, N)$ is obtained by averaging the results of 20 runs. It can be seen that

(i) For any given value of N, $\lambda_{1ns}(p, N)$ decreases to $-N$ as p increases from 0 to 1.
(ii) For any given value of $p \in (0, 1]$, $\lambda_{1ns}(p, N)$ decreases to $-\infty$ as N increases to $+\infty$.

The above results imply that, for any given coupling strength $c > 0$, we have

(i) For any given $N > T/c$, there exists a critical value $\overline{p}$ so that if $\overline{p} \leq p \leq 1$, then the small-world connected network will synchronize.
(ii) For any given $p \in (0, 1]$, there exists a critical value $\overline{N}$ so that if $N \geq \overline{N}$, then the small-world connected network will synchronize.

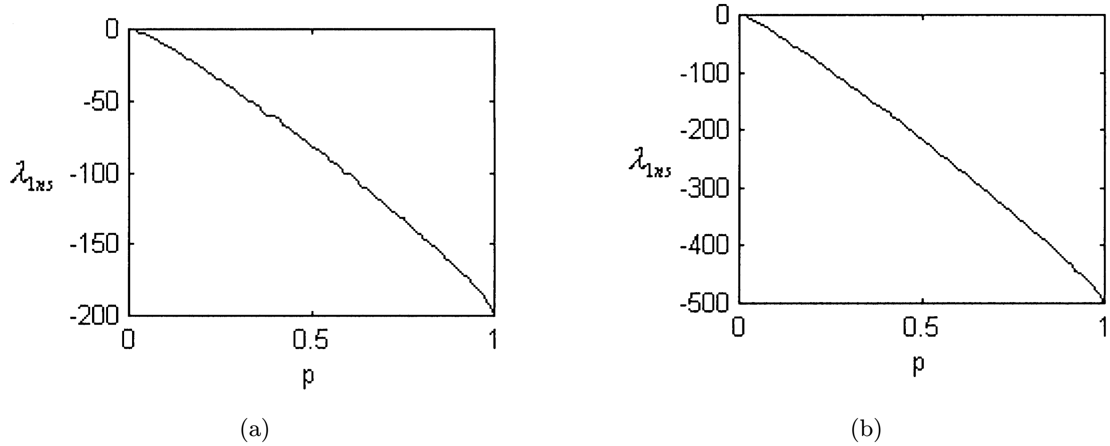

Fig. 1. Numerical values of $\lambda_{1ns}(p, N)$ as a function of the probability p: (a) $N = 200$; (b) $N = 500$.

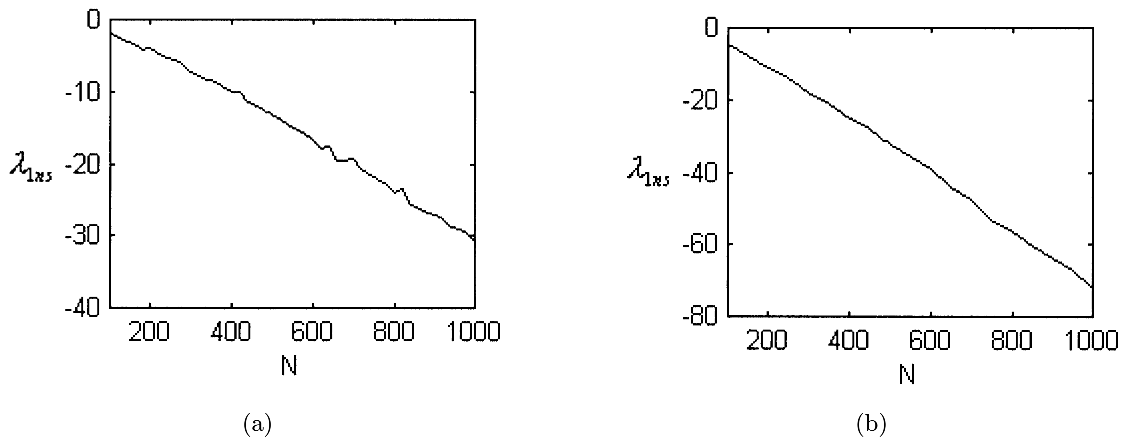

Fig. 2. Numerical values of $\lambda_{1ns}(p, N)$ as a function of the number of cells N: (a) $p = 0.05$; (b) $p = 0.1$.

4. Synchronization in a Network of Small-World Coupled Chua's Circuits

As an example, we now study synchronization in a network of small-world connected Chua's circuits. In the dimensionless form, a single Chua's circuit is described by [Chua *et al.*, 1993]:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_2 - x_1 + f(x_1)) \\ x_1 - x_2 + x_3 \\ -\beta x_2 - \gamma x_3 \end{pmatrix} \tag{11}$$

where $f(\cdot)$ is a piecewise-linear function,

$$f(x_1) = \begin{cases} -bx_1 - a + b & x_1 > 1 \\ -ax_1 & |x_1| \le 1 \\ -bx_1 + a - b & x_1 < -1 \end{cases} \tag{12}$$

in which $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $a < b < 0$. The state equations of the entire network are

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} \alpha(x_{i2} - x_{i1} + f(x_{i1})) + c \sum_{j=1}^{N} a_{ij} x_{j1} \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} - \gamma x_{i3} \end{pmatrix}, \quad i = 1, 2, \ldots, N. \tag{13}$$

For this network to synchronize, according to Lemma 1, we may take $T = -\alpha$. In simulations, the system parameters are chosen to be

$$\begin{gathered} \alpha = 10.0000, \quad \beta = 15.0000, \quad \gamma = 0.0385, \\ a = -1.2700, \quad b = -0.6800, \quad c = 1. \end{gathered} \tag{14}$$

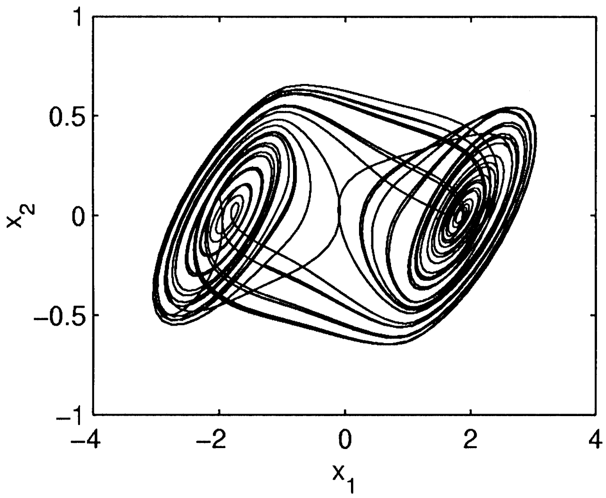

Fig. 3. Chaotic attractor of Chua's circuit (11), with parameters given in (14).

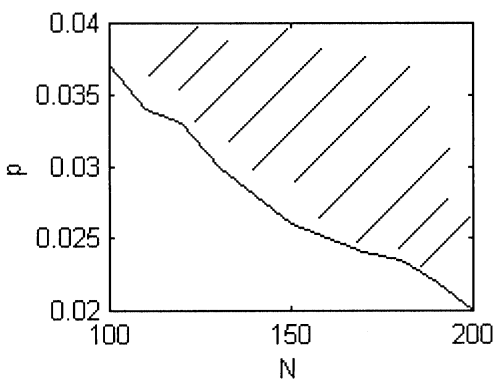

Fig. 4. Values of p and N achieving synchronization in the small-world network of Chua's circuits.

For this set of parameters, Chua's circuit (11) has a chaotic attractor, as shown in Fig. 3. The nearest-neighbor coupled Chua's network cannot synchronize for $N > 6$. According to Lemma 1, the small-world network will synchronize if

$$\lambda_{1ns}(p, N) \le \frac{a}{c} = -1.27 \tag{15}$$

Figure 4 shows the values of p and N which can achieve network synchronization. For example, for $N = 100$, 150 and 200, synchronization of the small-world connected network can be achieved, for $p > 0.0366$, $p > 0.0257$ and $p > 0.021$, respectively.

5. Conclusions

Starting with a nearest-neighbor coupled dynamical network, we can construct a small-world dynamical network by adding with probability p a connection between each of the other pair of cells. We found that, for any given coupling strength and a sufficiently large number of cells, synchronization in a network of linearly small-world coupled continuous-time dynamical systems can be achieved with a small value of p. In other words, the ability of achieving synchronization in an originally nearest-neighbor coupled system can be greatly enhanced by simply adding a small fraction of new connection, revealing an advantage of small-world network for chaos synchronization.

Acknowledgments

This work was supported by the UK Engineering and Physical Sciences Research Council through the EPSRC grant number GR/M97923/01, the Hong Kong GRC CERG Grant 9040565 and the National Natural Science Foundation of China through Grant Number 60174005.

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